Dynamics (AE1130-II) summary

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Table of contents

Introduction

Dynamics is known for being one of the harshest subjects in the entire BSc, however, the theories and the concepts are in fact pretty basic and intuitive since dynamics is a branch of classical mechanics. Therefore, it is easy to visualize and hence understand the concepts. The hard part of dynamics is solving problems. In order to master problem-solving skills, a person must understand the theory very well with the small details since the path that leads to the solution to the problems is usually going through these small details. Then, the student should have the right thought method to be able to combine different concepts and finally solve many problems. In this summary, I will explain the theory with details and solve some hard questions to show the reader how to think while solving problems.

Particle dynamics

We divide our dynamics course into two broad categories. Particle and rigid body dynamics.

In particle dynamics, we treat our objects as a single point, a particle therefore their shape does not matter, we treat all the shapes as a particle. For certain situations, this is a valid assumption, but not always, thats why we have rigid body dynamics and further but we won't go further than rigid body dynamics.

The situations where we can make particle assumptions are usually the situations where our objects is much smaller than the environment it is interacting with, however, when to make this assumption will be clearer when we solve questions. It can be said that the distinction can be made by intuition.

1.2: Rectilinear motion

Rectilinear motion refers to motion where the motion cannot be described by a single function, instead, multiple functions are needed to describe the motion. An example to this situation would be a in real life car trip.

Before starting, basic relationships should be known. They are:

- $v = \frac{ds}{dt}$
- $a = \frac{dv}{dt}$

Where $\boldsymbol{s}$ stands for displacement, $\mathbf{t}$ stands for time, $\boldsymbol{v}$ for velocity and $\boldsymbol{a}$ for acceleration. Remember that they are vectors.

However, we could also write a relationship that depends on displacement instead of time, since for certain situations we are more interested in how velocity and acceleration change depending on the displacement -such as simple harmonic motion.

These relationships are:

- $a = v\left(\frac{dv}{ds}\right)$ hence, acceleration at a specific position(displacement) can be found in velocity-displacement graph by multiplying the velocity by the gradient of this graph.
- $\frac{1}{2}(v_1^2 - v_0^2) = \int_{s_0}^{s_1} a\, ds$ by rearranging this formula the final velocity could be found $v_1 = \sqrt{\int_{s_0}^{s_1} a\, ds + 2v_0^2}$ hence, we would need the acceleration-displacement graph as well as the initial velocity of our particle to find the final velocity

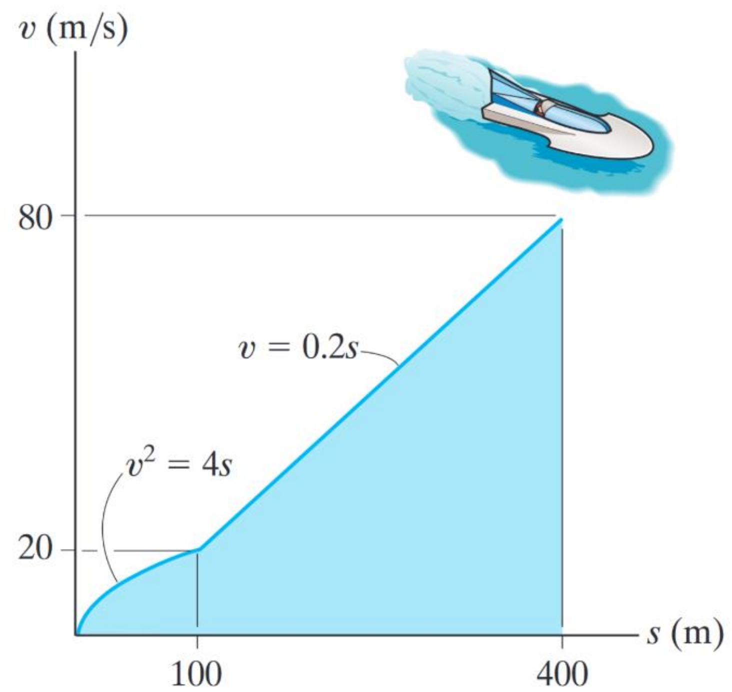
There is not much about rectilinear motion, the formulas given above should be applied directly. Let’s show some examples.

Problem solving:

12–23. If the effects of atmospheric resistance are accounted for, a freely falling body has an acceleration defined by the equation $a = 9.81[1 - v^2(10^{-4})]$ m/s^2, where v is in m/s and the positive direction is downward. If the body is released from rest at a *very high altitude*, determine (a) the velocity when $t = 5$ s, and (b) the body's terminal or maximum attainable velocity (as $t \to \infty$).

1) First thing we should always do is determine a coordinate system. For this question, we choose positive direction to be downwards and negative direction upwards.
2) Second thing that we should look for is the small hints in the text, for this question, it is that we started from rest and at a very high altitude
3) We are asked to find the velocity at 5 seconds after the drop and we know that $a = \frac{dv}{dt}$ hence the given expression could be written as $\frac{dv}{dt} = 9.81(1 - v^2 \cdot 10^{-4})$ and hence, the differential equation can be solved with basic calculus knowledge
4) The final equation is found to be $v = \frac{100(e^{0.1962t}-1)}{e^{0.1962t}+1}$
5) Thus, plugging in t = 5 gives v = $45.5ms^{-1}$ and t = ∞ gives v = $100ms^{-1}$

12–67. The boat travels along a straight line with the speed described by the graph. Construct the s–t and a–s graphs. Also, determine the time required for the boat to travel a distance $s = 400$ m if $s = 0$ when $t = 0$.



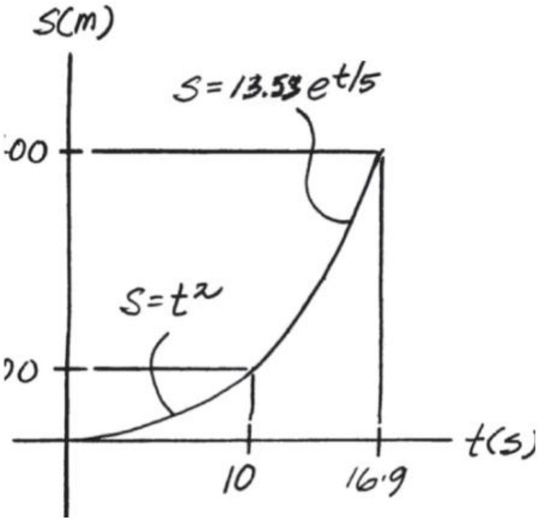
1) This question is in fact straightforward however, care must be put into the integral boundaries. Let us start with the first part of the question where we find the s-t graph. In this question we do not need to define a coordinate system since everything is given on a graph.
2) The motion is divided into two functions. Normally, rectilinear motion means motion that has so many functions however, for questions to be solved by hand easily, we are usually given 2-3 functions.
3) $v = \pm 2\sqrt{s}$ however, we pick the positive one because we can see on the graph that as the distance increases (positively) the velocity increases in the positive direction as well. Since we want $s - t$ relationship, we will use the fact that $v = \frac{ds}{dt}$, thus

   $s^{-\frac{1}{2}}ds = 2dt$

4) Solve the integral, $\int_0^{100} s^{-\frac{1}{2}} ds = \int_0^t 2dt \rightarrow s = t^2$ since we know that displacement is zero initially and thus $t = 10s$ when the function changes

5) Let us do the second function then $\frac{ds}{dt} = 0.2s \therefore \frac{ds}{s} = 0.2dt \int_{100}^{400} \frac{ds}{2} = \int_{10}^{t} 0.2dt$ thus $\ln 4 = 0.2t - 2 \therefore t = 16.9$ when s = 400m. We also figured out that the function is $s = \frac{100}{e^2} e^{0.2t}$

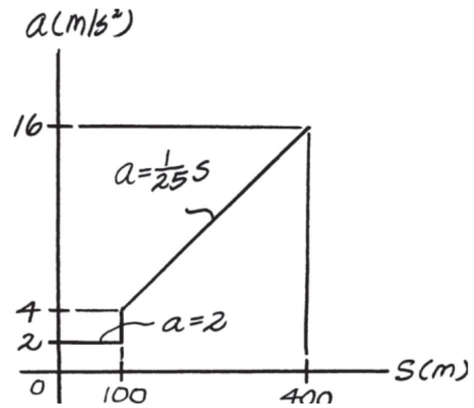
6) Combining all these information, we can sketch the s-t graph:



7) Now a-s graph can be found. Let's remind ourselves that $a = v\frac{dv}{ds}$

8) Start with the first function. $\frac{d}{ds}(v^2) = 2v\frac{dv}{ds} = 4 \rightarrow v\frac{dv}{ds} = a = 2$ hence it is a constant function

9) The second function can be written as $\frac{dv}{ds} = 0.2 \rightarrow v\frac{dv}{ds} = 0.2^2 s = a$ hence:



1.3: Curvilinear motion

Curvilinear motion is the motion that occurs when a particle is moving along a curved path. Do not confuse this with circular motion.

The displacement-velocity-acceleration relationships are the same as before.

Our universe has 3 dimensions, x-y-z, however, in many cases, we are able to reduce it to 1 or 2 dimensions when we are analyzing and solving the questions since usually there is nothing important going on in the third dimension.

For a 3 dimensional system, we could find displacement from the origin by:

$s = \sqrt{y^2 + z^2 + x^2}$

The same logic can be applied to velocity and acceleration calculations.

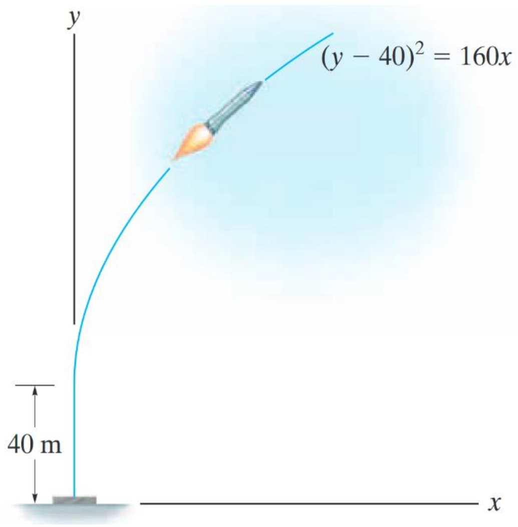
The last thing we should consider regarding the curvilinear motion is projectile motion. The projectile motion usually has 2 dimensions and x-y coordinate system is preferred. The basic idea behind it is what we learned in Chapter 1.2 however, now, in order to solve questions we have to combine information from 2 dimensions. There is a set of formulas derived from the basics that **only but only works when there is a constant acceleration** we call them SUVAT equations.

- $v = u + at$
- $s = \frac{(v+u)}{2}t$
- $s = ut + \frac{at^2}{2}$
- $v^2 = u^2 + 2as$

Where v is the final velocity, u is the initial velocity, t is time, s is displacement and a is acceleration.

Problem solving:

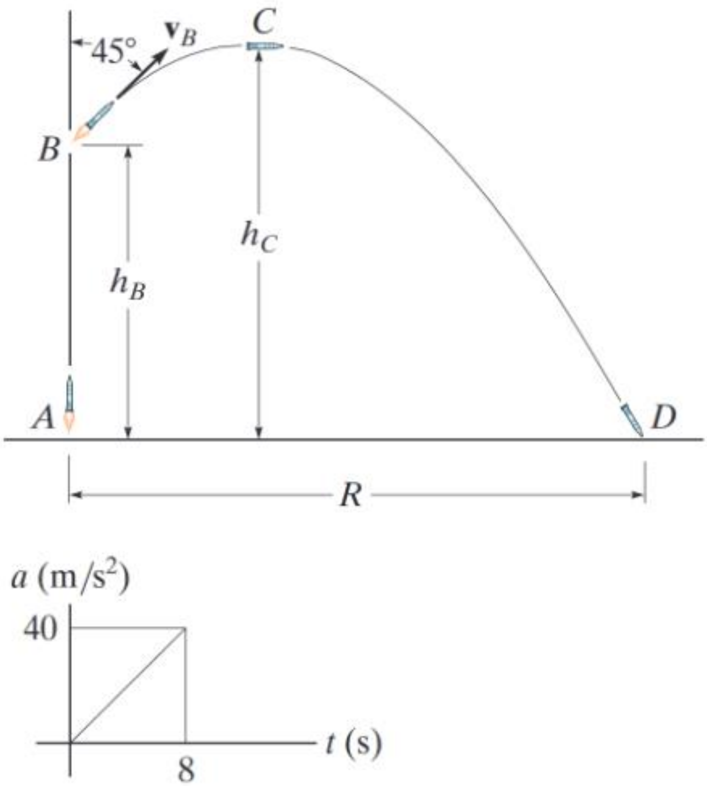
12–73. When a rocket reaches an altitude of 40 m it begins to travel along the parabolic path $(y - 40)^2 = 160x$, where the coordinates are measured in meters. If the component of velocity in the vertical direction is constant at $v_y = 180$ m/s, determine the magnitudes of the rocket's velocity and acceleration when it reaches an altitude of 80 m.



1) In this question we do not need to define a coordinate system as it is already given. One of the key information that we should notice is that the vertical velocity is constant and hence the vertical acceleration is zero. Additionally, we should notice that we are asked the total velocity and acceleration when the rocket reaches 80m, hence, we should use $\sqrt{v_x^2 + v_y^2}$ and the total acceleration will only be the horizontal acceleration.
2) Let's begin with writing the equation: $y^2 - 80y + 1600 = 160x$
3) We know that $\frac{dx}{dt} = v_x$ and similarly for y. Hence, let's differentiate the expression with respect to time, notice that we use implicit differentiation $2y\frac{dy}{dt} - 80\frac{dy}{dt} = 160\frac{dx}{dt}$ hence, if we plug in $y = 80$ and $\frac{dy}{dt} = 180$ we find that $\frac{dx}{dt} = 90ms^{-1}$
4) Thus, the total velocity is $\sqrt{90^2 + 180^2} = 201ms^{-1}$

5) To find the horizontal acceleration we should differentiate the entire expression with respect to time again: $2\left(\frac{dy}{dt}\right)^2 + 2y\frac{d^2y}{dt^2} - 80\frac{d^2y}{dt^2} = 160\frac{d^2x}{dt^2}$ since $\frac{d^2y}{dt^2} = 0$ we find $\frac{d^2x}{dt^2} = 405ms^{-2}$ which is also the total acceleration

***12–100.** The missile at A takes off from rest and rises vertically to B, where its fuel runs out in 8 s. If the acceleration varies with time as shown, determine the missile's height h_B and speed v_B. If by internal controls the missile is then suddenly pointed 45° as shown, and allowed to travel in free flight, determine the maximum height attained, h_C, and the range R to where it crashes at D.



1) First thing we should do is defining a coordinate system. Let us define the origin as where the rocket took off and we choose x-y coordinate system.
2) Important information that we have are that rocket was initially at rest, when it stops accelerating (upwards) it changes its motion and the first motions acceleration is $a = 5t$ by reading the graph and after changing its direction, it is free falling.
3) To find h_b and v_b we will use $a = 5t$
4) Since the initial altitude and velocity are 0 we can just integrate the acceleration. Hence: $v_b = \frac{5}{2}t^2$ and $h_b = \frac{5}{6}t^3$ and if we plug in $t = 8s$ we get $v_b = 160ms^{-1}$ and $h_b = 427m$
5) Now we can move on to the second part of the question. We know that our rocket is tilted 45 degrees hence we can find the initial velocities after the direction change. To find the max altitude, we should write the equation for vertical displacement $h_y = h_b + v_b \cos(45)t - \frac{9.81t^2}{2}$ and to find the maxima of this function, we could take the first derivative and find at what value of t that equation becomes zero, hence: $v_b \cos(45) - 9.81t = 0 \ \rightarrow t = 11.5s$ thus $h_y = 1079m$ at maximum altitude
6) Moving on to the last sub-question, the maximum range is achieved when the rocket falls on the ground hence, when its altitude is 0. Thus, $0 = 427 + v_b \cos(45)\, t - \frac{9.81t^2}{2} \rightarrow t = 26s$
7) The horizontal displacement can be written as $h_x = v_b \sin(45)\, t$ since there is no acceleration in the x direction. Plugging in t = 26s gives us $h_x = 2983m$

2.1: Normal&Tangential and polar coordinate systems

This lecture is particularly important as it is introducing the normal&tangential (n-t) and polar polar coordinate systems (c.s). Besides them, we have the x-y c.s which we are familiar with. Choosing the right c.s is essential for solving questions in dynamics in the easiest way possible. Technically all questions could be solved with all c.s however, it would require -much- more effort with some than others. Hence, when I mean the right c.s I mean the easiest c.s for that particiular question.

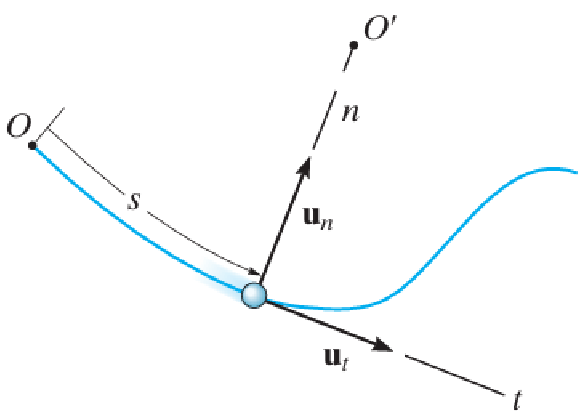
Previously we saw how we use the x-y c.s therefore I won't talk about it as it should be clear how and where it is used.

Today we will look at c.s we use when there is a curvilinear motion.

Let's begin with n-t c.s:

n-t:

n-t c.s has two axis where the "tangent" axis is tangent to the path, hence, the velocity vector is only this axis. Normal axis is normal to the path hence, also to the tangent axis therefore acceleration vector may be on both axis. The origin of this c.s is where our particle/point is and in this c.s the c.s is changing instantaneously. This means that the c.s is not static as it is in x-y c.s however, the c.s is changing as the particle propagates. This is because the tangent and hence the normal to the path is changing. When solving questions, always keep it in mind that this c.s is only valid instantaneously.



Keep in mind that the positive normal axis is always pointing at the center of the curve. The distance between the center and the particle is called the radius of curvature and is denoted by ρ. If the paths equation is given in terms of $y(x)$ we can define $\rho = \frac{\left(1+\left(\frac{dy}{dx}\right)^2\right)^{\frac{3}{2}}}{\left|\frac{d^2y}{dx^2}\right|}$

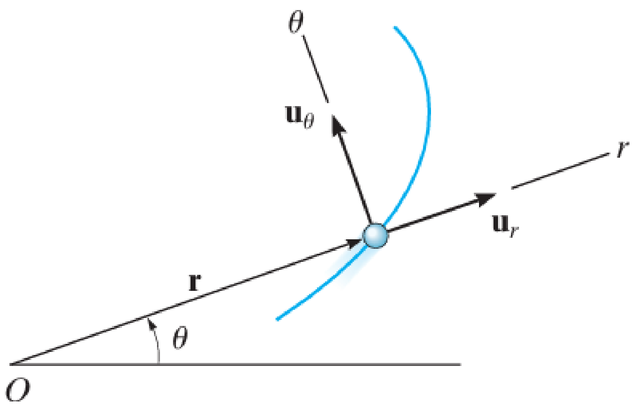
The velocity of the particle is v and only direction is only on the tangential axis.

Acceleration is defined as $a = \frac{dv}{dt}\boldsymbol{u_t} + \frac{v^2}{\rho}\boldsymbol{u_n}$ which makes sense since the acceleration on the tangential axis is change in speed of the particle and the normal acceleration is caused by the

We will mostly prefer this c.s where we have a curved path and we are not given an angular velocity. Also we would mostly prefer it when we are working with systems that have normal and tangential forces such as friction and normal reaction force.

Polar coordinate system:

Polar c.s is also used for curved paths however, we will prefer it when we are given information that is related to angular velocity or angular acceleration. This c.s has two axis, radial and axial axis. They are denoted by $\boldsymbol{u_r}$ and $\boldsymbol{u_\theta}$ respectively. This c.s is a static one, hence, it does not move with the particle unlike n-t c.s. The origin of the c.s is usually defined to be the point where the angular velocity is created.



Do not confuse the origin of the c.s with the center of the curve. In this c.s, the motion/path is achieved by moving on the radial axis as well as the angular velocity.

$v_r = \frac{dr}{dt}, v_\theta = r\frac{d\theta}{dt}$

$$a_r = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 \text{ }, a_\theta = r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}$$

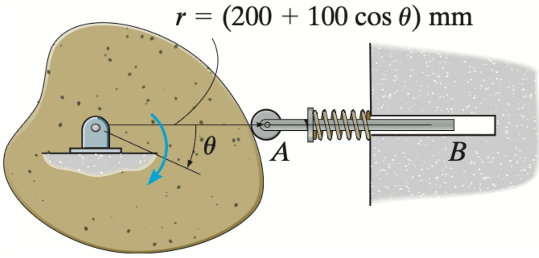
Now, it should be clearer why we prefer this c.s when we are given an angular velocity/acceleration.

Problem solving:

***12–112.** A particle moves along the curve $y = \sin x$ with a constant speed $v = 2$ m/s. Determine the normal and tangential components of its velocity and acceleration at any instant.

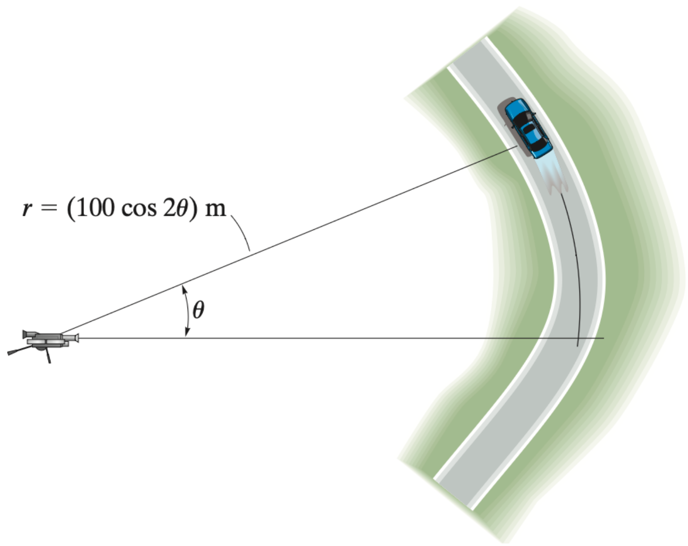
1) We know that our path is $y = \sin(x)$ hence a curved path and we are not given any angular value thus, it is clear that n-t c.s is the best option here.
2) We know that we only have velocity in the tangential and is constant hence at all times $v_t = 2ms^{-1}$
3) We also know that there are two acceleration contributions, tangential and normal. Tangential acceleration is the change in the speed of the particle and it is zero since we are moving at constant speed. Normal acceleration is found by $a_n = \frac{v^2}{\rho}$ however, we do not know the radius of curvature. By definition $\rho = \frac{(1+\cos^2(x))^{\frac{3}{2}}}{\sin(x)}$
4) Thus, $a = \frac{4\sin(x)}{(1+\cos^2(x))^{\frac{3}{2}}} ms^{-2}$

***12–184.** At the instant $\theta = 30°$, the cam rotates with a clockwise angular velocity of $\dot{\theta} = 5$ rad/s and, angular acceleration of $\ddot{\theta} = 6$ rad/s^2. Determine the magnitudes of the velocity and acceleration of the follower rod AB at this instant. The surface of the cam has a shape of a limaçon defined by $r = (200 + 100\cos\theta)$ mm.

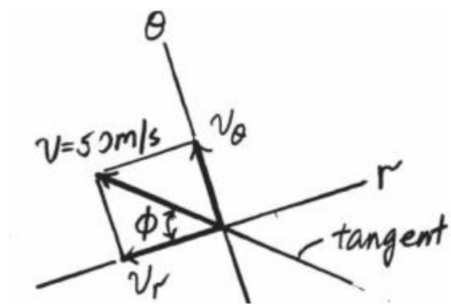


1) In this question, we have a curved path and we are given angular values therefore it hints us to use the polar c.s. We could say that the pin is our origin.
2) Now, we will meet with one of the most important concepts in dynamics. Constraints. Constraints basically block or lead the objects and particles in certain directions. We have general formulas however, for each case, we must consider their constraints. In this question, we can see that our rod is constrained to move in the angular direction hence it can only move in the radial direction. This means that we will ignore the velocity and acceleration in the angular direction as it is constrained and hence there is no movement there. We will only calculate the radial velocity and acceleration.
3) The harsh part was to notice the constraint, now we only have to differentiate. By radial velocity formula, $\frac{dr}{dt} = -100\sin(\theta)\frac{d\theta}{dt} \rightarrow -250mms^{-1} = -0.25ms^{-1}$
4) For acceleration we need $a_r = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 \rightarrow -100\cos(\theta)\left(\frac{d\theta}{dt}\right)^2 -$ $100\sin(\theta)\frac{d^2\theta}{dt^2} - 6^2(200 + 100\cos(\theta)) = -9.3ms^{-2}$
5) Notice how we used implicit differentiation

***12–192.** When $\theta = 15°$, the car has a speed of 50 m/s which is increasing at 6 m/s^2. Determine the angular velocity of the camera tracking the car at this instant.



1) We are asked to find the angular velocity hence it hints us to use polar c.s. Draw first



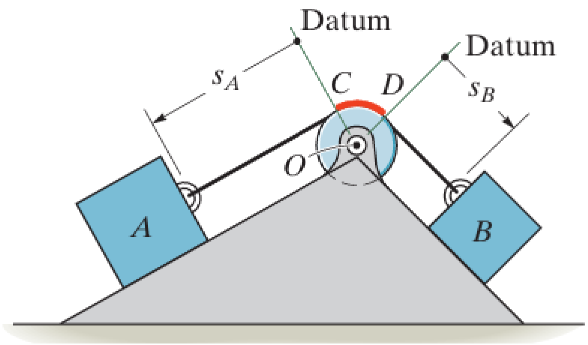
2) Now, we need to pick the key information such as $50ms^{-1}$ <u>speed</u>
3) Since we are given the speed, we need to figure out both the radial and angular velocity components (the axis), we are given the length "r" we can write the equation in terms of angular velocity and solve for it.
4) $v_r = \frac{dr}{dt} = -200\sin(2\theta)\frac{d\theta}{dt}$ and $v_\theta = 100\cos(2\theta)\frac{d\theta}{dt}$
5) Hence $50 = \frac{d\theta}{dt}\sqrt{(200\sin(2\theta))^2 + (100\cos(2\theta))^2} \rightarrow \frac{d\theta}{dt} = 0.378\frac{rad}{s}$

2.2: Dependent and relative motion

These two concepts are vastly important for upcoming lectures and in pretty much all dynamics questions, especially in rigid body dynamics.

Let's begin with dependent motion.

Dependent motion: Dependent motion arises when the motion of a particle is dependent on another particle. This is usually done by an inextensible chord. It is important that the chord is inextensible so that the motion can be transferred directly, otherwise, the chord would extend and then transfer the motion to the other particle.



Let us look at how to analyze such systems and I will give important hints.

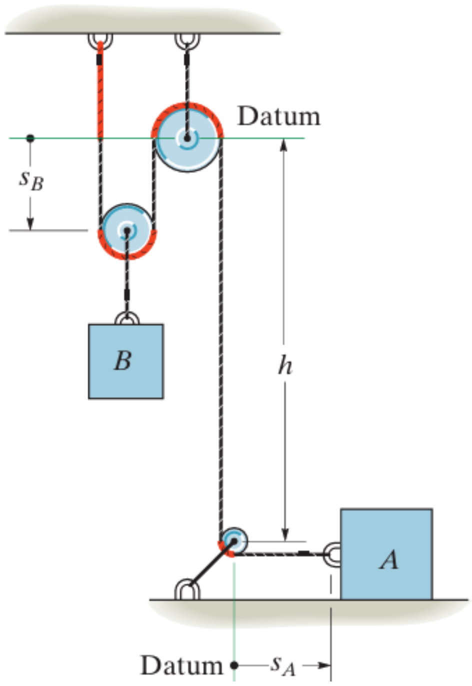
1) First thing we should determine a datum point, how to choose one will be told when I give hints
2) From this datum point, we take a distance to our particle
3) We do that in a way that we can sum these distances to equate it to the length of the chord
4) Then we take the derivative with respect to time, since the length of the chord is constant it is zero and the derivatives of the distances give us the speed at the particles/that point is moving
5) Thus, we can relate them to each other and see how one depends on the other

Now, I will give important tips that will be useful when solving difficult questions.

-The first thing you should do is to determine how many chords there are because, for each chord, we could write a separate equation that allows us to solve for more dependent values

-Another important step is to write the datum. We will write all our distances starting from the datum and all the distances should be perpendicular to that datum line. We must choose our datum so that it is at a static point or the datum is moving with the entire system. This needs to be satisfied otherwise the datum would be moving and hence, the found velocity would be relative, not absolute

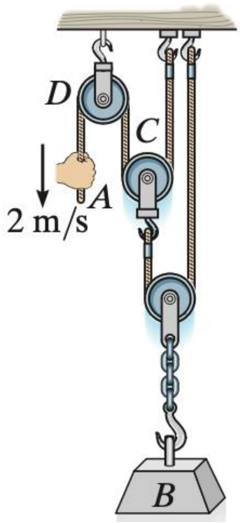
-In the system, there may be constant distances that do not change. When we cannot write our equation, we may use them to write the relationship and when differentiated, they disappear as they are constant lengths. Such lengths are indicated in red in the example below



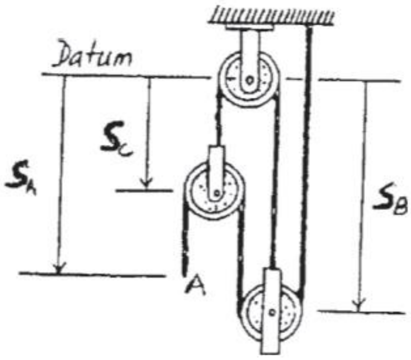
-When needed, you could write more than one datum point however, the requirements for the datum must still hold. We would write multiple datums when there is motion in more than one direction (since each datum can only be used for 1 direction motion, which is perpendicular to the datum line)

Problem solving:

***12–200.** If the end of the cable at A is pulled down with a speed of 2 m/s, determine the speed at which block B rises.



1) We should firstly find out how many ropes there are. We can clearly see that there are two
2) Now, we should determine a datum and write the appropriate distances to write the length. Notice that the length between the connection between the second and the third pulley is constant



3) Write the equation for the first rope: $s_A + s_c + (s_c + h) = l_1$ hence $v_c = -\frac{v_a}{2}$
4) Write the second equation: $s_b + (s_b - (s_c + h)) \rightarrow v_b = \frac{v_c}{2} \therefore v_b = \frac{-v_a}{4}$
5) Hence $v_b = -0.5ms^{-1}$

Relative motion:

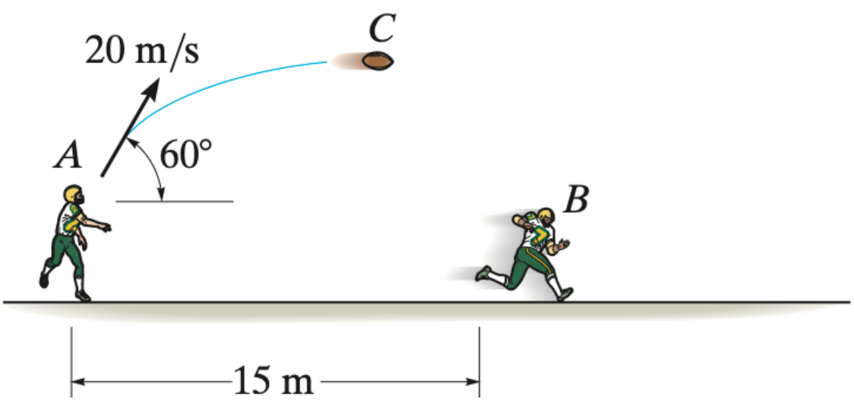
As the name suggests, relative motion describes the motion of two particles relative to one another. This concept is extremely important in dynamics especially when we will start with rigid body dynamics, our formulas will be derived from this concept.

The distance between particle B and A relative to B is written as $r_{b/a} = r_b - r_A$ we can similarly write the velocity and acceleration. In this equation r_b and r_A

This concept should be fairly easy to internalize. If not, you could think of it as how the cars that are going in the same direction as you appear to go slower and vice versa.

Problem solving:

***12–232.** At a given instant the football player at *A* throws a football *C* with a velocity of 20 m/s in the direction shown. Determine the constant speed at which the player at *B* must run so that he can catch the football at the same elevation at which it was thrown. Also calculate the relative velocity and relative acceleration of the football with respect to *B* at the instant the catch is made. Player *B* is 15 m away from *A* when *A* starts to throw the football.



1) Firstly we need to determine a c.s. In this question since there is no curved path x-y c.s seems the most appropriate. Let us take the origin to be where the person A stands. I prefer to take the positive x axis to be where the ball is thrown and positive y direction to be upwards.
2) The easiest way to find the required speed of the person B is by considering the relative motion between the ball and person B which I will refer as B.
3) First, we need to find the time it takes to fall on the ground hence B can catch it. Thus $0 = 20\sin(60)\,t - \frac{9.81}{2}t^2 \rightarrow t = \frac{40\sin(60)}{9.81} = 3.53s$
4) $r_{B/C} = r_B - r_C = 0$ for B to catch the ball and $v_{B/C} = v_B - 20\cos{(60)}$
5) Thus $r_{B/C} = 15 + 3.53(v_B - 20\cos(60)) = 0 \ \rightarrow v_B = 5.75ms^{-1}$
6) The relative velocity can be calculated by finding the total relative velocity hence both in x and y axis velocities are needed. In x axis: $v_{B/C} = 5.75 - 20\cos(60) = -4.25ms^{-1}$ and in y axis: $20\sin(60) = 17.3ms^{-1}$ since energy is conserved
7) Thus $\sqrt{4.25^2 + 17.3^2} = 17.8ms^{-1}$ and when we are asked such questions, we need to specify the angle as well. We do that by taking the tangent of the velocity vector hence $\tan(\theta) = \frac{17.3}{-4.25} \rightarrow \theta = 76.2°$ and we show the direction. Since the x-direction of the ball is to the right and y-direction is downwards, we write it as

8) It is obvious that there is no acceleration in the x-direction neither for the ball or B and the acceleration in y-direction only exist for the ball and it is clearly the Earth's gravitational acceleration $9.81ms^{-2}$ downwards

2.3: FBD, KD and EOM

This chapter is also one of the most basics and important part of dynamics. It should be familiar with you from high school, however, we must make it sure that we draw the Free Body Diagram, Kinetic Diagram and Equations of Motion correctly to be able to solve even complex questions with ease. Because, when these three points are written correctly, majority of the question is in fact solved.

- FBD makes us to figure out which forces are acting on the body
- Kinetic diagram tells us in which direction our body is moving at
- Equations of motion writes it down all that information mathematically so that we could solve the equation and get a numerical value

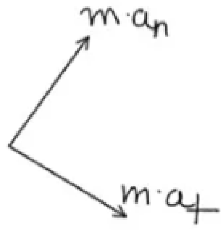
Just by doing these 3 steps, you could pass your dynamics exam without further solving the question.

Before starting how to perform these 3 steps, we should discuss the Newton's laws. Newton has 3 laws.

1) If the sum of the forces acting on an object is zero, the body is either moving at constant speed or is stationary
2) Sum of the forces acting on a body is equal to mass times acceleration of the body hence $\sum F = ma$
3) Every force has a counterforce acting on the exact opposite direction with the same magnitude. Pay attention that the force and the counterforce never acts on the same body

Now we can talk about how to perform the 3 steps.

1) First thing we should do is to determine a c.s. In previous chapters I told you how you should choose it therefore I will not repeat it again.
2) Second of all, we should draw a FBD. A box or a sketch of the shape of the body and then we could draw the forces acting on the body. Please pay attention to the length of the arrows
3) Kinetic diagram may be the only unfamiliar diagram. Kinetic diagram shows in which direction the body is accelerated at hence, the direction of the sum of the forces. Note that the KD's c.s is based on the c.s we chose. An example is:



4) EOM is simply writing sum of the forces acting on the body. Be careful at the direction the forces act as well as possible negative signs

Now I will talk about 2 important forces. Weight and friction.

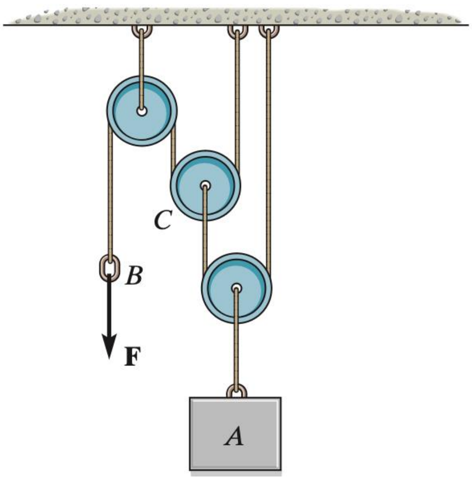
Weight: It is simply the mass of the object times the gravitational acceleration of the planet. On Earth, we usually take it as $g = 9.81ms^{-2}$ and weight is always acting towards the center of the planet hence, on the FBD it will always be downwards. For questions that are in space, you may have to use Newton's gravitational law.

Friction: Now, this is quite important because so many students are mixing up the type of the friction force and its properties. There are two types of friction forces. Static and kinetic friction. Static friction occurs when there is no relative motion between the two surfaces. It is so important to keep in mind that friction occurs when there is relative motion. Static friction may vary from zero to a maximum value. If we are told that the static friction is at its maximum value, then we could use the formula $F_{static} = \mu N$ where N is the normal force and μ is the static friction coefficient. Otherwise we should just write F_{static} in our EOM as the formula with coefficient is only true for the maximum static friction.

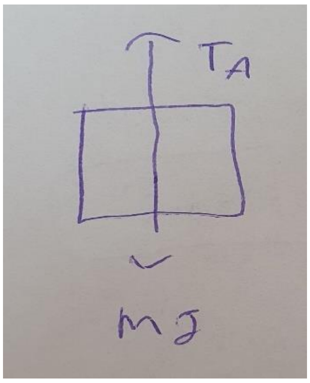
Kinetic friction does not have a similar property. It has no maximum value however, we use the formula with coefficient for kinetic friction. The difference is that the kinetic friction coefficient is smaller than the static friction coefficient. Kinetic friction is hence, always smaller than static friction. Later, when we work with rigid bodies, that information will be extremely important.

Problem solving:

***13–24.** If the supplied force $F = 150$ N, determine the velocity of the 50-kg block A when it has risen 3 m, starting from rest.



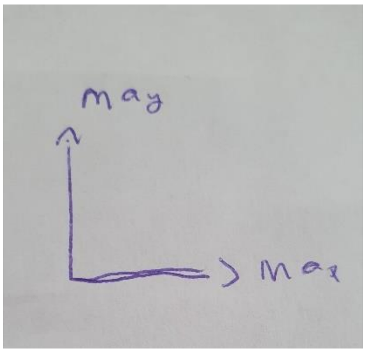
1) First of all we choose our c.s. In this question, it is clear that x-y c.s is the easiest one to work with. There is no motion in the x-direction as there is no force acting in the x-axes. I choose the upwards to be positive y-axes.
2) Second thing to do is draw the FBD.



3) When we track down the forces chord and forces acting on the middle pulley:

4) Since the pulley is in equilibrium, we can conclude that $T_A = 2 \cdot 2 \cdot 150 = 600N$

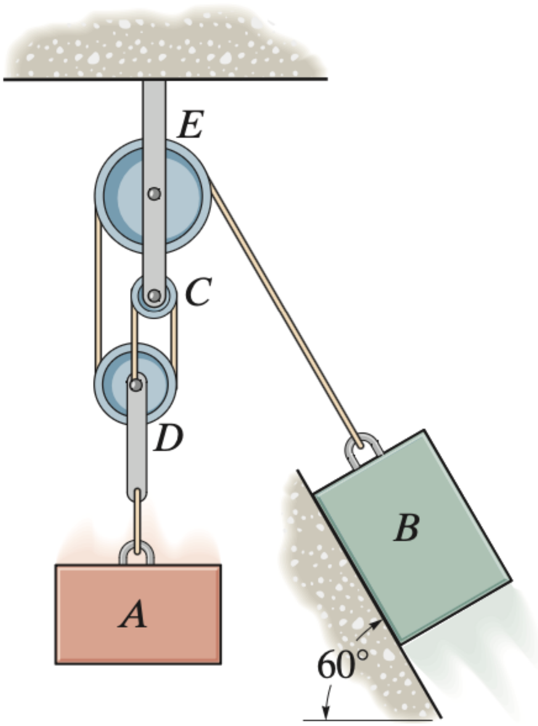
5) Now we need the KD. We know that the body is accelerating upwards hence



6) Now we can write the EOM for block A: $\sum F_y = T_A - mg = ma \ \rightarrow a = 2.19ms^{-2}$

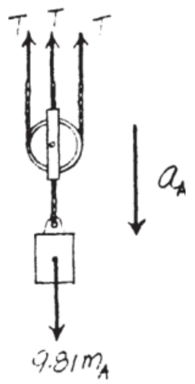
7) We need to find the time it took for the block to rise 3m and we know that our acceleration is constant ∴ we can use SUVAT: $3 = \frac{2.19\, t^2}{2} \rightarrow t = 1.66s$ ∴ $v = at \ \rightarrow$ $v = 3.62ms^{-1}$

***13–20.** Determine the required mass of block *A* so that when it is released from rest it moves the 5-kg block *B* 0.75 m up along the smooth inclined plane in $t = 2$ s. Neglect the mass of the pulleys and cords.

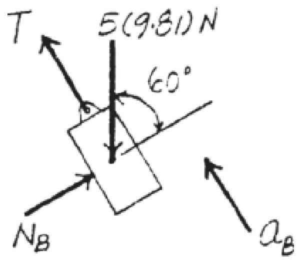


1) In this question it seems like x-y c.s is the easiest one to work with however, this time we will use two different c.s one for each block. I choose upward to be positive y-axes and right to be positive x-axes. For block A, I will use the typical x-y c.s however, for block B, I will tilt the c.s so that the x-axes would be aligned with the direction of the motion of block B. We usually do this trick to make further EOM calculations easier. We will always prefer to have our motion in the same direction as our c.s axis.
2) Now, let us draw the FBD for the blocks:

Block A:

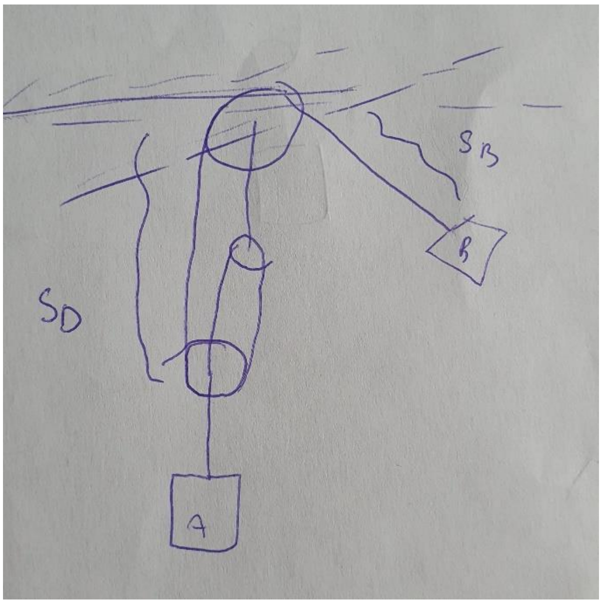


Block B:



3) Instead of drawing the KD, in this question I showed the direction of acceleration by writing "a" and showing the direction by an arrow. Instead of drawing the KD this could be done as well since there is no motion in other axes directions.
4) Now we can write the EOM. For block A: $\sum F_{A_y} = 9.81m_A - 3T = m_A a_A$ and $\sum F_{B_x} = T - 5 \cdot 9.81 \sin(60) = 5a_B$
5) We can clearly see that we need to find a_b to find T so we could solve for m_A and we will figure out a_A later. From the given kinematics information we could find a_b by using SUVAT since the acceleration is constant: $0.75 = \frac{4a}{2} \to a = 0.375ms^{-2}$ hence $T = 44.4N$

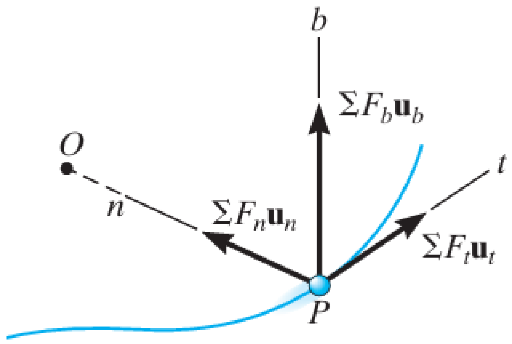
6) Now we need to find a_A to find the mass since we have one equation but two unknowns. Thus, we need another equation. What we should see is that there is dependent motion hence we could relate the acceleration of block A to the block B and figure out a_A thus:



7) Thus $l = s_A + 2(s_A - h) + s_B$ which gives us $a_A = -\frac{a_B}{3}$ (meaning block A is moving downwards) and if we plug in the numbers into EOM for block A we find $m_A = 13.7kg$

3.1: EOM on n-t c.s

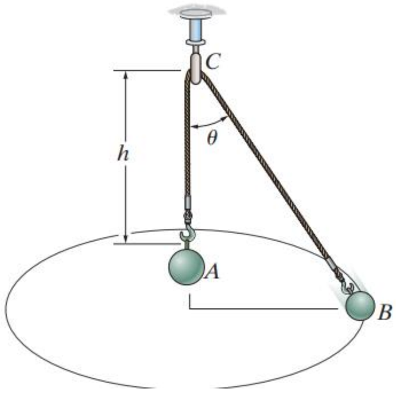
This chapter is pretty basic. It is exactly what we did in chapter 2.3 however, instead of using x-y c.s we will use n-t c.s. That is the only difference, the rest is exactly the same. To remind you, we will use this c.s:



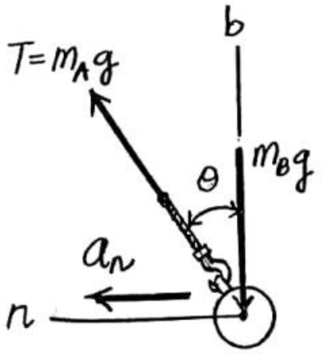
When we use n-t c.s we will have an extra standard procedure step compared to before. Even before choosing the c.s it may be very helpful to sketch the path then choose the c.s, draw FBD, draw KD and write EOM.

Problem solving:

13–67. Bobs A and B of mass m_A and m_B ($m_A > m_B$) are connected to an inextensible light string of length l that passes through the smooth ring at C. If bob B moves as a conical pendulum such that A is suspended a distance of h from C, determine the angle θ and the speed of bob B. Neglect the size of both bobs.

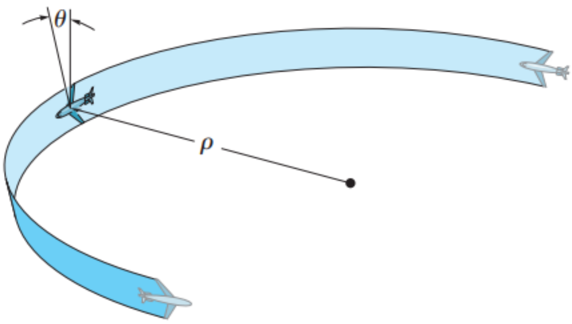


1) In this question x-y c.s may also be used however, since there is a curved path I choose to use n-t c.s. I will divide the system into two parts, A and B.
2) We know that A is in equilibrium therefore we can easily tell that $T = m_A g$ and let's draw the FBD for B

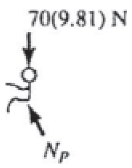


3) We know that the only acceleration is in the normal direction and in the z-direction the particle is in equilibrium hence $Tcos(\theta) = m_b g \rightarrow \theta = \arccos\left(\frac{m_b}{m_a}\right)$
4) $\sum F_{b_n} = Tsin(\theta) = \frac{mv^2}{\rho}$ and we can see that the radius of the circle/path is $p = \sin(\theta)(l-h)$ hence $v = \sqrt{\frac{g(l-h)(m_a^2 - m_b^2)}{m_a m_b}}$

13–83. The airplane, traveling at a constant speed of 50 m/s, is executing a horizontal turn. If the plane is banked at $\theta = 15^\circ$, when the pilot experiences only a normal force on the seat of the plane, determine the radius of curvature ρ of the turn. Also, what is the normal force of the seat on the pilot if he has a mass of 70 kg.



1) First of all I want to say that many students get confused with whether they should use sine or cosine in this question. To figure that out I will give a nice and easy tip
2) We have a curved path and clearly n-t c.s is the best option as we are not given any angular value. Let's draw the FBD of the pilot:



3) We know that the only acceleration is in the normal direction hence, in the z-axes, there should be no acceleration. Here, we need to figure out whether we should use sine or cosine to project normal force on the z-axes. In such situations what I do is to check what happens in extreme angles. In other words, when $\theta = 0 \; or \; 90$. When the bank angle is zero, we can see that the plane is inverted 90 degrees hence the pilot is not feeling any normal force to counter his weight (I try to imagine how the plane would look like, thats how I come up with this argument) on the other hand, when the bank angle is 90 degrees the plane is moving in the typical cruise way and hence, the pilot is receiving the maximum normal force. When sine is 90 degrees it is at its max

thus we should use sine for the z direction and cosine for normal direction. Thus

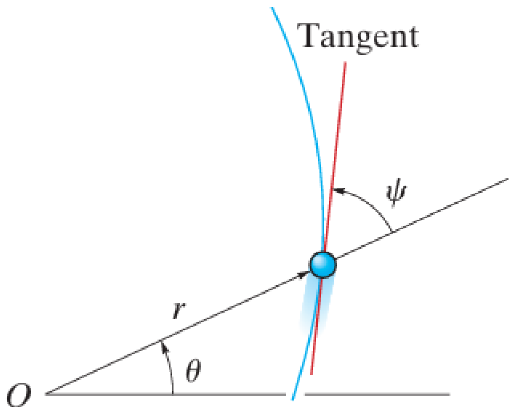
$\sum F_z = N\sin(\theta) - mg = 0 \rightarrow N = 2.65kN$

4) $\sum F_n = Ncos(\theta) = \frac{mv^2}{\rho} \rightarrow \rho = 68.3m$

3.2: EOM on polar c.s

Similarly from chapters 2.3 and 3.1, we will do the exact same thing on polar c.s however there is a small extra this time therefore I will not repeat the first part and just tell the new stuff.

As we know friction force is tangent and opposite to the direction of the particle and normal force is perpendicular to the tangent. How can we project these forces on the polar c.s axis then?



Firstly we sketch the path and draw a tangent to that path. Later, we draw our axis and we define the angle between the extended radial axis and the tangent line to be ψ where $\tan(\psi) = \frac{r}{\frac{dr}{d\theta}}$ thus we could project the normal force and friction force on our axis.

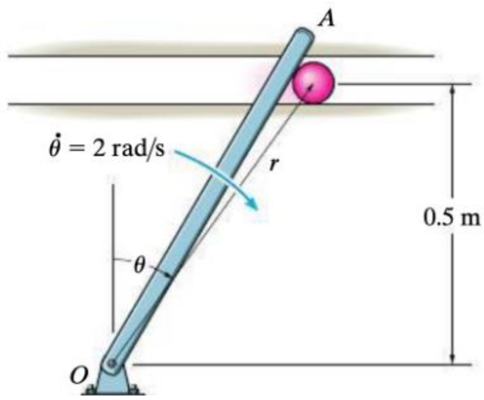
To understand this better, let's solve a problem.

Problem solving

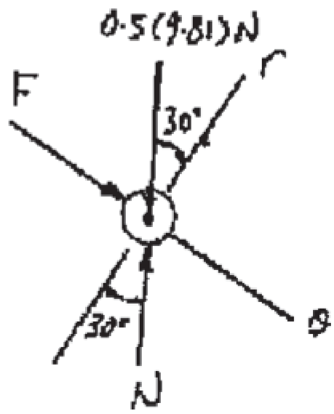
Solve problem 13-110:

13–109. The particle has a mass of 0.5 kg and is confined to move along the smooth horizontal slot due to the rotation of the arm *OA*. Determine the force of the rod on the particle and the normal force of the slot on the particle when $\theta = 30°$. The rod is rotating with a constant angular velocity $\dot{\theta} = 2$ rad/s. Assume the particle contacts only one side of the slot at any instant.

13–110. Solve Prob. 13–109 if the arm has an angular acceleration of $\ddot{\theta} = 3$ rad/s^2 when $\dot{\theta} = 2$ rad/s at $\theta = 30°$.



1) First thing we do is sketch the path of the particle. From the constraint it should be obvious that the particle is moving horizontally only hence, the tangent is a horizontal line which means the normal force is perpendicular to the slot
2) From the given angular values it is clear that we should be using polar c.s
3) Let us draw the FBD then



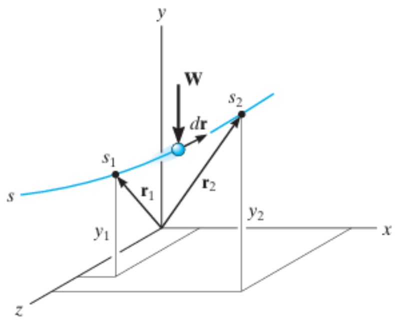
4) There is the weight, normal force, and the force from the rod acting on the particle. The force that comes from the rod is always perpendicular to the rod in other words it is on the angular axis
5) Let us calculate ψ first: We know that $r = \frac{0.5}{\cos(\theta)} \therefore \frac{dr}{d\theta} = \frac{\sin(\theta)}{2\cos^2(\theta)} \rightarrow \psi = 60°$ from the geometry, this means that the angle between the normal force and radial axis is $30°$
6) From the standard formula we could calculate the acceleration
7) $\sum F_r = Ncos(30) - mgcos(30) = ma_r$
8) $\sum F - mgsin(30) - Nsin(30) = ma_\theta$
9) By solving these equations we find that $N = 6.37N$ and $F = 2.93N$

3.3: Work, energy and power

Since all these concepts (such as what is energy, what is work etc.) should be quite familiar by now, I will just stay on the important parts and hints that I will give to solve problems faster and easier.

The work of a force is the dot product of the displacement and force vectors hence $dU = Fdr\cos(\theta)$ where θ is the angle between these two vectors.

Work of a weight is a bit different than work of a force. What we do is to determine a zero point, a datum first.



Then $U_{1-2} = -W\Delta y$ hence, if we go below our datum we get a positive work and vice versa. However, with the method I will show soon we will not have to bother with this at all as defining a datum may be confusing in complex problems.

The work of a spring is also different. Keep in mind that the formula I will show is not valid for circular springs. Work of a spring is defined as $U = \frac{ks^2}{2}$ where k is the spring constant and s is the spring displacement from the equilibrium position. Keep in mind that spring force is a type of a potential energy that cannot be negative.

Finally, probably the most important energy formula: Kinetic energy KE $= \frac{mv^2}{2}$. It would be possible that kinetic energy is the only real form of energy in terms of mechanical energy because all types of energy is in fact turned into kinetic energy when released.

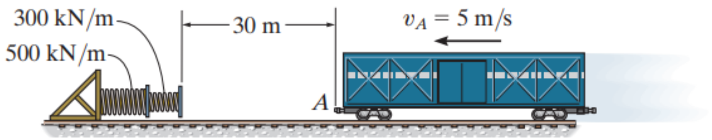
Now, I want you to forget about all the energy conservation stuff. What we will do is to write the energy situation initially and then in the final situation and equate them. Because the total energy must be the same, of course we are including lost energy such as energy lost to friction in our equation. When I solve an example it will be clearer how to do it. We will use this method for all questions that involve energy.

Another question that students usually struggle with is when to use the concept of energy. My advice is to write it all the time if it can be written, sometimes it is quite clear that we lack so much of data that we cannot really use it. However, even if you lack small data, always but always write the initial and final energy equation because you may use that information to solve another equation by combining it with other concepts. We only have a few distinct concepts therefore when we solve complex questions, my advice is to write all the concepts equations down and solve the puzzle basically.

Before we dive into some problems, I want to talk about power. Power is defined as $P = \frac{dU}{dt}$ and if we have constant velocity and a force then $P = Fv$. The efficiency of a machine can be found by $\mu = \frac{Power/energy\ out}{Power/energy\ in}$

Problem solving:

14–29. The train car has a mass of 10 Mg and is traveling at 5 m/s when it reaches A. If the rolling resistance is 1/100 of the weight of the car, determine the compression of each spring when the car is momentarily brought to rest.

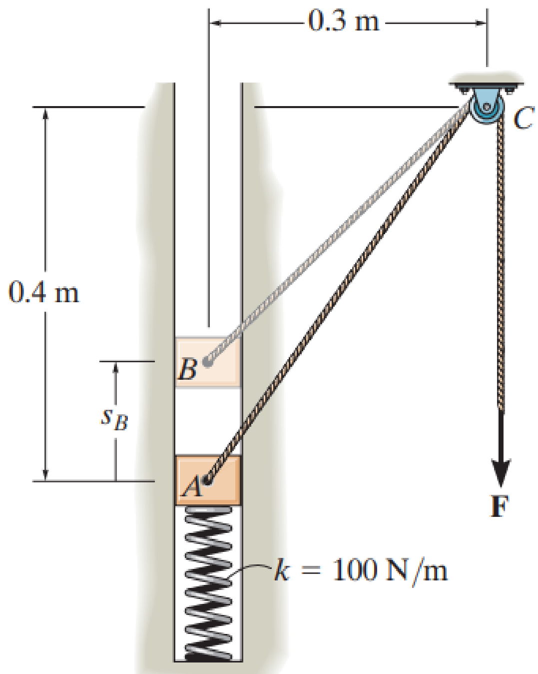


1) First of all we should try to imagine/animate the situation in our mind. What will happen is that the train will start slowing down due to friction and then it will hit the springs. It is very important to notice that when the springs are shrinking the friction still acts on the train hence friction keeps slowing down the train. Then, at one point the train will stop due to friction energy loss and spring storing the energy.
2) Second thing we do is to define a c.s and in this question, we have a linear motion therefore x-y c.s is clearly the best option. I define upwards to be positive y-axis and left to be the positive x-axis. Note that there is no motion in y-axis therefore it is useless.
3) In this question it is clear that we should involve energy because we are given distance, friction and spring. We could draw the FBD too just so we understand the situation better however, EOM would not be useful in this case since the acceleration of the train is changing due to the spring gets more and more shrunk. This would leave us with a nasty differential equation. To notice such details and prevent unnecessary work you should solve many questions and get familiar with dynamics
4) Let us write the energy equation. Initial energy only consists of the KE of the train hence: $\frac{10^5 \cdot 5^2}{2}$
5) Final energy is the energy stored in the springs and the energy dissipated into heat due to friction. Notice that we do not know the distance the springs shrank:

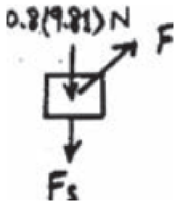
$\frac{3 \cdot 10^5 x_1^2}{2} + \frac{5 \cdot 10^5 x_2^2}{2} + 10^3 \cdot 10 \cdot \frac{9.81}{100}(30 + s)$ where s is the distance the train moved after the 30m distance which is also $s = x_1 + x_2$

6) It is clear that we need another equation to solve for x_1 and x_2. Here, we need he knowledge that the force that is exerted on the both springs is the same hence $3 \cdot 10^5 x_1 = 5 \cdot 10^5 x_2$ thus we could equate the initial and final energy equations and we find that $x_1 = 0.628m$ and $x_2 = 0.377m$

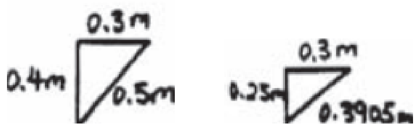
14–35. The block has a mass of 0.8 kg and moves within the smooth vertical slot. If it starts from rest when the *attached* spring is in the unstretched position at A, determine the *constant* vertical force F which must be applied to the cord so that the block attains a speed $v_B = 2.5$ m/s when it reaches B; $s_B = 0.15$ m. Neglect the size and mass of the pulley. *Hint:* The work of **F** can be determined by finding the difference Δl in cord lengths AC and BC and using $U_F = F\,\Delta l$.



1) Let's imagine how the system works. The force will pull the block and gravity + spring will slow down. I choose the typical x-y c.s since there is a linear motion
2) Let's draw the FBD



3) Just like in the previous question, using EOM would cause a nasty equation which we do not want. By looking at the given information, we could try to use energy. Also, notice the important hints given in the question that the slot is smooth which means there is no energy loss to friction and we know that there will be no motion in the x-axis due to the slot constraint although the force pulls. In reality there is a normal force acting from the slot but I did not show it in the FBD not to confuse the readers
4) Now, let's find the initial energy: We know that the force applied in the direction of vertical displacement does work. Let us calculate the difference in length to use the hint they gave us in the question



5) Thus, the initial energy is $0.11F$. Let's calculate final energy: We know that the block moved up 0.15m up and let's define our datum the point where the spring is unstretched. In the final position, the block gains gravitational potential energy, since the spring moves with the block it also gains spring potential energy and finally it has kinetic energy as it gained speed. Thus: $0.8 \cdot \frac{2.5^2}{2} + 0.8 \cdot 9.81 \cdot 0.15 + 100 \cdot \frac{0.15^2}{2}$
6) Equating the initial and final energy, we find F = 43.9N

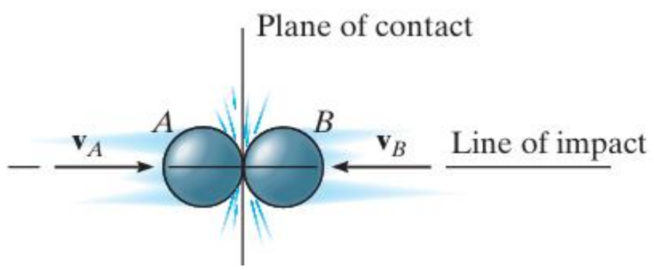
4.1: Linear momentum and impulse

Just like I did for energy, momentum is also assumed to be well-known therefore I will only go through the important formula and hints.

Momentum is simply "mv" and just like energy, it is also constant. Meaning that initial momentum + the applied impulse ($I = \int F dt$) is equal to the final momentum. We will always use this theorem when we use momentum concept. Note that this is true for that specific direction. Always remember that momentum is a vector since velocity is a vector quantity.

Most questions combine momentum with other concepts, especially with energy. Also, notice that if a force acts on an object, it always creates impulse. It is not like energy where some forces may not do work. However, sometimes some forces may be negligible. We call the non-negligible impulses "impulsive shots". Impulsive shots are usually very large forces. Non-impulsive shots are negligible. Usually in the questions it will be told whether they are negligible or not however, even if it is not told, what you could do is to compare the magnitude of the force to the other forces acting on the object or comparing it to the object's mass.

Last thing we should consider is impact:



First thing we should do is to figure out the line of impact. This should be obvious from the image. We basically have a formula for the collision of two objects. We will just use it and thats all. The formula is: $e = \frac{v_{B_2} - v_{A_2}}{v_{A_1} - v_{B_1}}$ where e is the coefficient of restitution.

I advise you to choose the object B to be the object that got hit and object A to be the object that hit.

It is very important to stick with your c.s otherwise you would surely make a plus-minus mistake.

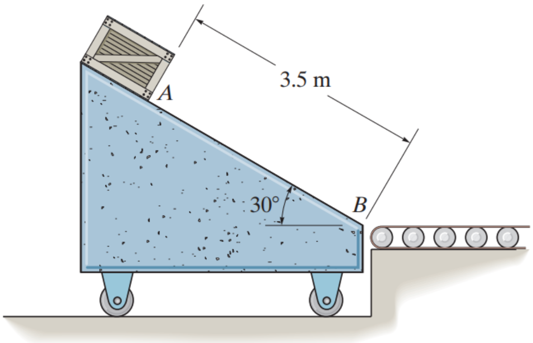
There are two special cases. However, you do not have to memorize them because they can easily be deduced. The special situations are when $e = 0$ or $e = 1$

When $e = 0$ it means that the objects are moving with the same velocity. Since they just collided and if they are moving with the same speed and direction, it means that they are now moving together. This is called plastic impact.

When $e = 1$ it means that the objects are reflected back (opposite direction) with the speed they came. This is called elastic impact.

Problem solving:

15–57. The free-rolling ramp has a mass of 40 kg. A 10-kg crate is released from rest at A and slides down 3.5 m to point B. If the surface of the ramp is smooth, determine the ramp's speed when the crate reaches B. Also, what is the velocity of the crate?



1) Firstly we should imagine what is going on. The crate moves down and to the right (along the ramp axis) and the ramp moves to the left. This is because of Newton's third law. The normal force that acts on the crate pushes the ramp to the left. Notice that this would mean there is a relative motion Since they are moving in the opposite direction, the crate would reach B even faster. Also, notice that the crate is moving to the right and down but the ramp only moves left.
2) Now we should establish a c.s and since the motion is linear, I choose x-y c.s. Normally we tilt the c.s when there is a tilted path however, in this question, I will not tilt it since it would complicate the motion of the ramp. I choose the positive x-axis to the right and the positive y-axis to the upward.
3) As I said previously, we should use as much concept as we could. I will try energy and momentum. Kinetics could also be an option however, instincts should tell us that this would overcomplicate things because there is a relative motion. However, in case our equations will not be enough, we could try kinetics too.
4) Initial energy: Everything is at rest but the crate is at position A, the center of gravity of the ramp does not change as its vertical position does not change and there is no force that does work therefore $U = 3.5\sin(30)\, m_A g$

5) Final energy: Finally, the crate is at position B which is our datum hence there is no longer any gravitational potential energy. However, the crate and the ramp have kinetic energy thus: $U = \frac{m_A v_A^2}{2} + \frac{m_B v_B^2}{2}$
6) Now, it should be clear that we need to relate the velocities. Here we could try momentum
7) Initial momentum is zero since the crate and the ramp are at rest
8) Final momentum can be divided into horizontal and vertical, however, since the ramp only moves horizontally we should use the horizontal momentum: $P = m_A V_A \cos(30) - m_B v_B$
9) There is no impulse because the normal forces are force pairs thus they cancel each other for the total system. Thus: $v_A = \frac{m_B v_B}{m_A \cos(30)}$
10) Equating the initial and final energy and writing the equation in terms of v_B only we find that $v_A = 5.4ms^{-1}$ and $v_B = 1.1ms^{-1}$

4.2: Angular momentum

This part is the exact same of the previous part but for angular values. Hence, there is a great similarity therefore I will not be repeating the same logic however, I will give important hints in the problem solving part.

Angular momentun "H"is defined as $H = \boldsymbol{r} \; x \; m\boldsymbol{v}$ in vector form however, in partical dynamics we will mostly use the scalar form $H = r \cdot mv$ where r and v are perpendicular to each other. Keep in mind that "r" is the arm and "v"is the velocity vector.

When a force acts on the object, it may create angular impulse. Yes, it may, it is not guranteed unlike linear impulse since if the force and the arm in the same direction, it does not create any angular impulse. Formula for angular impulse is $\int_{t_1}^{t_2} \boldsymbol{r} \; x \; \boldsymbol{F} \, dt$ where F is the force acting on the body. See how similar it is to linear impulse?

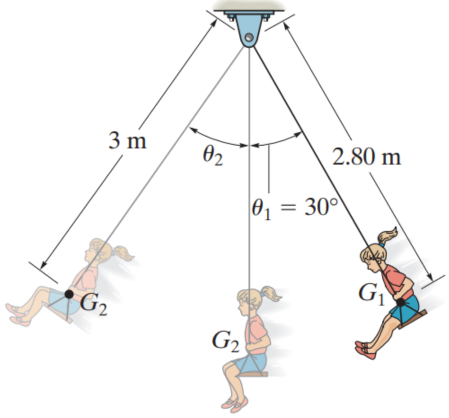
Again, very similarly to linear impulse formula, in all questions we will use this formula:
$H_1 + \int_{t_1}^{t_2} \boldsymbol{r} \; x \; \boldsymbol{F} \, dt = H_2$

Just like before, we will mostly combine angular momentum with energy to solve questions. Also do not forget that most of the time we will be using n-t c.s since all angular momentum questions will have curved paths.

Now, let's look at some problems. I will be giving important hints that you may see applications later in other questions.

Problem solving:

***15–104.** A child having a mass of 50 kg holds her legs up as shown as she swings downward from rest at $\theta_1 = 30°$. Her center of mass is located at point G_1. When she is at the bottom position $\theta = 0°$, she *suddenly* lets her legs come down, shifting her center of mass to position G_2. Determine her speed in the upswing due to this sudden movement and the angle θ_2 to which she swings before momentarily coming to rest. Treat the child's body as a particle.

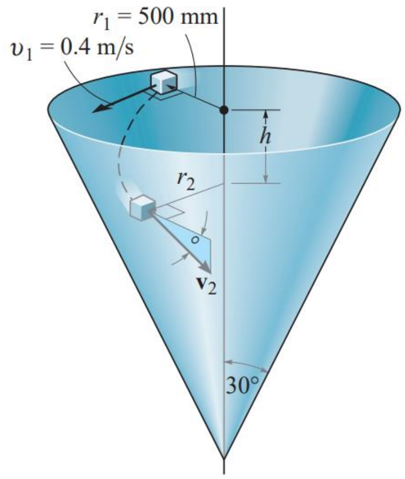


1) First of all, let us draw the path of the child. We can see that it is a curved path. However, what we should notice and will be vital to solve this question is that, at the lowest point of the arc, the distance from the datum to the center of gravity changes from G_1 to G_2 which means that at that instant, the length is both 2.8m and 3m. This concept may sound awkward at first however, whenever we are told that there is an instantaneous change, we should think like this and get used to this concept.
2) Since the path is curved, I will use n-t c.s. However, the issue we can see is that the acceleration and the radius of curvature is changing therefore EOM is not the way to go. What else do we know? Energy and momentum of course. So let us try to apply them
3) Firstly we need to find the velocity of the child when it reaches the lowest point but still did not move her legs hence, the radius is still 2.8m. Initially, the child is at rest and let us define our datum to be the lowest point. We know that the tension force does no work since it is always perpendicular to the displacement. Thus, there is no work done by any force (besides the weight of course) hence:

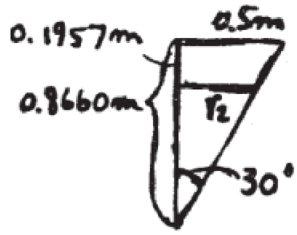
   $\frac{mv^2}{2} = 2.8mg(1 - \cos(30)$ Thus $v = 2.53ms^{-1}$ when the child is at the lowest point without moving the legs.

4) Now, the child moved her legs hence the radius has changed to 3m from 2.8m. This means her velocity has changed because the angular momentum is conserved. You may think that the forces create impulse however since the change happens instantaneously, there is no impulse also, both forces at the lowest point are parallel to the arm hence no impulse is created.
5) At the final position, the child is at rest again hence we can use energy method again for 3m radius this time. Using the exact same logic in step 3 gives us the angle 27°
6) Notice how we used energy to eliminate the varying impulse calculation for intermediate steps (from the highest to the lowest points)

15–111. A small block having a mass of 0.1 kg is given a horizontal velocity $v_1 = 0.4$ m/s of when $r_1 = 500$ mm. It slides along the smooth conical surface. Determine the distance h it must descend for it to reach a speed of $v_2 = 2$ m/s. Also, what is the angle of descent θ, that is, the angle measured from the horizontal to the tangent of the path?



1) Let us imagine the situation. The block is moving down while making circles and as it goes down, it gets faster. Again, it should be clear that EOM would not be very helpful in this question since there are so many unknowns therefore we should try our luck with momentum and energy approaches. However, first of all we should determine a c.s. Since we have a curved path again, we will be using n-t c.s with additional z-axis
2) First of all we should calculate the velocity at the second position. It should be clear that energy can be used here. There are two forces acting on the block. Normal force and the weight of the block. Normal force is always perpendicular to the displacement thus it does no work. Thus, energy is conserved and $\frac{mv_1^2}{2} + mgh = \frac{mv_2^2}{2}$ gives us $v = 0.196m$
3) Now, we are asked for the angle shown in the figure. What we should notice is that the angular momentum is conserved since weight is always downwards, it does not create any impulse and the normal force is acting in the z-axis and normal axis which again does not create any impulse as it is parallel to the arm. If we look at the initial angular momentum we can see that the arm and the velocity are perpendicular to each other hence, we can easily find the angular momentum by $H_1 = mv_1r_1$
4) Now, we want to find the r_2 to write the momentum in position 2. We could use geometry to find r_2



5) Now, we would have to project the velocity component onto the horizontal shown as it is perpendicular to the arm r_2 thus $H_2 = mv_2\cos(\theta)\,r_2 = H_1$ gives us $\theta = 75°$

Rigid body:

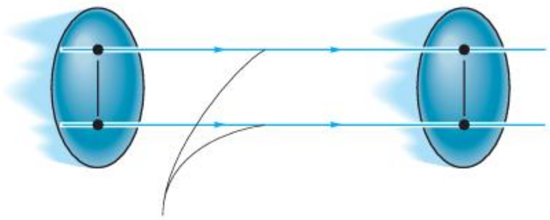
We are finally done with the particle physics part. I have good and bad news. Now, things will get very fun and interesting however it will also get much harder. Do not worry though, from now on I will be giving many more hints thus you will not have to spend so much time figuring out the tricks and the logic behind questions.

In particle physics, the shape of the object did not matter at all, it was always a point however, now the shape of the object matters very much. Also, now we have rotation around its own axis property of bodies. Rigid body dynamics can be applied to many in real life situations. Cars, bicycles, electrical bread cutters, anything with gears, an airplane etc.

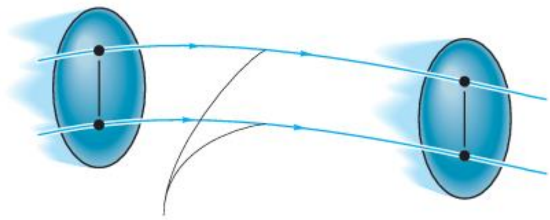
4.3: Kinematics of rigid body, general plane motion

Here, we will learn how to apply kinematics for rigid bodies. It is still quite similar to particle dynamics but slightly different. I will be giving important hints and facts therefore pay attention, please.

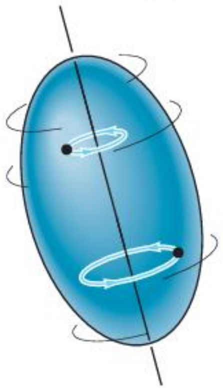
We have three distinct motion types. The figure below explains the situation pretty clearly.



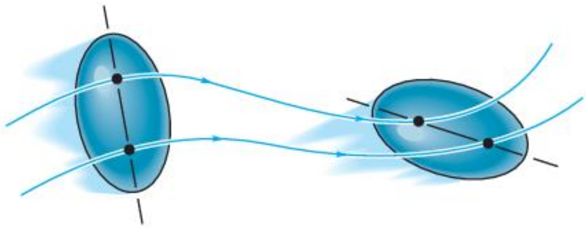
Path of rectilinear translation
(a)



Path of curvilinear translation
(b)



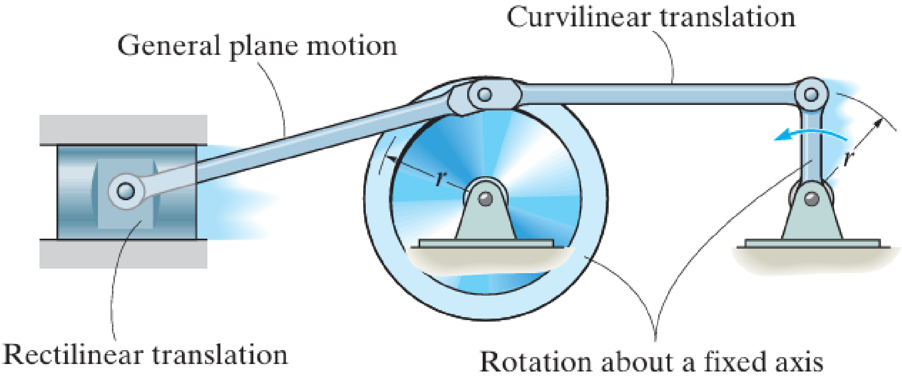
Rotation about a fixed axis
(c)



General plane motion
(d)

General plane motion is the combination of all these motion types and of course, it is the hardest one. When we solve questions, we will always assume the motion is a general plane motion since it is the general equation. Then, for the motion types that do not exist in our case, we will say that their velocity is zero hence we will achieve the right equation for the given motion. However, always be aware of the types of motions.

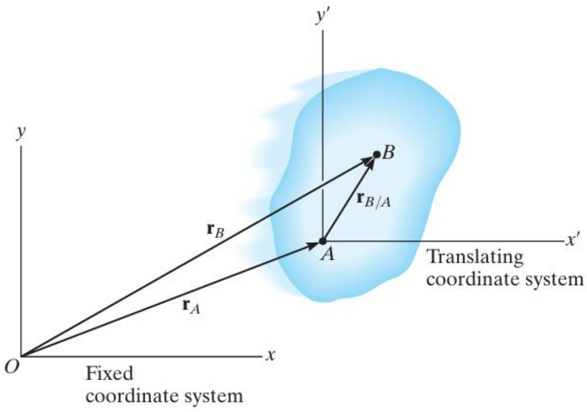
In most questions we will have combinations of different bodies with different motion types. See the image below.



Now, let us look at the formulation of different kinds of motions.

Translational:

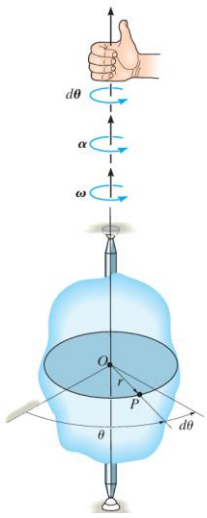
For such a body:



From now on we will always use <u>relative motion</u> formulas. Thus, we would formulate different points on the body such as: $\boldsymbol{r_B} = \boldsymbol{r_A} + \boldsymbol{r_{B/A}}$

However, on a rigid body, since the <u>distance between two points</u> such as point A and point B does <u>not change</u>, the velocity is $\boldsymbol{v_A} = \boldsymbol{v_B}$ and $\boldsymbol{a_A} = \boldsymbol{a_B}$

Rotational: For rotational motion, we will use angular values.

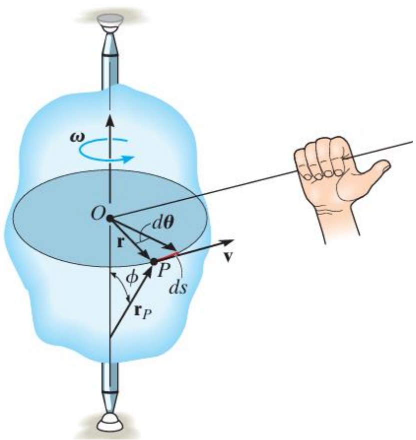


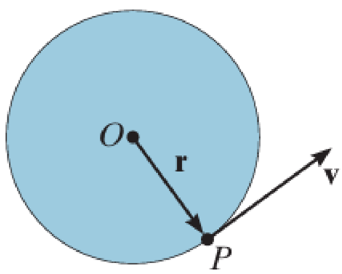
Now, we will add an important new part to our c.s. The direction of rotation. We will mostly work in 2D situations/situations where we could simplify the system into a 2D problem therefore this rotation direction refers to the rotation around the z-axis. I suggest you to always stick with defining counterclockwise (CCW) to be positive. Plus-minus mistakes are so common and often lead to bigger problems in complex problems therefore choosing CCW to be positive always helps solving questions as you would get used to the same calculation method.

Our formulas are $\omega = \frac{d\theta}{dt}$ and $a = \frac{d\omega}{dt}$

Thus, if we have a constant angular acceleration, we could use our SUVAT equations for the angular values as it was shown before.

In many situations we will be interested in the linear velocity of a point on the rigid body thus:





$$v = \omega r$$

Or in the vector form: $\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{r_P}$ Please pay attention to the cross-product order.

Have you noticed something from particle physics? Yes, again, we have our normal and tangential acceleration. Now, we learned how to relate the angular velocity to the linear velocity which I will explain why it's so important.

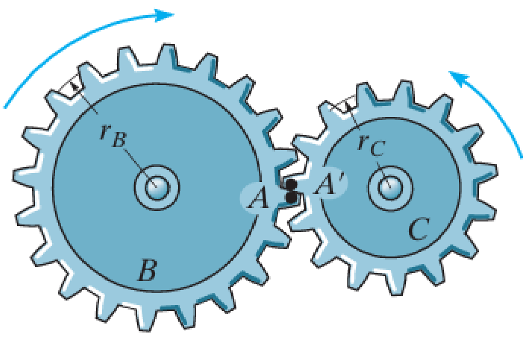
Our acceleration formulas are: $a_t = \alpha r$ and $a_n = \omega^2 r$ thus $\boldsymbol{a} = \boldsymbol{a_t} + \boldsymbol{a_n} = \boldsymbol{\alpha} \times \boldsymbol{r} - \omega^2 \boldsymbol{r}$

Notice that the angular velocity in this formula is not a vector

So, why do we need the linear velocity of a point on the rigid body?

In different questions there may be specific needs however, I will name the two most important reasons that we will commonly see in problems.

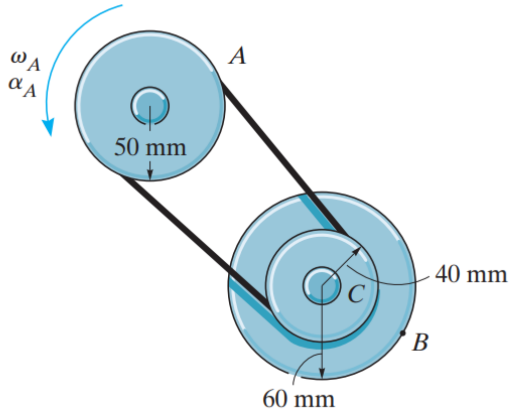
1) When we have gears such as the one below, when the gears are in contact, they must turn in the opposite directions, otherwise they would break. This concept will be used a lot in upcoming questions where we will have to consider if the body would break and formulate the situation accordingly. If it is breaking, it should be in the opposite direction basically. Now, think of the point A. When we analyze the point from either gear, we must have the same linear velocity, it clearly would not make any sense in reality if point A had two different velocities. However, the gears could have different angular velocities. Thus, we need the linear velocity since $\omega_B r_B = \omega_C r_C = v_A$. We do not need this information yet since we did not involve the forces into account however, do not forget that these two gears exert force on each other according to Newton's third law.



2) We will not really use this information yet, however, when we work with energy in rigid bodies, we will use linear velocity to calculate how much the point moved thus we could calculate the heat loss due to friction

Problem solving:

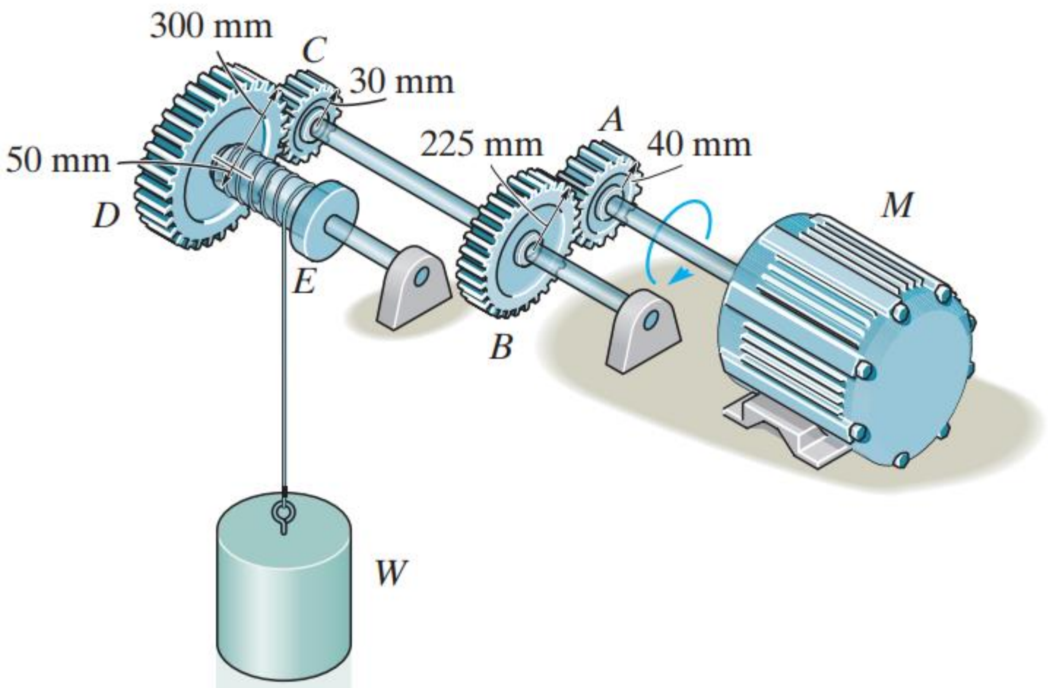
16–10. At the instant $\omega_A = 5$ rad/s, pulley A is given a constant angular acceleration $\alpha_A = 6$ rad/s^2. Determine the magnitude of acceleration of point B on pulley C when A rotates 2 revolutions. Pulley C has an inner hub which is fixed to its outer one and turns with it.



1) Firstly, we should notice that pulley C and B has the same angular velocity and acceleration, however, at their radius, the linear velocity and acceleration would be different.
2) To being with, lets find the initial angular velocity of pulley C. Since this is an inextensible chord, what we know is that the total length must stay the same which also means that the linear speed on the chord is same everywhere, otherwise it would be an extensible chord. Thus: $\omega_A r_A = \omega_C r_C$. Hence, we need ω_A after 2 revolutions and thus $2 \cdot 2\pi = 5t + \frac{6t^2}{2} \rightarrow t = 1.376$ since the other answer is negative which cannot be thus $\omega_A = 5 + 1.376 \cdot 6 = 13.3 rad/s$ Which gives us $\omega_C = \omega_B = 16.6 rad/s$
3) We know that the acceleration is constant in the cable is also the same everywhere(thus the tangential acceleration) hence applying the same logic, the relative acceleration between the center of the pulleys and at their maximum radial distance is simply equal to the angular acceleration (this was the logic in step 2 as well) because the center of the pulleys are stationary hence there is no translational motion of the pulley. Thus $\alpha_A r_A = \alpha_C r_C \rightarrow \alpha_C = \alpha_B = 7.5 rad/s^2$
4) Now, we could calculate the both tangential and normal acceleration of the point B, we do that separately since they asked for the magnitude of the acceleration. Using

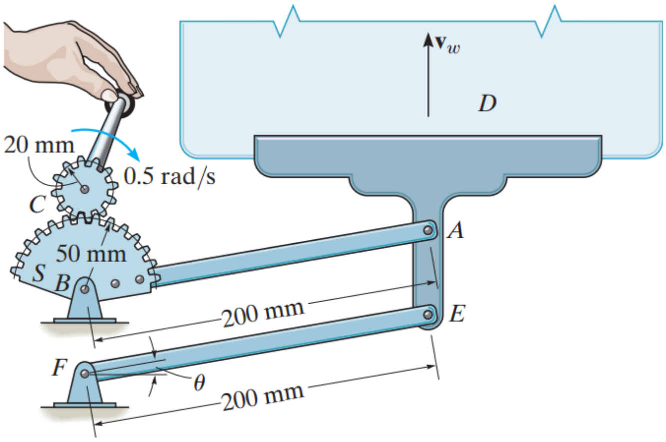
the standard formulas given $\alpha_{t_B} = 0.45ms^{-2}$ and $\alpha_{n_B} = 16.4809ms^{-2}$ $a = \sqrt{a_{t_B}^2 + a_{n_b}^2} = 16.48ms^{-2}$

16–30. Determine the distance the load W is lifted in $t = 5$ s using the hoist. The shaft of the motor M turns with an angular velocity $\omega = 100(4 + t)$ rad/s, where t is in seconds.



1) In this question, we can clearly see that every gear rotates the other gear system and hence, the final system rotates in the same direction as the motor, which is CCW. Thus, the load is lifted, not descended. It should be clear that we should apply the linear velocity principle again and again til we get the angular velocity of the final gear system. Also notice that we have to integrate final linear velocity to get the displacement. $\int 100(4 + t)dt = 400t + 50t^2$ and when $t = 5$ we find that $s_W = 2.9m$

16–38. The mechanism for a car window winder is shown in the figure. Here the handle turns the small cog C, which rotates the spur gear S, thereby rotating the fixed-connected lever AB which raises track D in which the window rests. The window is free to slide on the track. If the handle is wound at 0.5 rad/s, determine the speed of points A and E and the speed v_w of the window at the instant $\theta = 30°$.



1) First, let us imagine what is going on here. The gears turn and hence the connected pin rotates the lever also, most importantly, we know that the car window is constrained to move in the vertical direction only
2) From the linear velocity principle $20 \cdot 0.5 = 50 \cdot \omega_B \rightarrow \omega_B = 0.2rad/s$
3) Thus, from the formula, points A and E have the speed $0.2 \cdot 0.2 = 0.04ms^{-1}$
4) We know that the window only moves in the vertical direction hence we need the vertical components of the velocity. How could we find it? Imagine if we had a horizontal line and it is moved by an angular velocity, initially, the only velocity is in the vertical direction, right? That's eactly how we could project the horizontal and vertical velocity, by thinking a straight line moved by an angular velocity and checking its initial velocity direction. Thus, in this question $0.2 \cdot 0.2 \cos(30) = 0.035ms^{-1}$

5.1: Absolute motion and relative velocity analysis

This topic is extremely important because now, we will learn how to calculate linear and angular velocities in rigid bodies. For this, we have two methods. Absolute motion analysis and relative velocity analysis -the relative acceleration analysis will be shown in the next lecture.

Absolute motion:

Absolute motion is relating geometries and then differentiating the expression to find the wanted quantity. For instance, if we have a term $\frac{d\theta}{dt}$ it means angular velocity and $\frac{dx}{dt}$ means linear velocity. Absolute motion is easier to explain on examples therefore wait til the solving part however, here is the solution method we will follow:

1) Firstly, we have to determine a reference point. This point must be a fixed reference from an inertial frame.
2) We will choose our distance to wherever we want from our reference point.
3) Then, by using geometric relations, we will formulate this distance.
4) Finally, we will differentiate that expression and do not forget that we have to write that expression so that when we differentiate, we can relate our unknown and known expressions so that we can solve the equation.

When we differentiate our distance once, we get the velocity and when we differentiate twice, we get the acceleration.

Notice that we will be using implicit differentiation when we differentiate the expression.

Also, I would like to have your attention on the fact that our distance must be the distance between the reference point and the point we are investigating.

Relative motion analysis:

Relative velocity/acceleration analysis is going to be the method we will be using most of the time when we solve rigid body problems. This method is simply the relative velocity formula we learned before. We choose two different points on a body and use this formula:

$$\boldsymbol{v_A} = \boldsymbol{v_B} + \boldsymbol{v_{A/B}} \rightarrow \boldsymbol{v_A} = \boldsymbol{v_B} + \boldsymbol{\omega_{A/B}} \times \boldsymbol{r_{A/B}}$$

The angular velocity is the angular velocity of the body chosen. Notice that the points A and B have to be on the same body. Also, do not forget to choose the c.s and you have to place your c.s at point B since we are looking at the position of point A with respect to point B.

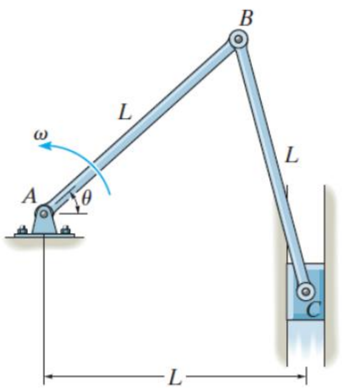
I know that this way it may take unnecessarily too much time since you would have to place a c.s and perform cross product therefore we would use the scalar version instead. Then, you should draw the distance and draw the velocity vector.

! We have not defined it most of the times therefore you may not be used to it however, do not forget to define the positive direction of rotation (CCW or CW). I advise to define CCW to be positive every time.

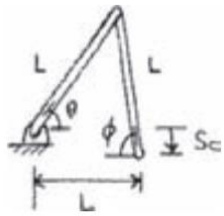
Lastly, I would like you to remind you that you should try to figure out small hints to determine v_A or v_B since most of the time we will not be given both. For instance, one of them may be zero.

Problem solving:

16–49. Bar AB rotates uniformly about the fixed pin A with a constant angular velocity $\boldsymbol{\omega}$. Determine the velocity and acceleration of block C, at the instant $\theta = 60°$.



1) First of all, what we should notice is that when $\theta = 60°$ there will be a equilateral triangle formed. Now, we should determine a fixed reference point. Clearly, the best choice is the pin A. We need the distance between block C and the reference point.
2) To write that distance, we need to define another angle:



3) Thus $s_c = L\sin(\emptyset) - Lsin(\theta)$
4) Since differentiating this expression would create an unknown expression: $\frac{d\emptyset}{dt}$ we have to relate $\emptyset$ and θ and this can be done by writing the horizontal length from the reference point to the block C. The total length is constant and its L hence:

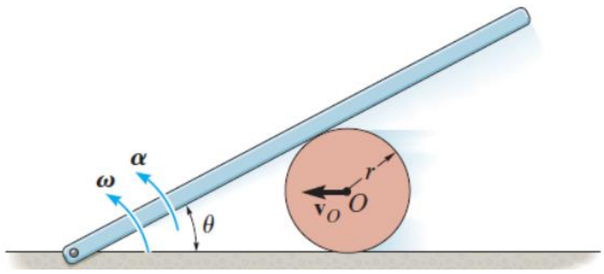
$Lcos(\emptyset) + Lsin(\theta) = L$

5) Thus, by differentiating this expression we could find that $\frac{d\emptyset}{dt} = -\omega$
6) Let us differentiate s_c: $\frac{ds_C}{dt} = Lcos(\emptyset) \cdot -\omega - Lcos(\theta)\omega$

7) Solving everything gives us $v_C = L\omega \uparrow$

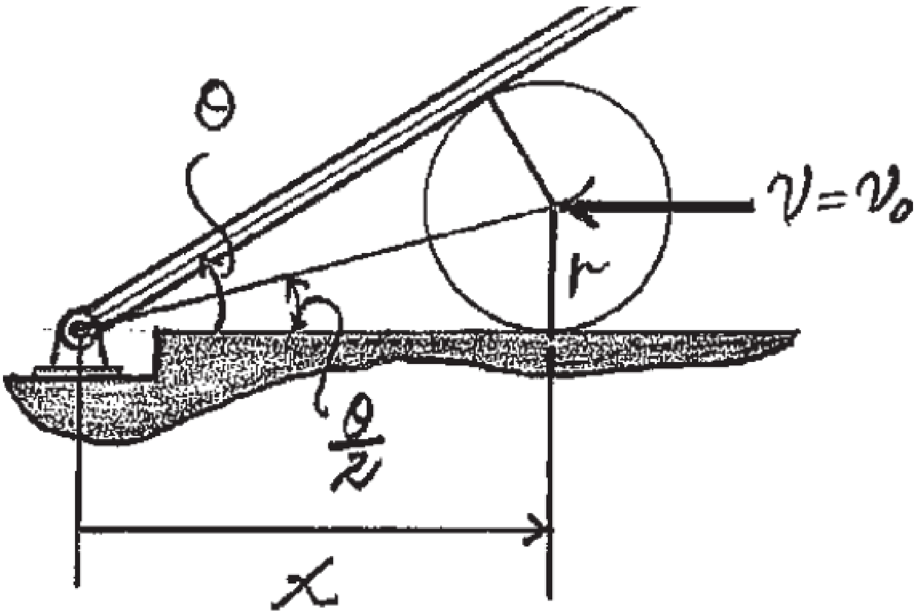
8) To find the acceleration, we have to differentiate s_C again and do not forget that $\frac{d\omega}{dt} = 0$ solving the equation again (It is just basic calculus therefore I will not show the working) $a = 0.58L\omega^2 \uparrow$

16–50. The center of the cylinder is moving to the left with a constant velocity $\mathbf{v}_0$. Determine the angular velocity $\boldsymbol{\omega}$ and angular acceleration $\boldsymbol{\alpha}$ of the bar. Neglect the thickness of the bar.



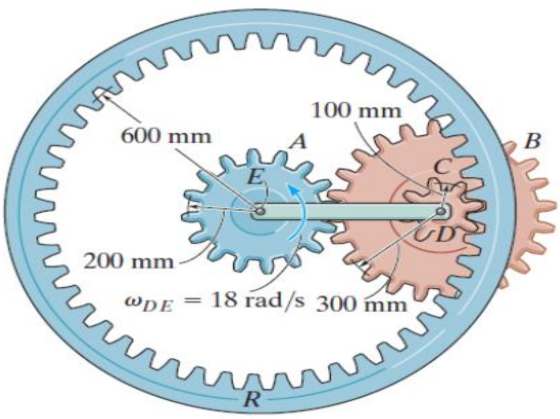
1) In this question we are given the linear velocity and asked for the angular one. Of course, it does not change anything for our procedure. Let us define the pin connection to be our reference point since it is fixed and intuitive.

2) Let us try to create some geometric relationships. Here, it all depends on how familiar you are with geometry:



3) I suggest this geometry. You can see that we have two right triangles with a common hypothenus. Let us call the hypotenuse "L" and let us divide θ into two different angles. If we take the sine of these angles we will notice that both of them are equal to $\frac{r}{L}$ hence the angles are equal. This means that the angles are $\frac{\theta}{2}$
4) Now, we need to relate the distance "x" ,which is the distance between the reference to the center of the cylinder, to the angle θ thus: $\tan\left(\frac{\theta}{2}\right) = \frac{r}{x}$
5) Notice that I did not use cosine since "L" is not constant and hence it would give us new terms if we differentiate.
6) Differentiate both hand sides: $\frac{dx}{dt} = -\frac{r}{2}\csc^2\left(\frac{\theta}{2}\right)\frac{d\theta}{dt}$ thus $\omega = \frac{2v_0}{r}\sin^2\left(\frac{\theta}{2}\right)$
7) Acceleration can be found by differentiating again and I will not show the calculation since it is basic calculus: $\alpha = \frac{2v_0^2}{r^2}\sin(\theta)\sin^2\left(\frac{\theta}{2}\right)$

16–73. The epicyclic gear train consists of the sun gear A which is in mesh with the planet gear B. This gear has an inner hub C which is fixed to B and in mesh with the fixed ring gear R. If the connecting link DE pinned to B and C is rotating at $\omega_{DE} = 18$ rad/s about the pin at E, determine the angular velocities of the planet and sun gears.



Prob. 16–73

1) You may be wondering how to know which analysis method to use. From the given system and what is known, you should figure out intuitively. If we are given a gear system for instance, it is very likely that relative motion analysis is used.
2) Here, we could easily find the velocity of point D by $v_D = v_E + \omega_{D/E} r_{D/E}$ which gives us $v_D = 0 + 18 \cdot 0.5 = 9ms^{-1} \uparrow$

3) To figure out the angular velocity of gear A, we would need the velocity at the connection point between gear B and A as it seems to be the only way to relate any velocity to gear A.
4) Since gear C and B are connected, they have the same angular velocity which means we would need to find the angular velocity of gear C. We know that at the connection point between the fixed gear and gear C is zero since fixed gear means it does not move.
5) Let us define that connection point "P": $v_P = v_C + \omega_C r_{p/C} \rightarrow 0 = 9 + 0.1\omega_C \rightarrow$ $\omega_C = -90 rad/s$
6) Let us define the connection point between gears A and B "P'"
7) $v_{P\prime} = v_B + \omega_C r_{P'/B} \rightarrow v_{P\prime} = 9 - 90 \cdot -0.3 = 36 ms^{-1} \uparrow$
8) Since at P' the linear velocity is the same from both gears side: $0.2\omega_E = 36 \rightarrow$ $\omega_E = 180 rad/s$

5.3: Instantaneous center of zero velocity and acceleration analysis

This chapter will be extremely important because we will learn about the kinematics of rigid bodies which will enable us to calculate acceleration and thus perform calculations on Force analysis. This lecture consists of Instantenous Center of Zero Velocity (IC) and relative acceleration analysis. IC will be useful when it comes to calculate the velocity of points/angular velocities when we do not have enough information to just apply our velocity analysis methods and to calculate the acceleration, we will need the angular velocity.

IC:

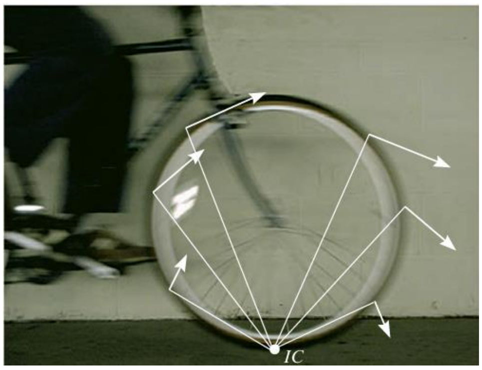
IC is a point on or outside of the body, where the velocity of that point is zero for that instant. It is very important to understand that the point does not have to be on the body and that it is zero only for that instant. The next instant that point may have velocity.

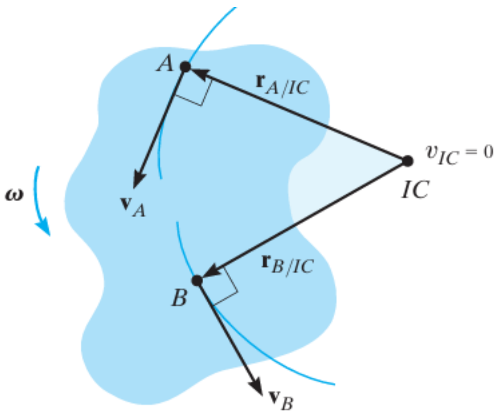
Why is IC important? Because when we write the relative velocity equation:
$\boldsymbol{v_B} = \boldsymbol{v_A} + \boldsymbol{\omega_{B/A}} \times \boldsymbol{r_{B/A}}$ if the point A is an IC, $\boldsymbol{v_B} = \boldsymbol{\omega_{B/A}} \times \boldsymbol{r_{B/A}}$ which simplifies the equation.

How to find IC?

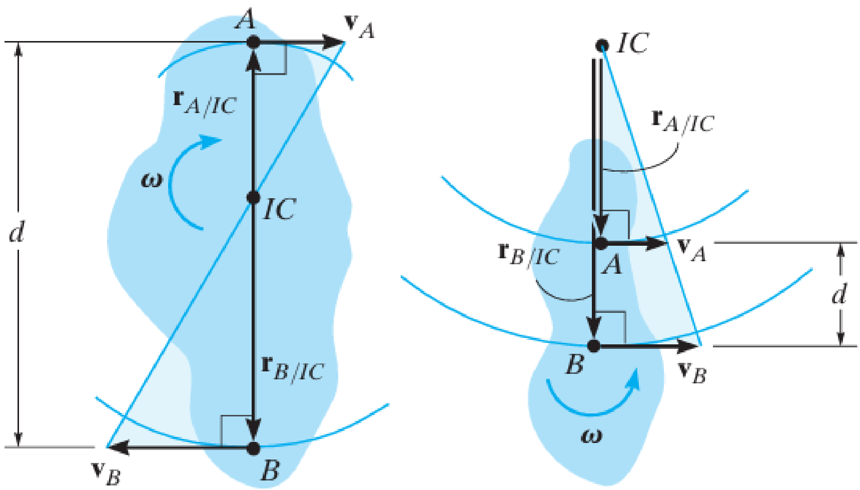
Later I will give some other tips (especially in the problem-solving part) on how to figure out the IC quickly and in tricks about it but let me show the formal method first.

To find the IC, let us first determine two velocities. Draw the velocity vectors. Then, from that point, draw an axis perpendicular to the vector, and do this for both velocities. The point where these two axes meet is where our IC is. For instance, on a wheel:





However, we have some special cases. The images below explain the procedure clearly, we will just use the property of geometry:



In the last example, you might have noticed a special case. What is the velocities are the same? This would mean that the body does not have a rotational motion and it is a purely translational motion. From the geometry (also if we formulate it) we could figure out that the IC is infinitely far away which also means that we do not have an IC point.

We will mostly use this method when we have bodies that are not circular -such as gears and wheels.

Now I will give very important tips for certain situations.

Wheels or wheel-like situations. If the wheel is rotating without slipping and the point of contact with the ground is stationary, the point of contact is the IC.

Why should it not be slipping? Because slipping means the body is rotating but it also has translational motion hence, the contact point is not really at rest because it is also translating.

Why the contact point should be at rest? Because then it would mean that the contact point has a velocity.

Another important point for figuring out where the IC is the inextensible chords. The point of contact of the body with the chord is an IC if the chord is not moving. Why? Because if the chord is inextensible it would mean that a point on the chord cannot move/have velocity otherwise it would extend.

!!! It is extremely important to note that accelerations are not intuitive and IC does not mean acceleration is also zero. Also know that there is no such thing for acceleration, a point where acceleration is zero.

Acceleration analysis:

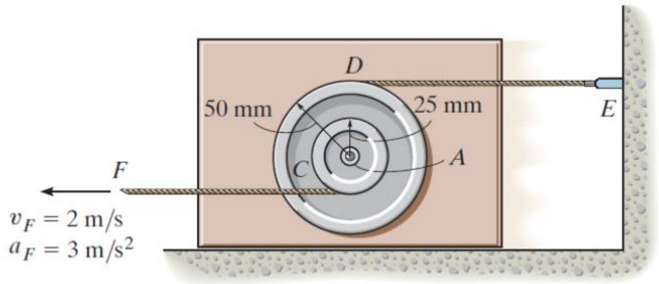
Acceleration analysis is pretty straightforward because it is what we already saw before.

$$\boldsymbol{a_B} = \boldsymbol{a_A} + \boldsymbol{\alpha} \times \boldsymbol{r_{B/A}} - \omega^2 \boldsymbol{r_{B/A}}$$

I want to point out that if IC is on the rope, there is no tangential acceleration (firs two terms) because of the reason explained before however, IC may have normal acceleration. This may sound nonsense and unintuitive but it is indeed possible.

Problem solving:

16–122. A single pulley having both an inner and outer rim is pin connected to the block at A. As cord CF unwinds from the inner rim of the pulley with the motion shown, cord DE unwinds from the outer rim. Determine the angular acceleration of the pulley and the acceleration of the block at the instant shown.

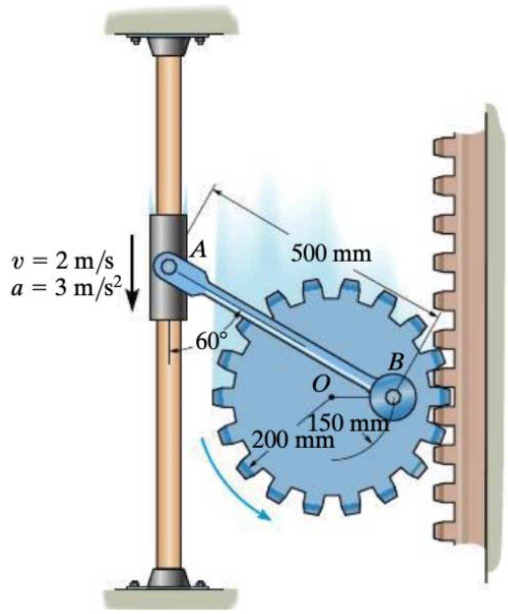


1) First of all we should start with the c.s. In this question, the standard x-y c.s and CCW moment is chosen to be positive.
2) Now, we have to find out the key points. First of all, we know that the acceleration of the block is the acceleration of the center of the pulley (also notice that the center point does not have a rotational motion, only translational). Finally, we know that the top rope is at rest hence point D is an IC however, the lower rope has a velocity so it is not an IC and point F has the velocity and acceleration given in the image.
3) To find the acceleration, we should find the angular velocity first. $v_F = v_D - 0.075\omega$ $\therefore \omega = -26.7 rad/s$
4) To write the acceleration between D and F, we need the normal acceleration at D (since we know that there is no tangential acceleration anyway) hence we need another equation as there are too many unknowns. Thus, we should write acceleration between C and D first: $\boldsymbol{a_C} = \boldsymbol{a_D} + \boldsymbol{\alpha} \times \boldsymbol{r_{C/D}} - \omega^2 \boldsymbol{r_{C/D}}$
5) $a_C = \begin{pmatrix} 0 \\ a_{D_y} \\ 0 \end{pmatrix} + \begin{pmatrix} 0.05\alpha \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 26.7^2 \cdot 0.05 \\ 0 \end{pmatrix}$ thus $a_{D_y} = -26.7^2 \cdot 0.05 = -17.8 rad/s^2$

6) $\boldsymbol{a_F} = \boldsymbol{a_D} + \boldsymbol{\alpha} \times \boldsymbol{r_{F/D}} - \omega^2 \boldsymbol{r_{F/D}} \rightarrow \begin{pmatrix} -3 \\ a \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -17.8 \\ 0 \end{pmatrix} + \begin{pmatrix} 0.075\alpha \\ 0 \\ 0 \end{pmatrix} +$

$\begin{pmatrix} 0 \\ 26.7^2 \cdot 0.075 \\ 0 \end{pmatrix} \therefore \alpha = -40 rad/s^2$ thus $a_C = \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix} ms^{-2}$

16–123. The collar is moving downward with the motion shown. Determine the angular velocity and angular acceleration of the gear at the instant shown as it rolls along the fixed gear rack.



1) Let us start with defining the c.s. I will use the standard x-y c.s. We should notice that the collard is constrained to move in the y-axis only. Also, we know that the connection between the gear and the rack is an IC. The gears velocity is measured at the center O and we can see that it is also only moving in the y-axis
2) It should be intuitive that we need to find v_B to find the asked quantities. However, to find v_B we need the angular velocity of AB hence we would need another equation. This hints us that we need to use our constraints. Whenever you are stuck with your equations, always try to write more equations involving the constraints of the question. This logic is the same as combining different concepts such as energy and impulse. Another way we could write v_B is relative velocity between O and B thus:

$$\boldsymbol{v_B} = \boldsymbol{v_A} + \boldsymbol{\omega_{AB}} \times \boldsymbol{r_{AB}} \rightarrow \begin{pmatrix} v_{B_x} \\ v_{B_y} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0.5\sin(60)\,\omega_{AB} \\ -0.5\cos(60)\,\omega_{AB} \\ 0 \end{pmatrix}$$

3) $\boldsymbol{v_O} = \boldsymbol{v_B} + \omega \times \boldsymbol{r_{O/B}} \rightarrow \begin{pmatrix} 0 \\ v \\ 0 \end{pmatrix} = \begin{pmatrix} v_{B_x} \\ v_{B_y} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -0.15\omega \\ 0 \end{pmatrix}$ now we would have to combine the equations: $\begin{pmatrix} 0 \\ v \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0.5\sin(60)\,\omega_{AB} \\ -0.5\cos(60)\,\omega_{AB} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -0.15\omega \\ 0 \end{pmatrix} \therefore \omega_{AB} = 0$

4) This means that $v_B = 2ms^{-1} \downarrow$

5) To find the angular velocity of the gear we could use the points B and IC since they are the only combination that is left: $\begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -0.05\omega \\ 0 \end{pmatrix} \rightarrow \omega = 40rad/s$

6) $\boldsymbol{a_O} = \boldsymbol{a_{IC}} + \boldsymbol{\alpha} \times \boldsymbol{r_{IC}} - 40^2\boldsymbol{r_{IC}} \rightarrow \begin{pmatrix} 0 \\ a_{O_y} \\ 0 \end{pmatrix} = \begin{pmatrix} a_{IC_x} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -0.2\alpha \\ 0 \end{pmatrix} + \begin{pmatrix} 320 \\ 0 \\ 0 \end{pmatrix} \therefore a_{IC_x} = -320rad/s^2$ and $a_{O_y} = -0.2\alpha$

7) $\boldsymbol{a_B} = \boldsymbol{a_A} + \boldsymbol{\alpha_{BA}} \times \boldsymbol{r_{BA}} \rightarrow \begin{pmatrix} a_{B_x} \\ a_{B_y} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \\ 0 \end{pmatrix} + \begin{pmatrix} 0.5\sin(60)\,\alpha_{AB} \\ -0.5\cos(60)\,\alpha_{AB} \\ 0 \end{pmatrix}$

8) $\boldsymbol{a_B} = \boldsymbol{a_O} + \boldsymbol{\alpha} \times \boldsymbol{r_{B/O}} - \omega^2\boldsymbol{r_{B/O}} \rightarrow \begin{pmatrix} 0.5\sin(60)\,\alpha_{AB} \\ -0.5\cos(60)\,\alpha_{AB} - 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -0.2\alpha \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -0.05\alpha \\ 0 \end{pmatrix} + \begin{pmatrix} -240 \\ 0 \\ 0 \end{pmatrix}$ solving the equation we find $\alpha_{AB} = -960rad/s^2$ and $\alpha = 8374rad/s^2$

5.3: EOM, translation

Since we learned how to calculate acceleration in rigid bodies, now we could involve forces too. In this lecture, I will give extremely important facts and methods that we will use throughout the rigid body lectures from now on. EOM of rigid bodies will be a 3 topic series.

The main logic is the same as particle dynamics. We draw the FBD and write the EOM. However, you may think, which acceleration value are we supposed to use since the acceleration on a rigid body may have different acceleration. For the sum of the forces, we will always use the acceleration of the center of mass.

However, we have a new addition to our EOM. Sum of the moments. We will formulate it as such: $\sum \boldsymbol{M_P} = \boldsymbol{r_{G*P}} \times m\boldsymbol{a_P} + I_P\boldsymbol{\alpha}$ where sum of the moments (both applied moments and moments created by forces) at point P is equal to the cross product of the displacement between from P to center of mass (COM) and acceleration at point P and scalar multiplication of the mass and scalar multiplication of second moment of inertia (MOI) times the angular acceleration of the body.

!!! Pay close attention to the values. From where to where they are because otherwise, you would compute wrong.

What you might have noticed is that if we choose our point P at COM, the cross-product term would be zero. What we will mostly do is to choose P either at COM or at the center or rotation. It does not matter at which point you choose it, what we do is trying to simplify the expression by choosing P at the best position possible.

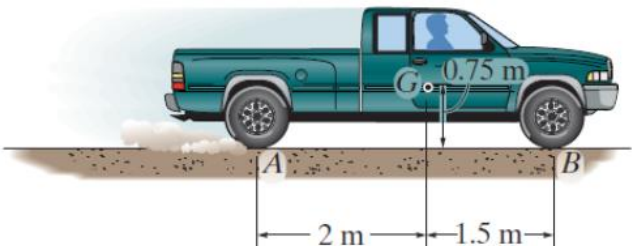
! Do not forget to apply the parallel axes theorem when you calculate MOI.

Radius of gyration is defined as $k = \sqrt{\frac{I}{m}}$ you can use it to calculate MOI quickly.

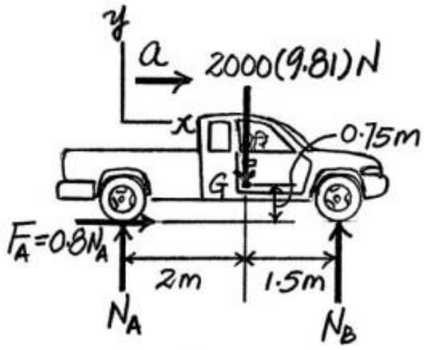
So, as the name suggests, this lecture is only for translational motion which also means sum of the moments is zero. Now, let us solve questions and I will give the important hints while solving them.

Problem solving:

17–29. Determine the shortest time possible for the rear-wheel drive, 2-Mg truck to achieve a speed of 16 m/s with a constant acceleration starting from rest. The coefficient of static friction between the wheels and the road surface is $\mu_s = 0.8$. The front wheels are free to roll. Neglect the mass of the wheels.



1) We need to establish a c.s first. I choose our standard x-y c.s
2) Second thing is of course drawing the FBD:



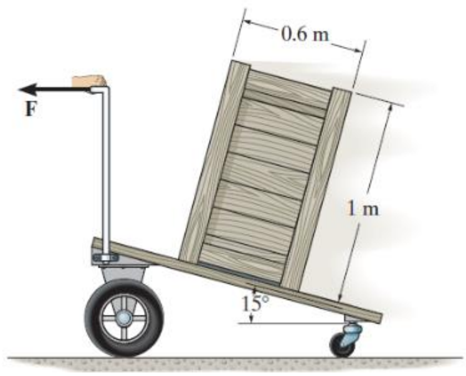
3) As you might have noticed, there is no friction acting on the front wheel. The reason is because there is no torque acting on the front wheel. In order to have static friction, we need the system to be in pure rolling (no slipping) and there must be a torque causing rotation of the object (wheel in this case). If there is no torque, then there is no static friction either. The second thing is why the direction of friction is to the right (direction of the car). This is simply because what makes the car move is static friction. We will discuss how to figure out the direction of friction much more in detail later. In this question, just know that the friction is the force that makes the car accelerate to the right
4) $\sum F_y = N_A + N_B - 2000 \cdot 9.81 = 0$ clearly no acceleration in the y axis
5) $\sum F_x = 0.8N_A = 2000a_x$

6) $\sum M_G = 0.75 \cdot 0.8N_A + 1.5N_B - 2N_A = 0$ since the car does not rotate

7) Solving the equations, we find $a_x = 4.059ms^{-2}$

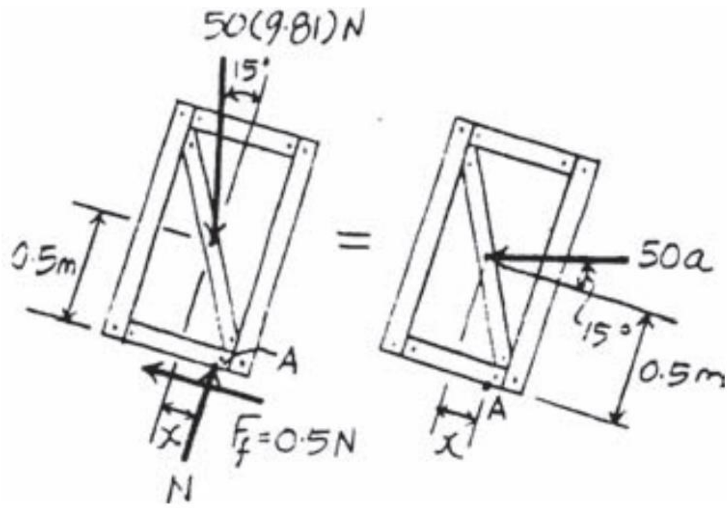
8) Since this acceleration is constant, we could use SUVAT: $\frac{16}{4.059} = 3.94s$

17–47. The uniform crate has a mass of 50 kg and rests on the cart having an inclined surface. Determine the smallest acceleration that will cause the crate either to tip or slip relative to the cart. What is the magnitude of this acceleration? The coefficient of static friction between the crate and cart is $\mu_s = 0.5$.



1) Let us define a c.s first. I will use the tilted x-y c.s since the motion of the crate is in that direction

2) Now, we should go with the standard procedure and draw the FBD and KD:



3) Drawing of the FBD should be fairly easy. The static friction is in the negative direction because since the cart moves to the left, a force must make the crate move in

the same direction. Static friction is the only force that can do that, otherwise the crate would just fall off.

4) The key point in this diagram is the direction of acceleration in KD. Remember that KD shows the direction of acceleration. In this question, we know that the cart accelerates to the left and since are trying to figure out the acceleration when the crater does not slip wrt the cart. This means that the cart and the crater must have the same acceleration. To see this, think of inertial frame again and try to animate their motion.
5) Let us assume the crater did not tip but slipped first. Then we would have to decompose the acceleration onto our c.s:
6) $\sum F_y = ma_{G_y} = N - 50 \cdot 9.81 \cdot \cos(15) = -50a \sin(15)$
7) $\sum F_x = ma_{G_x} = 50 \cdot 9.81 \sin(15) - 0.5N = -50a \cos(15)$
8) Solving them gives us $N = 448N$ and $a = 2.01ms^{-2}$
9) Now we will check whether the crate tips. To do that, we could simply check where the normal and friction force acts. Since we already know their magnitude, we would only have to figure out where on the crate they act. The point where they act is where the crate is rotating around since normal and friction force act at the contact point. This means that the furthest contact point possible is the edge of the crate. If the distance we get is further away than the edge, then it means our crate tipped over. If not, it means that we still have a contact point hence we did not tip over. I will choose the contact point as my point where I will take the moments around since it would cancel out friction and normal force, leaving the acceleration and weight terms left. The angular acceleration is zero because we are assuming that the crate did not tip over. Hence, there is no angular motion
10) $\sum M_A = 50 \cdot 9.81 \cdot \cos(15) \cdot x - 50 \cdot 9.81 \cdot \sin(15) \cdot 0.5 = 50\text{a} \cos(15) \cdot 0.5 + 50a \sin(15)\, x$
11) Solving this gives us $x = 0.25m$ which is before reaching the edge hence the crate did not tip but slipped first. Since at this acceleration value, the crate slips but does not tip.

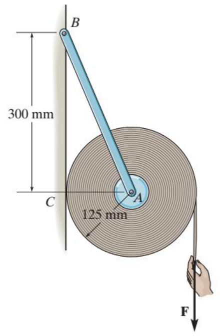
6.1: Kinetics of a rigid body, Rotation about a fixed axis

This chapter is pretty much the same as the previous chapter. The only difference is that we are now interested in fixed-axis rotational motion. The most important section will be the next section where we will talk about general plane motion, which means the combination of 5.3 and 6.1.

Let us dive into the problems

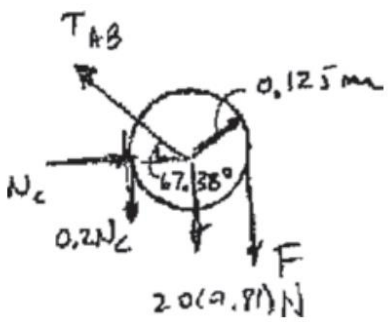
Problem solving:

17–63. The 20-kg roll of paper has a radius of gyration $k_A = 90$ mm about an axis passing through point A. It is pin supported at both ends by two brackets AB. If the roll rests against a wall for which the coefficient of kinetic friction is $\mu_k = 0.2$, determine the constant vertical force F that must be applied to the roll to pull off 1 m of paper in $t = 3$ s starting from rest. Neglect the mass of paper that is removed.



Prob. 17–63

1) First of all let us define a c.s, in this question the typical x-y c.s seems easy to work with. Again, to figure out which c.s to use you need to solve many questions and look at what you are given
2) Let us draw the FBD now:



3) Now, EOM should be written. What we know is that the COM of the roll does not move. The forces T_{AB} and T_{BA} are two force members hence they act in the direction of the rod which means we can decompose the force by using the angle $67.4°$
4) The second important point with the FBD we should notice is the direction of friction. We know that we have kinetic friction and friction always opposes motion. In this case COM does not move however, by looking at the system, we see that the force F pulls the paper down (which means it created a moment) and hence the paper at point C has to go up otherwise the paper would be extensible. Thus, the friction opposes the motion of that point and hence it points downwards.

5) $\sum F_x = N - Tcos(67.4) = 0$
6) $\sum F_y = Tsin(67.4) - 0.2N - 20 \cdot 9.81 - F = 0$
7) $\sum M_G = 1.25(0.2N - F) = I\alpha$ I chose at COM because it cancels out the most moment arms and hence simplifies the expression
8) We need one more known to solve the question and since we are given a kinematic relation, we could find the angular acceleration since we know that it is constant: $1 = \alpha \cdot \frac{3^2}{2} \rightarrow \alpha = 1.78 rad/s^2$
9) Solving the equations, we find $F = 22N$

6.2: Rigid body kinematics, general plane motion:

The final part of rigid body kinematics. Now, we will look at when all the acceleration values are non-zero. Hence, there is no new concept other than what we saw in 5.3 however, I will talk difference between kinetic and static friction and how to figure out their direction.

First of all, you should know that maximum static friction is always bigger than kinetic friction. Also, static friction may have a value between zero and a maximum value. Many students struggle with finding the direction of friction. When it is taught well, it is quite easy in fact.

First of all, always keep in mind that friction arises due to relative motion between the contact surfaces. Also, friction always opposes the relative motion, hence, friction acts in the opposite direction of the relative motion between the contact surfaces. Hence, all we have to do is to find the motion of the object wrt the surface. Before that, I would like to mention about slipping and non-slipping conditions. If an object is rolling without slipping, it means that it has a pure rolling motion and thus it has static friction. If an object is slipping, it means it has kinetic friction.

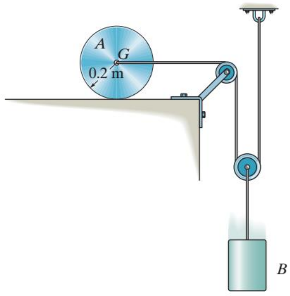
When we are trying to figure out the direction of friction, the method we follow is, we think of how the situation would be if there were no friction. Find the relative velocity of the contact point and then find the direction of friction.

What about when we do not know whether the object slips or not? In this case, what we do is to assume that the object does not slip, solve the equations for F_f which is the static friction. Then, compare this value to the maximum static force. If our static friction is smaller than the maximum static friction, then it is all fine and it means the object does not slip because there is no contradiction created. However, if friction force is bigger than the maximum static friction, there is a contradiction hence, it means that our initial assumption (that the object does not slip) is false and the object does slip. The same logic could be applied to kinetic friction analysis.

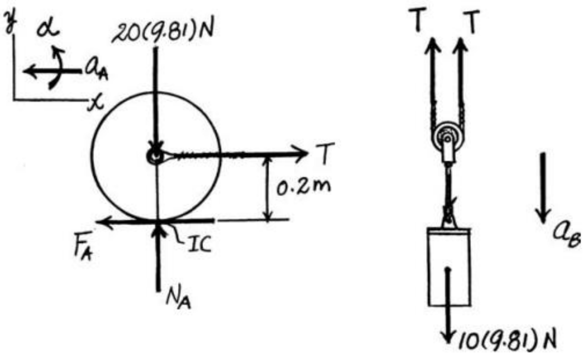
!!! Do not forget that in order to have static friction, we need to have a force or something that creates torque. If no torque is applied, there is no static friction either. This rule does not apply to kinetic friction though.

Problem solving:

*17–112. The 20-kg disk A is attached to the 10-kg block B using the cable and pulley system shown. If the disk rolls without slipping, determine its angular acceleration and the acceleration of the block when they are released. Also, what is the tension in the cable? Neglect the mass of the pulleys.

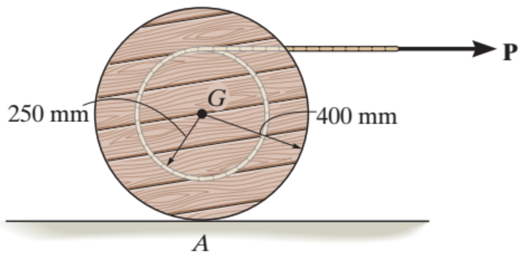


1) Standard x-y c.s would do the job. Let us draw the FBD and KD:

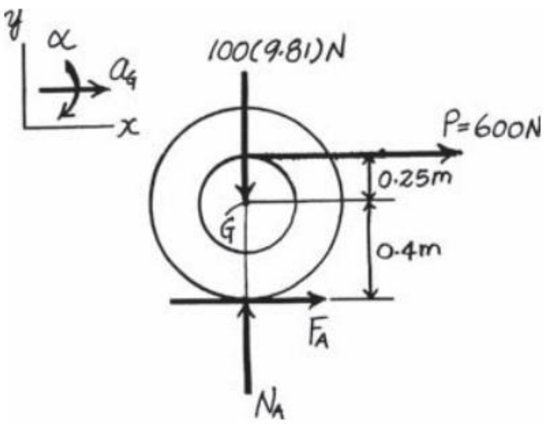


2) How did we determine the direction of friction? First of all, we know that it is static friction since the disk does not slip. Now, imagine if there was no friction. What would happen then? Because of the tension force, the disk would move to the right without rotating at all as it does not create moment (wrt the COM). Hence, the contact point would also move to the right, meaning, the friction opposes that and is to the left
3) There is of course no acceleration in the y-axis and only tension accelerates the disk
4) Let us look at the block B: $\sum F_y = ma_y : 2T - 98.1 = 10a_b$
5) $\sum M_{IC} : 0.2T = \alpha\left(20 \cdot 0.2^2 + \frac{1}{2}20 \cdot 0.2^2\right)$
6) To solve our equations, we need one more equation which is the relation between angular and linear acceleration. Previously we showed that $a = 0.2\alpha$ if we look at the relative acceleration between IC and center O
7) Solving all the equations, we get $T = 45.3N$

17–103. The spool has a mass of 100 kg and a radius of gyration $k_G = 0.3$ m. If the coefficients of static and kinetic friction at A are $\mu_s = 0.2$ and $\mu_k = 0.15$, respectively, determine the angular acceleration of the spool if $P = 600$ N.



1) Standard x-y c.s is good. Let us find the MOI first: $I = 0.3^2 \cdot 100 = 9kgm^2$
2) Following the standard procedure, let us draw the FBD and KD:



3) In this question friction is to the right because. Imagine if there was no friction. Then, the force P would create a moment and hence the spool would start rotating CW thus, at the contact point with the surface, the velocity would be to the left (notice we do not really care the velocity of the ground as it is zero hence, the relative velocity is simply the velocity of the spool at contact point). Thus the friction opposes leftwards motion and it is to the right
4) Since we do not know whether the spool slips, we have to check it. Let us assume it does not slip. $N = 981N$ and $\sum F_x = 600 + F_f = 100a_x$
5) $\sum M_g = 600(0.25) - F_f(0.4) = \alpha(100(100(0.3)^2)$
6) Since we assume it does not slip, at IC the acceleration is zero since the ground is stationary. Thus $a_x = 0.4\alpha$
7) Solving the equations, we get $F_f = 24N$ and maximum static friction is $196N$

8) Since our friction is smaller than the maximum static friction, we know that our assumption was right and the spool does not slip. The according angular acceleration is $15.6 rad/s^2$

6.3: Rigid body energy

Rigid body energy is pretty similar to particle dynamics energy. The only difference is that we may now have rotational kinetic energy. Previously we only had linear velocity thus linear kinetic energy but if a body rotates as well, we should also include its rotational kinetic energy. I will explain the concept on the formulas:

We still use our main energy equation: $T_1 + \sum U = T_2$ and in this case, our T (kinetic energy) is $T = \frac{1}{2}mv_G^2 + \frac{1}{2}I_G\omega^2$ notice how we perform our calculations at COM. However, there is a simplification exists, if we take our values at the center of rotation (the point where the body rotates around, for example, IC in a wheel rolling motion) $T = \frac{1}{2}I_O\omega^2$

Work done by gravity, spring, etc. All stay the exact same from before, however, I would like to go over work done by a force again and newly, work done by a moment.

Work done by a force: $U = F \cdot s \cdot \cos(\theta)$ but notice that on a rigid body, all the points do not travel the same amount hence, you should calculate how/how much the point the force acts on the body travels. The points travel ≠ travel of the COM

What is new is work done by purely moment applied. It is important to understand that this moment is not the moment created by a force but an external moment applied. Because, if you add moment created by the force and work done by the force, you would simply count the same work twice which would give a faulty answer.

Work done by a moment: $U = \int_{\theta_1}^{\theta_2} M d\theta$ hence we need to know over what angle the moment is applied.

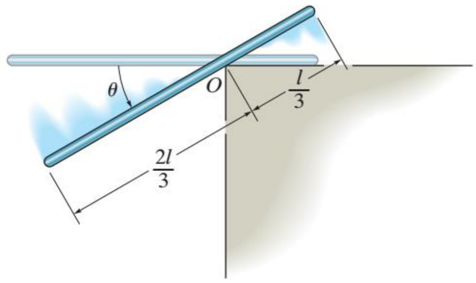
The last important point is to notice how some forces do no work. For instance, all forces act at the IC do no work since IC do does not move hence there is no displacement at IC. Normal forces, the forces that are perpendicular to the displacement also do no work.

How to know when to use concept of energy? As I said previously, look at what you are given and check whether there are any energy losses or energy calculations that are not possible to use. If so, energy may not be the best option. However, you could always write all concepts easily since there are 4 in total. Velocity-acceleration analysis, EOM, energy and momentum. Thus, if you are stuck and do not know what to do, you could write all of them down and see what works and how to combine the information.

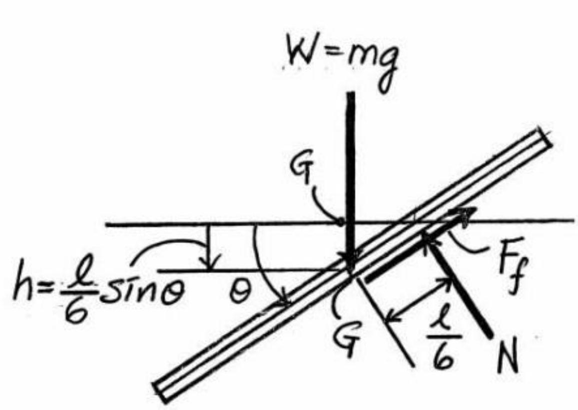
Problem solving:

18–34. The uniform bar has a mass m and length l. If it is released from rest when $\theta = 0°$, determine its angular velocity as a function of the angle θ before it slips.

18–35. The uniform bar has a mass m and length l. If it is released from rest when $\theta = 0°$, determine the angle θ at which it first begins to slip. The coefficient of static friction at O is $\mu_s = 0.3$.



1) Let us start from 18.34 since we need it to solve 18.35. First of all we need a c.s. If we look at the motion the bar, we can see that the contact point O is the center of rotation and the COM of the bar has a circular motion. Thus, n-t c.s seems the best option since it is an instantaneous c.s thus x-y c.s is not the best option and polar c.s is clearly not useful.
2) Let us draw the FBD:



3) It should be clear which forces act on the body. However, the direction of friction may not be intuitive. Think like this, from the standing point to the point where it creates an angle, the x-position of the right side of the bar decreased hence it moved to the left. This means that the contact point is moving the left and thus friction opposes this

motion and thus it is rightward. Since the contact point is O which is an edge, the normal force acts perpendicular to the bar and friction tangential.

4) We know that the forces besides gravity does no work since the bar does not slip, meaning the contact point is an IC. This hints us to use energy
5) Let us calculate I_O then to simplify the expression: $I_O = \frac{ml^2}{9}$
6) $\frac{mgl}{6}\sin(\theta) = \frac{1}{18}ml^2\omega^2$ thus $\omega = \sqrt{\frac{3gsin(\theta)}{l}}$
7) Now let's being the second part of the question. We are asked to find the angle when it starts slipping. But how can we relate slipping to our equations? Remember that after the static friction coefficient reaches its maximum value, then the friction changes from static to kinetic friction which also means slipping starts. Thus, if we find the angle when the static friction coefficient is at its max, we can solve the problem.
8) The only way we can relate friction is by writing the EOM, so let us begin:
9) $\sum F_n = ma_{G_n} = \mu N - mgsin(\theta)$ but $a_{G_n} = \frac{mv^2}{r} \therefore a_{G_n} = m\left(\frac{3gsin(\theta)}{l}\right)\left(\frac{l}{6}\right)$ hence $\mu N = 1.5mgsin(\theta)$
10) $\sum F_t = ma_{G_t} = mgcos(\theta) - N$ thus we need to find a_{G_t} to find N to find μ
11) From before we know that if we could find angular acceleration we could use relative acceleration analysis to find tangential acceleration since at IC the tangential acceleration is zero thus: $\sum M_O: \frac{mgcos(\theta)}{6} = \alpha\left(\frac{ml^2}{12} + \frac{ml^2}{36}\right) \therefore \alpha = \frac{3gcos(\theta)}{2l}$ hence we know that $a_{G_t} = \alpha\left(\frac{l}{6}\right)$
12) Solving the equations, we find that $\mu = \frac{1.5}{0.75}\tan(\theta) \therefore \theta = 8.5°$ as it can be seen from the relationship, the bigger the angle the bigger the static friction however since 0.3 is the maximum static friction coefficient, after that angle the bar would slip

7.1: Rigid body momentum and impulse

I will not go through the basics of momentum since it was explained previously. As you may have noticed already, in rigid body dynamics we always have an extra angular property addition on top of a linear one.

In rigid body dynamics, linear momentum is defined as $\boldsymbol{L} = m\boldsymbol{v_G}$ notice how the velocity is the velocity of the COM. The rest of the theories are the same as before.

Angular momentum is defined as: $\boldsymbol{H_P} = I_G\boldsymbol{\omega} + \boldsymbol{r_{G/P}} \times (m\boldsymbol{v_G})$ remember how the expression simplifies if we take our point at COM.

What about impulse?

Linear impulse is the same: $\int_{t_1}^{t_2} \boldsymbol{F}dt$

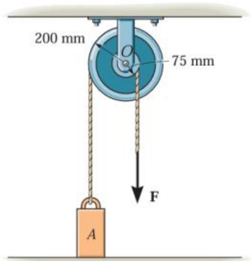
Angular impulse: $\int_{t_1}^{t_2} (\boldsymbol{r} \times \boldsymbol{F})dt = \int_{t_1}^{t_2} \boldsymbol{M}dt$

How do we add the impulse created by a force that also creates moment?

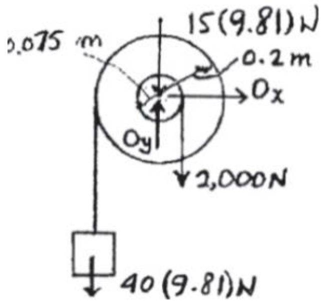
In energy, we learned that we should not add both linear and angular work created by a force. Because this makes us count the same work twice. However, in momentum, we add both of them. Because in momentum, a force both creates angular momentum and linear momentum. However, in energy, when we add the linear work done by a force, since the equation has angular velocity term, it includes both linear and angular work done.

Problem solving:

19–18. The double pulley consists of two wheels which are attached to one another and turn at the same rate. The pulley has a mass of 15 kg and a radius of gyration $k_O = 110$ mm. If the block at A has a mass of 40 kg, determine the speed of the block in 3 s after a constant force $F = 2$ kN is applied to the rope wrapped around the inner hub of the pulley. The block is originally at rest. Neglect the mass of the rope.



1) Firstly let us define a c.s. The standard x-y c.s works for this question. Draw the FBD



2) From the given information, it can be noticed that momentum is applicable in this question. The block has a linear momentum and the pulley has an angular momentum. Thus, two seperate equations should be written. Let us start with the block.
3) We know that the block gows upwards and tension force in the cable and the gravity does linear impulse: $T(3) - 40(9.81)(3) = 40v$
4) Tension and the force applied creates angular impulse. Notice that the tension acts upwards on the pulley: $2000(0.075)(3) - T(0.2)(3) = 0.11^2(15)\omega$
5) Now, we can see that angular and linear velocity must be related to solve the question. Also notice that since the chord is inextensible, the point where the chord is connected to the pulley has the same velocity as the block. Since the pulleys COM is stationary: $v = 0.2\omega$
6) Solving the equations we find that $v = 24.1ms^{-1}$

7.2: Conservation of momentum in rigid body dynamics

This section is just a small extension of the previous chapter however there are some extremely important hints and analysis methods you need to solve problems.

First of all, when do we know that the momentum is conserved? In other words when we know that the forces are impulsive?

If we are not told, what we can do is to check whether -usually object(s) crashing- the impact Happen very quickly. After solving problems, this will be more intuitive.

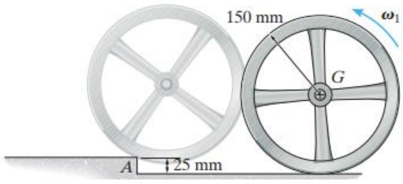
Second thing is the point where we will write the momentum equation from. At first it may seem like we would any point we want, or most likely the COM to avoid the second term of the momentum formula however, no. Every time we analyze a system, we will choose the pivot point -the point where the system rotates around.

Final important analysis method is following how the velocity changes at the pivot position during the impact. During the impact, the momentum will be conserved however, the (angular) velocity will of course change. Hence, at the same moment we get two different velocities. Do not forget to draw that diagram to perform the calculations errorless.

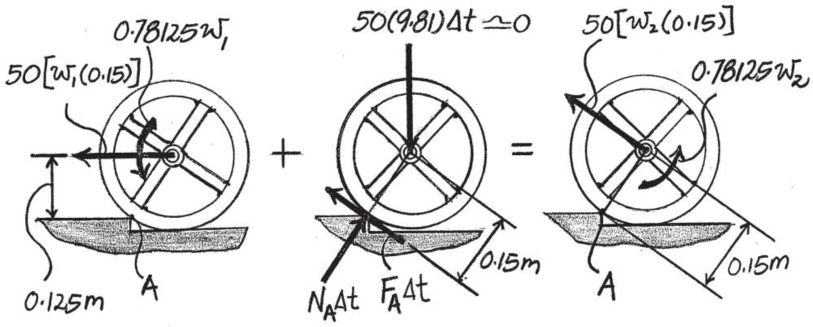
To understand how these concepts are applied, let us solve some questions for the last time.

Problem solving:

19–49. The wheel has a mass of 50 kg and a radius of gyration of 125 mm about its center of mass *G*. Determine the minimum value of the angular velocity ω_1 of the wheel, so that it strikes the step at *A* without rebounding and then rolls over it without slipping.



1) Firstly the c.s of course. I will use the standard x-y c.s. What we need to figure out from the minimum velocity information is that the wheel should stop right after it is over the bump. Also MOI $I_G = 0.125^2(50) = 0.781kgm^2$
2) We know that the wheel pivots about the point A. Also, we know that momentum is conserved since the wheel rolls over without slipping. Let us consider the velocity change hence the momentum conservation at the impact. The image shows the velocity before and after the impact:



3) What we should notice is that the arm length before is 0.125m since the arm is perpendicular (since the velocity is horizontal) meanwhile the arm in the second situation is 0.15m as it can be seen from the geometry.
4) Let us write the conservation of momentum then:
5) $0.781\omega_1 + 50(0.125)(0.15\omega_1) = 0.781\omega_2 + 50(0.15)(0.15\omega_2)$
6) You may ask why the linear velocity is calculated as 0.15ω it is because if we calculate the velocity of the wheel before and after the velocity change:

7) Before the velocity change, the wheel is just rolling and since it does not slip, we know that it can be calculated by 0.15ω and right at the pivot point, this velocity is still the same. After the change at the pivot point, it now changes as it is shown in the diagram. Since it is now pivoting at point A, the arm is again the radius perpendicular to the velocity hence 0.15ω. However, notice that they are not the same velocity.
8) To solve for ω_1 we need to find ω_2. Here, we could use the energy as it should be intuitive from the given. Since the wheel does not slip, there is no loss to energy either thus: $mgh = \frac{I_G\omega_2^2}{2} + \frac{m(0.15\omega_2)^2}{2}$ Solving the equations, we find $\omega_1 = 3.98 rad/s$