Differential Equations

Order: Order of the highest derivative in the equation.
Degree: The power to which the highest order derivative is raised.
Example:

For the below equation, Order: 2, Degree: 1

$$\frac{d^2y}{dx^2} + 5(\frac{dy}{dx})^3 - 4y = e^x$$

Linearity

For an equation to be linear, the **power of all derivatives should be be 1** and the **coefficients of the derivatives (on LHS) and the RHS should be functions of only the independent variable**.

Population Dynamics Example

The rate of increase/decrease of a population at time t is proportional to the size of the population at time t.

Increase: $\frac{dp}{dt} = kp$
Decrease: $\frac{dp}{dt} = -kp$

Types of Solutions

Explicit Solution

A function that when substituted for y, satisfies the equation for all 'x' in an interval 'I' is called an explicit solution to the equation. The interval 'I' is the domain of the solution.

Implicit Solution

A relation G(x, y) = 0 is said to be an implicit solution on an interval 'I' if it defines one or more explicit solutions.
Steps to solve:

- Does the relation satisfy the ODE?
  - G(x,y) = 0
  - $\frac{d}{dx}(G(x,y)) = \frac{d}{dx}(0)$
- Does the relation define one or more explicit solutions?

    - Make y the subject in G(x,y)
    - Find the domain
- Check the interval
    - Fix the domain

Existence and Uniqueness of a Solution

f(x, y) and $\frac{\partial f}{\partial y}$ should be continuous for a unique solution to exist.

Separation of Variables

1. Bring 'y' to the left and 'x' to the right.
2. Integrate both sides with respect to the variables on each side.

$$\frac{dy}{dx} = g(x) \cdot p(y)$$

$$\int \frac{1}{p(y)} dy = \int g(x) dx$$

$$P(y) = G(x) + C$$

Solving Linear Equations

*Refer above to remind yourself what linear differential equations are

First order linear equation can be expressed as:

$$a_1(x)\frac{dy}{dx} + a_o(x)y = b(x)$$

Changed to the standard form:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

$$\text{where P(x)=}\ \frac{a_o}{a_1}\ \text{and Q(x)=}\ \frac{b(x)}{a_1}$$

<u>Steps to solve:</u>

1. Write the equation in the standard form:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

2. Calculate the integrating factor μ of the formula:

$$\mu(x) = e^{\int P(x)dx}$$

3. Compute the solution:

$$y(x) = \frac{1}{\mu(x)} \int \mu(x)Q(x)dx$$

Existence and Uniqueness of a Solution

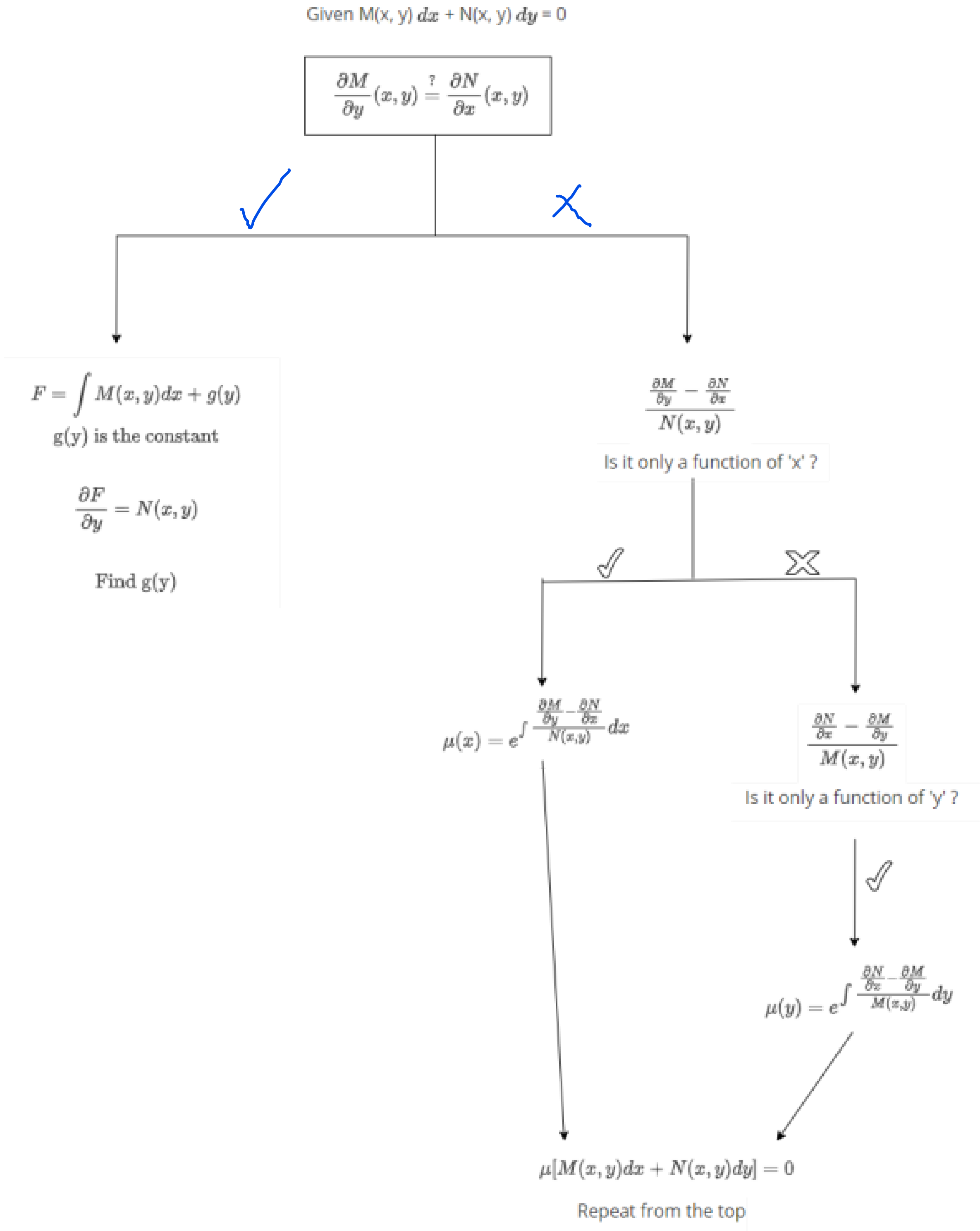
If P(x) and Q(x) are continuous on an interval (a, b) that contains the point 'x_o', then for any initial value 'y_o', then there exists a unique solution.

Solving Exact Equations

Given z = F(x, y),

$$dz = \frac{\partial F}{\partial x} \cdot dx + \frac{\partial F}{\partial y} \cdot dy$$

<u>Steps to solve:</u>



Substitution and Transformation

Homogeneous Equation

<u>Checking whether an equation is Homogeneous:</u>

If the right hand side of the equation $\frac{dy}{dx} = f(x,y)$ can be expressed as a function of the ratio $\frac{y}{x}$, the equation can be called *Homogeneous*.

- Shortcut: Prove that $f(tx, ty) = f(x,y)$, and the equation can be called *Homogeneous*.
- Shortcut: The power of each term should be the same (The power x^2 is 2, the power of xy is 2).
- Shortcut: Use your eyes.

Steps to solve:

1. Make the substitution: $v = \frac{y}{x}$

   $\frac{dy}{dx} = v + x\frac{dv}{dx}$

2. Hence, the homogeneous ODE changes to: $v + x\frac{dv}{dx} = G(v)$

   $G(v) = f(x, vx)$

3. Use Separable Method to solve the equation.

Equations of the form $\frac{dy}{dx} = G(ax + by)$

Steps to solve:

1. Make the substitution: $z = ax + by$
   $\frac{dz}{dx} = a + b\frac{dy}{dx}$
2. Blah, blah, blah (read steps from above)

Bernoulli Equation

If a first-order equation can be written in the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

where P(x) and Q(x) are continuous on an interval (a, b) and n is a real number, the equation can be called a *Bernoulli Equation*.

Steps to solve:

For n = 0, 1, use previous methods.

1. For $n \neq 1$, make the substitution: $v = y^{(1-n)}$.

   $\frac{dv}{dx} = (1-n)y^{-n}\frac{dy}{dx}$
2. The equation can then be written as: $\frac{1}{1-n}\frac{dv}{dx} + P(x)v = Q(x)$
3. Blah, blah, blah (read steps from above)

Real Life Applications

Newton's Law of Heating & Cooling

The rate of change of temperature T(t) is proportional to the difference between the temperature of the body T(t) and the surrounding temperature T_A.

$$\frac{dT}{dt} \propto (T(t) - T_A)$$

Compartment Analysis

Given that

R_{in}: Concentration of substance in inflow x input rate;

R_{out}: Concentration of substance in outflow x output rate = $\frac{x(t)}{V+(\text{input rate - output rate})t} \times$ output rate;

V: Volume and x(t): Concentration of substance at time 't'

$$\frac{dx}{dt} = R_{in} - R_{out}$$

$$\frac{dx}{dt} = R_{in} - \left(\frac{x(t)}{V + (\text{input rate - output rate})t} \times \text{output rate}\right)$$

Complementary and Particular Solution

Study Superposition yourself:
finding the particular solution using superposition

$ay'' + by' + cy = f(t)$

$y_c = C_1 y_1 + C_2 y_2 \longrightarrow$

① $y = e^{rt}$

Real: Distinct: $y_1 = e^{r_1 t}$ $y_2 = e^{r_2 t}$

Repeated: $y_1 = e^{rt}$ $y_2 = te^{rt}$

Complex: $y_c = e^{\alpha t}[C_1\cos(\beta t) + C_2\sin(\beta t)]$

② For $at^2y'' + bty' + cy = f(t)$,

$y = t^r$ $\therefore r^2(a) + r(b-a) + (c) = 0$

Real: Distinct: $y_1 = t^{r_1}$ $y_2 = t^{r_2}$

Repeated: $y_1 = t^r$ $y_2 = t^r \ln|t|$

Complex: $y_c = C_1 t^{\alpha}\cos(\beta\ln|t|) + C_2 t^{\alpha}\sin(\beta\ln|t|)$

$y_p \longrightarrow$

①

$f(t)$	y_p
t^3	At^3+Bt^2+Ct+D
e^t	e^t
$\sin(3t)$	$\sin(3t)+\cos(3t)$

② $y_p = v_1 y_1 + v_2 y_2$

where $v_1 = -\int \frac{f(t)\, y_2}{a[y_1 y_2' - y_1' y_2]}\, dt$

$v_2 = \int \frac{f(t)\, y_1}{a[y_1 y_2' - y_1' y_2]}\, dt$

Provided y_1, $y_2 = vy_1 \longrightarrow$ For $y'' + p(t)y' + q(t)y = 0$

$v = \int \frac{e^{-\int p(t)dt}}{y_1^2}\, dt$

Mechanical Oscillator

Mechanical Oscillator

$T = \frac{2\pi}{\omega}$ $f = \frac{1}{T}$

- Undamped: $\frac{d^2x}{dt^2} + \omega^2 x = 0$ $\omega^2 = \frac{k}{m}$

$$\boxed{\begin{aligned} x(t) &= C_1\cos(\omega t) + C_2\sin(\omega t) \\ x(t) &= A\sin(\omega t + \phi) \end{aligned}}$$

$x(t) = \underbrace{A\sin\varphi}_{C_1}\cos(\omega t) + \underbrace{A\cos\varphi}_{C_2}\sin(\omega t)$

$\frac{C_1}{C_2} = \tan(\varphi)$ $C_1^2 + C_2^2 = A^2$

- Damped: $\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x(t) = 0$ $2\lambda = \frac{\beta}{m}$ $F_d = -\beta\frac{dx}{dt}$

$r = -\lambda \pm \sqrt{\lambda^2 - \omega^2}$

① $\lambda^2 - \omega^2 > 0$ $x(t) = e^{-\lambda t}[C_1 e^{-\sqrt{\lambda^2-\omega^2}} + C_2 e^{\sqrt{\lambda^2-\omega^2}}]$ over damped system

② $\lambda^2 - \omega^2 = 0$ $x(t) = e^{-\lambda t}[C_1 + C_2 t]$ critically damped system

③ $\lambda^2 - \omega^2 < 0$ $x(t) = e^{-\lambda t}[C_1\cos(\sqrt{\omega^2-\lambda^2}\,t) + C_2\sin(\sqrt{\omega^2-\lambda^2}\,t)]$ under damped system

$e^{-\lambda t}$: Damping Friction

Laplace Transforms

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt$$

Table of Laplace Transforms:

A TABLE OF LAPLACE TRANSFORMS

$f(t)$	$F(s) = \mathscr{L}\{f\}(s)$	$f(t)$	$F(s) = \mathscr{L}\{f\}(s)$
1. $f(at)$	$\frac{1}{a}F\left(\frac{s}{a}\right)$	20. $\frac{1}{\sqrt{t}}$	$\frac{\sqrt{\pi}}{\sqrt{s}}$
2. $e^{at}f(t)$	$F(s-a)$	21. $\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{3/2}}$
3. $f'(t)$	$sF(s)-f(0)$	22. $t^{n-(1/2)}$, $n=1,2,\dots$	$\frac{1\cdot3\cdot5\cdots(2n-1)\sqrt{\pi}}{2^n s^{n+(1/2)}}$
4. $f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)$	23. t^r, $r>-1$	$\frac{\Gamma(r+1)}{s^{r+1}}$
	$-\cdots-sf^{(n-2)}(0)-f^{(n-1)}(0)$	24. $\sin bt$	$\frac{b}{s^2+b^2}$
5. $t^nf(t)$	$(-1)^nF^{(n)}(s)$	25. $\cos bt$	$\frac{s}{s^2+b^2}$
6. $\frac{1}{t}f(t)$	$\int_s^\infty F(u)\,du$	26. $e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$
7. $\int_0^t f(v)\,dv$	$\frac{F(s)}{s}$	27. $e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$
8. $(f*g)(t)$	$F(s)G(s)$	28. $\sinh bt$	$\frac{b}{s^2-b^2}$
9. $f(t+T)=f(t)$	$\frac{\int_0^T e^{-st}f(t)\,dt}{1-e^{-sT}}$	29. $\cosh bt$	$\frac{s}{s^2-b^2}$
10. $f(t-a)u(t-a)$, $a\ge0$	$e^{-as}F(s)$	30. $\sin bt - bt\cos bt$	$\frac{2b^3}{(s^2+b^2)^2}$
11. $g(t)u(t-a)$, $a\ge0$	$e^{-as}\mathscr{L}\{g(t+a)\}(s)$	31. $t\sin bt$	$\frac{2bs}{(s^2+b^2)^2}$
12. $u(t-a)$, $a\ge0$	$\frac{e^{-as}}{s}$	32. $\sin bt + bt\cos bt$	$\frac{2bs^2}{(s^2+b^2)^2}$
13. $\Pi_{a,b}(t)$, $0<a<b$	$\frac{e^{-sa}-e^{-sb}}{s}$	33. $t\cos bt$	$\frac{s^2-b^2}{(s^2+b^2)^2}$
14. $\delta(t-a)$, $a\ge0$	e^{-as}	34. $\sin bt\cosh bt - \cos bt\sinh bt$	$\frac{4b^3}{s^4+4b^4}$
15. e^{at}	$\frac{1}{s-a}$	35. $\sin bt\sinh bt$	$\frac{2b^2s}{s^4+4b^4}$
16. t^n, $n=1,2,\dots$	$\frac{n!}{s^{n+1}}$	36. $\sinh bt - \sin bt$	$\frac{2b^3}{s^4-b^4}$
17. $e^{at}t^n$, $n=1,2,\dots$	$\frac{n!}{(s-a)^{n+1}}$	37. $\cosh bt - \cos bt$	$\frac{2b^2s}{s^4-b^4}$
18. $e^{at}-e^{bt}$	$\frac{(a-b)}{(s-a)(s-b)}$	38. $J_\nu(bt)$, $\nu>-1$	$\frac{(\sqrt{s^2+b^2}-s)^\nu}{b^\nu\sqrt{s^2+b^2}}$
19. $ae^{at}-be^{bt}$	$\frac{(a-b)s}{(s-a)(s-b)}$		

*Unit step function, window function covered in the table [10-13]

*Extras: $\prod_{a,b}(t) = u(t-a) - u(t-b)$ $\qquad$ $\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}(\mathcal{L}\{f(t)\})$

Initial Value Problems:

I am just going to give the overall process:

1. Your equation is in 't'
2. Do laplace transform of the entire equation to get your equation in 's'

3. Make y(s) the subject and do laplace inverse of the entire equation to get equation in 't'
   *Tip: use partial fractions and completing the square wherever needed
4. Make y(t) the subject and DONE

Convolution Equation:

$$\mathcal{L}^{-1}\{F(s) \cdot G(s)\} = f(t) * g(t) = \int_0^t f(t-v)g(v)dv = \int_0^t g(t-v)f(v)dv$$

$$\mathcal{L}\{f(t) * g(t)\} = F(s) \cdot G(s)$$

Solving Linear System of Equations:

1. Do Laplace transform of your entire equation to get your equation in 's'
2. Your equation must look something like this:

$$a_1x(s) + a_2y(s) = a_3$$
$$b_1x(s) + b_2y(s) = b_3$$

3. Use Cramer's rule to find x(s),

$$\underbrace{\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}}_{\mathbf{A}} \begin{bmatrix} x(s) \\ y(s) \end{bmatrix} = \begin{bmatrix} a_3 \\ b_3 \end{bmatrix} \qquad \mathbf{A_1} = \begin{bmatrix} a_3 & a_2 \\ b_3 & b_2 \end{bmatrix}$$

$$x(s) = \frac{|\mathbf{A_1}|}{|\mathbf{A}|} = \frac{a_3b_2 - a_2b_3}{a_1b_2 - a_2b_1}$$

4. Do Laplace inverse of x(s) to get x(t)
5. Substitute x(t) into one of your equations (equations in the question), and find y(t)... you may need to use methods like the separable method
6. ~~Show off to your instructor that you have found x(t) and y(t)~~
7. ~~Wait for a 0 because you accidently wrote a + instead of a -~~