# Aerodynamics I

**Note 1** It is assumed that you have a clear understanding of (i) 1+1 (ii) each chapter of the course, as these notes merely serve as a medium of recall

**Note 2** I do not affirm that everything required can be found here

**Last Updated** Fall 2024

# Contents

# 1 Fundamentals

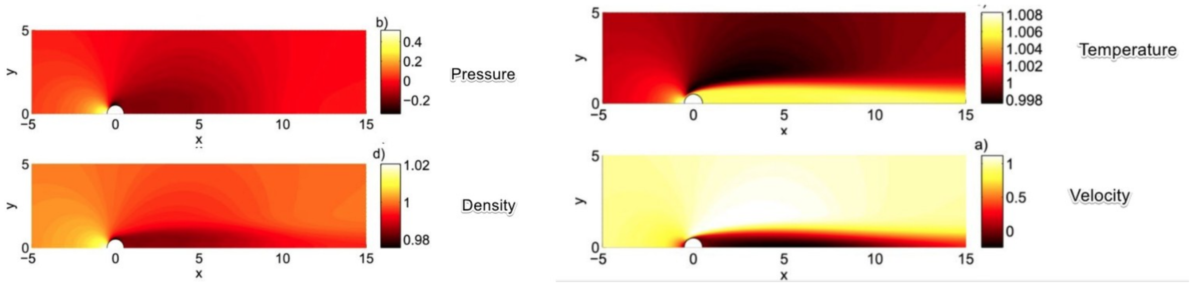
## 1.1 Variables

There are four variables that govern a flow field:

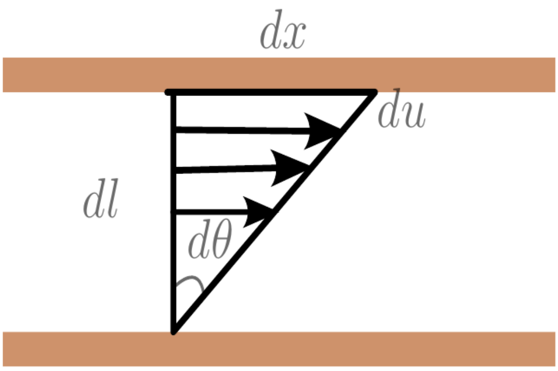
- Pressure
- Density
- Temperature
- Velocity

It should be mentioned that while temperature is a measure of kinetic energy, it is intrinsic: It deals with molecule to molecule. Velocity is extrinsic: It deals with a collection of molecules.

If the above is understood, the below figures should also be understood.




Close to velocity, there is another variable of importance: Viscosity.

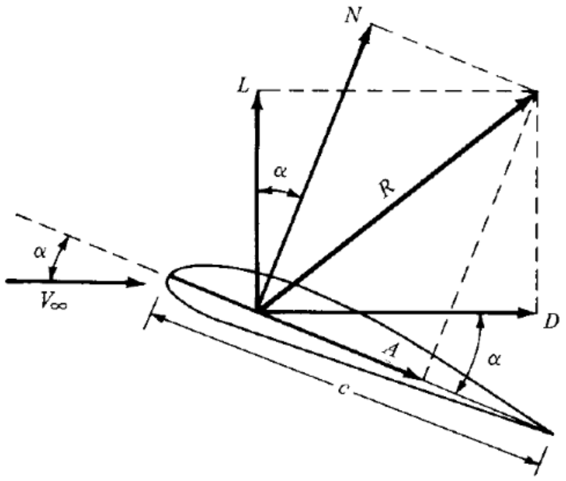



$$\tau \propto \frac{d\theta}{dt} \propto \frac{\frac{dx}{dl}}{dt} \propto \frac{du}{dl}$$

$$\tau = \mu \frac{du}{dl}$$

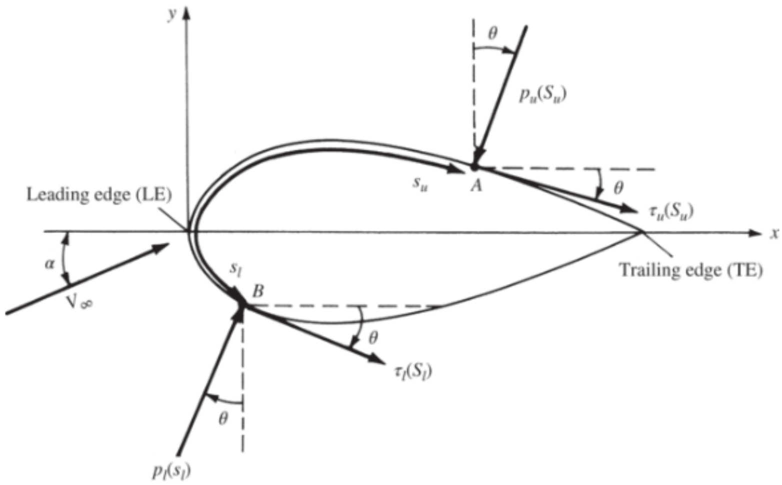
## 1.2 Force Distribution

Look at the plane flying in the air. The wing of the plane is acted upon by lift and drag forces or the resultant force.




Through baby algebra we get: $L = N\cos\alpha - A\sin\alpha$ and $D = N\sin\alpha + A\cos\alpha$. The meaning of each letter should be known to you by now. If not, please drop the course.

The resultant force is a result of pressure and shear stresses.




Through baby algebra we get:

$$N = -\int_{\mathrm{LE}}^{\mathrm{TE}} (p_u\cos\theta + \tau_u\sin\theta)ds_u + \int_{\mathrm{LE}}^{\mathrm{TE}} (p_l\cos\theta - \tau_l\sin\theta)ds_l$$

$$A = \int_{\mathrm{LE}}^{\mathrm{TE}} (-p_u\sin\theta + \tau_u\cos\theta)ds_u + \int_{\mathrm{LE}}^{\mathrm{TE}} (p_l\sin\theta + \tau_l\cos\theta)ds_l$$

$$\begin{aligned} M_{\mathrm{LE}} = &\int_{\mathrm{LE}}^{\mathrm{TE}} [(p_u\cos\theta + \tau_u\sin\theta)x - (p_u\sin\theta - \tau_u\cos\theta)y]ds_u \\ &+ \int_{\mathrm{LE}}^{\mathrm{TE}} [(-p_l\cos\theta + \tau_l\sin\theta)x + (p_l\sin\theta + \tau_l\cos\theta)y]ds_l \end{aligned}$$

In the process of understanding aerodynamics, we define certain dimensionless quantities called coefficients.

$$q_\infty = \frac{1}{2}\rho_\infty V^2$$

$$C_L = \frac{L}{q_\infty S} \qquad C_D = \frac{D}{q_\infty S}$$

$$C_N = \frac{N}{q_\infty S} \qquad C_A = \frac{A}{q_\infty S}$$

$$C_M = \frac{M}{q_\infty S l}$$

$$C_p = \frac{p - p_\infty}{q_\infty} \qquad C_f = \frac{\tau}{q_\infty}$$

Since we've developed dimensionless quantities, we might as well re-write our $N$, $A$, $M_{LE}$ equations.

$$c_l = c_N \cos\alpha - c_a \sin\alpha \quad \text{and} \quad c_d = c_n \sin\alpha + c_a \cos\alpha$$

We do this by first defining $ds^2 = dx^2 + dy^2$ (remember to take a positive angle from the $x$ axis in the direction of positive AoA), then we divide the equations with $q_\infty S$.

$$c_n = \frac{1}{c}\left[\int_0^c (C_{p,l} - C_{p,u})dx + \int_0^c \left(c_{f,u}\frac{\mathrm{d}y_u}{\mathrm{d}x} + c_{f,l}\frac{\mathrm{d}y_l}{\mathrm{d}x}\right)dx\right]$$

$$c_a = \frac{1}{c}\left[\int_0^c \left(c_{p,u}\frac{\mathrm{d}y_u}{\mathrm{d}x} - c_{p,l}\frac{\mathrm{d}y_l}{\mathrm{d}x}\right)dx + \int_0^c (c_{f,u} + c_{f,l})dx+\right]$$

$$\begin{aligned} c_{m_{LE}} = \frac{1}{c^2}\Bigg[ &\int_0^c (C_{p,u} - C_{p,l})x dx - \int_0^c \left(c_{f,u}\frac{\mathrm{d}y_u}{\mathrm{d}x} + c_{f,l}\frac{\mathrm{d}y_l}{\mathrm{d}x}\right)x dx \\ &+ \int_0^c \left(C_{p,u}\frac{\mathrm{d}y_u}{\mathrm{d}x} + c_{f,u}\right)y_u dx + \int_0^c \left(-C_{p,l}\frac{\mathrm{d}y_l}{\mathrm{d}x} + c_{f,l}\right)y_l dx\Bigg] \end{aligned}$$

Note the inclusion of $\frac{\mathrm{d}y}{\mathrm{d}x}$. This is manually added to account for the change in $y$ with $x$.

**Center of Pressure**

Since we love defining stuff, let's define a point through which the resultant force is said to act. We'll call this the center of pressure.

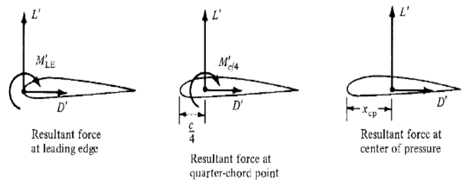
$$M_{LE} = -N x_{cp}$$

Hence, about our center of pressure, the aerodynamic moment is zero.

## Example: Evaluating Moments

Suppose that the resultant force acts at the leading edge. If we want to push this force to the center of pressure, what is $M_{LE}$?

$$M_{LE} = -Lx_{cp} = M_{\frac{c}{4}} - L\frac{c}{4}$$




### 1.3 Flows

Two flows are similar if they have

- similar geometric shapes
- same parameters:
  - $M = \frac{u}{c}$
  - $Re = \frac{\rho u L}{\mu}$

I shall introduce you to a new word: the mean free path ($\lambda$). This is the distance a molecule travels before it encounters a collision with another molecule.

If $\lambda >$ body, the flow is molecular. If $\lambda <<$ body, the flow is continuum. We assume—for obvious reasons—the latter.

Flows can be differentiated based on mass diffusion, viscosity, and thermal conduction.

Speaking of viscosity, we have two: viscous ($Re$ is small) and inviscous ($Re$ is large). While there is no truly inviscid fluid, we may assume that flow far above our wing is inviscid... for the sake of calculations.

# 2 A Touch of Math

The below aerodynamics assumes that the flow is in continuum. The Eulerian approach is used.

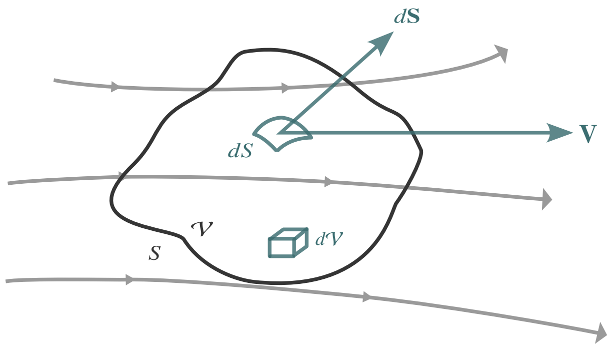
## 2.1 Understanding Mass

In aerodynamics, it is better to use the following definition of mass: $m = \iiint\limits_V \rho dV$

But sometimes, we might begin with $m = \rho V$ and later expand to the above.

## 2.2 The Continuity Equation

Draw the below shape in open air. This will be our control volume. We observe what goes in, and what goes out.




Since mass is coming in and leaving the control volume, I want to find the time rate of decrease of mass inside the control volume. Why? I make the decisions. Not you. On a serious note, you can decide to take the time rate of increase of mass, the only difference is in the polarity.

$$-\dot{m} = -\frac{\partial}{\partial t}\oiiint\limits_{\mathcal{V}} \rho d\mathcal{V} \xrightarrow{\text{For our Eulerian method, } \mathcal{V} \text{ remains the same}} -\dot{m} = -\oiiint\limits_{V} \frac{\partial \rho}{\partial t}$$

Now let's find the mass flowing through (into and out of) the control volume.

The vector normal to $dS$ is $\mathbf{n}$ (Remember that $d\mathbf{S} = \mathbf{n}dS$). The velocity of a fluid particle could be in any direction. So $\mathbf{V} \cdot \mathbf{n}$ represents the component of the velocity in $\mathbf{n}$. We shall call this $V_n$. For air entering, the dot product would be negative; for air leaving, the dot product would be positive.

$$d\dot{m} = \frac{1}{dt}\Big(\rho(V_n dt)dS\Big) = \rho(\mathbf{V} \cdot \mathbf{n})dS = \rho(\mathbf{V} \cdot d\mathbf{S}) = (\rho\mathbf{V}) \cdot d\mathbf{S} \quad \longrightarrow \quad \dot{m} = \oiint \rho\mathbf{V} \cdot d\mathbf{S}$$

Now let's utilize the fact that the mass flowing through the control volume is equal to the time rate of decrease of mass inside the control volume.

$$\oiint_S \rho \mathbf{V} \cdot d\mathbf{S} = -\oiiint_{\mathcal{V}} \frac{\partial \rho}{\partial t} d\mathcal{V} \qquad \longrightarrow \qquad \oiiint_{\mathcal{V}} \left[ \frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{V} \right] d\mathcal{V} = 0$$

The above Chocolate3 equation is called the *continuity equation*.

The only way for the above integral to be zero for an arbitrary control volume is for the integrand to be zero:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{V} = 0$$

## 2.3 The Momentum Equation

My lovely momentum is back. Momentum is defined as $\mathbf{F} = \dfrac{\mathrm{d}(m\mathbf{V})}{\mathrm{d}t}$.

Let's play with the derivative.

$$\begin{aligned}
\frac{\mathrm{d}(m\mathbf{V})}{\mathrm{d}t} &= m\frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t} + \mathbf{V}\frac{\mathrm{d}m}{\mathrm{d}t} \\
&= \frac{\partial}{\partial t} \oiiint_{\mathcal{V}} \rho \mathbf{V} d\mathcal{V} + \oiint_S (\rho \mathbf{V} \cdot d\mathbf{S})\mathbf{V}
\end{aligned}$$

Let's work out the forces acting on the control volume.

$$\begin{aligned}
F &= \text{gravity, magnetic, ...} + \text{Pressure} + \text{Viscous Forces} \\
&= \oiiint_{\mathcal{V}} \rho \mathbf{f} d\mathcal{V} - \oiiint_{\mathcal{V}} \nabla P d\mathcal{V} + \mathcal{F}_{\text{viscous}}
\end{aligned}$$

Note that pressure acts over a surface ($dS$), I skipped that step and applied the gradient theorem.

Compiling everything to one equation gives the *momentum equation*.

Considering just one component of velocity, say the **u** component, we can simplify the equation to

$$\oiiint_{\mathcal{V}} \left[ \frac{\partial (\rho u)}{\partial t} + \nabla \cdot (\rho u \mathbf{V}) - \rho f_x + \frac{\partial P}{\partial x} - (\mathcal{F}_{\text{viscous}})_x \right] d\mathcal{V} = 0$$

## 2.4 Substantial Derivative

First we must familiarize ourselves with an identity. Let $a$ be a scalar and $\mathbf{b}$ be a vector.

$$
\begin{aligned}
\nabla \cdot (a\mathbf{b}) &= \frac{\mathrm{d}(a\mathbf{b})}{\mathrm{d}x} + \frac{\mathrm{d}(a\mathbf{b})}{\mathrm{d}y} + \frac{\mathrm{d}(a\mathbf{b})}{\mathrm{d}z} \\
&= a\left[\frac{\mathrm{d}\mathbf{b}}{\mathrm{d}x} + \frac{\mathrm{d}\mathbf{b}}{\mathrm{d}y} + \frac{\mathrm{d}\mathbf{b}}{\mathrm{d}z}\right] + \mathbf{b}\left[\frac{\mathrm{d}a}{\mathrm{d}x} + \frac{\mathrm{d}a}{\mathrm{d}y} + \frac{\mathrm{d}a}{\mathrm{d}z}\right] \\
\nabla \cdot (a\mathbf{b}) &= a(\nabla \cdot \mathbf{b}) + \mathbf{b} \cdot \nabla a
\end{aligned}
$$

Take $\rho$. Remember that $\rho$ is a function of $(x,\ y, z,\ t)$ where $(x,\ y,\ z)$ are functions of $t$. Let's find its derivative with respect to $t$.

$$
\begin{aligned}
\frac{\mathrm{d}\rho}{\mathrm{d}t} &= \left(\frac{\partial \rho}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial \rho}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial \rho}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t}\right) + \frac{\partial \rho}{\partial t} \\
&= \left(\frac{\partial \rho}{\partial x}V_x + \frac{\partial \rho}{\partial y}V_y + \frac{\partial \rho}{\partial z}V_z\right) + \frac{\partial \rho}{\partial t} \qquad \text{where } \mathbf{V} \text{ is velocity} \\
&= (\nabla\rho \cdot \mathbf{V}) + \frac{\partial \rho}{\partial t} \\
\frac{D\rho}{Dt} &= \nabla\rho \cdot \mathbf{V} + \frac{\partial \rho}{\partial t}
\end{aligned}
$$

The capital $D$ indicates a substantial/material derivative: the time rate of change of a property of a fluid element as it moves through space.

Let's now re-write our equations to include the substantial derivative. I will be ignoring many steps as these can be done by baby algebra.

1. **Continuity Equation**

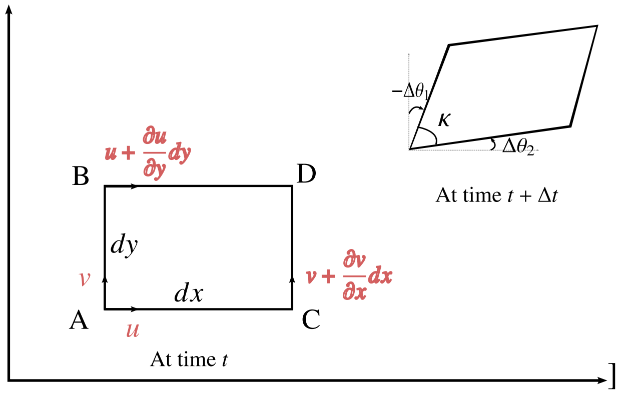
$$
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = \frac{\partial \rho}{\partial t} + \rho(\nabla \cdot \mathbf{V}) + \nabla\rho \cdot \mathbf{V} \qquad \longrightarrow \qquad \frac{D\rho}{Dt} + \rho(\nabla \cdot \mathbf{V}) = 0
$$

2. **Momentum Equation**

$$
\begin{aligned}
\frac{\partial \rho u}{\partial t} + \nabla \cdot (\rho u \mathbf{V}) - \rho f_x + \frac{\partial P}{\partial x} - (\mathcal{F}_{\text{viscous}})_x &= \rho\left(\frac{\partial u}{\partial t} + \nabla u \cdot \mathbf{V}\right) + u\left(\cancelto{0}{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V})}\right) - \rho f_x + \frac{\partial P}{\partial x} - (\mathcal{F}_{\text{viscous}})_x \\
&\Rightarrow \rho\frac{Du}{Dt} - \rho f_x + \frac{\partial P}{\partial x} - (\mathcal{F}_{\text{viscous}})_x = 0
\end{aligned}
$$

## 2.5 Distortion, Vorticity, Strain Rate

Take a fluid element of the shape of a rectangle. This is ABCD in the below diagram. After a certain amount of time, the fluid has distorted. We will be playing with coordinates now.




Let's find the new locations of points A, B and C.

$$
\begin{aligned}
(x_A,\ y_A)_{t+\Delta t} &= (x_A,\ y_A)_t + (u,\ v)\Delta t \\
(x_B,\ y_B)_{t+\Delta t} &= (x_B,\ y_B)_t + (u + \frac{\partial u}{\partial y}dy,\ v)\Delta t \\
(x_C,\ y_C)_{t+\Delta t} &= (x_C,\ y_C)_t + (u,\ v + \frac{\partial v}{\partial x}dx)\Delta t
\end{aligned}
$$

Let's now find the tangent of our two thetas.

$$
\tan(-\Delta\theta_1) \approx -\Delta\theta_1 = \frac{\left(u + \frac{\partial u}{\partial y}dy - u\right)\Delta t}{dy} \qquad \tan(\Delta\theta_2) \approx \Delta\theta_2 = \frac{\left(v + \frac{\partial v}{\partial x}dx - v\right)\Delta t}{dx}
$$

$$
\frac{\Delta\theta_1}{\Delta t} = -\frac{\partial u}{\partial y} \qquad \frac{\Delta\theta_2}{\Delta t} = \frac{\partial v}{\partial x}
$$

These resemble *angular velocities*. Since we are lazy, we consider the average for the angular velocity for our plane. And writing for 3D renders:

$$
\vec{\omega} = \frac{1}{2}\begin{bmatrix} \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \\ \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \\ \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \end{bmatrix}
$$

We do not like the presence of the half, so we define a new variable ($\vec{\xi}$) such that $\vec{\xi} = 2\vec{\omega} = \nabla \times \mathbf{V}$. We call $\vec{\xi}$ *vorticity*.

Look back at our deformed fluid element. Observe the angle $\kappa$. We call this the strain ($\epsilon$), and say that strain is positive when $\kappa$ is decreasing.

$$\text{Strain} = -\frac{\Delta\kappa}{\Delta t} = -(\Delta\theta_1 - \Delta\theta_2) = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

Writing the strain in 3D renders:

$$\vec{\epsilon} = \begin{bmatrix} \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\ \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \\ \frac{\partial v}{\partial z} + \frac{\partial w}{\partial x} \end{bmatrix}$$

All of this can be summarized by a Jacobian matrix:

$$J = \frac{\partial(u,\ v,\ w)}{\partial(x,\ y,\ z)} = \begin{bmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{bmatrix}$$

The diagonal of this matrix represents the divergence of velocity (dilatation), while the upper and lower triangles have partial fractions for the vorticity and strain.

## 2.6 Circulation

$\vec{\xi}$ deals with a tiny little fluid element. To find $\vec{\xi}$ over an entire surface, we integrate it over the surface. We call this integral $\Gamma$, the *circulation*.

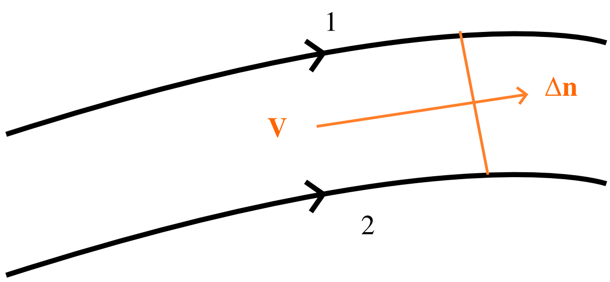
$$\Gamma = -\oint \mathbf{V} \cdot d\mathbf{r} \quad \text{or} \quad \Gamma = -\oiint \vec{\xi} \cdot d\mathbf{S}$$

In mathematics, the line integral is positive counterclockwise, but in aerodynamics, we ascribe positive to a clockwise motion. We thereby introduce a negative sign.

If a $dS$ region is chosen that is very small, so small that $\vec{\xi} \cdot \mathbf{n}$ is constant, we arrive at $\vec{\xi} \cdot \mathbf{n} = -\frac{\mathrm{d}\Gamma}{\mathrm{d}S}$.

## 2.7 Stream Functions in 2D

Take the below two streamlines 1 and 2. They are separated by a distance of $\Delta\mathbf{n}$. There is flow moving between them at a velocity of $\mathbf{V}$.




The direction of $\Delta\mathbf{n}$ is towards the top left.

Let's take $\mathbf{V}_i$ that is tangent to the curves at any point. Then $d\mathbf{r} \times \mathbf{V}_i = 0$. From this, we arrive at

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{v}{u} \quad \xrightarrow{solving} \quad f(x,y) = c$$

where $c$ is the constant of integration. We call $f$ the stream function.

Since $c$ is just some constant, we will give $f$ a better definition: $c$ will now represent the mass flow rate and $f$ will be re-written as $\bar{\psi}$. If the bar is present, we call it the mass flow rate, if the bar is absent, we call it the volume flow rate.

Now it would make sense to say that $\Delta\bar{\psi} = c_2 - c_1 = \dot{m}$.

$$\lim_{\Delta\vec{n}\to 0} \frac{\Delta\bar{\psi}}{\Delta\vec{n}} = \frac{\mathrm{d}\bar{\psi}}{\mathrm{d}\vec{n}} = \rho\vec{V}$$

This tells us that we can find $\rho V$ by differentiating $\psi$ in the direction normal to $\vec{n}$

$$\therefore d\bar{\psi} = \rho(udy - vdx)$$

The differential of $\bar{\psi}$ is $d\bar{\psi} = \bar{\psi}_x dx + \bar{\psi}_y dy$. Comparing the two equations we've got, we obtain

$$\rho u = \frac{\partial\bar{\psi}}{\partial y} \qquad \rho v = -\frac{\partial\bar{\psi}}{\partial x}$$

For incompressible flows, we can write

$$u = \frac{\partial\psi}{\partial y} \qquad v = -\frac{\partial\psi}{\partial x}$$

## 2.8 Velocity Potential

I will introduce you to an identity: $\nabla \times (\nabla\phi) = 0$. From this, *for irrotational flows*, we can write that $\mathbf{V} = \nabla\phi$—since $\vec{\xi} = \mathbf{0}$.

A better definition: For an irrotational flow, there exists a scalar function ($\phi$) such that $\mathbf{V}$ is a gradient of this function. Such a flow is also called a potential flow.

We are going to invent a new symbol: $\nabla^2$. Where, for a scalar $c$, $\nabla \cdot \nabla c = \nabla^2 c$. Therefore, $\nabla^2\phi = 0$. This is known as *laplace's equation*.

Proof:

$$\frac{\partial\rho}{\partial t} + \nabla \cdot (\rho\mathbf{V}) = 0$$

For a steady, incompressible flow,

$$
\begin{aligned}
\cancel{\frac{\partial \rho}{\partial t}} + \rho(\nabla \cdot \mathbf{V}) + \cancel{\mathbf{V} \cdot \nabla \rho} &= 0 \\
\nabla \cdot \mathbf{V} &= 0 \qquad \text{for a finite } \rho \\
\nabla \cdot \nabla \phi &= 0 \\
\nabla^2 \phi &= 0
\end{aligned}
$$

# 3 Incompressible and Inviscid Flow

## 3.1 Bernoulli's Equation

Let's begin with the *momentum equation* in its *substantial derivative* form—since I'm lazy, I shall consider only the $x$ component.

$$\rho\frac{Du}{Dt} = -\frac{\partial P}{\partial x} + \rho f_x + (\mathcal{F}_{\text{viscous}})_x$$

Grab your pens and start marking these assumptions: (1) Incompressible flow ($\nabla\rho = \vec{0}$) (2) Steady flow ($\frac{\partial}{\partial t} = 0$) (3) No body forces (4) No viscous forces.

$$\rho\frac{Du}{Dt} = -\frac{\partial P}{\partial x} \qquad \longrightarrow \qquad \left(\rho\left(\cancel{\frac{\partial u}{\partial t}} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial P}{\partial x}\right) \times dx$$

Recall the streamline equations ($d\mathbf{s} \times \mathbf{V} = 0$):

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{w}{u} \qquad\qquad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{v}{u}$$

Substituting this in our little equation renders

$$\rho\left(\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz\right) = -\frac{\partial P}{\partial x}dx \qquad \longrightarrow \qquad udu = -\frac{1}{\rho}\frac{\partial P}{\partial x}dx$$

Now we do some little math plays.

$$\frac{1}{2}\,\mathrm{d}u^2 = -\frac{1}{\rho}\frac{\partial P}{\partial x}dx \qquad \xrightarrow{\text{Summing all other components}} \qquad \frac{1}{2}\mathrm{d}(V^2) = -\frac{1}{\rho}\mathrm{d}P$$

Note that instead of writing $||\mathbf{V}||^2$, I am writing $V^2$ to make the equation look prettier.

$$\Delta P + \frac{1}{2}\rho(\Delta V)^2 = c$$

Using bernoulli's equation, we can re-write $c_p$ as $c_p = 1 - \left(\frac{V}{V_\infty}\right)^2$. This can be derived via baby algebra.

## 3.2 Laplace's Equation

For an incompressible and steady flow, from the continuity equation, we know that $\nabla \cdot \mathbf{V} = 0$.

Let's substitute the streamline equations in this. To accomodate for my laziness, we will consider just 2D.

$$\nabla \cdot \mathbf{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial x}\right) = 0$$

If the flow is irrotational, we know that $\nabla \times \mathbf{V} = 0$. Let's substitute the streamline equation in this.

$$\nabla \times \mathbf{V} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{\partial}{\partial x}\left(-\frac{\partial \psi}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial y}\right) = 0$$

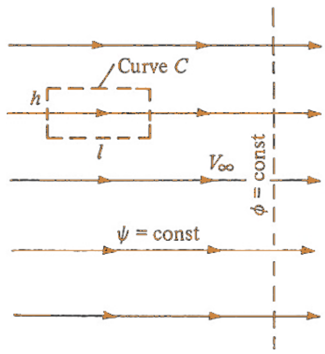
From these, we can conclude the following for an incompressible and irrotational flow:

1. The flow must have $\phi$ and $\psi$ that satisfies $\nabla^2\phi = 0$ and $\nabla^2\psi = 0$.
2. Any solution of $\nabla^2\phi = 0$ represents $\phi$ or $\psi$.

## Example: Uniform Flow

Find $\psi$, $\phi$ and $\Gamma$ such that $\nabla \cdot \mathbf{V} = 0$ and $\nabla \times \mathbf{V} = \vec{0}$

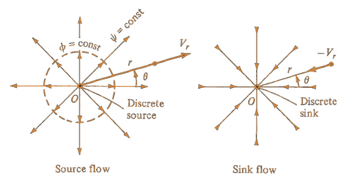
1. Set your boundary conditions for freestream:
   $u = \phi_x = \psi_y = V_\infty$ and $v = \phi_y = -\psi_x = 0$.
2. Set your surface conditions: $\mathbf{V} \cdot \mathbf{n} = 0$ or $\frac{\partial \phi}{\partial n} = 0$ or $\frac{\partial \psi}{\partial s} = 0$.
3. From our boundary conditions for freestream, we get $\phi(x, y) = V_\infty x + f(y)$ and $\phi(x, y) = c + g(x)$. By analysis, we get $\phi(x, y) = V_\infty x$.
4. From our boundary conditions for freestream, we can also get $\psi(x, y) = V_\infty y$.
5. $\Gamma = -V_\infty l - 0(h) + V_\infty l + (o)h = 0$. Or, $\Gamma = -\oiint_S \cancelto{0}{\xi} \cdot d\mathbf{S} = 0$.

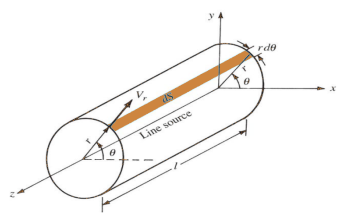



## Example: Source and Sink

Find $\psi$, $\phi$ and $\Gamma$ such that $\nabla \cdot \mathbf{V} = 0$ and $\nabla \times \mathbf{V} = \vec{0}$. Take $V_r = \frac{c}{r}$ and $V_\theta = 0$.

1. Find $c$:
   $\dot{m} = \rho V_r l r \int_0^{2\pi} d\theta = (\rho V_r l r)(2\pi)$
   $\Lambda = \frac{\dot{\forall}}{l} = \frac{\dot{m}}{\rho l} = 2\pi r V_r \quad \Rightarrow \quad c = \frac{\Lambda}{2\pi}$
2. Find $\phi$:
   $$\left.\begin{aligned} V_r = \phi_r \quad &\Rightarrow \quad \phi = \tfrac{\Lambda}{2\pi}\ln r + f(\theta) \\ V_\theta = \tfrac{1}{r}\phi_\theta \quad &\Rightarrow \quad \phi = \text{constant} + g(r) \end{aligned}\right\} \phi = \frac{\Lambda}{2\pi}\ln r$$
3. Find $\psi$:
   $$\left.\begin{aligned} V_r = \tfrac{1}{r}\psi_\theta \quad &\Rightarrow \quad \psi = \tfrac{\Lambda}{2\pi}\theta + f(r) \\ V_\theta = -\psi_r \quad &\Rightarrow \quad \psi = \text{constant} + f(\theta) \end{aligned}\right\} \psi = \frac{\Lambda}{2\pi}\theta$$
4. Find $\Gamma$:
   $$\Gamma = -\int_0^{2\pi} \cancelto{0}{V_\theta} d\theta = -\iint \cancelto{0}{\xi} \cdot d\mathbf{S} = 0$$

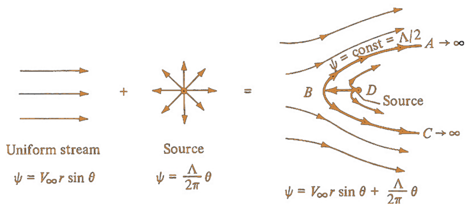


## Example: Superposition

Find $\psi$, $V_r$ and $V_\theta$ when the previous two flows are merged into one.

From the previous examples, we know that
$\psi_1 = V_\infty y = V_\infty r\sin\theta$ and $\psi_2 = \frac{\Lambda}{2\pi}\theta$
From superposition, we know that $\psi = \psi_1 + \psi_2$.
Therefore, $\psi = V_\infty r\sin\theta + \frac{\Lambda}{2\pi}\theta$
$V_\theta = -\psi_r = -v_\infty \sin\theta$ $\qquad V_r = \frac{1}{r}\psi_\theta = v_\infty\cos\theta + \frac{\Lambda}{2\pi r}$



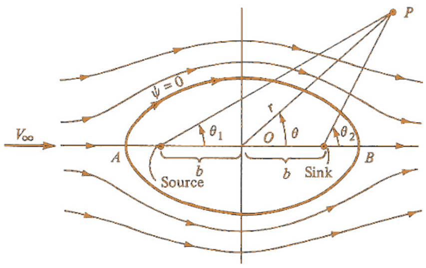

## Example: Rankine Oval

Find $\psi$ when the previous example is reflected across the y-axis and a sink and a source is on the other side.

Let's assume that the sink and source have the same strength. Why? My wish.
Following the previous example,

$$\psi = (V_\infty r\sin\theta) + \left(\frac{\Lambda}{2\pi}\theta_1\right) + \left(-\frac{\Lambda}{2\pi}\theta_2\right)$$

$$\psi = V_\infty r\sin\theta - \frac{\Lambda}{2\pi}\Delta\theta$$




## Example: Double Flow

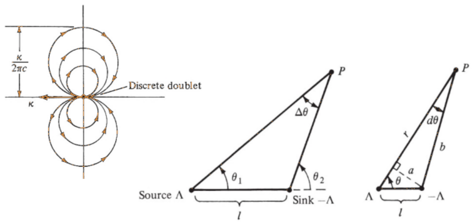
Find $\psi$ when a source and a sink are place at a distance $l$ ($l$ is veryyy small) from each other

First, study the right triangle carefully. I will be using much of its geometry.
From our previous examples, we know that

$$\psi = -\frac{\Lambda}{2\pi}\Delta\theta = -\frac{\Lambda}{2\pi}\frac{l\sin\theta}{r - l\cos\theta}$$

If $l \to 0$, but $\Lambda l$ = constant, I can define $\Lambda l = \kappa$.

$$\psi = -\frac{\kappa}{2\pi}\frac{\sin\theta}{r}$$



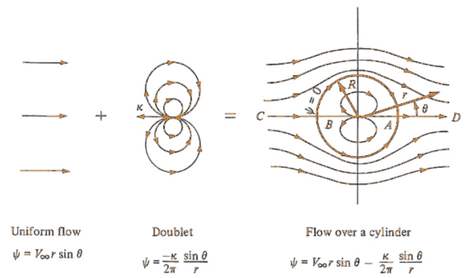
## Example: Non-Lifting Flow over a Cylinder

Find $\psi$, $V_r$ and $V_\theta$ when the above example is subjected to a uniform flow.

From the previous examples, $\psi = V_\infty r \sin\theta - \dfrac{\kappa}{2\pi}\dfrac{\sin\theta}{r}$
Let $\psi = 0$, we get that $r^2 = \frac{\kappa}{2\pi V_\infty}$. This is the circle separating the internal and external flow. I shall call this $R^2$.
$\therefore\ \psi = V_\infty r \sin\theta\left(1 - \frac{R^2}{r^2}\right)$

$V_r = \frac{1}{r}\psi_\theta = V_\infty \cos\theta\left(1 - \frac{R^2}{r^2}\right)$

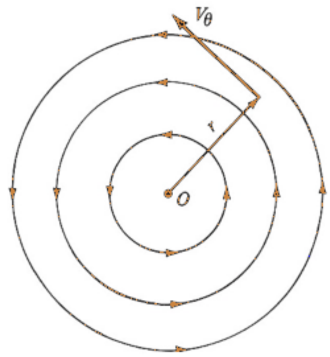
$V_\theta = -\psi_r = -V_\infty \sin\theta\left(1 + \frac{R^2}{r^2}\right)$
Something I'd like you to notice is the direction of flow around the circle.




## Example: Vortex Flow

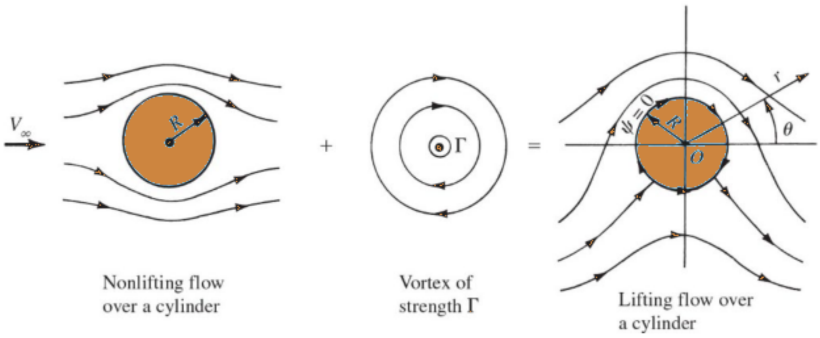
Find $\psi$ and $\phi$ such that $V_r = 0\ \wedge\ V_\theta = \frac{c}{r}$

Let's calculate $\Gamma$ at any $r$, $\Gamma = -\int V_\theta r d\theta = -V_\theta 2\pi r$.
Something to note here is that, at the origin, $\xi \to \infty$.
Using our velocity relations, we can obtain $\phi = -\frac{\Gamma}{2\pi}\theta$ and $\psi = \frac{\Gamma}{2\pi}\ln r$

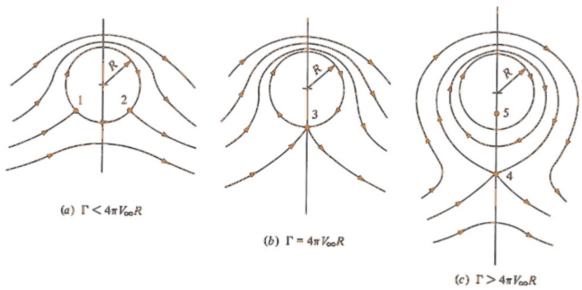


## Example: Lifting Force over a Circular Cylinder

Find $\psi$, $L$, and the stagnation points.




Our lifting-cylinder is subject to freestream flow, a doublet, and a vortex. The vortex's velocity vectors are strong enough to force the velocity vectors of the doublet to follow one direction.
$\psi = \psi_1 + \psi_2 = (V_\infty r \sin\theta)\left(1 - \frac{R^2}{r^2}\right) + \frac{\Gamma}{2\pi} \ln r$
If we say that $\psi_2 = \psi_2 - \text{constant}$ (constant of integration), where this constant is when $r = R$, we can say
$\psi = \psi_1 + \psi_2 = (V_\infty r \sin\theta)\left(1 - \frac{R^2}{r^2}\right) + \frac{\Gamma}{2\pi} \ln \frac{r}{R}$
Using the velocity relations, we can find $V_r$ and $V_\theta$; from this, we can find the stagnation points.
We will find that our stagnation points depend on the value of $\Gamma$ with respect to $4\pi R V_\infty$:

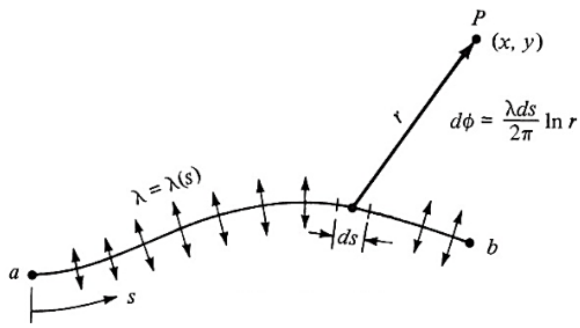



Once again, recall the vortex's influence on the doublet, and this should explain the 5 points.
To find $L$,

1. Find $V_\theta$ at $R$
2. Find $c_p$ via $c_p = 1 - \left(\frac{V_\theta}{V_\infty}\right)^2$ knowing that $V_r = 0$ at $R$
3. Find $c_l$ using the looong integral formula. You will find that $c_l = \frac{\Gamma}{RV_\infty}$
4. Find $L$ knowing that $D = 0$ since our body is symmetric. You will find that $L' = \rho_\infty V_\infty \Gamma$ where $L'$ is the lift per unit span. **This $L' \propto \Gamma$ relation is called the Kutta-Joukowski theorem.**

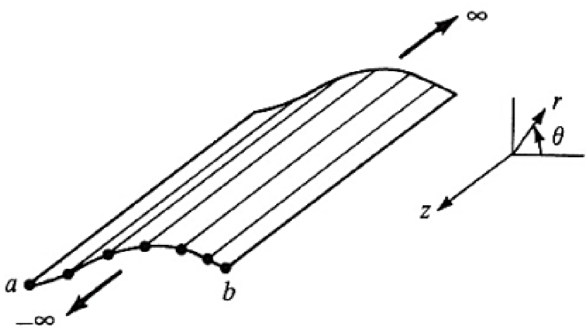
## 3.3 Numerical Source Plane Method

Let's find $\phi$ for the below curve.



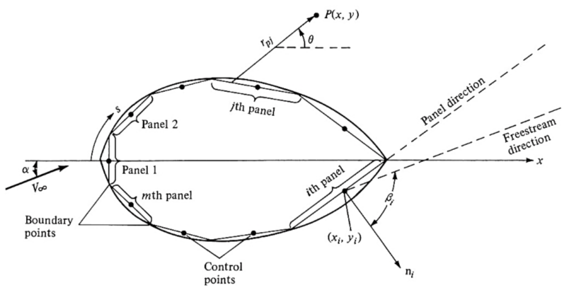

Letting $\lambda(s)$ be the source strength per unit length, from the previous section's examples, we can state that $d\phi = \dfrac{\lambda ds}{2\pi} \ln r$.

Now look at the curve in 3D; it is a sheet.




Integrating to get $\phi$ would give $\phi = \displaystyle\int_a^b \frac{\lambda ds}{2\pi} \ln r$.

Having done this, let's move on to the real thing: modelling the flow over any given shape. For derivation, we shall consider an airfoil.




I have broken down the airfoil into panels/lines. I call these straight lines panels. At the start of this section, we had found $\phi$ for these panels.

Before derivation, I want to make clear a few things:

- Each panel has a constant $\lambda$ and $s$. For our approximation, $s$ represents the length of these lines.

- Every panel has a different $\lambda$ and $s$.
- The midpoint of each panel is called the control point.
- I want the normal velocity at the panels to be zero. To make things easier, normal velocity at the control points to be zero.

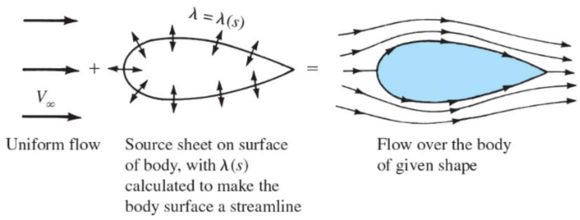
Enough dealing with an imaginary point, now instead of some point $P$, I will use another plane as a reference, plane $i$. So $\phi$ at some panel $j$ with $i$ as a reference is

$$\phi = \sum_{j=1}^{n} \frac{\lambda_j}{2\pi} \int_j \ln r_{ij} ds_j$$

Here, $r_{ij}^2 = (x_i - x_j)^2 + (y_i - y_j)^2$.

Geometry and this definition of $r_{ij}$ is important, as evluating the integral requies $x_i$ and $x_j$ be re-written to introduce $s_j$, and this can be done only by introducing the angles $\beta$ and the angle of the plane with the $x$ axis.[1]

Flow over an airfoil looks like this:




Recall that I wanted the normal velocity at the control point to be zero:

$$V_n = \frac{\partial(V_{\infty,\,n} + V_n)}{\partial n_i} = V_\infty \cos\beta_i + \frac{\lambda_i}{2} + \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{\lambda_j}{2\pi} \int_j \frac{\partial}{\partial n_i}\left(\ln r_{ji}\right) ds_j = 0$$

The term where $j = i$ was removed from the series, and if the integral is evaluated, it will be found that it would equal $\pi$. The integral is too long, so I will not include it here, but it is quite simple.

Similarly, an expression for $V_s$ can also be found. This is an exercise left for the reader. $V_s$ can be used to find $c_p$ via $c_p = 1 - \left(\frac{V_s}{V_\infty}\right)^2$.

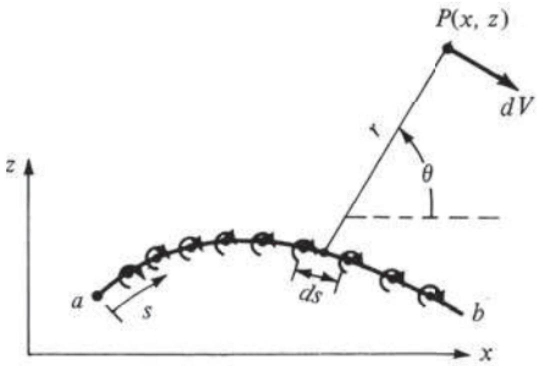
To judge the accuracy of our approximation, we must ensure that, for a closed body, the sum of all source and sink strengths must equal zero; otherwise the body would be producing or absorbing mass from the flow, and this, by common sense, is impossible. Mathematically speaking, $\sum_{j=1}^{n} \lambda_j s_j = 0$.

[1] For those who wish to try, replace $x$ and $y$ with the something like $x_j = x_{\text{starting of j}} + s_j \cos\theta_{\text{angle between plane and axis}}$, relate $\beta$ with this $\theta$ at $\hat{\mathbf{n}}$.

# 4 Incompressible Flow Over Airfoils

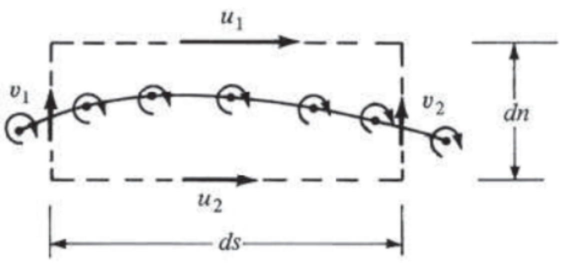
## 4.1 The Vortex Sheet

Take a curve, fill it with vortices, model it. Also define $\Gamma = \gamma ds$.



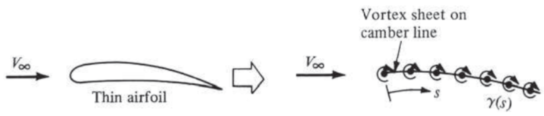

From what we've learnt, we can state without baby steps: $\phi = -\frac{1}{2\pi}\int \theta\gamma ds$

$\Gamma$ is getting bored, so let's make a rectangle for it to play:



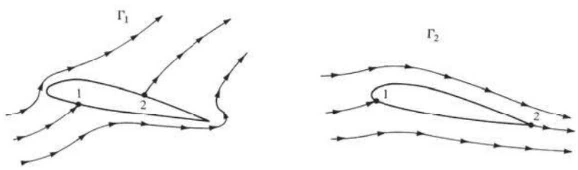

$$\Gamma = (u_1 - u_2)ds + (v_1 - v_2)dn$$
$$\lim_{dn \to 0} \Gamma \to (u_1 - u_2)ds = \gamma dS$$
$$\therefore \; \gamma = u_1 - u_2$$

If we consider an airfoil that is very thin, then we can approximate it as:




## 4.2 The Kutta Condition

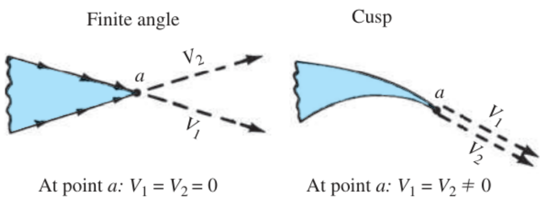
Someone was playing with the above approximation and found an unusual case with a certain $\Gamma$:




The picture on the left never happens in reality: Air does not re-circulate about the trailing edge, instead it leaves the trailing edge like the right image.

We must ensure that whoever follows our approximations must not encounter this.

Let's zoom into the trailing edge,




These are two types of trailing edges we can have. For flow to never re-circulate, the above conditions must be met.

But how is $V_1 = V_2$? We use bernoulli for this: Take bernoulli at the trailing edge veryyy close to the top and bottom:

$$P_a + \frac{1}{2}\rho V_1^2 = P_a + \frac{1}{2}\rho V_2^2 \quad \rightsquigarrow \quad V_1 = V_2$$

Mathematically, I can say $\gamma(\text{TE}) = V_1 - V_2 = 0$. This is called the Kutta condition.

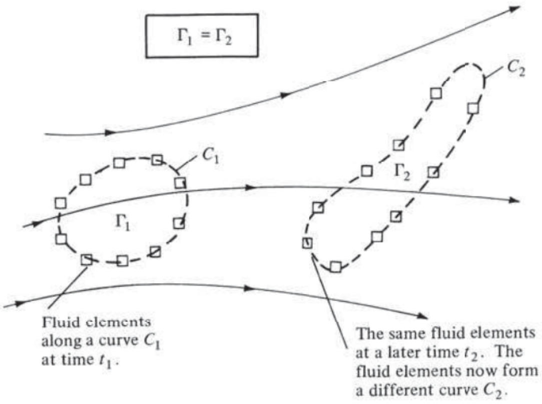
In words,

- $\Gamma$ is such that the flow leaves TE smoothly.
- If the TE angle is finite, TE is a stagnation point.
- If TE is cusped, top and bottom velocities are finite, equal and are in the same direction.

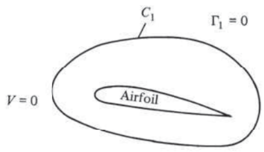
## 4.3 Kelvin's Circulation Theorem

Our math says that flow must leave TE smoothly. How does this happen in reality? Answer: Vortices.

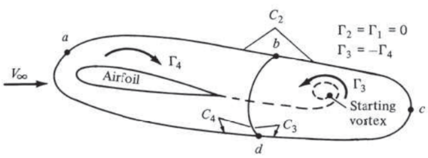
To better understand this, realize that $\Gamma$ for the same fluid elements at any time is always the same. In other words, $\frac{D\Gamma}{Dt} = 0$.




This can be proven mathematically[2] and by common sense. But this requires that the curve and the fluid move together in time, and that nothing enters or leaves.




(a) Fluid at rest relative to the airfoil




(b) Picture some moments after the start of the flow

When an airfoil moves through air, flow tends to curl around the trailing edge (we had seen this in the previous section, and to avoid this, we stated that $\gamma(\text{TE}) = 0$). There is high $\vec{\xi}$ currently at the TE.

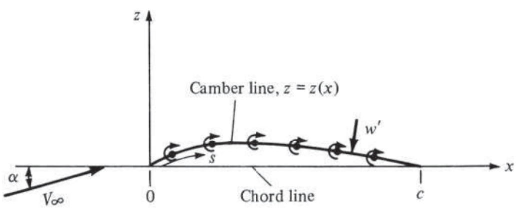
This curl region is flushed downstream from TE as the airfoil moves; this is similar to our vortex, and hence we shall call this the starting vortex.

From the picture to the left, we can state that $\Gamma_1 = \Gamma_2 = \Gamma_3 + \Gamma_4 = 0$, $\therefore$ $\Gamma_3 = -\Gamma_4$. *Math tells us that there must be an equal and opposite reaction to the starting vortex around the airfoil.* With time, $\vec{\xi}$ from the TE is constantly fed into the starting vortex, and one point, $\Gamma_3$ will be strong enough such that flow will leave the TE smoothly. At this point, $\vec{\xi}$ at the TE be zero and the starting vortex will no longer grow in strength; i.e., a steady circulation will exist around the airfoil.

## 4.4 Thin Airfoil Theory: The Symmetric Airfoil

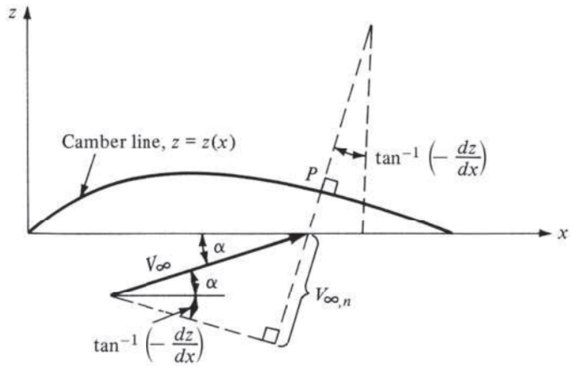
We are now going to work with a symmetric thin airfoil. Thin because we want the airfoil approximation mentioned previously.

Let us assume that our camber line represents the streamline of the flow around the airfoil. Define $w'(s)$ such that it represents the velocity normal to the camber line due to the vortices.



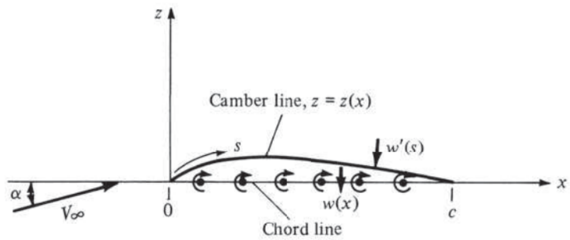

[2]chain rule, momentum equation, $\rho$ is constant

For the camber line to be a streamline of the flow, $V_n = 0$. Hence, $V_{\infty,\,n} + w'(s) = 0$.



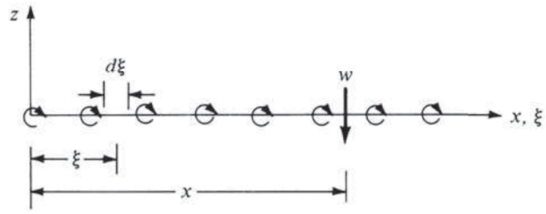

By the above geometry, $V_{\infty,\,n} = V_\infty \sin\left[\alpha + \tan^{-1}\left(-\frac{\mathrm{d}z}{\mathrm{d}x}\right)\right]$

To find $w'(s)$, we will make another approximation: Let the camber and chord lines be very close to each other such that the vortices can be assumed to be on the chord line. This is why the section is titled 'symmetric airfoil'.




Now, we define $w(x)$ such that $w'(s) \approx w(x)$.

Since we kept talking about $\vec{\xi}$ at the trailing edge (which is now at $x = c$), let's replace $x$ with $\vec{\xi}$.




The math now becomes too complex, so I will skip it. If you want to know the math, please change majors.

$$w(x) = -\int_0^c \frac{\gamma(\xi)d\xi}{2\pi(x-\xi)}$$

Now we get an equation I like to call the *fundamental equation of thin airfoil theory*. It simply states that the camber line is a streamline of the flow.

$$\frac{1}{2\pi}\int_0^c \frac{\gamma(\xi)d\xi}{x-\xi} = V_\infty\left(\alpha - \frac{\mathrm{d}z}{\mathrm{d}x}\right)$$

$$\boxed{\begin{array}{c}\frac{dz}{dx}=0\\ x=\frac{c}{2}(1-\cos\theta_0),\ \xi=\frac{c}{2}(1-\cos\theta)\end{array}} \longrightarrow \gamma(\theta) = 2\alpha V_\infty \frac{(1+\cos\theta)}{\sin\theta}$$

The solution can be confirmed by setting $\theta = \pi$. (The solution should be $\gamma(\pi) = 0$ for the Kutta condition to be met.)

Finding $c_l$ via $\rho_\infty V_\infty \int \gamma(\xi) d\xi$ gives $c_l = 2\pi\alpha$ and $\frac{\mathrm{d}c_l}{\mathrm{d}\alpha} = 2\pi$.

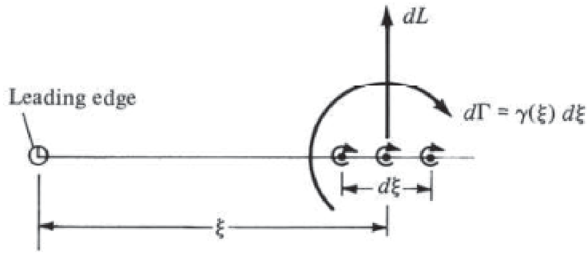
Moment analysis gives

$$M'_{LE} = -\int_0^c \xi dL = -\rho_\infty V_\infty \int_0^c \xi\gamma(\xi) d\xi = -q_\infty c^2 \frac{\pi\alpha}{2}$$

$$c_{m,le} = -\frac{c_l}{4}$$

$$c_{m,\ \frac{c}{4}} = c_{m,\ le} + \frac{c_l}{4}$$

$$c_{m,\ \frac{c}{4}} = 0$$




This tells us that the center of pressure and aerodynamic center is at the quarter-chord point for a symmetric airfoil.

## 4.5 The Cambered Airfoil

We are now going to extend thin airfoil theor for a cambered airfoil.

A cambered airfoil has a $\mathrm{d}z/\mathrm{d}x$. Sparing you the math, here is the solution to the integral:

$$\gamma(\theta) = 2V_\infty\left(A_0\frac{1+\cos\theta}{\sin\theta} + \sum_{n=1}^{\infty} A_n \sin n\theta\right) \text{ where } A_0 = \alpha - \frac{1}{\pi}\int_0^\pi \frac{\mathrm{d}z}{\mathrm{d}x}\,\mathrm{d}\theta_0 \ \text{ and } A_n = \frac{2}{\pi}\int_0^\pi \frac{\mathrm{d}z}{\mathrm{d}x}\cos n\theta_0\,\mathrm{d}\theta_0$$

$A_0$ depends on $\alpha$ and the shape of the camber line while $A_n$ depends only on the shape of the camber line.

From the calculation of $\Gamma$, you will obtain

$$c_l = 2\pi\left(\alpha + \frac{1}{\pi}\int_0^\pi \frac{\mathrm{d}z}{\mathrm{d}x}(\cos\theta_0 - 1)\,\mathrm{d}\theta_0\right) \quad \text{and} \quad \frac{\mathrm{d}c_l}{\mathrm{d}\alpha} = 2\pi$$

We can also find $\alpha$ where $L = 0$: $\alpha_{L=0} = -\frac{1}{\pi}\int_0^\pi \frac{\mathrm{d}z}{\mathrm{d}x}\,\mathrm{d}\theta_0$

Finding the coefficients of moment,

$$c_{m,\ le} = -\left[\frac{c_l}{4} + \frac{\pi}{4}(A_1 - A_2)\right] \qquad x_{cp} = \frac{c}{4}\left[1 + \frac{\pi}{c_l}(A_1 - A_2)\right]$$

$$c_{m,\ \frac{c}{4}} = \frac{\pi}{4}(A_2 - A_1) \qquad x_{ac} = -\frac{m_0}{a_0} + 0.25$$

What is $m_0$ and $a_0$? I have no idea. I am losing my mind.

## 4.6 Viscous Flow

To obtain skin-friction drag on an airfoil, let's approximate it as the skin-friction drag on a flat plate at $\alpha = 0$.

This approximation is best for thin airfoils at small $\alpha$. Once again, this is only for low speed flow.

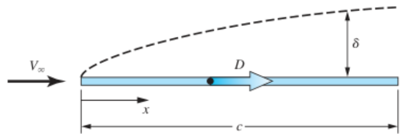
In Aerodynamics II, you will visit an exact analytical solution for the laminar boundary layer flow over a flat plate. For now, we will only look at the result of the derivation.

**Laminar Flow**

The boundary layer thickness is

$$\delta = \frac{5x}{\sqrt{\mathrm{Re}_x}}$$

where $\mathrm{Re}_x = \frac{\rho_\infty V_\infty x}{\mu_\infty}$. This $x$ is measured from the leading edge. Integrating the plate for shear stress gives $D_f$ or $2D_{f,\text{ top}}$. Another result from the derivation is




$$c_f = \frac{1.328}{\sqrt{\mathrm{Re}_x}}$$

By this, $D_f$ can be found.

**Tubulent Flow**

The boundary layer thickness is

$$\delta = \frac{0.37x}{\mathrm{Re}_x^{\frac{1}{5}}}$$

This $x$ is measured from the leading edge. The coefficient of friction is

$$c_f = \frac{0.074}{\mathrm{Re}_x^{\frac{1}{5}}}$$

It is common sense that due to the boundary layer, the flow can transition from laminar to turbulent. The point of transition is marked by $x_{\text{cr}}$. Re at this point is called the critical Reynolds number.

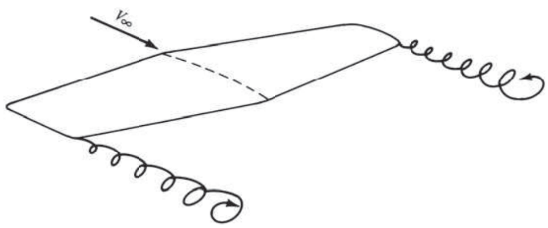
When a plate has a transition over it, the drag can be found by

$$D_{\text{total}} = D_{\text{laminar}} + (D_{\text{turbulent, total}} - D_{\text{turbulent, cr}}) \quad \rightsquigarrow \quad c_{f\,\text{total}} = \frac{x_{\text{cr}}}{c} c_{f\,\text{laminar}} + \left( c_{f\,\text{turbulent, total}} - \frac{x_{\text{cr}}}{c} c_{f\,\text{turbulent, cr}} \right)$$

# 5 Incompressible Flow over Finite Wings

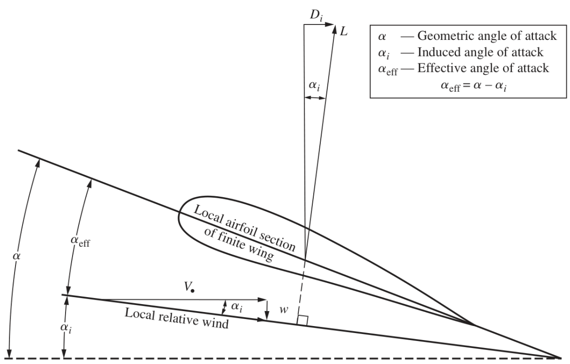
## 5.1 Downwash and Induced Drag

Flow over a wing is different from that over an airfoil. At the ends of the wing, flow is forced from the high pressure region just underneath the tips to the low pressure region on the top. This 'forcing' causes the flow to curl around the wing tips.




This curling causes streamlines over the wing's surface to bend towards the wing's end. This curled flow is called a wing tip vortex.

The vortices induce a small velocity component in the downward direction at the wing; this is called *downwash* ($w$). Let's put this in geometry:




Freestream has been pushed down by $\alpha_i$ due to the downwash. $L$ is oriented to the local relative wind, and hence is inclined by $\alpha_i$. A component of this lift is $D_i$, called the *induced drag*.
Since we are working with an inviscid and incompressible flow, $D_i$ is finite.
We define $c_d$ as $cd = \frac{D_f + D_p}{q_\infty S}$ due to viscous effects. Now, we define $C_{D,\,i} = \frac{D_i}{q_\infty S}$. The total drag being $C_D = c_d + C_{D,\,i}$ for a finite wing.

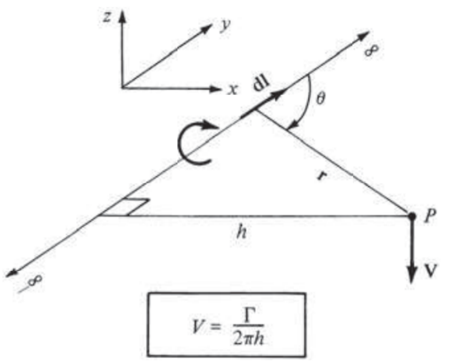
## 5.2 Vortex Filament, Biot-Savart Law, and Helmholtz's Theorems

Let us return to our vortex filament. Generalizing it, our filament can now be curved. Let us keep it such that the circulation taken about any path enclosing the filament to be a constant value $\Gamma$.

For now, accpet that velocity can be found by

$$d\mathbf{v} = \frac{\Gamma}{4\pi}\frac{d\mathbf{l}\times\mathbf{r}}{||\mathbf{r}||^3}$$

The derivation involved is too long, and I might update these notes to cover that another time. This equation is called the *Biot-Savart Law*.




$$\begin{aligned}\mathbf{V} &= \int_{-\infty}^{\infty}\frac{\Gamma}{4\pi}\frac{d\mathbf{l}\times\mathbf{r}}{||\mathbf{r}||^3}\\ &= \frac{\Gamma}{4\pi}\int_{-\infty}^{\infty}\frac{\sin\theta}{r^2}dl\\ V &= \frac{\Gamma}{2\pi h}\end{aligned}$$

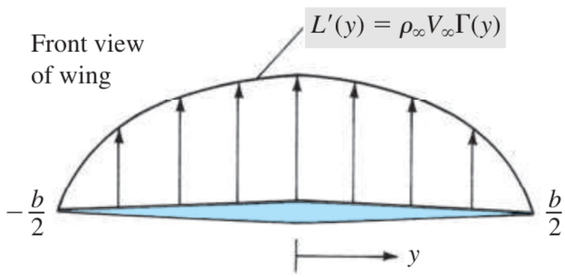
1. The strength of a vortex filament is constant along its length.
2. A vortex filament cannot end in a fluid; it must extend to the boundaries of the fluid or form a closed path.

These two points are called Helmholtz's vortex theorems.

## 5.3 Prandtl's Classical Lifting-Line Theory

On a wing, consider a location $y_1$ where the lift per unit span is $L'(y_1)$. Now take $y_2$ where we have $L'(y_2)$. Both these $L'$ are generally different due to varying $c$, $\alpha$, airfoil shape along the wing.

The lift distribution is found to be as below

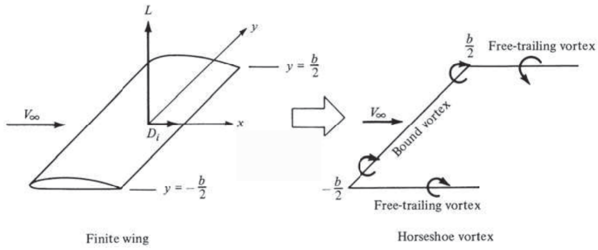



Note that $\Gamma$ is a function of $y$.

Prandtl reasoned that a vortex filament of strength $\Gamma$ that is shomehow bounded to a fixed location in a flow will experience a force $L'$ from the Kutta-Joukowski theorem.

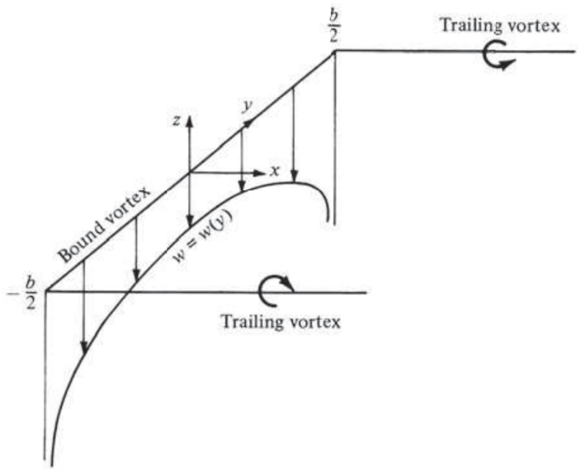
A bound vortex is different from a free vortex–the free vortex moves with the same fluid elements throughout the flow.

To model the above mentioned $\Gamma(y)$ distribution, let's replace the finite wing with a bound vortex. Following Helmholtz's second theorem, our vortex filament will continue as two free vortices trainling downstream from the wing tips to infinity. (The vortex filament is bent in a U shape or a horseshoe.)




The graphs we look now are obtained by throwing the aforementioned $\mathbf{v}$ in some software.

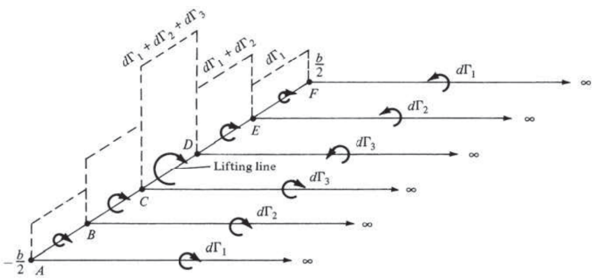
There will be $w$ induced from $-\frac{b}{2}$ to $\frac{b}{2}$ by the horseshoe vortex.




This $w(y)$ is due to the trailing vortices; the bounded vortex induces no velocity along itself. From the aforementioned $\mathbf{v}$ of a vortex filament, we find $w(y) = -\frac{\Gamma}{4\pi(\frac{b}{2}+y)} - \frac{\Gamma}{4\pi(\frac{b}{2}-y)}$

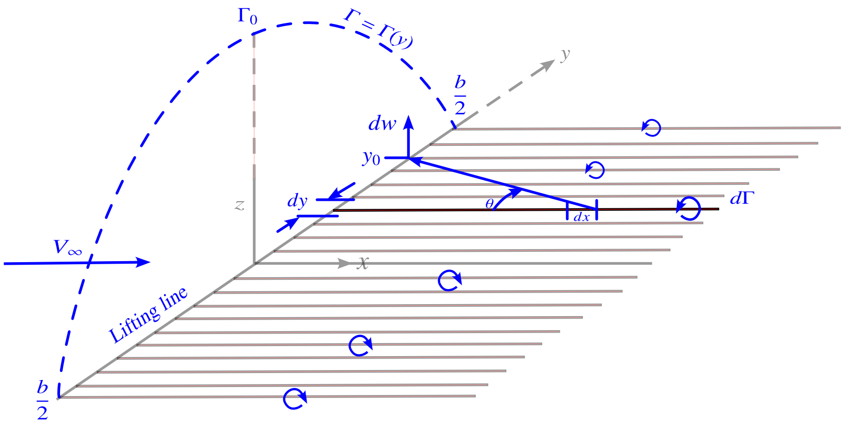
If we only look at a single horseshoe vortex, it does not give give a realistic model of a finite wing; espcially the observation that $w$ reaches $\infty$ at the tips. To surmount this, let's surperimpose a large number of horshoe vortices, each with a different length of the bound vortex, but with all the bound vortices coincident along a single line called the *lifting line*.

Here is an idea of what the above paragraph means:




Here, we have superimposed 3 horseshoe vortices. Also note that the strength of each trailing vortex is equal to the change in circulation along the lifting line.

Oops... a mathematican came and superimposed an infinite number of horseshoe vortices... Each has a strength of $d\Gamma$. Ignore the math in the below picture, just look at the graph.




Since we are taking an infinite number, $\Gamma(y)$ is now represented by a continuous line representing a continuous distribution. And now the finite number of trailing vortices have become a continuous vortex sheet. The total strenght of this sheet across the span of the wind is zero as it contains pairs of trailing vortices of equal strength in opposite directions.

Look back at the image, and focus on the math. We are singling out an infinitesimal small segment of the lifting line $dy$ at $y$. The strenght of the trailing vortex at $y$ must equal the change in criculation ($d\Gamma$) along the lifting line.

For the trailing vortex at $y$, the any segment $dx$ of this vortex will induce velocity ($dw$) at any point, say $y_0$. This $dw$ can be found by the Biot-Savart's law.

$$w(y_0) = -\frac{1}{4\pi}\int_{-\frac{b}{2}}^{\frac{b}{2}} \frac{\frac{\mathrm{d}\Gamma}{\mathrm{d}y}\,\mathrm{d}y}{y_0 - y}$$

The induced angle of attack ($\alpha_i$) from the previous section was found to be $\alpha_i(y_0) = \tan^{-1}\left(\frac{-w(y_0)}{V_\infty}\right)$

In terms of $\Gamma$, this is $\alpha_i(y_0) = \frac{1}{r\pi V_\infty}\int_{-\frac{b}{2}}^{\frac{b}{2}} \frac{\frac{\mathrm{d}\Gamma}{\mathrm{d}y}\,\mathrm{d}y}{y_0 - y}$

In terms of $\alpha_{\text{eff}}$ (angle of attack actually seen by the local airfoil), $c_l$ is found to be $c_l$ =

$a_0(a_{\text{eff}}(y_0) - a_{L=0})$. $a_0$ can be replaced with $2\pi$ by the thin airfoil theory.

Doing some further math gives us a major equation:

$$c_l = \frac{2\Gamma(y_0)}{V_\infty c(y_0)}$$
$$2\pi(a_{\text{eff}}(y_0) - a_{L=0}) = \frac{2\Gamma(y_0)}{V_\infty c(y_0)}$$

$$a_{\text{eff}} = \frac{\Gamma(y_0)}{\pi V_\infty c(y_0)} + a_{L=0}$$

Substituting $\alpha_i(y_0)$ and $\alpha_{\text{eff}}$ in $\alpha = \alpha_{\text{eff}} + \alpha_i$ gives

$$\alpha(y_0) = \frac{\Gamma(y_0)}{\pi V_\infty c(y_0)} + \alpha_{L=0}(y_0) + \frac{1}{4\pi V\infty} \int_{-\frac{b}{2}}^{\frac{b}{2}} \frac{\frac{d\Gamma}{dy}\, dy}{y_0 - y}$$

This is the *fundamental equation of Prandtl's lifting-line theory*.

This equation gives us 3 main aerodynamic characteristics of a finite wing:

1. The lift distribution is obtained from the Kutta-Joukowski theorem
2. The total life is obtained by integrating the Kutta-Joukowski theorem over the span
3. The induced drag is obtained by

$$D_i' = L_i' \sin \alpha_i \approx L_i' \alpha_i$$
$$D_i = \int_{-\frac{b}{2}}^{\frac{b}{2}} L'(y)\alpha_i(y)\, dy = \rho_\infty V_\infty \int_{-\frac{b}{2}}^{\frac{b}{2}} \Gamma(y)\alpha_i(y)\, dy$$
$$C_{D,\,i} = \frac{D_i}{q_\infty S} = \frac{2}{V_\infty S \int_{-\frac{b}{2}}^{\frac{b}{2}} \Gamma(y)\alpha_i(y)\, dy}$$

## 5.4 Elliptical Lift Distribution

If we consider $\Gamma$ as $\Gamma(y) = \Gamma_0 \sqrt{1 - \left(\frac{2y}{b}\right)^2}$,

- $\Gamma_0$ is the circulation at the origin
- $\Gamma$ varies elliptically with distance $y$ along the span; hence an elliptical lift/circulation distribution
- $\Gamma$ at the tips is 0

Let's find the aerodynamic properties of a finite wing with such a distribution. Why? Please don't ask me why.

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}y} = -\frac{4\Gamma_0}{b^2}\frac{y}{\sqrt{1-\frac{4y^2}{b^2}}}$$

$$\therefore\ w(y_0) = \frac{\Gamma_0}{\pi b^2}\int_{-\frac{b}{2}}^{\frac{b}{2}} \frac{y}{(y_0-y)\sqrt{1-\frac{4y^2}{b^2}}}\,\mathrm{d}y$$

From substitutions and integral identities,

$$w(\theta_0) = -\frac{\Gamma_0}{2b}$$

This states that downwash is constant over the span for an elliptical lift distribution.

$$\alpha_i = -\frac{w}{V_\infty} = \frac{\Gamma_0}{2bV_\infty} = \frac{C_L}{\pi\ \mathrm{AR}}$$

The induced angle of attack is also constant over the span. And, it is related to AR.

$$\Gamma_0 = \frac{2V_\infty S C_L}{b\pi}$$

We can get $C_{D,\ i}$ from substituting our new equations in the equation of the previous chapter

$$C_{D,\ i} = \frac{C_L^2}{\pi\ \mathrm{AR}}$$

Here, AR is the aspect ratio; found by $\frac{b^2}{S}$

For an elliptical lift distribution,

$$\frac{\mathrm{d}C_L}{\mathrm{d}\alpha} = a = \frac{a_0}{1+\frac{a_0}{\pi\ \mathrm{AR}}}$$

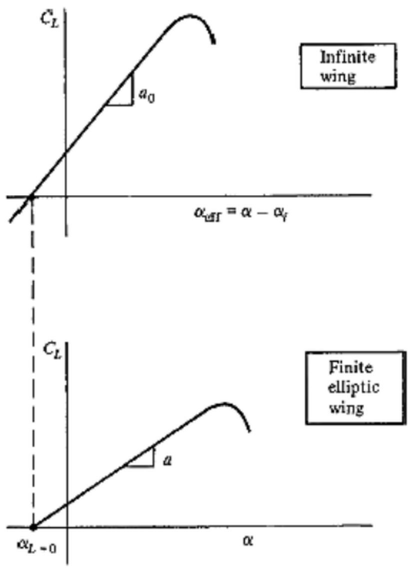
where $a_0$ is the lift slope of an airfoil. This equation came from simple line equations:

Lift slope:

$$\frac{dC_L}{d(\alpha - \alpha_i)} = a_0$$

$$C_L = a_0(\alpha - \alpha_i) + \text{const}$$

$$C_L = a_0\left(\alpha - \frac{C_L}{\pi \text{AR}}\right) + \text{const}$$




For a general lift distribution,

$$\frac{\mathrm{d}C_L}{\mathrm{d}\alpha} = a = \frac{a_0}{1 + \left(\frac{a_0}{\pi\,\text{AR}}\right)(1+\tau)} \qquad\qquad C_{D_i} = \frac{C_L^2}{\pi\,\text{AR}}(1+\delta)$$

A note on units:

- When using $C_L = a(\alpha - \alpha_{L=0})$, the units of $a$ must be $\frac{1}{\text{deg}}$ or $\frac{1}{\text{rad}}$. When using $2\pi$ from thin airfoil theory, the units here are $2\pi\ \frac{1}{rad}$
- When using $a = \frac{a_0}{1+\frac{a_0}{\pi\,\text{AR}}}$, the units of $a_0$ must be $\frac{1}{\text{rad}}$