Solutions to the Waves and Electromagnetism exam of August 2015

August 10, 2016

Question 1

Period of a Simple Harmonic Oscillator is given by:

$$T = 2\pi\sqrt{\frac{m}{k}} \tag{1}$$

And the spring constant k can be found by:

$$k = \frac{F}{\Delta y} = \frac{mg}{\Delta y} \tag{2}$$

Combining the above two equations gives us the desired result:

$$T = 2\pi\sqrt{\frac{\Delta y}{g}} \tag{3}$$

Question 2

Use the conservation of energy applicable to SHM:

$$\frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}kA^2 \tag{4}$$

For the maximum velocity case at zero displacement:

$$\frac{1}{2}mv_{max}^2 = \frac{1}{2}\frac{k}{A}^2 \tag{5}$$

Above equality can be solved for the spring constant k. Then application of conservation of energy to the case where $x = x_1$ we can find the speed:

$$v = \sqrt{\frac{k}{m}(A^2 - x_1^2)} \tag{6}$$

Question 3

Displacement as a function of time for a damped oscillator is given by:

$$x(t) = A_0 e^{-\frac{b}{2m}t} cos(\omega' t) \tag{7}$$

Hence the amplitude as a function of time is:

$$A(t) = A_0 e^{-\frac{b}{2m}t} \tag{8}$$

Solving the above equation for the linear damping coefficient b gives:

$$b = -\frac{2m}{t}\ln(\frac{A(t)}{A_0}) \tag{9}$$

Question 4

Note how it is not possible to find the velocity of the wave by simple differentiation with respect to time. Hence, it is convenient to make use of the wave equation, which can be seen as a form of Newton's second law ($F = ma$) for small amplitude waves:

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} \tag{10}$$

Hence the speed of the wave v should equal:

$$v = \sqrt{\frac{\frac{\partial^2 y}{\partial t^2}}{\frac{\partial^2 y}{\partial x^2}}} \tag{11}$$

Question 5

For a string with restriction of motion at two ends the fundamental frequency is:

$$f = \frac{1}{2l}v = \frac{1}{2l}\sqrt{\frac{F_t}{\mu}} \tag{12}$$

Where μ is the mass per unit length of the chord:

$$\mu = \frac{m}{l} \tag{13}$$

Question 6

Wavelength is $4L$ as $L = \lambda_4$ since the pipe only contains one antinode and one node. The displacement antinode is ofcourse located at the pipe's open end as movement of the air is restricted at the closed end.

Question 7

Combination of two waves of different frequencies leads to a combined wave with beat (constructive interference) frequency of:

$$f_{beat} = \|f_1 - f_2\| \tag{14}$$

Hence if one frequency is decreased, the beat frequency could either decrease or increase depending on the frequencies relative positions in the frequency spectrum.

Question 8

Intensity is proportional to $\frac{1}{r^2}$ as the surface area over which the energy is spread increases with the square of the distance (i.e. it is a sphere). Hence:

$$I_2 = I_1(\frac{r_1}{r_2})^2 \tag{15}$$

Question 9

Let us compute the relative intensity of 76 trombones with respect to 1 trombone:

$$\beta_{relative} = 10\log(\frac{76I}{I}) = 10\log(76) \tag{16}$$

Thus the desired intensity level is simply:

$$\beta_2 = \beta_1 + \beta_{relative} = 70 + 10\log(76) \tag{17}$$

Question 10

As stated in question 7, the beat frequency is:

$$f_{beat} = \|f_1 - f_2\| \tag{18}$$

Now the frequency of the emitted sound arriving back at the police car is altered by both the fact that the approaching car observes more wave fronts per second as the stationary car does as well as the fact that the distance between emitted wave fronts from the approaching car is less. Hence we have to account for 2 Doppler effects. The frequency the approaching car observes is:

$$f_{car} = \frac{v}{\lambda} = \frac{v_{sound} + v_{car}}{\lambda} = f_1(1 + \frac{v_{car}}{v_{sound}} \tag{19}$$

Now the frequency arriving back at the police car is:

$$f_2 = \frac{v}{\lambda'} = \frac{v}{\lambda - v_{car}T} = \frac{f_{car}}{1 - \frac{v_{car}}{v_{sound}}} \tag{20}$$

Hence the beat frequency becomes:

$$f_{beat} = \|f_1 - f_1(\frac{v_{sound} + v_{car}}{v_{sound} - v_{car}})\| \tag{21}$$

Question 11

This question is easily solved by application of trigonometry:

$$\tan(\theta) = \frac{h}{vt} \tag{22}$$

Moreover:

$$\theta = \arcsin(\frac{v_{snd}}{v_{plane}}) \tag{23}$$

Then solving for the altitude h gives:

$$h = vt\tan(\theta) = vt\tan(\arcsin(\frac{v_{snd}}{v_{plane}})) \tag{24}$$

Question 12

Magnitude of Electric field at origin follows from vector addition. Hence:

$$\|E_o\| = \|E_A + E_B\| \tag{25}$$

Application of Coulumb's law for electrostatics gives that:

$$E_i = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2} \tag{26}$$

Hence the magnitude of the electric field becomes:

$$E_o = \frac{Q}{4\pi\epsilon_0}\sqrt{(\frac{1}{l^2})^2 + (\frac{1}{4l^2})^2} = \frac{Q}{16\pi\epsilon_0 l^2}\sqrt{17} \tag{27}$$

Question 13

Note that this revolves about a square, hence we can apply Gauss' Law to the cube enclosing q and divide the result by 6.

$$\phi_E = \frac{q}{6\epsilon_0} \tag{28}$$

Question 14

Electric field is zero if:

$$\frac{\partial V}{\partial x} = 0 \quad \text{and} \quad \frac{\partial V}{\partial x} = 0 \tag{29}$$

because the electric field is the negative of the rate of change of the electric potential in any direction.

Question 15

Note how the electric potential due to a point charge can be seen as the work that is required to bring a given charge this close to the point charge. Hence it is negative for a negative charge as work is done by the electric field on the charge as it is brought to this position. As the negative point charge is in this case larger than the positive point charge, we can already intuitively reason that the given charge must be located closer to the positive charge than the negative charge to have zero potential. Now let us express our words into an equation:

$$\frac{1}{4\pi\epsilon_0}\frac{Q_A}{r_A} + \frac{1}{4\pi\epsilon_0}\frac{Q_B}{r_B} = 0 \tag{30}$$

Or

$$\frac{q}{x} = \frac{3q}{d-x} \tag{31}$$

Hence we can solve for x:

$$x = \frac{d}{4} \tag{32}$$

Question 16

The fact that the capacitors are in series gives us that the accumulated voltage drop over the capacitors must equal the source voltage:

$$V_0 = \frac{Q_1}{C_1} + \frac{Q_2}{C_2} \tag{33}$$

However, due to the fact that the piece of wire between the two capacitors is isolated we also have that:

$$Q = Q_1 = Q_2 \tag{34}$$

Such that we can solve the first equation and find:

$$Q = (C_1 + C_2)V_0 \tag{35}$$

Question 17

Apply Kirchoff's rules. The two loops across the circuit gives us that:

$$\mathcal{E}_1 = I_1R_1 + I_2R_2 \tag{36}$$

$$\mathcal{E}_1 = I_1R_1 + I_3R_3 + \mathcal{E}_2 \tag{37}$$

We also know that by the conservation of charge:

$$I_1 = I_2 + I_3 \tag{38}$$

We can express the two currents on the right in terms of I_1 by use of the loop rules and solve above equation for I_1. The desired I_3 can then easily be found by substitution of I_1 into the expression of I_3 as a function of I_1.

Question 18

We need to find a proper value of the resistor R_{sh} such that the proper ratio of current flows through the galvanometer allowing for full deflection at $10A$. As the resistances will be applied in parallel we can reason that:

$$IR_{eq} = I_g R_g \tag{39}$$

Such that the required equivalent resistance must be equal to:

$$R_{eq} = R_g(\frac{I_g}{I}) \tag{40}$$

Now the equivalent resistance is simply:

$$R_{eq} = \frac{R_{sh}R_g}{R_{sh} + R_g} \tag{41}$$

Thus we can solve for R_{sh} and find that:

$$R_{sh} = R_g \frac{I_g}{I - I_g} \tag{42}$$

Question 19

Just apply the definition of the magnetic field here. Force is to the left as wire is displaced from it's equilibrium, to the left.

Question 20

Note how the magnetic field vector for a current-carrying wire is always perpendicular to the line connecting a point and the axis of the wire. The 'horizontal' components of the magnetic field cancel yet the vertical components complement each other.

Question 21

The magnetic field of an infinitisemal amount of charge is given by the Biot-Savart law:

$$d\mathbf{B} = \frac{\mu_0 I}{4\pi} \frac{d\mathbf{l} \times \hat{\mathbf{r}}}{r^2} \tag{43}$$

We can argue that the two straight components of the wire do not add to the magnetic field at C. Hence we only consider the arc elements. Note how the line r connecting a wire element and point C is always perpendicular. Hence the Biot-Savart law reduces to:

$$dB = \frac{\mu_0 I}{4\pi} \frac{dl}{r^2} \tag{44}$$

Let us now integrate over both arcs. We first consider the arc of radius R_1.

$$B = \int dB = \int \frac{\mu_0 I}{4\pi}\frac{dl}{r^2} = \int_0^\theta \frac{\mu_0 I}{4\pi}\frac{R_1 d\theta}{R_1^2} = \frac{\mu_0 I\theta}{4\pi}\frac{1}{R_1} \tag{45}$$

Note how the integral for the arc of radius R_2 will be exactly the same. However, we must also consider the facxt that the magnetic field due to arc of radius R_1 is upwards whereas the other is downwards. Hence the final magnitude of the magnetic field becomes:

$$B = \frac{\mu_0 I\theta}{4\pi}(\frac{1}{R_1} - \frac{1}{R_2}) \tag{46}$$

Question 22

By definition of the magnetic dipole vector, the torque experienced by a loop of current-carrying wire inside a magnetic field is given by:

$$\mathbf{T} = \boldsymbol{\mu} \times \mathbf{B} \tag{47}$$

Where the magnetic dipole vector of the given loop is simply:

$$\boldsymbol{\mu} = AI\hat{\mathbf{n}} = \pi\frac{d^2}{4}I\hat{\mathbf{n}} \tag{48}$$

Carrying out the cross product and computing the length of this vector yields the desired torque magnitude.

Question 23

It is of course much greater, as ferromagnetic material contains domains which will align with the magnetic field present and greatly increase its magnitude.

Question 24

Magnetic field through A increases in the upward direction, hence this is counteracted (Lenz's law) by a clockwise EMF to produce a downwards magnetic field. Net magnetic field through B is always zero. Magnetic field through C is increasing downwards, hence a counterclockwise EMF is produced.

Question 25

The output voltage will be equal to zero if all impedance of the circuit resides in the resistance Z_1 since this is the only possibility for the voltage to drop. The total impedance of the circuit is:

$$Z = \sqrt{Z_1^2 + (X_L - X_C)^2} \tag{49}$$

If $Z = Z_1$, then $X_L = X_C$. Hence the frequency must satisfy the resonating frequency:

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{1}{LC}} \tag{50}$$