Rewriting the equations of motion

The equations of motion are quite difficult to deal with. To get some useful data out of them, we need to make them a bit simpler. For that, we first linearize them. We then simplify them. And after that, we set them in a non-dimensional form.

1 Linearization

1.1 The idea behind linearization

Let's suppose we have some non-linear function $f(\mathbf{X})$. Here, $\mathbf{X}$ is the **state** of the system. It contains several **state variables**. To linearize $f(\mathbf{X})$, we should use a multi-dimensional Taylor expansion. We then get

$$f(\mathbf{X}) \approx f(\mathbf{X_0}) + f_{X_1}(\mathbf{X_0})\Delta X_1 + f_{X_2}(\mathbf{X_0})\Delta X_2 + \ldots\ldots + f_{X_n}(\mathbf{X_0})\Delta X_n + \text{higher order terms.} \quad (1.1)$$

Here, $\mathbf{X_0}$ is the **initial point** about which we linearize the system. The linearization will only be valid close to this point. Also, the term ΔX_i indicates the deviation of variable X_i from the initial point $\mathbf{X_0}$.

When applying linearization, we always neglect higher order terms. This significantly simplifies the equation. (Although it's still quite big.)

1.2 Linearizing the states

Now let's apply linearization to the force and moment equations. We start at the right side: the states. We know from the previous chapter that

$$F_x = m(\dot{u} + qw - rv), \quad (1.2)$$
$$F_y = m(\dot{v} + ru - pw), \quad (1.3)$$
$$F_z = m(\dot{w} + pv - qu). \quad (1.4)$$

So we see that $F_x = f(\dot{u}, q, w, r, v)$. The state vector now consists of five states. By applying linearization, we find that

$$F_x = m(\dot{u_0} + q_0 w_0 - r_0 v_0) + m(\Delta\dot{u} + q_0\Delta w + w_0\Delta q - r_0\Delta v - v_0\Delta r), \quad (1.5)$$
$$F_y = m(\dot{v_0} + r_0 u_0 - p_0 w_0) + m(\Delta\dot{v} + r_0\Delta u + u_0\Delta r - p_0\Delta w - w_0\Delta p), \quad (1.6)$$
$$F_z = m(\dot{w_0} + p_0 v_0 - q_0 u_0) + m(\Delta\dot{w} + p_0\Delta v + v_0\Delta p - q_0\Delta u - u_0\Delta q). \quad (1.7)$$

We can apply a similar trick for the moments. This would, however, give us quite big expressions. And since we don't want to spoil too much paper (safe the rainforests!), we will not derive those here. Instead, we will only examine the final result in the end.

1.3 Linearizing the forces

Now let's try to linearize the forces. Again, we know from the previous chapter that

$$F_x = -W\sin\theta + X, \quad (1.8)$$
$$F_y = W\sin\psi\cos\theta + Y, \quad (1.9)$$
$$F_z = W\cos\psi\cos\theta + Z, \quad (1.10)$$

where the **weight** $W = mg$. We see that this time $F_x = f(\theta, X)$. Also, $F_y = f(\psi, \theta, Y)$ and $F_Z = f(\psi, \theta, Z)$. It may seem that linearization is easy this time. However, there are some problems.

The problems are the forces X, Y and Z. They are not part of the state of the aircraft. Instead they also depend on the state of the aircraft. And they don't only depend on the current state, but on the entire history of states! (For example, a change in angle of attack could create disturbances at the wing. These disturbances will later result in forces acting on the tail of the aircraft.)

How do we put this into equations? Well, we say that X is not only a function of the velocity u, but also of all its derivatives $\dot{u}, \ddot{u}, \ldots$. And the same goes for v, w, p, q and r. This gives us an infinitely big equation. (Great....) But luckily, experience has shown that we can neglect most of these time derivatives, as they aren't very important. There are only four exceptions. $\dot{v}$ strongly influences the variables Y and N. Also, $\dot{w}$ strongly influences Z and M. We therefore say that

$$\begin{aligned} F_x &= f(\theta, X) \qquad \text{with} \qquad X = f(u, v, w, p, q, r, \delta_a, \delta_e, \delta_r, \delta_t), && (1.11) \\ F_y &= f(\psi, \theta, Y) \qquad \text{with} \qquad Y = f(u, v, w, \dot{v}, p, q, r, \delta_a, \delta_e, \delta_r), && (1.12) \\ F_z &= f(\psi, \theta, Z) \qquad \text{with} \qquad Z = f(u, v, w, \dot{w}, p, q, r, \delta_a, \delta_e, \delta_r, \delta_t). && (1.13) \end{aligned}$$

When creating the Taylor expansion, we have to apply the chain rule. We then find that

$$\begin{aligned} F_x(\mathbf{X}) &\approx F_x(\mathbf{X_0}) - W\cos\theta_0\Delta\theta + X_u\Delta u + X_v\Delta v + X_w\Delta w + X_p\Delta p + \ldots + X_{\delta_t}\Delta\delta_t, && (1.14) \\ F_y(\mathbf{X}) &\approx F_y(\mathbf{X_0}) - W\sin\psi_0\sin\theta_0\Delta\theta + W\cos\psi_0\cos\theta_0\Delta\psi + Y_{\dot{v}}\Delta\dot{v} + \ldots + Y_{\delta_r}\Delta\delta_r, && (1.15) \\ F_z(\mathbf{X}) &\approx F_z(\mathbf{X_0}) - W\cos\psi_0\sin\theta_0\Delta\theta - W\sin\psi_0\cos\theta_0\Delta\psi + Z_{\dot{w}}\Delta\dot{w} + \ldots + Z_{\delta_t}\Delta\delta_t. && (1.16) \end{aligned}$$

Now that's one big Taylor expansion. And we haven't even written down all terms of the equation. (Note the dots in the equation.) By the way, the term X_u indicates the derivative $\partial X/\partial u$. Similarly, $X_v = \partial X/\partial v$, and so on.

You may wonder what δ_a, δ_e, δ_r and δ_t are. Those are the settings of the aileron, elevator, rudder and thrust. These settings of course influence the forces acting on the aircraft. We will examine those coefficients later in more detail. (You may also wonder, why doesn't Y depend on the thrust setting δ_t? This is because we assume that the direction of the thrust vector lies in the plane of symmetry.)

2 Simplification

2.1 Symmetry and asymmetry

Let's try to simplify that monstrocity of an equation of the previous part. To do that, we have to apply several tricks. The most important one, is that of symmetry and asymmetry.

We can make a distinction between symmetric and asymmetric forces/deviations. The symmetric deviations (the deviations which don't break the symmetry) are u, w and q. The symmetric forces/moments are X, Z and M. Similarly, the asymmetric deviations are v, p and r. The asymmetric forces/moments are Y, L and N.

It can now be shown that there is no coupling between the symmetric and the asymmetric properties. (That is, as long as the deviations are small.) In other words, X is uneffected by v, p and r. Thus $X_v = X_p = X_r = 0$. The same trick works for the other forces and moments as well. This causes a lot of terms to disappear in the force equations.

2.2 Simplifying the force equations

There is also another important trick we use, when simplifying the force equations. We assume that the aircraft is flying in a steady symmetric flight. This means that

$$\begin{array}{llllllll} u_0 \neq 0 & \dot{u}_0 = 0 & p_0 = 0 & \dot{p}_0 = 0 & \varphi_0 = 0 & \dot{\varphi}_0 = 0 & X_0 \neq 0 & \dot{X}_0 = 0, \\ v_0 = 0 & \dot{v}_0 = 0 & q_0 = 0 & \dot{q}_0 = 0 & \theta_0 \neq 0 & \dot{\theta}_0 = 0 & Y_0 = 0 & \dot{Y}_0 = 0, \\ w_0 \neq 0 & \dot{w}_0 = 0 & r_0 = 0 & \dot{r}_0 = 0 & \psi_0 \neq 0 & \dot{\psi}_0 = 0 & Z_0 \neq 0 & \dot{Z}_0 = 0. \end{array} \tag{2.1}$$

This greatly simplifies the $F_x(\mathbf{X_0})$, $F_y(\mathbf{X_0})$ and $F_z(\mathbf{X_0})$ terms.

Now it is finally time to apply all these simplifications and tricks. It will give us the force equations for small deviations from a steady symmetric flight. These equations are

$$-W \cos\theta_0 \theta + X_u u + X_w w + X_q q + X_{\delta_e} \delta_e + X_{\delta_t} \delta_t = m(\dot{u} + w_0 q), \tag{2.2}$$

$$W \cos\theta_0 \psi + Y_v v + Y_{\dot{v}} \dot{v} + Y_p p + Y_r r + Y_{\delta_a} \delta_a + Y_{\delta_r} \delta_r = m(\dot{v} + u_0 r - w_0 p), \tag{2.3}$$

$$-W \sin\theta_0 \theta + Z_u u + Z_w w + Z_{\dot{w}} \dot{w} + Z_q q + Z_{\delta_e} \delta_e + Z_{\delta_t} \delta_t = m(\dot{w} - u_0 q). \tag{2.4}$$

Of these three equations, the first and the third correspond to symmetric motion. The second equation corresponds to asymmetric motion.

You may wonder, where did all the Δ's go to? Well, to simplify our notation, we omitted them. So in the above equation, all variables indicate the displacement from the initial position $\mathbf{X_0}$.

Finally, there is one more small simplification we could do. We haven't fully defined our reference system yet. (We haven't specified where the X axis is in the symmetry plane.) Now let's choose our reference system. The most convenient choice is in this case the stability reference frame F_S. By choosing this frame, we have $u_0 = V$ and $w_0 = 0$. (V is the velocity.) This eliminates one more term.

2.3 The moment equations

In a similar way, we can linearize and simplify the moment equations. We won't go through that tedious process. By now you should more or less know how linearization is done. We'll just mention the results. They are

$$L_v v + L_p p + L_r r + L_{\delta_a} \delta_a + L_{\delta_r} \delta_r = I_x \dot{p} - J_{xz} \dot{r}, \tag{2.5}$$

$$M_u u + M_w w + M_{\dot{w}} \dot{w} + M_q q + M_{\delta_e} \delta_e + M_{\delta_t} \delta_t = I_y \dot{q}, \tag{2.6}$$

$$N_v v + N_{\dot{v}} \dot{v} + N_p p + N_r r + N_{\delta_a} \delta_a + N_{\delta_r} \delta_r = I_z \dot{r} - J_{xz} \dot{p}. \tag{2.7}$$

Of these three equations, only the second one corresponds to symmetric motion. The other two correspond to asymmetric motion.

2.4 The kinematic relations

The kinematic relations can also be linearized. (This is, in fact, not that difficult.) After we apply the simplifications, we wind up with

$$\dot{\varphi} = p + r \tan\theta_0, \tag{2.8}$$

$$\dot{\theta} = q, \tag{2.9}$$

$$\dot{\psi} = \frac{r}{\cos\theta_0}. \tag{2.10}$$

Of these three equations, only the second one corresponds to symmetric motion. The other two correspond to asymmetric motion.

3 Setting the equations in a non-dimensional form

3.1 The dividing term

Aerospace engineers often like to work with non-dimensional coefficients. By doing this, they can easily compare aircraft of different size and weight. So, we will also try to make our equations non-dimensional. But how do we do that? We simply divide the equations by a certain value, making them non-dimensional.

The question remains, by what do we divide them? Well, we divide the force equations by $\frac{1}{2}\rho V^2 S$, the symmetric moment equation by $\frac{1}{2}\rho V^2 S\bar{c}$, the asymmetric moment equations by $\frac{1}{2}\rho V^2 S b$, the symmetric kinematic equation by $V/\bar{c}$ and the asymmetric kinematic equations by V/b. Here, S is the wing surface area, $\bar{c}$ is the mean chord length, and b is the wing span. (Note that we use $\bar{c}$ for symmetric equations, while we use b for asymmetric equations.)

3.2 Defining coefficients

Dividing our equations by a big term won't make them look prettier. To make them still readable, we need to define some coefficients. To see how we do that, we consider the term $X_u u$. We have divided this term by $\frac{1}{2}\rho V^2 S$. We can now rewrite this term to

$$\frac{X_u u}{\frac{1}{2}\rho V^2 S} = \frac{X_u}{\frac{1}{2}\rho V S}\frac{u}{V} = C_{X_u}\hat{u}. \tag{3.1}$$

In this equation, we have defined the **non-dimensional velocity** $\hat{u}$. There is also the coefficient $C_{X_u} = X_u/(\frac{1}{2}\rho V S)$. This coefficient is called a **stability derivative**.

We can apply the same trick to other terms as well. For example, we can rewrite the term $X_w w$ to

$$\frac{X_w w}{\frac{1}{2}\rho V^2 S} = \frac{X_w}{\frac{1}{2}\rho V S}\frac{w}{V} = C_{X_\alpha}\alpha, \tag{3.2}$$

where the **angle of attack** α is approximated by $\alpha = w/V$. We can also rewrite the term $X_q q$ to

$$\frac{X_q q}{\frac{1}{2}\rho V^2 S} = \frac{X_q}{\frac{1}{2}\rho V^2 S\bar{c}}\frac{\bar{c}}{V}\frac{d\theta}{dt} = C_{X_q} D_c\theta. \tag{3.3}$$

This time we don't only see a new coefficient. There is also the **differential operator** D_c. Another differential operator is D_b. D_c and D_b are defined as

$$D_c = \frac{\bar{c}}{V}\frac{d}{dt} \qquad \text{and} \qquad D_b = \frac{b}{V}\frac{d}{dt}. \tag{3.4}$$

In this way, a lot of coefficients can be defined. We won't state the definitions of all the coefficients here. (There are simply too many for a summary.) But you probably can guess the meaning of most of them by now. And you simply have to look up the others.

3.3 The equations of motion in matrix form

So, we could now write down a new set of equations, with a lot of coefficients. However, we know that these equations are linear. So, we can put them in a matrix form. If we do that, we will find two interesting matrix equations. The equations for the symmetric motion are given by

$$\begin{bmatrix} C_{X_u} - 2\mu_c D_c & C_{X_\alpha} & C_{Z_0} & C_{X_q} \\ C_{Z_u} & C_{Z_\alpha} + (C_{Z_{\dot{\alpha}}} - 2\mu_c)D_c & -C_{X_0} & 2\mu_c + C_{Z_q} \\ 0 & 0 & -D_c & 1 \\ C_{m_u} & C_{m_\alpha} + C_{m_{\dot{\alpha}}} D_c & 0 & C_{m_q} - 2\mu_c K_Y^2 D_c \end{bmatrix} \begin{bmatrix} \hat{u} \\ \alpha \\ \theta \\ \frac{q\bar{c}}{V} \end{bmatrix} = \begin{bmatrix} -C_{X_{\delta_e}} & -C_{X_{\delta_t}} \\ -C_{Z_{\delta_e}} & -C_{Z_{\delta_t}} \\ 0 & 0 \\ -C_{m_{\delta_e}} & -C_{m_{\delta_t}} \end{bmatrix} \begin{bmatrix} \delta_e \\ \delta_t \end{bmatrix}. \tag{3.5}$$

You may note that, instead of using the subscript M, we use the subscript m. This is just a writing convention. You also haven't seen the variable K_Y^2 yet. It is defined as $K_Y^2 = \frac{I_y}{m\bar{c}^2}$. Also, $\mu_c = \frac{m}{\rho S \bar{c}}$.

The equations for the asymmetric motion are given by

$$\begin{bmatrix} C_{Y_\beta} + (C_{Y_{\dot{\beta}}} - 2\mu_b)D_b & C_L & C_{Y_p} & C_{Y_r} - 4\mu_b \\ 0 & -\frac{1}{2}D_b & 1 & 0 \\ C_{l_\beta} & 0 & C_{l_p} - 4\mu_b K_X^2 D_b & C_{l_r} + 4\mu_b K_{XZ} D_b \\ C_{n_\beta} + C_{n_{\dot{\beta}}} D_b & 0 & C_{n_p} + 4\mu_b K_{XZ} D_b & C_{n_r} - 4\mu_b K_Z^2 D_b \end{bmatrix} \begin{bmatrix} \beta \\ \varphi \\ \frac{pb}{2V} \\ \frac{rb}{2V} \end{bmatrix} = \begin{bmatrix} -C_{Y_{\delta_a}} & -C_{Y_{\delta_r}} \\ 0 & 0 \\ -C_{l_{\delta_a}} & -C_{l_{\delta_r}} \\ -C_{n_{\delta_a}} & -C_{n_{\delta_r}} \end{bmatrix} \begin{bmatrix} \delta_a \\ \delta_r \end{bmatrix}. \tag{3.6}$$

Again, note that, instead of using the subscripts L and N, we have used l and n. Also, the **slip angle** β is defined as $\beta = v/V$.

3.4 Equations of motion in state-space form

We can also put our equation in state-space form, being

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u} \qquad \text{and} \qquad \mathbf{y} = C\mathbf{x} + D\mathbf{u}. \tag{3.7}$$

Here, A is the **state matrix**, B is the **input matrix**, C is the **output matrix** and D is the **direct matrix**. Since the system is time-invariant, all these matrices are constant. Also, $\mathbf{x}$ is the **state vector**, $\mathbf{u}$ is the **input vector** and $\mathbf{y}$ is the **output vector**.

The state-space form has several advantages. First of all, the parameters can be solved for at every time t. (The complicated equations for this are known.) Second, computers are very good at performing simulations, once a situation has been described in state-space form.

After some interesting matrix manipulation, the state-space form of the symmetric motions can be derived. The result is

$$\begin{bmatrix} \dot{\hat{u}} \\ \dot{\alpha} \\ \dot{\theta} \\ \frac{\dot{q}\bar{c}}{V} \end{bmatrix} = \begin{bmatrix} x_u & x_\alpha & x_\theta & 0 \\ z_u & z_\alpha & z_\theta & z_q \\ 0 & 0 & 0 & \frac{V}{\bar{c}} \\ m_u & m_\alpha & m_\theta & m_q \end{bmatrix} \begin{bmatrix} \hat{u} \\ \alpha \\ \theta \\ \frac{q\bar{c}}{V} \end{bmatrix} + \begin{bmatrix} x_{\delta_e} & x_{\delta_t} \\ z_{\delta_e} & z_{\delta_t} \\ 0 & 0 \\ m_{\delta_e} & m_{\delta_t} \end{bmatrix} \begin{bmatrix} \delta_e \\ \delta_t \end{bmatrix}. \tag{3.8}$$

There are quite some strange new coefficients in this equation. The equations, with which these coefficients are calculated, can be looked up. However, we will not mention those here.

You may notice that, in the above equation, we only have the state matrix A and the input matrix B. The matrices C and D are not present. That is because they depend on what output you want to get out of your system. So we can't generally give them here. They are often quite trivial though.

Similarly, the state-space form of the asymmetric motions can be found. This time we have

$$\begin{bmatrix} \dot{\beta} \\ \dot{\varphi} \\ \frac{\dot{p}b}{2V} \\ \frac{\dot{r}b}{2V} \end{bmatrix} = \begin{bmatrix} y_\beta & y_\varphi & y_p & y_r \\ 0 & 0 & \frac{2V}{b} & 0 \\ l_\beta & 0 & l_p & l_r \\ n_\beta & 0 & n_p & n_r \end{bmatrix} \begin{bmatrix} \beta \\ \varphi \\ \frac{pb}{2V} \\ \frac{rb}{2V} \end{bmatrix} + \begin{bmatrix} 0 & y_{\delta_r} \\ 0 & 0 \\ l_{\delta_a} & l_{\delta_r} \\ n_{\delta_a} & n_{\delta_r} \end{bmatrix} \begin{bmatrix} \delta_a \\ \delta_r \end{bmatrix}. \tag{3.9}$$

And that concludes this collection of oversized equations.