Introduction

In this summary we examine the flight dynamics of aircraft. But before we do that, we must examine some basic ideas necessary to explore the secrets of flight dynamics.

1 Basic concepts

1.1 Controlling an airplane

To control an aircraft, control surfaces are generally used. Examples are elevators, flaps and spoilers. When dealing with control surfaces, we can make a distinction between primary and secondary flight control surfaces. When **primary control surfaces** fail, the whole aircraft becomes uncontrollable. (Examples are elevators, ailerons and rudders.) However, when **secondary control surfaces** fail, the aircraft is just a bit harder to control. (Examples are flaps and trim tabs.)

The whole system that is necessary to control the aircraft is called the **control system**. When a control system provides direct feedback to the pilot, it is called a **reversible system**. (For example, when using a mechanical control system, the pilot feels forces on his stick.) If there is no direct feedback, then we have an **irreversible system**. (An example is a fly-by-wire system.)

1.2 Making assumptions

In this summary, we want to describe the flight dynamics with equations. This is, however, very difficult. To simplify it a bit, we have to make some simplifying assumptions. We assume that ...

- There is a **flat Earth**. (The Earth's curvature is zero.)
- There is a **non-rotating Earth**. (No Coriolis accelerations and such are present.)
- The aircraft has **constant mass**.
- The aircraft is a **rigid body**.
- The aircraft is **symmetric**.
- There are no **rotating masses**, like turbines. (Gyroscopic effects can be ignored.)
- There is **constant wind**. (So we ignore turbulence and gusts.)

2 Reference frames

2.1 Reference frame types

To describe the position and behavior of an aircraft, we need a **reference frame** (RF). There are several reference frames. Which one is most convenient to use depends on the circumstances. We will examine a few.

- First let's examine the **inertial reference frame** F_I. It is a right-handed orthogonal system. Its origin A is the center of the Earth. The Z_I axis points North. The X_I axis points towards the **vernal equinox**. The Y_I axis is perpendicular to both the axes. Its direction can be determined using the right-hand rule.

- In the **(normal) Earth-fixed reference frame** F_E, the origin O is at an arbitrary location on the ground. The Z_E axis points towards the ground. (It is perpendicular to it.) The X_E axis is directed North. The Y_E axis can again be determined using the right-hand rule.
- The **body-fixed reference frame** F_b is often used when dealing with aircraft. The origin of the reference frame is the center of gravity (CG) of the aircraft. The X_b axis lies in the symmetry plane of the aircraft and points forward. The Z_b axis also lies in the symmetry plane, but points downwards. (It is perpendicular to the X_b axis.) The Y_b axis can again be determined using the right-hand rule.
- The **stability reference frame** F_S is similar to the body-fixed reference frame F_b. It is rotated by an angle α_a about the Y_b axis. To find this α_a, we must examine the **relative wind vector** $\mathbf{V_a}$. We can project this vector onto the plane of symmetry of the aircraft. This projection is then the direction of the X_S axis. (The Z_S axis still lies in the plane of symmetry. Also, the Y_S axis is still equal to the Y_b axis.) So, the relative wind vector lies in the $X_S Y_S$ plane. This reference frame is particularly useful when analyzing flight dynamics.
- The **aerodynamic (air-path) reference frame** F_a is similar to the stability reference frame F_S. It is rotated by an angle β_a about the Z_S axis. This is done, such that the X_a axis points in the direction of the relative wind vector $\mathbf{V_a}$. (So the X_a axis generally does not lie in the symmetry plane anymore.) The Z_a axis is still equation to the Z_S axis. The Y_a axis can now be found using the right-hand rule.
- Finally, there is the **vehicle reference frame** F_r. Contrary to the other systems, this is a left-handed system. Its origin is a fixed point on the aircraft. The X_r axis points to the rear of the aircraft. The Y_r axis points to the left. Finally, the Z_r axis can be found using the left-hand rule. (It points upward.) This system is often used by the aircraft manufacturer, to denote the position of parts within the aircraft.

2.2 Changing between reference frames

We've got a lot of reference frames. It would be convenient if we could switch from one coordinate system to another. To do this, we need to rotate reference frame 1, until we wind up with reference frame 2. (We don't consider the translation of reference frames here.) When rotating reference frames, **Euler angles** ϕ come in handy. The Euler angles ϕ_x, ϕ_y and ϕ_z denote rotations about the X axis, Y axis and Z axis, respectively.

We can go from one reference frame to any other reference frame, using at most three Euler angles. An example transformation is $\phi_x \to \phi_y \to \phi_z$. In this transformation, we first rotate about the X axis, followed by a transformation about the Y axis and the Z axis, respectively. The order of these rotations is very important. Changing the order will give an entirely different final result.

2.3 Transformation matrices

An Euler angle can be represented by a **transformation matrix** $\mathbb{T}$. To see how this works, we consider a vector $\mathbf{x^1}$ in reference frame 1. The matrix $\mathbb{T}_{21}$ now calculates the coordinates of the same vector $\mathbf{x^2}$ in reference frame 2, according to $\mathbf{x^2} = \mathbb{T}_{21}\mathbf{x^1}$.

Let's suppose we're only rotating about the X axis. In this case, the transformation matrix $\mathbb{T}_{21}$ is quite simple. In fact, it is

$$\mathbb{T}_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi_x & \sin\phi_x \\ 0 & -\sin\phi_x & \cos\phi_x \end{bmatrix}. \tag{2.1}$$

Similarly, we can rotate about the Y axis and the Z axis. In this case, the transformation matrices are, respectively,

$$\mathbb{T}_{21} = \begin{bmatrix} \cos\phi_y & 0 & -\sin\phi_y \\ 0 & 1 & 0 \\ \sin\phi_y & 0 & \cos\phi_y \end{bmatrix} \qquad \text{and} \qquad \mathbb{T}_{21} = \begin{bmatrix} \cos\phi_z & \sin\phi_z & 0 \\ -\sin\phi_z & \cos\phi_z & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2.2}$$

A sequence of rotations (like $\phi_x \to \phi_y \to \phi_z$) is now denoted by a sequence of matrix multiplications $\mathbb{T}_{41} = \mathbb{T}_{43}\mathbb{T}_{32}\mathbb{T}_{21}$. In this way, a single transformation matrix for the whole sequence can be obtained.

Transformation matrices have interesting properties. They only rotate points. They don't deform them. For this reason, the matrix columns are orthogonal. And, because the space is not stretched out either, these columns must also have length 1. A transformation matrix is thus orthogonal. This implies that

$$\mathbb{T}_{21}^{-1} = \mathbb{T}_{21}^{T} = \mathbb{T}_{12}. \tag{2.3}$$

2.4 Transformation examples

Now let's consider some actual transformations. Let's start at the body-fixed reference frame F_b. If we rotate this frame by an angle α_a about the Y axis, we find the stability reference frame F_S. If we then rotate it by an angle β_a about the Z axis, we get the aerodynamic reference frame F_a. So we can find that

$$\mathbf{x^a} = \begin{bmatrix} \cos\beta_a & \sin\beta_a & 0 \\ -\sin\beta_a & \cos\beta_a & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{x^S} = \begin{bmatrix} \cos\beta_a & \sin\beta_a & 0 \\ -\sin\beta_a & \cos\beta_a & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\alpha_a & 0 & \sin\alpha_a \\ 0 & 1 & 0 \\ -\sin\alpha_a & 0 & \cos\alpha_a \end{bmatrix} \mathbf{x^b}. \tag{2.4}$$

By working things out, we can thus find that

$$\mathbb{T}_{ab} = \begin{bmatrix} \cos\beta_a\cos\alpha_a & \sin\beta_a & \cos\beta_a\sin\alpha_a \\ -\sin\beta_a\cos\alpha_a & \cos\beta_a & -\sin\beta_a\sin\alpha_a \\ -\sin\alpha_a & 0 & \cos\alpha_a \end{bmatrix}. \tag{2.5}$$

We can make a similar transformation between the Earth-fixed reference frame F_E and the body-fixed reference frame F_b. To do this, we first have to rotate over the **yaw angle** ψ about the Z axis. We then rotate over the **pitch angle** θ about the Y axis. Finally, we rotate over the **roll angle** φ about the X axis. If we work things out, we can find that

$$\mathbb{T}_{bE} = \begin{bmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ \sin\varphi\sin\theta\cos\psi - \cos\varphi\sin\psi & \sin\varphi\sin\theta\sin\psi + \cos\varphi\cos\psi & \sin\varphi\cos\theta \\ \cos\varphi\sin\theta\cos\psi + \sin\varphi\sin\psi & \cos\varphi\sin\theta\sin\psi - \sin\varphi\cos\psi & \cos\varphi\cos\theta \end{bmatrix}. \tag{2.6}$$

Now that's one hell of a matrix ...

2.5 Moving reference frames

Let's examine some point P. This point is described by vector $\mathbf{r^E}$ in reference frame F_E and by $\mathbf{r^b}$ in reference frame F_b. Also, the origin of F_b (with respect to F_E) is described by the vector $\mathbf{r_{Eb}}$. So we have $\mathbf{r^E} = \mathbf{r_{Eb}} + \mathbf{r^b}$.

Now let's examine the time derivative of $\mathbf{r^E}$ in F_E. We denote this by $\left.\frac{d\mathbf{r^E}}{dt}\right|_E$. It is given by

$$\left.\frac{d\mathbf{r^E}}{dt}\right|_E = \left.\frac{d\mathbf{r_{Eb}}}{dt}\right|_E + \left.\frac{d\mathbf{r^b}}{dt}\right|_E. \tag{2.7}$$

Let's examine the terms in this equation. The middle term of the above equation simply indicates the movement of F_b, with respect to F_E. The right term is, however, a bit more complicated. It indicates the change of $\mathbf{r^b}$ with respect to F_E. But we usually don't know this. We only know the change of $\mathbf{r^b}$ in F_b. So we need to transform this term from F_E to F_b. Using a slightly difficult derivation, it can be shown that

$$\left.\frac{d\mathbf{r^b}}{dt}\right|_E = \left.\frac{d\mathbf{r^b}}{dt}\right|_b + \mathbf{\Omega_{bE}} \times \mathbf{r^b}. \tag{2.8}$$

The vector $\mathbf{\Omega_{bE}}$ denotes the **rotation vector** of F_b with respect to F_E. Inserting this relation into the earlier equation gives us

$$\left.\frac{d\mathbf{r^E}}{dt}\right|_E = \left.\frac{d\mathbf{r_{Eb}}}{dt}\right|_E + \left.\frac{d\mathbf{r^b}}{dt}\right|_b + \mathbf{\Omega_{bE}} \times \mathbf{r^b}. \tag{2.9}$$

This is quite an important relation, so remember it well. By the way, it holds for every vector. So instead of the position vector $\mathbf{r}$, we could also take the velocity vector $\mathbf{V}$.

Finally, we note some interesting properties of the rotation vector. Given reference frames 1, 2 and 3, we have

$$\mathbf{\Omega_{12}} = -\mathbf{\Omega_{21}} \qquad \text{and} \qquad \mathbf{\Omega_{31}} = \mathbf{\Omega_{32}} + \mathbf{\Omega_{21}}. \tag{2.10}$$