Deriving the equations of motion

The flight dynamics of an aircraft are described by its **equations of motion** (EOM). We are going to derive those equations in this chapter.

1 Forces

1.1 The basic force equation

To derive the equations of motion of an aircraft, we start by examining forces. Our starting point in this is Newton's second law. However, Newton's second law only holds in an inertial reference system. Luckily, the assumptions we have made earlier imply that the Earth-fixed reference frame F_E is inertial. (However, F_b is not an inertial reference frame.) So we will derive the equations of motion with respect to F_E.

Let's examine an aircraft. Newton's second law states that

$$\mathbf{F} = \int d\mathbf{F} = \frac{d}{dt}\left(\int \mathbf{V_p}\, dm\right), \tag{1.1}$$

where we integrate over the entire body. It can be shown that the right part of this equation equals $\frac{d}{dt}(\mathbf{V_G} m)$, where $\mathbf{V_G}$ is the velocity of the center of gravity of the aircraft. If the aircraft has a constant mass, we can rewrite the above equation into

$$\mathbf{F} = m\frac{d\mathbf{V_G}}{dt} = m\mathbf{A_G}. \tag{1.2}$$

This relation looks very familiar. But it does imply something very important. The acceleration of the CG of the aircraft does not depend on how the forces are distributed along the aircraft. It only depends on the magnitude and direction of the forces.

1.2 Converting the force equation

There is one slight problem. The above equation holds for the F_E reference frame. But we usually work in the F_b reference frame. So we need to convert it. To do this, we can use the relation

$$\mathbf{A_G} = \left.\frac{d\mathbf{V_G}}{dt}\right|_E = \left.\frac{d\mathbf{V_G}}{dt}\right|_b + \mathbf{\Omega_{bE}} \times \mathbf{V_G}. \tag{1.3}$$

Inserting this into the above equation will give

$$\mathbf{F} = m\left.\frac{d\mathbf{V_G}}{dt}\right|_b + m\,\mathbf{\Omega_{bE}} \times \mathbf{V_G} = m\begin{bmatrix}\dot{u} + qw - rv \\ \dot{v} + ru - pw \\ \dot{w} + pv - qu\end{bmatrix}. \tag{1.4}$$

By the way, in the above equation, we have used that

$$\mathbf{V_G} = \begin{bmatrix}u \\ v \\ w\end{bmatrix} \qquad \text{and} \qquad \mathbf{\Omega_{bE}} = \begin{bmatrix}p \\ q \\ r\end{bmatrix}. \tag{1.5}$$

Here, u, v and w denote the velocity components in X, Y and Z direction, respectively. Similarly, p, q and r denote rotation components about the X, Y and Z axis, respectively.

1.3 External forces

Let's take a look at the forces $\mathbf{F}$ our aircraft is subject to. There are two important kinds of forces: gravity and aerodynamic forces. The gravitational force $\mathbf{F_{gravity}}$ is, in fact, quite simple. It is given by

$$\mathbf{F^E_{gravity}} = \begin{bmatrix} 0 & 0 & mg \end{bmatrix}^T, \tag{1.6}$$

where g is the **gravitational acceleration**. The superscript E indicates that the force is given in the F_E reference frame. However, we want the force in the F_b reference frame. Luckily, we know the transformation matrix $\mathbb{T}_{bE}$. We can thus find that

$$\mathbf{F^b_{gravity}} = \mathbb{T}_{bE}\mathbf{F^E_{gravity}} = mg \begin{bmatrix} -\sin\theta \\ \sin\varphi\cos\theta \\ \cos\varphi\cos\theta \end{bmatrix}. \tag{1.7}$$

The aerodynamic forces $\mathbf{F_{aero}}$ are, however, a lot more difficult. For now, we won't examine them in depth. Instead, we simply say that

$$\mathbf{F^b_{aero}} = \begin{bmatrix} X^b & Y^b & Z^b \end{bmatrix}^T. \tag{1.8}$$

By combining this knowledge with the equation of motion for forces, we find that

$$m \begin{bmatrix} \dot{u} + qw - rv \\ \dot{v} + ru - pw \\ \dot{w} + pv - qu \end{bmatrix} = mg \begin{bmatrix} -\sin\theta \\ \sin\varphi\cos\theta \\ \cos\varphi\cos\theta \end{bmatrix} + \begin{bmatrix} X^b \\ Y^b \\ Z^b \end{bmatrix}. \tag{1.9}$$

2 Moments

2.1 Angular momentum

Before we're going to look at moments, we will first examine angular momentum. The **angular momentum** of an aircraft $\mathbf{B_G}$ (with respect to the CG) is defined as

$$\mathbf{B_G} = \int d\mathbf{B_G} = \mathbf{r} \times \mathbf{V_P}\, dm, \tag{2.1}$$

where we integrate over every point P in the aircraft. We can substitute

$$\mathbf{V_P} = \mathbf{V_G} + \left.\frac{d\mathbf{r}}{dt}\right|_b + \mathbf{\Omega_{bE}} \times \mathbf{r}. \tag{2.2}$$

If we insert this, and do a lot of working out, we can eventually find that

$$\mathbf{B_G} = \mathbb{I}_G \mathbf{\Omega_{bE}}. \tag{2.3}$$

The parameter $\mathbb{I}_G$ is the **inertia tensor**, with respect to the CG. It is defined as

$$\mathbb{I}_G = \begin{bmatrix} I_x & -J_{xy} & -J_{xz} \\ -J_{xy} & I_y & -J_{yz} \\ -J_{xz} & -J_{yz} & I_z \end{bmatrix} = \begin{bmatrix} \int(r_y^2 + r_z^2)dm & -\int(r_x r_y)dm & -\int(r_x r_z)dm \\ -\int(r_x r_y)dm & \int(r_x^2 + r_z^2)dm & -\int(r_y r_z)dm \\ -\int(r_x r_z)dm & -\int(r_y r_z)dm & \int(r_x^2 + r_y^2)dm \end{bmatrix}. \tag{2.4}$$

We have assumed that the XZ-plane of the aircraft is a plane of symmetry. For this reason, $J_{xy} = J_{yz} = 0$. This simplifies the inertia tensor a bit.

2.2 The moment equation

It is now time to look at moments. We again do this from the inertial reference frame F_E. The moment acting on our aircraft, with respect to its CG, is given by

$$\mathbf{M_G} = \int d\mathbf{M_G} = \int \mathbf{r} \times d\mathbf{F} = \int \mathbf{r} \times \frac{d\left(\mathbf{V_p}\, dm\right)}{dt}, \tag{2.5}$$

where we integrate over the entire body. Luckily, we can simplify the above relation to

$$\mathbf{M_G} = \left.\frac{d\mathbf{B_G}}{dt}\right|_E. \tag{2.6}$$

The above relation only holds for inertial reference frames, such as F_E. However, we want to have the above relation in F_b. So we rewrite it to

$$\mathbf{M_G} = \left.\frac{d\mathbf{B_G}}{dt}\right|_b + \mathbf{\Omega_{bE}} \times \mathbf{B_G}. \tag{2.7}$$

By using $\mathbf{B_G} = \mathbb{I}_G \mathbf{\Omega_{bE}}$, we can continue to rewrite the above equation. We eventually wind up with

$$\mathbf{M_G} = \mathbb{I}_G \left.\frac{d\mathbf{\Omega_{bE}}}{dt}\right|_b + \mathbf{\Omega_{bE}} \times \mathbb{I}_G \mathbf{\Omega_{bE}}. \tag{2.8}$$

In matrix-form, this equation can be written as

$$\mathbf{M_G} = \begin{bmatrix} I_x\dot{p} + (I_z - I_y)qr - J_{xz}(pq + \dot{r}) \\ I_y\dot{q} + (I_x - I_z)pr + J_{xz}(p^2 - r^2) \\ I_z\dot{r} + (I_y - I_x)pq + J_{xz}(qr - \dot{p}) \end{bmatrix}. \tag{2.9}$$

Note that we have used the fact that $J_{xy} = J_{yz} = 0$.

2.3 External moments

Let's take a closer look at $\mathbf{M_G}$. Again, we can distinguish two types of moments, acting on our aircraft. There are moments caused by gravity, and moments caused by aerodynamic forces. Luckily, the moments caused by gravity are zero. (The resultant gravitational force acts in the CG.) So we only need to consider the moments caused by aerodynamic forces. We denote those as

$$\mathbf{M^b_{G,aero}} = \begin{bmatrix} L & M & N \end{bmatrix}^T. \tag{2.10}$$

This turns the moment equation into

$$\begin{bmatrix} I_x\dot{p} + (I_z - I_y)qr - J_{xz}(pq + \dot{r}) \\ I_y\dot{q} + (I_x - I_z)pr + J_{xz}(p^2 - r^2) \\ I_z\dot{r} + (I_y - I_x)pq + J_{xz}(qr - \dot{p}) \end{bmatrix} = \begin{bmatrix} L \\ M \\ N \end{bmatrix}. \tag{2.11}$$

3 Kinematic relations

3.1 Translational kinematics

Now that we have the force and moment equations, we only need to find the kinematic relations for our aircraft. First, we examine translational kinematics. This concerns the velocity of the CG of the aircraft with respect to the ground.

The velocity of the CG, with respect to the ground, is called the **kinematic velocity $\mathbf{V_k}$**. In the F_E reference system, it is described by

$$\mathbf{V_k} = \begin{bmatrix} V_N & V_E & -V_Z \end{bmatrix}^T. \tag{3.1}$$

In this equation, V_N is the velocity component in the Northward direction, V_E is the velocity component in the eastward direction, and $-V_Z$ is the vertical velocity component. (The minus sign is present because, in the Earth-fixed reference system, V_Z is defined to be positive downward.)

However, in the F_b reference system, the velocity of the CG, with respect to the ground, is given by

$$\mathbf{V_G} = \begin{bmatrix} u & v & w \end{bmatrix}^T. \tag{3.2}$$

To relate those two vectors to each other, we need a transformation matrix. This gives us

$$\mathbf{V_k} = \mathbb{T}_{Eb}\mathbf{V_G} = \mathbb{T}_{bE}^T\mathbf{V_G}. \tag{3.3}$$

This is the translational kinematic relation. We can use it to derive the change of the aircraft position. To do that, we simply have to integrate the velocities. We thus have

$$x(t) = \int_0^t V_N\,dt, \qquad y(t) = \int_0^t V_E\,dt \quad \text{and} \quad h(t) = \int_0^t -V_Z\,dt. \tag{3.4}$$

3.2 Rotational kinematics

Now let's examine rotational kinematics. This concerns the rotation of the aircraft. In the F_E reference system, the rotational velocity is described by the variables $\dot{\varphi}$, $\dot{\theta}$ and $\dot{\psi}$. However, in the F_b reference system, the rotational velocity is described by p, q and r. The relation between these two triples can be shown to be

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\sin\theta \\ 0 & \cos\varphi & \cos\theta\sin\varphi \\ 0 & -\sin\varphi & \cos\theta\cos\varphi \end{bmatrix} \begin{bmatrix} \dot{\varphi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}. \tag{3.5}$$

This is the rotational kinematic relation. It is interesting to note that, if $\varphi = \theta = \psi = 0$, then $p = \dot{\varphi}$, $q = \dot{\theta}$ and $r = \dot{\psi}$. By the way, we can also invert the above relation. We would then get

$$\begin{bmatrix} \dot{\varphi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 1 & \sin\varphi\tan\theta & \cos\varphi\tan\theta \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi/\cos\theta & \cos\varphi/\cos\theta \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix}. \tag{3.6}$$