Chapter 8

Problems and Solutions Section 8.1 (8.1 through 8.7)

8.1 Consider the one-element model of a bar discussed in Section 8.1. Calculate the finite element of the bar for the case that it is free at both ends rather than clamped.

Solution: The finite element for a rod is derived in section 8.1. Since u_1 is not restrained equations (8.7) and (8.11) are the finite element matrices.

8.2 Calculate the natural frequencies of the free-free bar of Problem 8.1. To what does the first natural frequency correspond? How do these values compare with the exact values obtained from methods of Chapter 6?

Solution:

$$K = \frac{EA}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad M = \frac{\rho Al}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$M^{-1}K = \frac{6E}{\rho l^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$\lambda_{1,2} = 0, \dfrac{12E}{\rho l^2}$ and the corresponding eigenvectors are

$\mathbf{x}_1 = [1 \quad 1]^T/\sqrt{2}$ and $\mathbf{x}_2 = [1 \quad -1]^T/\sqrt{2}$

Therefore, $\omega_1 = 0,\ \omega_2 = \sqrt{\dfrac{12E}{\rho l^2}}$

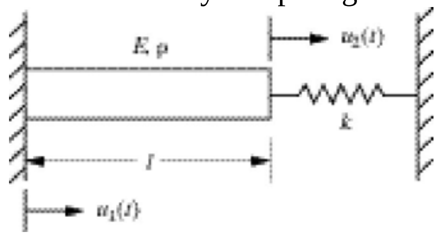
The first natural frequency corresponds to the rigid body mode, or pure translation.

From the solution to problem 6.8,

$$\omega_1 = 0,\ \omega_2 = \sqrt{\frac{\pi^2 E}{\rho l^2}}$$

The first natural frequency is predicted exactly while the second is 10.2% high. A point of interest is that, due to symmetry, the first mode of a clamped-free rod of length $l/2$ has the same natural frequency as the second mode of a free-free rod of length l.

8.3 Consider the system of Figure P8.3, consisting of a spring connected to a clamped-free bar. Calculate the finite element model and discuss the accuracy of the frequency prediction of this model by comparing it with the method of Chapter 6.



Solution:

The finite element for the clamped-free rod is given by (8.14) as

$$\frac{\rho A l}{3}\ddot{u}_2(t) + \frac{EA}{l}u_2(t) = 0$$

The spring has the effect of adding stiffness K at u_2. Thus,

$$\frac{\rho A l}{3}\ddot{u}_2(t) + \left(\frac{EA}{l} + K\right)u_2(t) = 0$$

From (1.16)

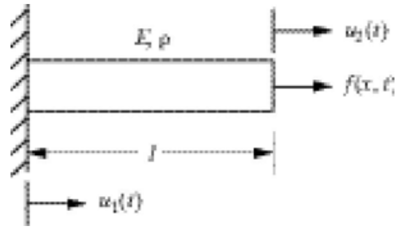
$$\omega = \sqrt{\frac{3(Kl + EA)}{\rho A l}}$$

Next consider the first natural frequency as predicted from the distributed parameter approach of chapter 6. In particular Table 6.1 gives the frequency equation for this system as $\lambda_n \cot\lambda_n = -(Kl/EA)$ where $\lambda_n = \omega_n l/c$, $c^2 = E/\rho$. Approximating $\cot x = 1/x - x/3$ the frequency equation of Table 6.1 becomes

$\lambda_n (1/\lambda_n - \lambda_n/3) = -(kl/EA)$ or for n=1 $\omega^2 l^2/c^2 = 3(1 + kl/EA)$

which upon solving for ω is identical to the one element FEM frequency derived above.

8.4 Consider a clamped-free bar with a force *f(t)* applied in the axial direction at the free end as illustrated in Figure P8.4. Calculate the equations of motion using a single-element finite element model.



Solution:

The finite element equation of motion for an unforced clamped-free bar is given by equation (8.14). Using (8.13) it can be seen that the forced equation is

$$\frac{\rho A l}{3}\ddot{u}_2(t) + \frac{EA}{l}u_2(t) = f(t)$$

8.5 Compare the solution of a cantilevered bar modeled as a single finite element with that of the distributed-parameter method summarized in Figure 8.1 truncated at three modes by calculating (a) $u(x,t)$ and (b) $u(l/2,t)$ for a 1-m aluminum beam at t = 0.1, 1, and 10s using both methods. Use the initial condition $u(x,0) = 0.1x$ m and $u_t(x,0) = 0$.

Solution: (8.5, 8.6)

For the finite element of the bar

$\rho = 2700$ kg/m^3, $E = 7 \times 10^{10}$ N/m^2

The unforced equation of motion is then

$$\ddot{u}_2(t) + 7.78 \times 10^7 u_2(t) = 0$$

From window 8.2

$u_2(t)=.1\cos(8.819\times10^3 t)$

Using the shape functions for the bar

$u(\text{x,t}) = u_2(t)x=.1x\cos(8.819\times10^3 t)$

For the continuous model truncated at 3 modes, (see example6.3.1)

$\omega_{1,2,3}$ = 8000 rad/s, 24000 rad/s, 40000 rad/s and the mode shapes are

$$X_1(x) = \sin\left(\frac{\pi}{2}\frac{x}{l}\right) \quad , \; X_4(x) = \sin\left(\frac{7\pi}{2}\frac{x}{l}\right)$$
$$X_2(x) = \sin\left(\frac{3\pi}{2}\frac{x}{l}\right) \quad , \; X_5(x) = \sin\left(\frac{9\pi}{2}\frac{x}{l}\right)$$
$$X_3(x) = \sin\left(\frac{5\pi}{2}\frac{x}{l}\right)$$

The solution is given by (6.27) as

$$u(x,t) = \sum_{n=1}^{\infty}(c_n \sin\omega_n t + d_n \cos\omega_n t)X_n(x)$$

Since we are given $\dot{u}(x,t) = 0,\; c_n = 0$

$$u(x,t) = \sum_{n=1}^{\infty} a_n \cos(\omega_n t)X_n(x)$$

Considering the initial condition $u(x,0) = .1x$

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin\frac{(2n-1)\pi}{2}\frac{x}{l} = .1x$$

Multiplying by $\sin\frac{(2m-1)\pi}{2}\frac{x}{l}$ and integrating from $x = 0$ to $x = l$,

$$\int_0^l .1x\sin\left(\frac{(2m-1)\pi}{2}\frac{x}{l}\right)dx = a_n\int_0^l \sin\left(\frac{(2n-1)\pi}{2}\frac{x}{l}\right)\sin\left(\frac{(2m-1)\pi}{2}\frac{x}{l}\right)dx$$

$$\int_0^l .1x\sin\left(\frac{(2m-1)\pi}{2}\frac{x}{l}\right)dx = a_n\left(\frac{l^2}{2}\right)$$

$$a_n = \frac{2}{l^2}\int_0^l .1x\sin\left(\frac{(2n-1)\pi}{2}\frac{x}{l}\right)dx$$

a_1=.08106, a_2=-.009006, a_3=.003242, a_4=0.001654, a_5=.001001

from (6.63)

$$\omega_n = \frac{(2n-1)\pi}{2}\sqrt{\frac{E}{\rho}},\ \omega_1 = \frac{\pi}{2}\sqrt{\frac{E}{\rho}} = 7998\text{rad/s},\ \omega_2 = \frac{3\pi}{2}\sqrt{\frac{E}{\rho}} = 23994\text{rad/s},$$

$$\omega_3 = \frac{5\pi}{2}\sqrt{\frac{E}{\rho}} = 39990\text{rad/s},\ \omega_4 = \frac{7\pi}{2}\sqrt{\frac{E}{\rho}} = 55987\text{rad/s}$$

$$\omega_5 = \frac{9\pi}{2}\sqrt{\frac{E}{\rho}} = 71982\text{rad/s}$$

Substitution into 6.27 yields

$$\omega(x,t) = .08106\cos(7998t)\sin\left(\frac{\pi}{2}\frac{x}{l}\right) - .00901\cos(23994t)\sin\left(\frac{3\pi}{2}\frac{x}{l}\right)$$
$$+ .00324\cos(39990t)\sin\left(\frac{5\pi}{2}\frac{x}{l}\right) - .00165\cos(55987t)\sin\left(\frac{7\pi}{2}\frac{x}{l}\right)$$
$$+ .001001\cos(71982t)\sin\left(\frac{9\pi}{2}\frac{x}{l}\right)$$

Note that for problem 8.5 the last two terms are neglected.

8.5

$$u(x,t)\Big|_{t=.1} = -.021205\sin\left(\frac{\pi x}{2}\right) - .00643\sin\left(\frac{3\pi x}{2}\right) - .00314\left(\frac{5\pi x}{2}\right)$$

$$u(x,t)\Big|_{t=1} = .07133\sin\left(\frac{\pi x}{2}\right) - .00077\sin\left(\frac{3\pi x}{2}\right) - .00255\left(\frac{5\pi x}{2}\right)$$

$$u(x,t)\Big|_{t=10} = .01900\sin\left(\frac{\pi x}{2}\right) - .00587\sin\left(\frac{3\pi x}{2}\right) - .003\left(\frac{5\pi x}{2}\right)$$

$$u(x,t)\Big|_{t=.1,\,x=.5} = -.01732$$

$$u(x,t)\Big|_{t=1,\,x=.5} = .05169$$

$$u(x,t)\Big|_{t=10,\,x=.5} = .01546$$

8.6 Repeat Problem 8.5 using a five-mode model. Can you draw any conclusions?

Solution:

$$u(x,t)\big|_{t=.1} = -.0212\sin\left(\frac{\pi x}{2}\right) - .00643\sin\left(\frac{3\pi x}{2}\right) - .00314\left(\frac{5\pi x}{2}\right)$$
$$- .00153\sin\left(\frac{7\pi x}{2}\right) - .00069\sin\left(\frac{9\pi x}{2}\right)$$

$$u(x,t)\big|_{t=1} = .07133\sin\left(\frac{\pi x}{2}\right) - .0077\sin\left(\frac{3\pi x}{2}\right) - .00255\left(\frac{5\pi x}{2}\right)$$
$$+ .00129\sin\left(\frac{7\pi x}{2}\right) - .00026\sin\left(\frac{9\pi x}{2}\right)$$

$$u(x,t)\big|_{t=10} = .01900\sin\left(\frac{\pi x}{2}\right) + .00587\sin\left(\frac{3\pi x}{2}\right) - .00300\left(\frac{5\pi x}{2}\right)$$
$$- .00146\sin\left(\frac{7\pi x}{2}\right) + .00085\sin\left(\frac{9\pi x}{2}\right)$$

$$u(x,t)\big|_{t=.1,\,x=.5} = -.01672$$
$$u(x,t)\big|_{t=1,\,x=.5} = .05060$$
$$u(x,t)\big|_{t=10,\,x=.5} = .01709$$

For the finite element solution from (8.17)

$$u(x,t) = .1x\cos(8819.2t)$$

$u(x,t)\big|_{t=.1} = -.06445x$ $\quad$ $u(x,t)\big|_{t=.1,\,x=.5} = -.03222$

$u(x,t)\big|_{t=1} = -.07515x$ $\quad$ $u(x,t)\big|_{t=1,\,x=.5} = -.03758$

$u(x,t)\big|_{t=10} = .06047x$ $\quad$ $u(x,t)\big|_{t=10,\,x=.5} = .03024$

Conclusion: Not nearly enough elements were used to accurately determine the 1st natural frequency. Since the 1st mode dominates the response (this can be seen by comparing the coefficients, a_n), it must be determined well in order to predict the rod's response.

8.7 Repeat Problem 8.5 using only the first mode in the series solution and the initial condition $u(x,0) = 0.1\sin(\pi x/2l)$, $u_t(x,0) = 0$. For this initial condition, the first mode is exact. Why?

Solution:

Using the same procedure as in problem 8.5, the solution is

$$u(x,t) = .1\sin\left(\frac{\pi x}{2}\right)\cos(7998t)$$

$$u(x,t)\big|_{t=.1} = -.02616\sin\left(\frac{\pi x}{2}\right) \qquad u(x,t)\big|_{t=.1,x=.5} = -.01850$$

$$u(x,t)\big|_{t=1} = .08800\sin\left(\frac{\pi x}{2}\right) \qquad u(x,t)\big|_{t=1,x=.5} = .06223$$

$$u(x,t)\big|_{t=10} = .02344\sin\left(\frac{\pi x}{2}\right) \qquad u(x,t)\big|_{t=10,x=.5} = .01657$$

The finite element solution is unchanged. Again there is horrible agreement between the finite element model and the distributed parameter model.

The fist mode is exact because the initial condition is in the first mode. All coefficients, a_n, for modes other than the first mode are zero.

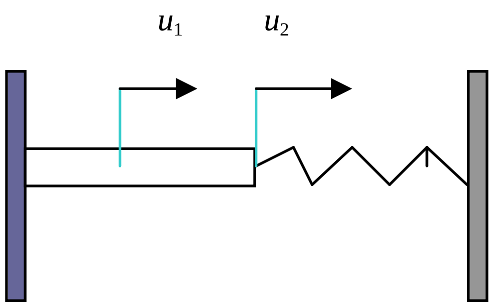
Problems and Solutions Section 8.2 (8.8 through 8.20)

8.8 Consider the bar of Figure P8.3 and model the bar with two elements. Calculate the frequencies and compare them with the solution obtained in Problem 8.3. Assume material properties of aluminum, a cross-sectional area of 1 m, and a spring stiffness of 1×10^6 N/m.

Solution: The finite element model for the two-element bar is

$$M\ddot{\mathbf{u}}(t) + K\mathbf{u}(t) = \mathbf{0}$$

where $\mathbf{u}(t) = [u_1 \quad u_2]^T$



$$M = \frac{\rho A l}{12}\begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix} \quad K = \frac{2EA}{l}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

As in problem 8.3, the spring adds a stiffness K to degree of freedom 2. The equation of motion is then

$$\frac{\rho A l}{12}\begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}\ddot{\mathbf{u}}(t) + \frac{2EA}{l}\begin{bmatrix} 2 & -1 \\ -1 & 1 + \dfrac{Kl}{2EA} \end{bmatrix}\mathbf{u}(t) = \mathbf{0}$$

The natural frequencies can be found by eigenanalysis. Using the material properties of aluminum

$\rho = 2700 \text{kg/m}^3$, $E = 7 \times 10^{10}$Pa

$\omega_1 = 129.0$ rad/s

$\omega_2 = 368.4$ rad/s

The solution obtained in problem 8.4 is $\omega_1 = 149.1$ rad/s.

8.9 Repeat Problem 8.8 with a three-element model. Calculate the frequencies and compare them with those of Problem 8.8.

Solution:

The finite element model of the 3 element rod for equal length elements is (from equation (8.25))

$$\frac{\rho A l}{18}\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix}\ddot{\mathbf{u}} + \frac{3EA}{l}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\mathbf{u} = \mathbf{0}$$

With the spring stiffness included, the global stiffness becomes

$$K = \frac{3EA}{l}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1+\dfrac{Kl}{3EA} \end{bmatrix}$$

Solving for the natural frequencies gives ω_1 = 125.85 rad/s, ω_2=333.1 rad/s, and ω_3 = 591.7 rad/s

The natural frequencies predicted in 8.9 should be better than those predicted in 8.8. You can compare them to the results of 2 element model by using VTB8_2 and loading the file p8_3_10.con.

8.10 Consider Example 8.2.2. Repeat this example with node 2 moved to $\ell/2$ so that the mesh is uniform. Calculate the natural frequencies and compare them to those obtained in the example. What happens to the mass matrix?

Solution: (8.10, 8.11)

The equation of motion can be shown to be

$$\frac{\rho Al}{12}\begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}\ddot{\mathbf{u}} + \frac{2EA}{l}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\mathbf{u} = \mathbf{0}$$

$$\omega_1 = \frac{1.16114}{l}\sqrt{\frac{E}{\rho}} = 8204.8 \text{ rad/s}$$

$$\omega_2 = \frac{5.6293}{l}\sqrt{\frac{E}{\rho}} = 28663 \text{ rad/s}$$

The first natural frequency is slightly improved (closer to the distributed parameter 'true' value) while the second natural frequency has become worse.

	Truth	Example 8.22	Problem 8.10	Example 8.2.1
ω_1	$1.571\frac{1}{l}\sqrt{\frac{E}{\rho}}$	$1.643\frac{1}{l}\sqrt{\frac{E}{\rho}}$	$1.611\frac{1}{l}\sqrt{\frac{E}{\rho}}$	$1.579\frac{1}{l}\sqrt{\frac{E}{\rho}}$
ω_2	$4.712\frac{1}{l}\sqrt{\frac{E}{\rho}}$	$5.196\frac{1}{l}\sqrt{\frac{E}{\rho}}$	$5.629\frac{1}{l}\sqrt{\frac{E}{\rho}}$	$5.167\frac{1}{l}\sqrt{\frac{E}{\rho}}$

The natural frequencies found using the 3 element model are much better than the 2 element model.

8.11 Compare the frequencies obtained in Problem 8.10 with those obtained in Section 8.2 using three elements.

Solution:

See the solution for problem 8.10.

8.12 As mentioned in the text, the usefulness of the finite element method rests in problems that cannot readily be solved in closed form. To this end, consider a section of an air frame sketched in Figure P8.13 and calculate a two-element finite model of this structure (i.e., find *M* and *K*) for a bar with

Solution:

$$A(x) = \frac{\pi}{4}\left[h_1^2 + \left(\frac{h_2 - h_1}{l}\right)^2 x^2 + 2h_1\left(\frac{h_2 - h_1}{l}\right)x\right]$$

Two methods exist for creating a finite element model for this wing. The first is to assume each element has a constant cross section. The second is to derive elements based on the variable cross section. If enough elements are used, constant cross section elements can yield acceptable results. However, since in this example only two elements are used, it is better to use a variable cross section element. Both solutions are given.

A: <u>Variable cross section elements</u>

Following the procedure of section 8.1, the shape function of the first element is given by

$$u(x,t) = \left(1 - \frac{2x}{l}\right)u_1(t) + \frac{2x}{l}u_2(t)$$

The strain energy for element 1 is given by

$$V_1(t) = \int_0^{l/2} EA(x)\left[\frac{\partial u_1(x,t)}{\partial x}\right]^2 dx$$

$$= \frac{E\pi}{48l}[(7h_1^2 + 4h_1h_2 + h_2^2)u_1^2(t) - (14h_1^2 + 8h_1h_2 + 2h_2^2)u_1(t)u_2(t)$$
$$+ (7h_1^2 + 4h_1h_2 + h_2^2)u_2^2(t)]$$

However, since $u_1(t) = 0$,

$$V_1(t) = \frac{E\pi}{48l}(7h_1^2 + 4h_1h_2 + h_2^2)u_2^2(t)$$

For element 2, the shape function is

$$u_2(x,t) = 2\left(1 - \frac{x}{l}\right)u_2(t) + \left(\frac{2x}{l} - 1\right)u_3(t)$$

The strain energy for element 2 is then given by

$$V_2(t) = \frac{1}{2}\int_{l/2}^{l} EA(x)\left[\frac{\partial u_2(x,t)}{\partial x}\right]^2 dx$$

$$= \frac{E\pi}{48l}\left(h_1^2 + 4h_1h_2 + 7h_2^2\right)\left(u_2^2(t) + 2u_2(t)u_3(t) + u_3^2(t)\right)$$

The total strain energy is then

$$V(t) = \frac{E\pi}{48l}\left((f_1 + f_2)u_2^2(t) - 2f_2u_2(t)u_3(t) + f_2u_3^2(t)\right)$$

where $f_1 = 7h_1^2 + 4h_1h_2 + h_2^2$ and $f_2 = h_1^2 + 4h_1h_2 + 7h_2^2$

In matrix form this is

$$V(t) = \frac{1}{2}\begin{bmatrix}u_2(t) & u_3(t)\end{bmatrix}K\begin{bmatrix}u_2(t) & u_3(t)\end{bmatrix}^T$$

where

$$K = \frac{E\pi}{24l}\begin{bmatrix} f_1 \times f_2 & -f_2 \\ -f_2 & f_2 \end{bmatrix}$$

The kinetic energy of element 1 is given by

$$T_1(t) = \int_0^{l/2} A(x)\rho\left[\frac{\partial u_1(x,t)}{\partial x}\right]^2 dx$$

$$= \frac{l\pi\rho}{1920}(16h_1^2 + 18h_1h_2 + 6h_2^2)\dot{u}_2^2(t)$$

(since $\dot{u}_1(t) = 0$, terms including $\dot{u}_1(t)$ have been dropped)

Similarly, the kinetic energy of element 2 is

$$T_2(t) = \int_{l/2}^{l} A(x)\rho\left[\frac{\partial u_2(x,t)}{\partial x}\right]^2 dx = \frac{l\pi\rho}{1920}[(6h_1^2 + 18h_1h_2 + 16h_2^2)\dot{u}_2^2$$

$$+ (3h_1^2 + 8h_1h_2 + 31h_2^2)\dot{u}_2\dot{u}_3 + (h_1^2 + 8h_1h_2 + 31h_2^2)\dot{u}_3^2]$$

The total kinetic energy can be written

$$T(t) = \frac{l\pi\rho}{1920}[(22h_1^2 + 36h_1h_2 + 22h_2^2)\dot{u}_2^2 + (3h_1^2 + 14h_1h_2 + 23h_2^2)\dot{u}_2\dot{u}_3$$
$$+ (h_1^2 + 8h_1h_2 + 31h_2^2)\dot{u}_3^2] = \frac{1}{2}\left[\dot{u}_2\dot{u}_3\right]M\left[\dot{u}_2\dot{u}_3\right]^T$$

where

$$M = \frac{l\pi\rho}{1920}\begin{bmatrix} 44h_1^2 + 72h_1h_2 + 44h_2^2 & 3h_1^2 + 14h_1h_2 + 23h_2^2 \\ 3h_1^2 + 14h_1h_2 + 23h_2^2 & 2h_1^2 + 16h_1h_2 + 62h_2^2 \end{bmatrix}$$

B: <u>Constant cross section elements</u>

The average cross section area of element 1 is

$$A_1 = \frac{\pi}{48}(7h_1^2 + 4h_1h_2 + h_2^2)$$

and the average cross section area of element 2 is

$$A_2 = \frac{\pi}{48}(h_1^2 + 4h_1h_2 + 7h_2^2)$$

Finding the potential energy again yields the same global stiffness matrix as for the variable cross section model.

The kinetic energy can then be found by

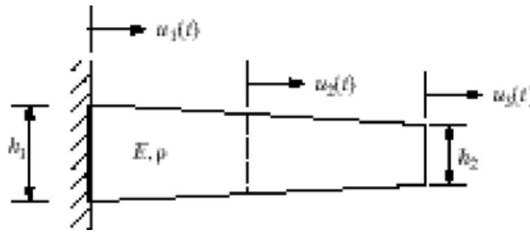
$$T(t) = \frac{1}{2}\int_0^{l/2} A_1\rho\left[\frac{\partial u_1(x,t)}{\partial x}\right]^2 dx + \frac{1}{2}\int_{l/2}^{l} A_2\rho\left[\frac{\partial u_2(x,t)}{\partial x}\right]^2 dx$$
$$= \frac{1}{2}\left[\dot{u}_2\dot{u}_3\right]M\left[\dot{u}_2\dot{u}_3\right]^T$$

where

$$M = \frac{\rho l}{12}\begin{bmatrix} 2(A_1 + A_2) & A_2 \\ A_2 & 2A_2 \end{bmatrix}$$

which is not identical to the mass matrix derived using variable cross section elements.

8.13 Let the bar in Figure P8.13 be made of aluminum 1 m in length with h_1 = 20 cm and h_2 = 10 cm. Calculate the natural frequencies using the finite element model of Problem 8.12.



Solution:

$E = 7 \times 10^{10}$Pa, ρ = 2700 kg/m^3

h_1 = .2m, h_2 = .1m, l = 1m

Using the variable cross section elements

$$K = \begin{bmatrix} 2.566 \times 10^9 & -8.705 \times 10^8 \\ -8.705 \times 10^8 & 8.705 \times 10^8 \end{bmatrix}$$

and

$$M = \begin{bmatrix} 16.081 & 2.783 \\ 2.783 & 4.506 \end{bmatrix}$$

The natural frequencies are then ω_1 = 7414 rad/s and ω_2 = 20368 rad/s

The constant cross sectional area mass matrix is

$$M = \begin{bmatrix} 16.493 & 2.798 \\ 2.798 & 5.596 \end{bmatrix}$$

which give ω_1 = 7092 rad/s, ω_2 = 18636 rad/s

8.14 Repeat Problems 8.12 and 8.13 using a three-element four-node finite element model.

Solution:

The shape functions for 3 evenly spaced elements are

$$u_1(x,t) = \left(1 - \frac{3x}{2l}\right)u_1(t) + \frac{3x}{l}u_2(t)$$
$$u_2(x,t) = 2\left(1 - \frac{3x}{2l}\right)u_2(t) + \left(\frac{3x}{l} - 1\right)u_3(t)$$
$$u_3(x,t) = 3\left(1 - \frac{x}{l}\right)u_3(t) + 2\left(\frac{3x}{2l} - 1\right)u_4(t)$$

Integrating to find the strain energy, the strain energies in matrix notation are

$$V_1(t) = \frac{1}{2}\begin{bmatrix} u_1 & u_2 \end{bmatrix} K_1 \begin{bmatrix} u_1 & u_2 \end{bmatrix}^T$$
$$V_2(t) = \frac{1}{2}\begin{bmatrix} u_2 & u_3 \end{bmatrix} K_2 \begin{bmatrix} u_2 & u_3 \end{bmatrix}^T$$
$$V_3(t) = \frac{1}{2}\begin{bmatrix} u_3 & u_4 \end{bmatrix} K_3 \begin{bmatrix} u_3 & u_4 \end{bmatrix}^T$$

where

$$K_1 = \frac{E\pi}{36l}(19h_1^2 + 7h_1h_2 + h_2^2)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$
$$K_2 = \frac{E\pi}{36l}(7h_1^2 + 13h_1h_2 + 7h_2^2)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$
$$K_3 = \frac{E\pi}{36l}(h_1^2 + 7h_1h_2 + 19h_2^2)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Writing the total strain energy in matrix form, the global stiffness matrix is

$$K = \frac{E\pi}{36l}\begin{bmatrix} f_1 + f_2 & -f_2 & 0 \\ -f_2 & f_2 + f_3 & -f_3 \\ 0 & -f_3 & f_3 \end{bmatrix}$$

where

$f_1 = 19h_1^2 + 7h_1h_2 + h_2^2$, $f_2 = 7h_1^2 + 13h_1h_2 + 7h_2^2$ and $f_3 = h_1^2 + 7h_1h_2 + 19h_2^2$

The kinetic energy of each element in matrix form is

$$T_1(t)=\frac{1}{2}\begin{bmatrix}\dot{u}_1 & \dot{u}_2\end{bmatrix}M_1\begin{bmatrix}\dot{u}_1 & \dot{u}_2\end{bmatrix}^T,\quad T_2(t)=\frac{1}{2}\begin{bmatrix}\dot{u}_2 & \dot{u}_3\end{bmatrix}M_2\begin{bmatrix}\dot{u}_2 & \dot{u}_3\end{bmatrix}^T,$$

$$T_3(t)=\frac{1}{2}\begin{bmatrix}\dot{u}_3 & \dot{u}_4\end{bmatrix}M_3\begin{bmatrix}\dot{u}_3 & \dot{u}_4\end{bmatrix}^T$$

where

$$M_1=\frac{l\pi\rho}{3240}\begin{bmatrix}76h_1^2+13h_1h_2+h_2^2 & \frac{1}{2}\left(63h_1^2+24h_1h_2+3h_2^2\right)\\ \frac{1}{2}\left(63h_1^2+24h_1h_2+3h_2^2\right) & 51h_1^2+33h_1h_2+6h_2^2\end{bmatrix}$$

$$M_2=\frac{l\pi\rho}{3240}\begin{bmatrix}31h_1^2+43h_1h_2+16h_2^2 & \frac{1}{2}\left(23h_1^2+44h_1h_2+23h_2^2\right)\\ \frac{1}{2}\left(23h_1^2+44h_1h_2+23h_2^2\right) & 16h_1^2+43h_1h_2+31h_2^2\end{bmatrix}$$

$$M_3=\frac{l\pi\rho}{3240}\begin{bmatrix}6h_1^2+33h_1h_2+51h_2^2 & \frac{1}{2}\left(3h_1^2+24h_1h_2+63h_2^2\right)\\ \frac{1}{2}\left(3h_1^2+24h_1h_2+63h_2^2\right) & h_1^2+13h_1h_2+76h_2^2\end{bmatrix}$$

Evaluating and assembling the mass and stiffness matrices gives:

$$K=\begin{bmatrix}9.285 & -3.726 & 0\\ -3.729 & 5.987 & -2.2602\\ 0 & -2.2602 & 2.2602\end{bmatrix}\times 10^9$$

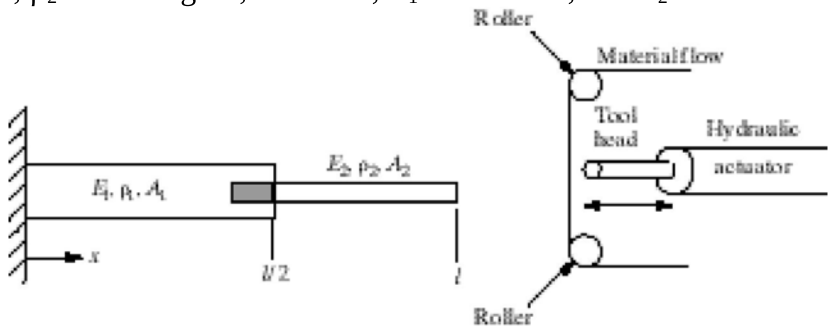
$$M=\begin{bmatrix}13.1423 & 2.6573 & 0\\ 2.6573 & 8.4299 & 1.6101\\ 0 & 1.6101 & 2.7751\end{bmatrix}$$

ω_1 = 10406 rad/s, ω_2 = 27309 rad/s, ω_3 = 47797 rad/s

Note that a ten element model yields

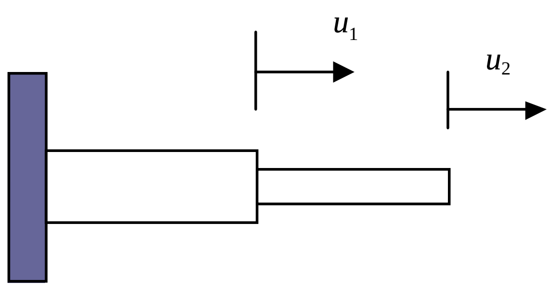
ω_1 = 10316 rad/s, ω_2 = 25183 rad/s

8.15 Consider the machine punch of Figure P8.15. This punch is made of two materials and is subject to an impact in the axial direction. Use the finite element method with two elements to model this system and estimate (calculate) the first two natural frequencies. Assume $E_1 = 8 \times 10^{10}$ Pa, $E_2 = 2.0 \times 10^{11}$ Pa, $\rho_1 = 7200$ kg/m^3, $\rho_2 = 7800$ kg/m^3, $l = 0.2$ m, $A_1 = 0.009$ m^2, and $A_2 = 0.0009$ m^2.



Solution: The total strain energy of the system is

$$V(t) = \frac{1}{2}\left\{u_1^2 \frac{2E_1A_1}{l} + \frac{2E_1A_1}{l}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}^T \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right\}$$



The vector of derivatives of the potential energy gives

$$\begin{bmatrix} \dfrac{\partial V}{\partial u_1} \\ \dfrac{\partial V}{\partial u_2} \end{bmatrix} = \frac{2}{l}\begin{bmatrix} E_1A_1 + E_2A_2 & -E_2A_2 \\ -E_2A_2 & E_2A_2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

The stiffness matrix is then

$$K = \frac{2}{l}\begin{bmatrix} E_1A_1 + E_2A_2 & -E_2A_2 \\ -E_2A_2 & E_2A_2 \end{bmatrix}$$

In similar fashion, the total kinetic energy is

$$T(t) = \frac{1}{2}\left\{\dot{u}_1^2 \frac{\rho_1A_1l}{6} + \frac{l}{12}\begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{bmatrix}^T \begin{bmatrix} 2\rho_2A_2 & \rho_2A_2 \\ \rho_2A_2 & 2\rho_2A_2 \end{bmatrix}\begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{bmatrix}\right\}$$

The mass matrix is then

$$M = \frac{l}{12}\begin{bmatrix} 2(\rho_2 A_2 + \rho_1 A_1) & \rho_2 A_2 \\ \rho_2 A_2 & 2\rho_2 A_2 \end{bmatrix}$$

$E_1 = 8 \times 10^{10}\,\text{Pa},\ \rho_1 = 7200\text{kg/m}^3,\ E_2 = 2.0 \times 10^{11}\,\text{Pa},\ \rho_2 = 7800\text{kg/m}^3,$

$l = .2 A_1 = .0009,\ A_2 = .0001$

$$K = \begin{bmatrix} 9.2 & -2 \\ -2 & 2 \end{bmatrix} \times 10^8 \quad M = \begin{bmatrix} .242 & .013 \\ .013 & .026 \end{bmatrix}$$

ω_1 = 47556.1 rad/s, ω_2 = 101975 rad/s

8.16 Recalculate the frequencies of Problem 8.15 assuming that it is made entirely of one material and size (i.e., $E_1 = E_2$, $\rho_1 = \rho_2$, and $A_1 = A_2$), say steel, and compare your results to those of Problem 8.15.

Solution:

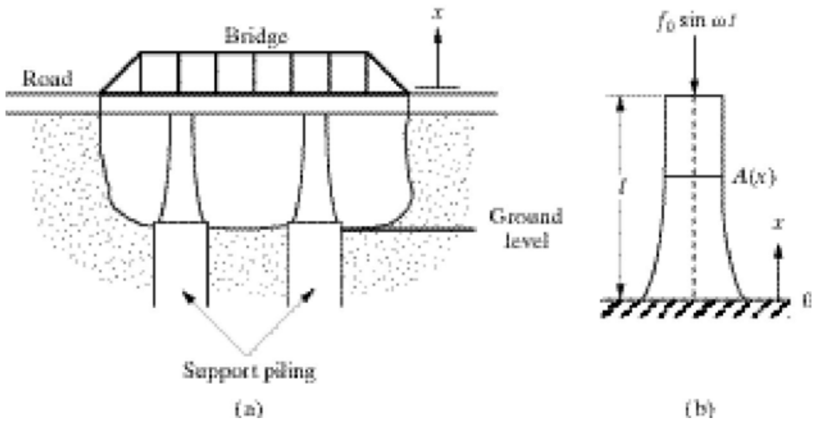
Assume $A_1 = A_2$, $E_1 = E_2$, $\rho_1 = \rho_2$

$$K = \begin{bmatrix} 4 & -2 \\ -2 & 2 \end{bmatrix} \times 10^8 \quad M = \begin{bmatrix} .052 & .013 \\ .013 & .026 \end{bmatrix}$$

ω_1 = 40798.6 rad/s, ω_2 = 142525 rad/s

The first natural frequency decreased. This example illustrates how a punch can be modified to raise the first natural frequency by changing the base material.

8.17 A bridge support column is illustrated in Figure P8.17. The column is made of concrete with a cross-sectioned area defined by $A(x) = A_0 e^{-x/l}$, where A_0 is the area of the column at ground. Consider this pillar to be cantilevered (i.e., fixed) at ground level and to be excited sinusoidally at its tip in the longitudinal direction due to traffic over the bridge. Calculate a single-element finite element model of this system and compute its approximate natural frequency.



Solution:

$A(x) = A_0 e^{-x/l}$

The potential energy is

$$V(t) = \frac{E}{2}\int_0^l A(x)\left[\frac{\partial u(x,t)}{\partial x}\right]^2 dx$$

where $u(x,t) = \left(1 - \frac{x}{l}\right)u_1(t) + \frac{x}{l}u_2(t)$

$$V(t) = \frac{EA_0}{2l}\frac{e-1}{e}\left(u_1(t) - u_2(t)\right)^2$$

$$= \frac{EA}{2l}\frac{e-1}{e}u_2^2(t)$$

The stiffness is then

$$K = \frac{EA}{l}\frac{(e-1)}{e}$$

Likewise, the kinetic energy is

$$T(t) = \frac{1}{2}\int_0^l \rho A\left[\frac{\partial u(x,t)}{\partial x}\right]^2 dx = \frac{Al\rho}{2e}(2e-5)\dot{u}_2^2(t)$$

The mass is then

$$M = \frac{Al\rho}{e}(2e - 5)$$

The first natural frequency is then approximately

$$\omega_1 = \sqrt{\frac{K}{M}} = \sqrt{\frac{E(e-1)}{(2e-5)l^2\rho}} = \frac{1.984}{l}\sqrt{\frac{E}{\rho}}$$

8.18 Redo Problem 8.17 using two elements. What would happen if the "traffic" frequency corresponds with one of the natural frequencies of the support column?

Solution: The shape functions for a 2 element model are

$$u_1(x,t) = \left(1 - \frac{2x}{l}\right)u_1(t) + \frac{2x}{l}u_2(t)$$

$$u_2(x,t) = 2\left(1 - \frac{x}{l}\right)u_2(t) + \left(\frac{2x}{l} - 1\right)u_3(t)$$

The total stain energy in matrix form is

$$V(t) = \frac{1}{2}\begin{bmatrix} u_2 & u_3 \end{bmatrix} K \begin{bmatrix} u_2 & u_3 \end{bmatrix}^T$$

where

$$K = \frac{4A\left(\sqrt{e} - 1\right)E_0}{el}\begin{bmatrix} 1 + \sqrt{e} & -1 \\ -1 & 1 \end{bmatrix}$$

Likewise the mass matrix can be found from the total potential energy to be

$$M = \frac{Al\rho}{e}\begin{bmatrix} 8\left(e - 1 - \sqrt{e}\right) & 10 - 6\sqrt{e} \\ 10 - 6\sqrt{e} & 8 - 13\sqrt{e} \end{bmatrix}$$

and the natural frequencies are then

$$\omega_1 = \frac{1.939}{l}\sqrt{\frac{E}{\rho}} \text{ rad/s}, \quad \omega_2 = \frac{5.605}{l}\sqrt{\frac{E}{\rho}} \text{ rad/s}$$

If the traffic frequency corresponds to a natural frequency of a pillar, the bridge might fail.

8.19 Problems 8.17 and 8.18 represent approximations. As pointed out in Problem 8.18, it is important to know the natural frequencies of this column as precisely as possible. Hence consider modeling this column as a uniform bar of average cross section, calculate the first few natural frequencies, and compare them to the results in Problem 8.17 and 8.18. Which model do you think is closest to reality?

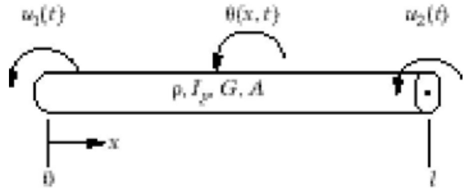
Solution:

The natural frequencies of a rod with constant cross sectional area are independent of the area. Therefore the first 2 natural frequencies are

$$\omega_1 \frac{1.571}{l}\sqrt{\frac{E}{\rho}} \text{ rad/s}, \quad \omega_2 = \frac{4.712}{l}\sqrt{\frac{E}{\rho}} \text{ rad/s}$$

It is doubtful that these results are better since we know from the finite element model that the varying cross sectional area does have an effect.

8.20 Torsional vibration can also be modeled by finite elements. Referring to Figure P8.20, calculate a single-element mass and stiffness matrix for the torsional vibration following the steps of Section 8.1. (*Hint*: $\theta(x,t) = c_1(t)\theta + c_2(t)$, $T(t) = \frac{1}{2}\int_0^l \rho I_\rho \left[\theta_t(x,t)\right]^2 dx$ and $V(t) = \frac{1}{2}\int_0^l GI_\rho \left[\theta_t(x,t)\right]^2 dx$.)



Solution:

From equation (6.64), The static (time independent) displacement of the torsional rod element must satisfy

$$\frac{\partial \tau}{\partial x} = 0 = GJ\frac{\partial^2 \theta(x,t)}{\partial x^2}$$

which has the same form as equation (8.1). This can be integrated to yield

$\theta(x) = C_1\theta + C_2$

At $x = 0$

$\theta(0) = \theta_1(t) = C_2$

Likewise, at $x = l$

$\theta(l) = \theta_2(t) = C_1 l + C_2$

$$C_1 = \frac{\theta_2(t) - C_2}{l} = \frac{\theta_2(t) - \theta_1(t)}{l}$$

Substituting the values of C_1 and C_2 into the shape function yields

$$\theta(x,t) = \left(1 - \frac{x}{l}\right)\theta_1(t) + \left(\frac{x}{l}\right)\theta_2(t)$$

Evaluating the strain energy yields

$$V(t) = \frac{GJ}{2l}\left(\theta_1^2 - 2\theta_1\theta_2 + \theta_2^2\right)$$
$$= \frac{1}{2}\left[\theta_1(t) \quad \theta_2(t)\right]K\left[\theta_1(t) \quad \theta_2(t)\right]^T$$

where the stiffness matrix is defined by

$$K = \frac{GJ}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Likewise, evaluating the kinetic energy yields

$$\begin{aligned} T(t) &= \frac{1}{2}\frac{\rho A l}{3}\left(\dot{\theta}_1^2 + \dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2\right) \\ &= \frac{1}{2}\left[\dot{\theta}_1(t) \quad \dot{\theta}_2(t)\right] M \left[\dot{\theta}_1(t) \quad \dot{\theta}_2(t)\right]^T \end{aligned}$$

where the mass matrix is defined by

$$M = \frac{\rho A l}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Problems and Solutions Section 8.3 (8.21 through 8.33)

8.21 Use equations (8.47) and (8.46) to derive equation (8.48) and hence make sure that the author and reviewer have not cheated you.

Solution:

$$u(x,t) = C_1(t)x^3 + C_2(t)x^2 + C_3(t)x + C_4(t) \qquad (8.46)$$

$$\begin{aligned} u(0,t) &= u_1(t) \qquad & u_x(0,t) &= u_2(t) \\ u(l,t) &= u_3(t) \qquad & u_x(l,t) &= u_4(t) \end{aligned} \qquad (8.47)$$

Substituting(8.46) into (8.47)

$$u(0,t) = C_4(t) = u_1(t)$$
$$u_x(0,t) = C_3(t) = u_2(t)$$
$$u(l,t) = C_1(t)l^3 + C_2(t)l^2 + C_3(t)l + C_4(t) = u_3(t)$$
$$u_x(l,t) = 3C_1(t)l + 2C_2(t)l + C_3(t) = u_4(t)$$

This gives

$$C_1 = \frac{1}{l^3}\left(2(u_1 - u_3) + l(u_2 + u_4)\right)$$
$$C_2 = \frac{1}{l^2}\left(3(u_3 - u_1) - l(u_4 + 2u_2)\right)$$
$$C_3 = u_2$$
$$C_4 = u_1$$

8.22 It is instructive, though tedious, to derive the beam element deflection given by equation (8.49). Hence derive the beam shape functions.

Solution:

Substituting (8.48) into (8.46) gives

$$u(x,t) = \left[1 - 3\frac{x^2}{l^2} + 2\frac{x^3}{l^3}\right]u_1(t) + l\left[\frac{x}{l} - 2\frac{x^2}{l^2} + \frac{x^3}{l^3}\right]u_2(t)$$
$$+ \left[3\frac{x^2}{l^2} - 2\frac{x^3}{l^3}\right]u_3(t) + l\left[\frac{-x^2}{l^2} + \frac{x^3}{l^3}\right]u_4(t)$$

8.23 Using the shape functions of Problem 8.22, calculate the mass and stiffness matrices given by equations (8.53) and (8.56). Although tedious, this involves only simple integration of polynomials in *x*.

Solution:

$$T(t) = \frac{1}{2}\int_0^l \rho A\big(u_t(x,t)\big)^2 dx$$

$$= \frac{1}{2}\dot{\mathbf{u}}^T M \dot{\mathbf{u}}$$

where

$$\mathbf{u} = \begin{bmatrix} u_1(t) & u_2(t) & u_3(t) & u_4(t) \end{bmatrix}^T$$

And M is given by equation (8.35).

Similarly

$$V(t) = \frac{1}{2}\int_0^l EI\big[u_{xx}(x,t)\big]^2 dx$$

$$= \frac{1}{2}\mathbf{u}^T K \mathbf{u}$$

where K is given by (8.56)

8.24 Calculate the natural frequencies of the cantilevered beam given in equation (8.69) using l = 1 m and compare your results with those listed in Table 6.1.

Solution:

$$M = \frac{\rho A}{840}\begin{bmatrix} 312 & 0 & 54 & -6.5 \\ 0 & 2 & 6.5 & -.75 \\ 54 & 6.5 & 156 & -11 \\ -6.5 & -.75 & -11 & 1 \end{bmatrix}$$

$$K = 8EI\begin{bmatrix} 24 & 0 & -12 & 3 \\ 0 & 2 & -3 & \frac{1}{2} \\ -12 & -3 & 12 & -3 \\ 3 & \frac{1}{2} & -3 & 1 \end{bmatrix}$$

Following the procedures of section 4.2

$$\omega_1 = 3.5177\sqrt{\frac{EI}{\rho A}},\ \omega_2 = 22.2215\sqrt{\frac{EI}{\rho A}}$$

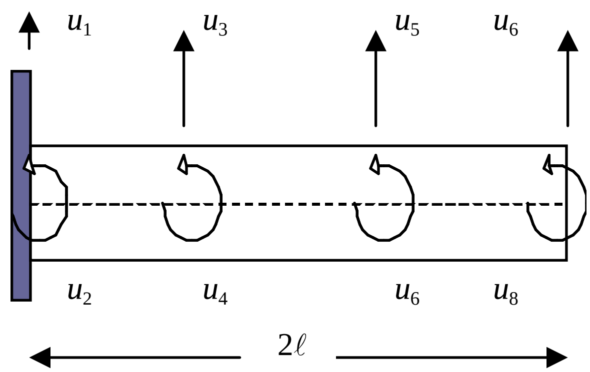
$$\omega_3 = 75.1571\sqrt{\frac{EI}{\rho A}},\ \omega_4 = 218.138\sqrt{\frac{EI}{\rho A}}$$

From continuous theory, the natural frequencies of a cantilevered beam are $\omega_i = \beta_i\sqrt{\frac{EI}{\rho A}}$ where $\beta_1 = 3.51601,\ \beta_2 = 22.0345,\ \beta_3 = 61.6972,\ \beta_4 = 120.9019.$

The predictions of the first two natural frequencies are quite accurate while the predictions of the third and fourth natural frequencies are terrible.

8.25 Calculate the finite element model of a cantilevered beam one meter in length using three elements. Calculate the natural frequencies and compare them to those obtained in Problem 8.23 and with the exact values listed in Table 6.4.

Solution: Define u_i using the following figure;



The equation for element one is

$$\frac{\rho Al}{420}\begin{bmatrix} 156 & -22l \\ -22l & 4l^2 \end{bmatrix}\begin{bmatrix} \ddot{u}_3 \\ \ddot{u}_4 \end{bmatrix} + \frac{EI}{l^3}\begin{bmatrix} 12 & -6l \\ -6l & 4l^2 \end{bmatrix}\begin{bmatrix} u_3 \\ u_4 \end{bmatrix} = \mathbf{0}$$

The equation for element two is

$$\frac{\rho Al}{420}\begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix}\begin{bmatrix} \ddot{u}_3 \\ \ddot{u}_4 \\ \ddot{u}_5 \\ \ddot{u}_6 \end{bmatrix} + \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}\begin{bmatrix} u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix} = \mathbf{0}$$

The equation for element 3 is the same as for element 2 but with the vector

$[u_3\ u_4\ u_5\ u_6]^T$ replaced with $[u_5\ u_6\ u_7\ u_8]^T$.

Combining the elemental equation using the superposition of the like coordinates yields

$$
\frac{\rho A l}{420}\begin{bmatrix} 312 & 0 & 54 & -13l & 0 & 0 \\ 0 & 8l^2 & 13l & -3l^2 & 0 & 0 \\ 54 & 13l & 312 & 0 & 54 & -13l \\ -13l & -3l^2 & 0 & 8l^2 & 13l & -3l^2 \\ 0 & 0 & 54 & 13l & 156 & -22l \\ 0 & 0 & -13l & -3l^2 & -22l & 4l^2 \end{bmatrix}\begin{bmatrix} \ddot{u}_3 \\ \ddot{u}_4 \\ \ddot{u}_5 \\ \ddot{u}_6 \\ \ddot{u}_7 \\ \ddot{u}_8 \end{bmatrix}
$$

$$
+\frac{EI}{l^3}\begin{bmatrix} 24 & 0 & -12 & 6l & 0 & 0 \\ 0 & 8l^2 & -6l & 2l^2 & 0 & 0 \\ -12 & -6l & 24 & 0 & -12 & 6l \\ 6l & 2l^2 & 0 & 8l^2 & -6l & 2l^2 \\ 0 & 0 & -12 & -6l & 12 & -6l \\ 0 & 0 & 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}\begin{bmatrix} u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{bmatrix} = \mathbf{0}
$$

which can also be written in the form

$$
\frac{\rho A l}{420}\begin{bmatrix} 312 & 0 & 54 & -13 & 0 & 0 \\ 0 & 8 & 13 & -3 & 0 & 0 \\ 54 & 13 & 312 & 0 & 54 & -13 \\ -13 & -3 & 0 & 8 & 13 & -3 \\ 0 & 0 & 54 & 13 & 156 & -22 \\ 0 & 0 & -13 & -3 & -22 & 4 \end{bmatrix}\begin{bmatrix} \ddot{u}_3 \\ l\ddot{u}_4 \\ \ddot{u}_5 \\ l\ddot{u}_6 \\ \ddot{u}_7 \\ l\ddot{u}_8 \end{bmatrix}
$$

$$
+\frac{EI}{l^3}\begin{bmatrix} 24 & 0 & -12 & 6 & 0 & 0 \\ 0 & 8 & -6 & 2 & 0 & 0 \\ -12 & -6 & 24 & 0 & -12 & 6 \\ 6 & 2 & 0 & 8 & -6 & 2 \\ 0 & 0 & -12 & -6 & 12 & -6 \\ 0 & 0 & 6 & 2 & -6 & 4 \end{bmatrix}\begin{bmatrix} u_3 \\ lu_4 \\ u_5 \\ lu_6 \\ u_7 \\ lu_8 \end{bmatrix} = \mathbf{0}
$$

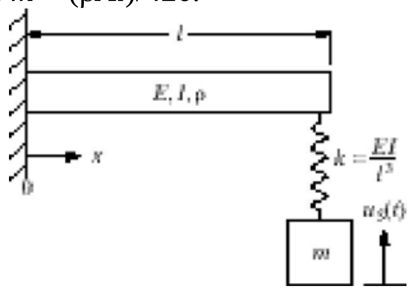
Following the procedure of example 8.3.3

$$
\omega_1 = .3907\frac{1}{l^2}\sqrt{\frac{EI}{\rho A}}\,,\ \omega_2 = 2.456\frac{1}{l^2}\sqrt{\frac{EI}{\rho A}}
$$

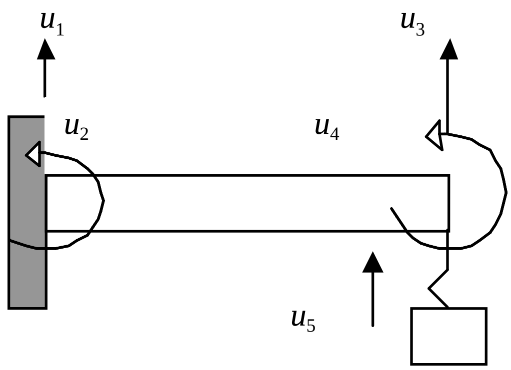
$$
\omega_3 = 6.941\frac{1}{l^2}\sqrt{\frac{EI}{\rho A}}\,,\ \omega_4 = 15.63\frac{1}{l^2}\sqrt{\frac{EI}{\rho A}}
$$

$$
\omega_5 = 29.42\frac{1}{l^2}\sqrt{\frac{EI}{\rho A}}\,,\ \omega_6 = 58.64\frac{1}{l^2}\sqrt{\frac{EI}{\rho A}}
$$

8.26 Consider the cantilevered beam of Figure P8.26 attached to a lumped spring-mass system. Model this system using a single finite element and calculate the natural frequencies. Assume $m = (\rho A l)/420$.



Solution: Define u_i using the following figure:



The model for the spring mass system is

$$\begin{bmatrix} 0 & 0 \\ 0 & m \end{bmatrix}\begin{bmatrix} \ddot{u}_3 \\ \ddot{u}_5 \end{bmatrix} + \frac{EI}{l^3}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} u_3 \\ u_5 \end{bmatrix} = \mathbf{0}$$

The single element model for the beam is

$$\frac{\rho A l}{420}\begin{bmatrix} 156 & -22l \\ -22l & 4l^2 \end{bmatrix}\begin{bmatrix} \ddot{u}_3 \\ \ddot{u}_4 \end{bmatrix} + \frac{EI}{l^3}\begin{bmatrix} 12 & -6l \\ -6l & 4l^2 \end{bmatrix}\begin{bmatrix} u_3 \\ u_4 \end{bmatrix} = \mathbf{0}$$

Superimposing like coordinates yields

$$\frac{\rho A l}{420}\begin{bmatrix} 156 & -22l & 0 \\ -22l & 4l^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \ddot{u}_3 \\ \ddot{u}_4 \\ \ddot{u}_5 \end{bmatrix} + \frac{EI}{l^3}\begin{bmatrix} 13 & -6l & -1 \\ -6l & 4l^2 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} u_3 \\ u_4 \\ u_5 \end{bmatrix} = \mathbf{0}$$

The equation of motion may also be written

$$\frac{\rho A l^4}{420 EI}\begin{bmatrix} 156 & -22 & 0 \\ -22 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \ddot{u}_3 \\ l\ddot{u}_4 \\ \ddot{u}_5 \end{bmatrix} + \begin{bmatrix} 13 & -6 & -1 \\ -6 & 4 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} u_3 \\ lu_4 \\ u_5 \end{bmatrix} = \mathbf{0}$$

The eigenvalue/eigenvector problem is then

$(A-\lambda I)\mathbf{v}=0$

where

$$A = M^{-1}K\frac{\rho A l^4}{420EI},\ \lambda = \frac{\rho A l^4}{420EI}\omega^2$$

$$A = \begin{bmatrix} -.5714 & .4571 & -.0286 \\ -4.6429 & 3.5143 & -.1571 \\ -1 & 0 & 0 \end{bmatrix}$$

$$\lambda_1 = .0294,\ \lambda_2 = 1,\ \lambda_3 = 2.9134$$

$$\omega_1 = 3.52\frac{1}{l^2}\sqrt{\frac{EI}{\rho A}}$$

$$\omega_2 = 20.49\frac{1}{l^2}\sqrt{\frac{EI}{\rho A}}$$

$$\omega_3 = 34.98\frac{1}{l^2}\sqrt{\frac{EI}{\rho A}}$$

8.27 Repeat Problem 8.26 using two finite elements for the beam and compare the frequencies.

Solution:

A two element model of a cantilevered beam has been created in example 8.3.3.

Superimposing like coordinates for this example with the spring mass model yields

$$\frac{\rho A l}{840}\begin{bmatrix} 312 & 0 & 54 & -6.5l & 0 \\ 0 & 2l^2 & 6.5l & -.75l^2 & 0 \\ 54 & 6.5l & 156 & -11l & 0 \\ -6.5l & -.75l^2 & -11l & l^2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} \ddot{u}_3 \\ \ddot{u}_4 \\ \ddot{u}_5 \\ \ddot{u}_6 \\ \ddot{u}_7 \end{bmatrix}$$

$$+\frac{8EI}{l^3}\begin{bmatrix} 24 & 0 & -12 & 3l & 0 \\ 0 & 2l^2 & -3l & \frac{1}{2}l^2 & 0 \\ -12 & -3l & 12+\frac{1}{8} & -3l & -\frac{1}{8} \\ 3l & \frac{1}{2}l^2 & -3l & l^2 & 0 \\ 0 & 0 & -\frac{1}{8} & 0 & \frac{1}{8} \end{bmatrix}\begin{bmatrix} u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{bmatrix} = \mathbf{0}$$

Note that the coordinate vector for the spring mass system has changed from $[u_3\ u_5]^T$ to $[u_5\ u_6]^T$.

As in (8.26), the equations may be written in the form

$$\frac{\rho A l^4}{6720 EI}\begin{bmatrix} 312 & 0 & 54 & -6.5 & 0 \\ 0 & 2 & 6.5 & -.75 & 0 \\ 54 & 6.5 & 156 & -11 & 0 \\ -6.5 & -.75 & -11 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} \ddot{u}_3 \\ l\ddot{u}_4 \\ \ddot{u}_5 \\ l\ddot{u}_6 \\ \ddot{u}_7 \end{bmatrix}$$

$$+\frac{8EI}{l^3}\begin{bmatrix} 24 & 0 & -12 & 3 & 0 \\ 0 & 2 & -3 & \frac{1}{2} & 0 \\ -12 & -3 & 12+\frac{1}{8} & -3 & -\frac{1}{8} \\ 3 & \frac{1}{2} & -3 & 1 & 0 \\ 0 & 0 & -\frac{1}{8} & 0 & \frac{1}{8} \end{bmatrix}\begin{bmatrix} u_3 \\ lu_4 \\ u_5 \\ lu_6 \\ u_7 \end{bmatrix} = \mathbf{0}$$

The eigenvalue/eigenvector problem is then

$$(A - \lambda I)\mathbf{v} = 0$$

where

$$A = M^{-1}K\frac{\rho A l^4}{6720 EI}, \ \lambda = \frac{\rho A l^4}{6720 EI}\omega^2$$

$$A = \begin{bmatrix} .2878 & .0640 & -.2907 & .0868 & -.0004 \\ 3.6700 & 2.1247 & -5.9000 & 1.5919 & -.0062 \\ .9274 & .2094 & -1.0368 & .3516 & -.0041 \\ 17.8241 & 4.8163 & -20.7187 & 6.6253 & -.0519 \\ 0 & 0 & -.0625 & 0 & .0625 \end{bmatrix}$$

$$\lambda_1 = .0427, \ \lambda_2 = .2455, \ \lambda_3 = .2772, \ \lambda_4 = .9173, \ \lambda_5 = 2.6614$$

$$\omega_1 = \frac{3.50}{l^2}\sqrt{\frac{EI}{\rho A}}, \ \omega_2 = \frac{20.12}{l^2}\sqrt{\frac{EI}{\rho A}}, \ \omega_3 = \frac{22.73}{l^2}\sqrt{\frac{EI}{\rho A}}$$

The one element (3 DOF) model predicted the first 2 natural frequencies well. The prediction of the third natural frequency was extremely poor using only one element.

8.28 Calculate the natural frequencies of a clamped-clamped beam for the physical parameters l = 1m, $E = 2 \times 10^{11}$ N/m^2, ρ = 7800 kg/m^3, $I = 10^{-6}$ m^4, and $A = 10^{-2}$ m^2, using the beam theory of Chapter 6 and a four-element finite element model of the beam.

Solution:

Using VTB8_1

$$M = \begin{bmatrix} 14.49 & 0 & 2.5071 & -.151 & 0 & 0 \\ 0 & .0232 & .0151 & -.0087 & 0 & 0 \\ 2.507 & .151 & 14.49 & 0 & 2.507 & -.151 \\ -.151 & -.0087 & 0 & .0232 & .151 & -.0087 \\ 0 & 0 & 2.5071 & .151 & 14.49 & 0 \\ 0 & 0 & -.151 & -.0087 & 0 & .0232 \end{bmatrix}$$

and

$$K = 1 \times 10^5 \begin{bmatrix} 3072 & 0 & -1536 & 192 & 0 & 0 \\ 0 & 64 & -192 & 16 & 0 & 0 \\ -1536 & -192 & 3072 & 0 & -1536 & 192 \\ 192 & 16 & 0 & 64 & -192 & 16 \\ 0 & 0 & -1536 & -192 & 3072 & 0 \\ 0 & 0 & 192 & 16 & 0 & 64 \end{bmatrix}$$

Remember to zero the *x* translations since we are not interested in the extensional deformations. The natural frequencies are then found to be

ω_1= 1134 rad/s, ω_2 = 3152 rad/s, ω_3 = 6253 rad/s, ω_4 = 11830 rad/s, ω_5 = 19565 rad/s, ω_6 = 31524 rad/s

From distributed theory

ω_1= 1132.9 rad/s, ω_2 = 3122.9 rad/s, ω_3 = 6122.2 rad/s, ω_4 = 10120 rad/s, ω_5 = 15118 rad/s, ω_6 = 21115 rad/s

8.29 Repeat Problem 8.28 with two elements and compare the frequencies with the four-element model. Calculate the frequencies of a clamped-clamped beam using one element. Any comment?

Solution:

Since only two of the six degrees of freedom are free, the mass and stiffness matrices are simply

$$M = \frac{2\rho A \frac{l}{2}}{420}\begin{bmatrix} 156 & 0 \\ 0 & 4\left(\frac{l}{2}\right)^2 \end{bmatrix}$$

and

$$K = \frac{2EI}{\left(\frac{l}{2}\right)^3}\begin{bmatrix} 12 & 0 \\ 0 & 4\left(\frac{l}{2}\right)^2 \end{bmatrix}$$

where l = 1 m. The natural frequencies are then

$$\omega_1 = \sqrt{\frac{\frac{192EI}{l^3}}{\frac{156\rho Al}{420}}} = 22.736\frac{1}{l^2}\sqrt{\frac{EI}{\rho A}} = 1151 \text{ rad/s}$$

$$\omega_2 = \sqrt{\frac{\frac{16EI}{l^3}}{\frac{\rho Al}{420}}} = 81.96\frac{1}{l^2}\sqrt{\frac{EI}{\rho A}} = 4151 \text{ rad/s}$$

If you are only interested in the first natural frequency, a two degree of freedom model is adequate. However, the six degree of freedom model is much more accurate and can better predict the second mode. (In general, a finite element model must have twice as many degrees of freedom as the number of modes you want to predict).

8.30 Estimate the first natural frequency of a clamped-simply supported beam. Use a single finite element.

Solution: Since we are using only one element, we need only take the finite element matrix for a single element and strike out the rows and columns corresponding to the fixed degrees of freedom to get the global matrices. This yields

$$M = \frac{4l^3 \rho A}{420}, \quad K = \frac{4l^2 EI}{l^3}$$

Since there is only a single degree of freedom

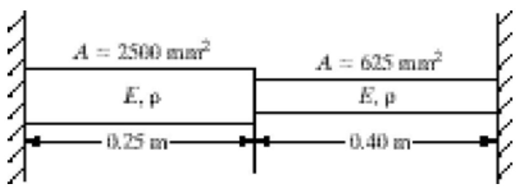
$$\omega_n = \sqrt{\frac{K}{M}} = \sqrt{420}\frac{1}{l^2}\sqrt{\frac{EI}{\rho A}} = 20.49\frac{1}{l^2}\sqrt{\frac{EI}{\rho A}} \text{ rad/s}$$

Distributed theory yields

$$\omega_n = 15.42\frac{1}{l^2}\sqrt{\frac{EI}{\rho A}}$$

One degree of freedom is not enough to predict the first natural frequency.

8.31 Consider the stepped beam of Figure P8.31 clamped at each end. Both pieces are made of aluminum. Use two elements, one for each step, and calculate the natural frequencies.



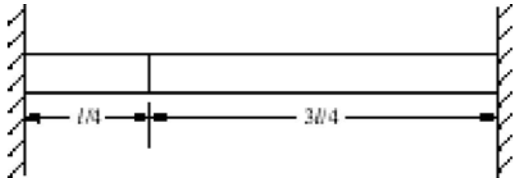
Solution: Only a single degree of freedom is free. The mass and stiffness matrices are therefore scalars.

$$K = \frac{E_1 A_1}{l_1} + \frac{E_2 A_2}{l_2} = 809375000 \text{ N/m}$$

$$M = \frac{1}{3}\left(\frac{\rho_1 A_1}{l_1} + \frac{\rho_2 A_2}{l_2}\right) = 10.41 \text{ kg}$$

$$\omega = \sqrt{\frac{K}{M}} = 8819.2 \text{ rad/s}$$

8.32 Use a two-element model of nonuniform length to estimate the first few natural frequencies of a clamped-clamped beam. Use the spacing indicated in Figure P8.32. Compare the result to the actual frequencies and to those of Problem 8.28 and 8.29.



Solution: Since it has been shown in example 8.3.3 that the variable *l* can be factored outside of the mass and stiffness matrices, we can substitute the percentage of total length of each element into the mass and stiffness matrices and get the correct natural frequencies.

$$M = \frac{\rho A(.25l)}{420}\begin{bmatrix} 156 & -22\times .25 \\ -22\times .25 & 4\times .25^2 \end{bmatrix} + \frac{\rho A(.75l)}{420}\begin{bmatrix} 156 & 22\times .75 \\ 22\times .75 & 4\times .75^2 \end{bmatrix}$$

$$= \frac{\rho A l}{420}\begin{bmatrix} 156 & 11 \\ 11 & 1.75 \end{bmatrix}$$

Similarly,

$$K = \frac{EI}{(.25l)^3}\begin{bmatrix} 12 & -6\times .25 \\ -6\times .25 & 4\times .25^2 \end{bmatrix} + \frac{EI}{(.75l)^3}\begin{bmatrix} 12 & 6\times .75 \\ 6\times .75 & 4\times .75^2 \end{bmatrix}$$

$$= \frac{EI}{l^3}\begin{bmatrix} 796.4 & -85.3 \\ -85.3 & 21.33 \end{bmatrix}$$

$$\omega = \sqrt{eig\left(\tilde{M}^{-1}\tilde{K}\right)}\frac{1}{l^2}\sqrt{\frac{EI}{\rho A}}$$

where $\tilde{M}$ and $\tilde{K}$ represent the mass and stiffness matrices with the variables *E*, *I*, *l*, ρ and *A* factored out.

$$\omega_1 = 25.31\frac{1}{l^2}\sqrt{\frac{EI}{\rho A}},\ \omega_2 = 132.6\frac{1}{l^2}\sqrt{\frac{EI}{\rho A}}$$

This is not nearly as good as the two element model where ω_1 was found to be

$$\omega_1 = 22.74\frac{1}{l^2}\sqrt{\frac{EI}{\rho A}}$$

as opposed to the "actual" (from distributed parameter theory) value of

$$\omega_1 = 22.37\frac{1}{l^2}\sqrt{\frac{EI}{\rho A}}$$

8.33 Calculate the first natural frequency of a clamped-pinned beam using first one, then two elements.

Solution:

From problem 8.30, using one element yields

$$\omega_1 = 20.49\frac{1}{l^2}\sqrt{\frac{EI}{\rho A}}$$

Using the vibration toolbox and the method described in 8.3.3 (also in the README.8 file) the two element model yields

$$\omega_1 = 15.56\frac{1}{l^2}\sqrt{\frac{EI}{\rho A}}$$

$$\omega_2 = 58.41\frac{1}{l^2}\sqrt{\frac{EI}{\rho A}}$$

$$\omega_3 = 155.6\frac{1}{l^2}\sqrt{\frac{EI}{\rho A}}$$

Problems and Solutions Section 8.4 (8.34 through 8.43)

8.34 Refer to the tapered bar of Figure P8.13. Calculate a lumped-mass matrix for this system and compare it to the solution of Problem 8.13. Since the beam is tapered, be careful how you divide up the mass.

Solution: The lumped mass at node 2 should be the total mass between $x = .25$ and $x = .75$. Therefore

$$M_2 = 2700\int_{.25}^{.75}\frac{\pi}{4}\left[h_1^2 + \left(\frac{h_2 - h_1}{l}\right)^2 x^2 + 2h_1\left(\frac{h_2 - h_1}{l}\right)x\right]dx$$
$$= 26.5$$

likewise for node 3

$$M_3 = 2700\int_{.75}^{1}\frac{\pi}{4}\left[h_1^2 + \left(\frac{h_2 - h_1}{l}\right)^2 x^2 + 2h_1\left(\frac{h_2 - h_1}{l}\right)x\right]dx$$
$$= 7.289$$

The mass matrix is then

$$M = \begin{bmatrix} 26.5 & 0 \\ 0 & 7.289 \end{bmatrix}$$

and the natural frequencies are

$\omega_1 = 6670$ rad/s and $\omega_2 = 13106$ rad/s.

For the distributed mass system

$\omega_1 = 7414$ rad/s and $\omega_2 = 20368$ rad/s.

The first natural frequency found by the distributed mass model is slightly better than the lumped mass model when compared to the three element distributed mass model derived in problem 13.

8.35 Calculate and compare the natural frequencies obtained for a tapered bar by using first, the consistent-mass matrix (Problem 8.12), and second, the lumped-mass matrix (Problem 8.34).

Solution:

See solution for Problem 8.34.

8.36 Consider again the machine punch of Problem 8.16 and Figure P8.15. Calculate the natural frequencies of this system using a lumped-mass matrix and compare the results to those obtained with the consistent-mass matrix.

Solution:

The lumped mass matrix is

$$M = \begin{bmatrix} \frac{\rho_1 A_1 l_1}{2} + \frac{\rho_2 A_2 l_2}{2} & 0 \\ 0 & \frac{\rho_2 A_2 l_2}{2} \end{bmatrix}$$

$$= rl \begin{bmatrix} A_1 + A_2 & 0 \\ 0 & A_2 \end{bmatrix}$$

$$= \begin{bmatrix} .078 & 0 \\ 0 & .039 \end{bmatrix}$$

The natural frequencies are

ω_1 = 38756 rad/s and ω_2 = 93565 rad/s.

The results for the consistent mass matrix were

ω_1 = 40798.6 rad/s and ω_2 = 142525 rad/s.

The first natural frequency is within 5% for both predictions. For this case, the inconsistent mass matrix is adequate for the 1st mode.

8.37 Consider again the bridge support of Figure P8.17 discussed in connection with Problem 8.17. Develop a four-element finite element model of this structure using a lumped-mass approximation and calculate the natural frequencies. Use constant area elements.

Solution:

We will use elements which each have constant cross section by finding the average area for each element. Elements are numbered from one to four from bottom to top.

$$A_1 = \frac{1}{.25l}\int_0^{.25l} A(x)dx = \frac{A_0}{.25l}\left(-le^{-\frac{x}{l}}\right)\Bigg|_0^{.25l}$$

$$= -4A_0\left(e^{-.25} - 1\right) = .8848A_0$$

likewise

$$A_2 = .6891A_0,\ A_3 = .5367A_0,\ A_4 = .4179A_0$$

Assembling the stiffness matrix yields

$$K = \frac{EA_0}{.25l}\begin{bmatrix} 1.5739 & -.6891 & 0 & 0 \\ -.6891 & 1.2258 & -.5367 & 0 \\ 0 & -.5367 & .9546 & -.4179 \\ 0 & 0 & -.4179 & .4179 \end{bmatrix}$$

To find the mass matrix, we will assume again that the elements have constant cross section. This yields

$$M = \frac{\rho A_0 l}{8}\begin{bmatrix} 1.5739 & 0 & 0 & 0 \\ 0 & 1.2258 & 0 & 0 \\ 0 & 0 & .9546 & 0 \\ 0 & 0 & 0 & .4179 \end{bmatrix}$$

The natural frequencies are then

$$\omega_1 = 1.86\frac{1}{l}\sqrt{\frac{E}{\rho}},\ \omega_2 = 4.50\frac{1}{l}\sqrt{\frac{E}{\rho}},\ \omega_3 = 6.62\frac{1}{l}\sqrt{\frac{E}{\rho}},\ \omega_4 = 7.78\frac{1}{l}\sqrt{\frac{E}{\rho}},$$

8.38 Consider the torsional vibration problem illustrated in Figure P8.20 and discussed in Problem 8.20. Calculate a lumped-mass matrix for the single element.

Solution:

The total mass moment of inertia would be divided between the two degrees of freedom.

Therefore

$$M = \frac{1}{2}\begin{bmatrix} I_p & 0 \\ 0 & I_p \end{bmatrix}$$

8.39 Estimate the first three natural frequencies of a clamped-free bar of length l in torsional vibration by using a lumped-mass model and four elements.

Solution:

The stiffness matrix is

$$K = \frac{4G\gamma}{l}\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

The mass matrix is

$$M = \frac{\rho J l}{4}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

The natural frequencies are then

$$\omega_1 = 1.56\frac{1}{l}\sqrt{\frac{G\gamma}{\rho J}},\ \omega_2 = 4.445\frac{1}{l}\sqrt{\frac{G\gamma}{\rho J}},\ \omega_3 = 6.65\frac{1}{l}\sqrt{\frac{G\gamma}{\rho J}},\ \omega_4 = 7.8463\frac{1}{l}\sqrt{\frac{G\gamma}{\rho J}}$$

From table 6.3, it can be seen that the first two natural frequencies predicted by the finite element model are good approximations.

8.40 Calculate the natural frequencies of a pinned-pinned beam of length *l* using one element and the consistent-mass matrix of equation (8.73).

Solution:

The mass matrix is

$$M = \frac{\rho A l^3}{48}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and the stiffness matrix is

$$K = \frac{EI}{l}\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$$

Finding the natural frequencies gives

$$\omega_1 = 9.798\frac{1}{l^2}\sqrt{\frac{EI}{\rho A}},\ \ \omega_2 = 16.971\frac{1}{l^2}\sqrt{\frac{EI}{\rho A}}$$

The first natural frequency from distributed theory is

$$\omega_n = 9.869\frac{1}{l^2}\sqrt{\frac{EI}{\rho A}}$$

8.41 Calculate the natural frequencies of a pinned-pinned beam of length *l* using one element and the lumped-mass matrix of equation (8.73). Compare your results to those obtained with at consistent-mass matrix of Problem 8.40.

Solution:

The consistent mass matrix is

$$M = \frac{\rho A l^3}{420}\begin{bmatrix} 4 & -3 \\ -3 & 4 \end{bmatrix}$$

which gives

$$\omega_1 = 10.96\frac{1}{l^2}\sqrt{\frac{EI}{\rho A}},\ \ \omega_2 = 50.20\frac{1}{l^2}\sqrt{\frac{EI}{\rho A}}$$

which is worse than the inconsistent mass matrix results. (See solution 8.40)

8.42 Calculate a three-element finite element model of a cantilevered beam (see Problem 8.25) using a lumped mass that includes rotational inertia. Also calculate the system's natural frequencies and compare them with those obtained with a consistent-mass matrix of Problem 8.25 and with the values obtained by the methods of Chapter 6.

Solution:

The mass matrix is $M = \rho A l \, \mathrm{diag}\left(1, \frac{1}{24}, 1, \frac{1}{24}, \frac{1}{2}, \frac{1}{48}\right)$ using the $[u_1 \; lu_2]$ convention for the displacement vector.

The natural frequencies are then

$$\omega_i = a_i \frac{1}{l^2}\sqrt{\frac{EI}{\rho A}}$$

$$a_i = .368,\ 2.00,\ 4.98,\ 10.7,\ 14.5,\ 17.1$$

This is not as good as the consistent mass matrix results. From distributed parameter theory $a_1 = .3911$.

8.43 Repeat Problem 8.42 using a lumped-mass matrix that neglects the rotational degree of freedom. Discuss any problems you encounter when trying to solve the related eigenvalue problem.

Solution:

$$M = \rho A l \, \mathrm{diag}\left(1, 0, 1, 0, \frac{1}{2}, 0\right)$$

The singularity of the mass matrix does not allow a solution to be found.

Problems and Solutions Section 8.5 (8.44 through 8.49)

8.44 Derive a consistent-mass matrix for the system of Figure 8.9. Compare the natural frequencies of this system with those calculated with the lumped-mass matrix computed in Section 8.5.

Solution: Using the vibration toolbox

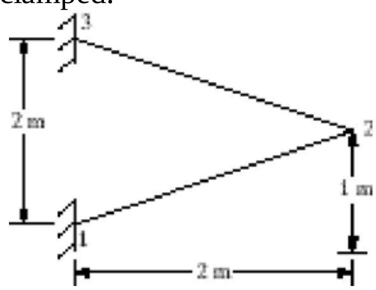
$$M = \rho Al\begin{bmatrix} .6857 & 0 \\ 0 & .7238 \end{bmatrix}$$

The natural frequencies are then

$$\omega_1 = .8311\frac{1}{l}\sqrt{\frac{E}{\rho}} \text{ and } \omega_1 = 1.479\frac{1}{l}\sqrt{\frac{E}{\rho}}$$

These are higher than those predicted with the inconsistent mass matrix

8.45 Consider the two beam system of Figure P8.45. Use VTB8_1 to create a two-element, rod/beam element model and compute the first three natural frequencies. Use $A = 0.0004$ m^2, $I = 1.33 \times 10^{-8}$ m^4, and the properties of aluminum. Assume that nodes 1 and 3 are clamped.



Solution:

```
%scipt file for problem 8.45
node=[0 0;1 .5;2 1;1  1.5;0 2];
ncon=[1 2 69e10 .004 1.33e-8 0 2700;
      2 3 69e10 .004 1.33e-8 0 2700;
      3 4 69e10 .004 1.33e-8 0 2700;
      4 5 69e10 .004 1.33e-8 0 2700];
zero=[1 1;
      1 2;
      1 3;
      5 1;
      5 2;
      5 3];
conm=[];
force=[];
save VTB8_45.con
```

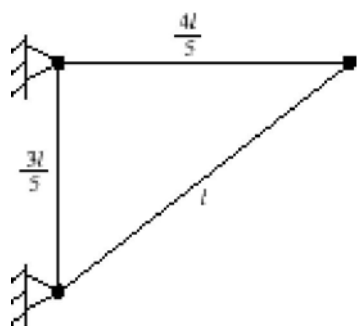
Running this yields that the first three natural frequencies are given as 377.5, 8763.7 and 10951.2 rad/s.

8.46 Follow the procedure of Problem 8.45 using two elements for each beam. Compare the natural frequencies and mode shapes of the four element model produced here to those of the two-element model of Problem 8.45. State which model is better and why.

Solution: Use the script file from 8.45 ending in VTB8_46.con
The first five natural frequencies are 286.8, 419.1, 1074.5, 1510.8, and 2838.9 rad/s. The result from the four element model is probably better because the additional elements allow the first few modes to be found in more detail. Notice the difference in the result for the first mode. The first mode is primarily a rotation of the joint between the two beams. The two element model shows this to be the only significant motion (load the .out data file to observe the mode shape vector). The four element model shows that the middle of each beam displaces and rotates as well.

The eight element model predicts the first five natural frequencies to be 284.3, 413.0 ,925.6, 1147.3, and 1959.7 rad/s, the first four of which agree well with the four element model results.

8.47 Determine a finite element model of the three-bar truss of Figure P8.47 using a lumped-mass matrix.



Solution:

Using VTB8_1

$$K = \frac{EA}{l}\begin{bmatrix} 1.89 & .48 \\ .48 & .36 \end{bmatrix}$$

The inconsistent mass matrix is

$$M = \rho Al\begin{bmatrix} .9 & 0 \\ 0 & .9 \end{bmatrix}$$

8.48 Determine a finite element model for the three-bar truss of Figure P8.47 using a consistent-mass matrix.

Solution:

Using VTB8_1 the consistent mass matrix is

$$M = \rho Al \begin{bmatrix} .6137 & -.0183 \\ -.0183 & .6549 \end{bmatrix}$$

However, this mass matrix is created using beam/rod elements. Using simple rod elements gives a consistent mass matrix

$$M = \rho Al \begin{bmatrix} .48 & .16 \\ .16 & .12 \end{bmatrix}$$

8.49 Compare the frequencies obtained for the system of Problem 8.48 with those of Figure P8.47.

Solution:

The natural frequencies using the consistent mass matrix are

$\omega_1 = 1.7321$ $\omega_2 = 2.1651$

The natural frequencies using the inconsistent mass matrix are

$\omega_1 = .4966$ $\omega_2 = 1.5012$

These results are terribly inconclusive, but since we have seen in previous examples that the consistent mass matrix generally yields the better results, one would expect the same to be true in this case.

Problems and Solutions Section 8.6 (8.50 through 8.54)

8.50 Consider the machine punch of Figure P8.15. Recalculate the fundamental natural frequency by reducing the model obtained in Problem 8.16 to a single degree of freedom using Guyan reduction.

Solution:

From the results of 8.16

$$K = \begin{bmatrix} 4 & -2 \\ -2 & 2 \end{bmatrix} \times 10^8, \quad M = \begin{bmatrix} .052 & .013 \\ .013 & .026 \end{bmatrix}$$

From (8.104)

$$Q^T MQ = .052 + .013 + .013 + .026 = .104$$

From (8.105)

$$Q^T KQ = (4 - 2) \times 10^8 = 2 \times 10^8$$

$$\omega = \sqrt{\frac{2 \times 10^8}{.104}} = 43852.9 \text{ rad/s}$$

which is a poor prediction of the first natural frequency. If we reorder *K* and *M* (reducing to coordinate 2) we get

$$Q^T MQ = .026 + .013 + .013 = .052$$

$$Q^T KQ = (2 - 1) \times 10^8 = 1 \times 10^8$$

$$\omega = 43852.9 \text{ rad/s}$$

which is the same result as reducing to coordinate 1.

8.51 Compute a reduced-order model of the three-element model of a cantilevered bar given in Example 8.3.2 by eliminating u_2 and u_3 using Guyan reduction. Compare the frequencies of each model to those of the distributed model given in Window 8.1.

Solution:

$$M = \frac{\rho Al}{18}\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$K = \frac{3EI}{l}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

Let $\tilde{M}$ and $\tilde{K}$ be the matrices with the coefficients factored out.

$$\tilde{M}_{11} = 4,\ \tilde{M}_{21} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \tilde{M}_{12}^T,\ M_{22} = \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\tilde{K}_{11} = 2,\ \tilde{K}_{21} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \tilde{K}_{11}^T,\ K_{22} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

Using equations (8.104) and (8.105)

$$\tilde{M}_r = Q^T MQ = 14$$

$$\tilde{K}_r = Q^T KQ = 1$$

and

$$\omega_n = \sqrt{\frac{\frac{3EA}{l}}{\frac{14\rho Al}{18}}} = 1.964\frac{1}{l}\sqrt{\frac{E}{\rho}}$$

as compared to the distributed model value of

$$\omega_1 = 1.57\frac{1}{l}\sqrt{\frac{E}{\rho}}$$

8.52 Consider the system defined by the matrices

$$M = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad K = \begin{bmatrix} 20 & -1 & 0 & 0 \\ -1 & 20 & -3 & 0 \\ 0 & -3 & 20 & -17 \\ 0 & 0 & 17 & 17 \end{bmatrix}$$

Use mass condensation to reduce this to a two-degree-of-freedom system with a nonsingular mass matrix.

Solution:

Following the same procedure as example 8.6.1

$$M_r = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \text{ and } K_r = \begin{bmatrix} 19.95 & -.15 \\ -.15 & 36.55 \end{bmatrix}$$

8.53 Recall the punch press problem modeled in Figure 4.28 and treated in Example 4.8.3. The mass and stiffness matrices are given by

$$M = \begin{bmatrix} 0.4\times 10^3 & 0 & 0 \\ 0 & 2.0\times 10^3 & 0 \\ 0 & 0 & 8.0\times 10^3 \end{bmatrix} \quad K = \begin{bmatrix} 30\times 10^4 & 30\times 10^4 & 0 \\ 30\times 10^4 & 38\times 10^4 & 8\times 10^4 \\ 0 & 8\times 10^4 & 88\times 10^4 \end{bmatrix}$$

Recalling that the only external force acting on the machine is at the $x_1(t)$ coordinate, reduce this to a single-degree-of-freedom system using Guyan reduction to remove x_2 and x_3. Compare this single frequency with those of Example 4.8.3.

Solution:

Following the same procedure as example 8.6.1

$M_r = 1.7385\times 10^3, K_r = 5.8537\times 10^4$ and the natural frequency is

$$\omega_n = \sqrt{\frac{K_r}{M_r}} = 5.803\,\text{rad/s}$$

Example 4.8.3 gave the first natural frequency as ω_1 = 5.387 rad/s which is within 10% of the Guyan reduced prediction.

8.54. Consider the beam example given in Example 7.6.2. Using the values given there (An aluminum beam: 0.5128 m x 25.5 mm x 3.2 mm, $E = 6.9\times10^{10}$ N/m^2 , ρ = 2715 kg/m^3, A = 8.16 m^2 and $I = 6.96\times10^{-11}$ m^4), compute the first 4 natural frequencies as accurately as possible and compare them to both the analytical values and the measured values.