Problems and Solutions Section 7.2 (7.1-7.5)

7.1 A low-frequency signal is to be measured by using an accelerometer. The signal is physically a displacement of the form 5 sin (0.2*t*). The noise floor of the accelerometer (i.e. the smallest magnitude signal it can detect) is 0.4 volt/g. The accelerometer is calibrated at 1 volt/g. Can the accelerometer measure this signal?

Solution:

From the problem statement:

$$x(t) = 0.5\sin(0.2t) \quad \text{m}$$
$$\dot{x}(t) = 0.1\cos(0.2t) \quad \text{m/s}$$
$$\ddot{x}(t) = -0.02\sin(0.2t) \quad \text{m/s}^2$$

The peak acceleration is:

$$\pm 0.2\,\text{m/s}^2\left[\frac{1g}{9.8\,\text{m/s}^2}\right] = \pm 0.0204g$$

Accelerometer calibration is $1\text{V}/g$, therefore the peak output of the accelerometer is:

$$\pm 0.0204g\left[\frac{1\text{V}}{g}\right] = \pm 0.0204\text{V}$$

Since the noise floor on the accelerometer is 0.4 V, then this acceleration cannot be measured.

7.2 Referring to Chapter 2, calculate the response of a single-degree-of-freedom system to a unit impulse and then to a unit triangle input lasting T second. Compare the two responses. The differences correspond to the differences between a "perfect" hammer hit and a more realistic hammer hit, as indicated in Figure 7.2. Use $\zeta = 0.01$ and $\omega = 4$ rad/s for your model.

Solution:

System: $\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = f(t)$ (Letting m = 1)

(i) $f(t) = \delta(t)$, a unit impulse

$$x(t) = e^{-\zeta\omega_n t}\sin(\omega_d t) \qquad \omega_d = \omega_n\sqrt{1-\zeta^2}$$

(ii) $f(t) = \dfrac{t}{T}u(t) - \dfrac{2}{T}(t-T)u(t-T) + \dfrac{1}{T}(t-2T)u(t-2T)$

$u(t\text{-}a)$ = unit step at $t = a$.

$$x(t) = \frac{1}{T}\{r(t) - 2r(t-T) + r(t-2T)\}$$

From table of Laplace transforms:

$$r(t) = \frac{1}{\omega_n^2}\left\{t - \frac{2\zeta}{\omega_n} + \frac{1}{\omega_d}e^{-\zeta\omega_n t}\sin(\omega_d t + \theta)\right\}u(t) \qquad \cos\theta = 2\zeta^2 - 1$$

$$x(t) \approx \frac{1}{T\omega_n^3}\Big\{\left[\omega_n t - 2\zeta - e^{-\zeta\omega_n t}\sin(\omega_d t\right]u(t)$$
$$-2\left[\omega_n(t-T) - 2\zeta - e^{-\zeta\omega_n(t-T)}\sin(\omega_d(t-T))\right]u(t-T)$$
$$+\left[\omega_n(t-2T) - 2\zeta - e^{-\zeta\omega_n(t-2T)}\sin(\omega_d(t-2T))\right]u(t-2T)\Big\}$$

since $\omega_n \approx \omega_d$ and $\theta \approx \pi/2$

7.3 Compare the Laplace transform of $\delta(t)$ with the Laplace transform of the triangle input of Figure 7.2 and Problem 7.2.

Solution:

(i) $f(t) = \delta(t)$, unit impulse

$F(s) = 1$

(ii) $f(t) = \frac{t}{T}u(t) - \frac{2}{T}(t-T)u(t-T) + \frac{1}{T}(t-2T)u(t-2T)$, unit triangle with period T.

$$F(s) = \frac{1}{T}\left\{\int_0^\infty te^{-st}dt - 2\int_T^\infty (t-T)e^{-st}dt + \int_{2T}^\infty (t-2T)e^{-st}dt\right\}$$

$$F(s) = \frac{1}{Ts^2}\left\{1 + e^{-sT} + e^{s2T}\right\}$$

7.4 Plot the error in measuring the natural frequency of a single-degree-of-freedom system of mass 10 kg and stiffness 350 N/m if the mass of the excitation device (shaker) is included and varies from 0.5 to 5 kg.

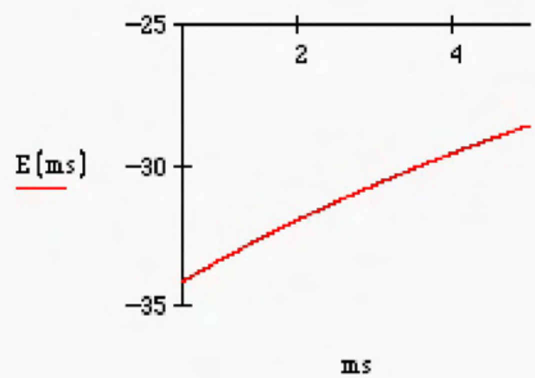
Solution:

$m = 10$ kg
$k = 350$ N/m
$0.5 \leq m_s \leq 5.0$ kg

$$\text{Error} = \sqrt{\frac{k}{m + m_s}} - \sqrt{\frac{k}{m}}$$

$k := 350$ $\quad m := 10$

$$E(ms) := \sqrt{\frac{k}{m + ms}} - \sqrt{\frac{k}{m}}$$



7.5 Calculate the Fourier transform of $f(t)$ = 3 sin 2t – 2 sin t – cos t and plot the spectral coefficients.

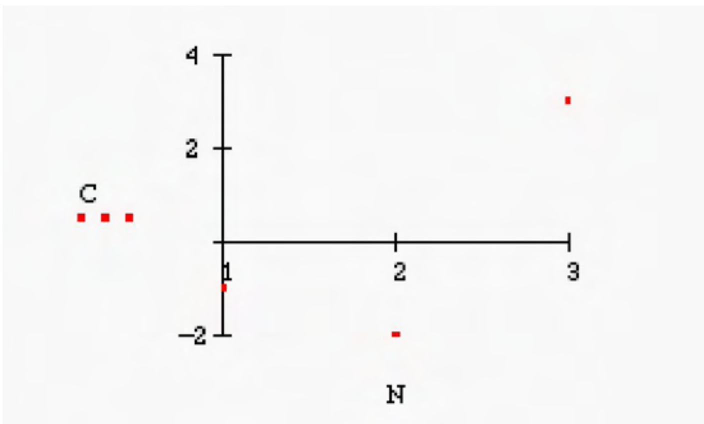
Solution:

$F(t)$ = 3 sin(2t) - 2sin(t) - cos(t)

ω_T = 1 rad/sec

a_1 = -1 b_1 = -2 b_2 = 3

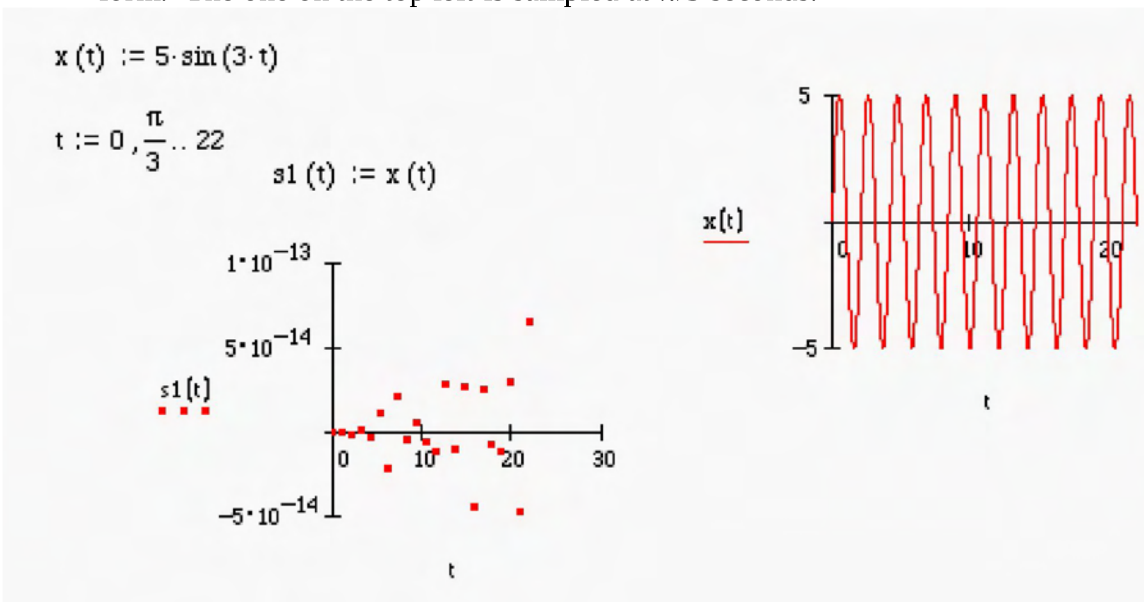
a_n = 0, n = 2, 3, …. b_n = 0, n = 3, 4, 5, …..



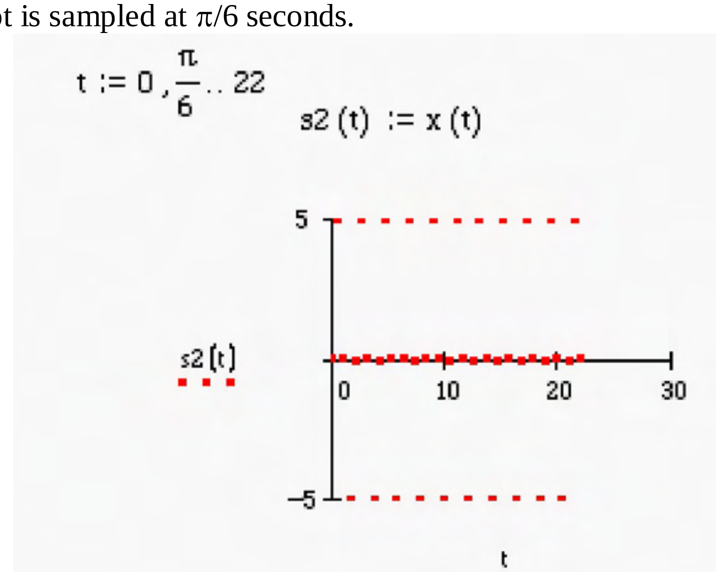
Problems and Solutions Section 7.3 (7.6-7.9)

7.6 Represent 5 sin 3*t* as a digital signal by sampling the signal at π/3, π/6 and π/12 seconds. Compare these three digital representations.

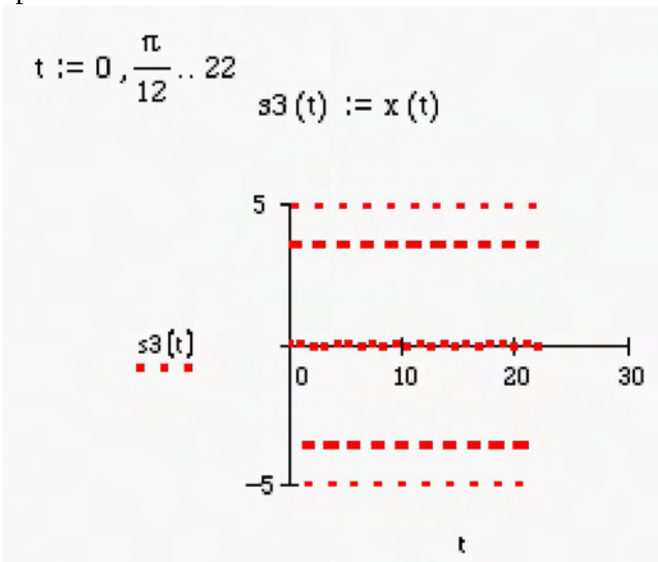
Solution: Four plots are shown. The one at the top far right is the exact wave form. The one on the top left is sampled at π/3 seconds.



The next plot is sampled at π/6 seconds.



The next plot is sampled at π/12 seconds.



None of the plots give the shape of a sine wave. However if the s3 is connected by lines, the wave shape is close.

7.7 Compute the Fourier coefficient of the signal |1120 sin (120 πt)|.

Solution:

$f(t)$ = |120sin(120πt)| (absolute value of the sine wave)

To calculate the Fourier series:

T = 1/120 sec ω_T = 240 π rad/sec

$$a_o = 240 \int_0^{1/120} 120\sin(120\pi t)dt$$

$$a_o = 480/\pi$$

$$a_n = 240 \int_0^{1/120} 120\sin(120\pi t)\cos(240\pi nt)dt$$

$$a_n = \frac{480}{\pi(1-4n^2)}$$

$$b_n = 240 \int_0^{1/120} 120\sin(120\pi t)\sin(240\pi nt)dt$$

$$b_n = 0$$

$$f(t) = \frac{240}{\pi}\left\{1 + \sum_{n=1}^{\infty} \frac{2}{1-4n^2}\cos(240\pi n)t\right\}$$

7.8 Consider the periodic function

$$x(t) = \begin{cases} -5 & 0 < t < \pi \\ 5 & \pi < t < 2\pi \end{cases}$$

and $x(t) = (t + 2\pi)$. Calculate the Fourier coefficients. Next plot $x(t)$: $x(t)$ represented by the first term in the Fourier series, $x(t)$ represented by the first two terms of the series, and $x(t)$ represented by the first three terms of the series. Discuss your results.

Solution: For the Fourier Series: $T = 2\pi$ $\omega_T = 1$

$$a_0 = 0$$

$$a_n = \frac{2}{2\pi}\left\{\int_0^{\pi} -5\cos(nt)dt + \int_{\pi}^{2\pi} 5\cos(nt)dt\right\}$$

$$\Rightarrow a_n = 0$$

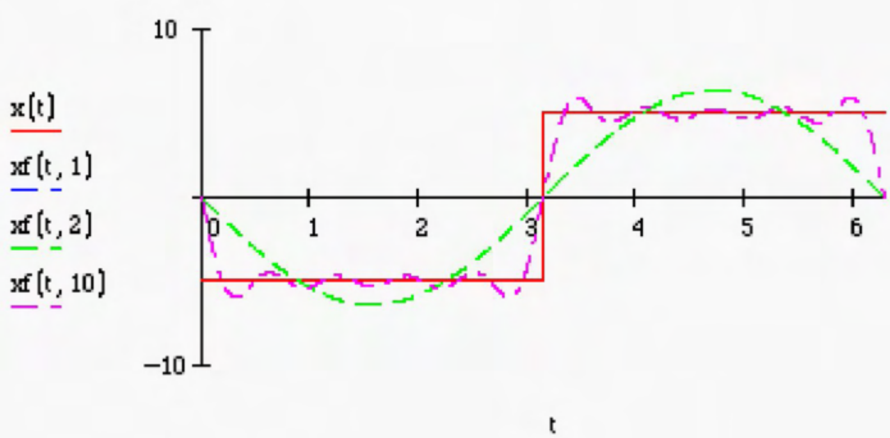
$$b_n = \frac{2}{2\pi}\left\{\int_0^{\pi} -5\sin(nt)dt \int_{\pi}^{2\pi} 5\sin(nt)dt\right\} \Rightarrow$$

$$b_n = \frac{-5}{\pi n}[1 - 2\cos(n\pi) + \cos(2n\pi)]$$

$$x(t) = -\frac{5}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}(1 - 2\cos(n\pi) + \cos(2n\pi)\sin(nt)$$

$x(t) := -5 + 10\cdot\Phi(t - \pi)$ The funtion for one cycle (0<t<2pi)

$$xf(t,N) := \frac{-5}{\pi}\cdot\sum_{n=1}^{N}\frac{(-2\cdot\cos(n\cdot\pi) + 1 + \cos(2\cdot n\cdot\pi))}{n}\cdot\sin(n\cdot t)$$



7.9 Consider a signal $x(t)$ with maximum frequency of 500 Hz. Discuss the choice of record length and sampling interval.

Solution:

For a signal with maximum frequency of 500 Hz, the sampling rate, f_s, should be

$$f_s > 2(500) = 1000 \text{ Hz}$$

Due to Shannon's sampling theorem. A better choice would be

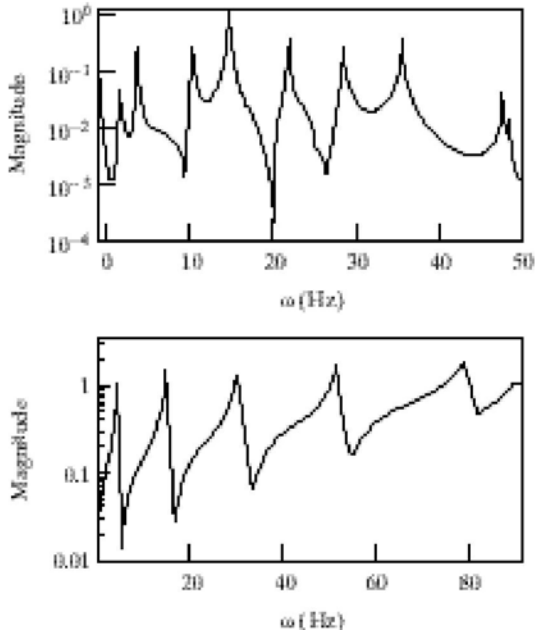
$$f_s = 2.5(500) = 1250 \text{ Hz}$$

Thus, the minimum sampling rate is 0.001 sec. and the suggested rate is 0.0008 sec.
Lower sampling rates will produce aliasing.

The record length N is usually a power of 2, such as 512, 1024, 2048, etc.
Windowing is performed to reduce leakage.

Problems and Solutions for Section 7.4 (7.10-7.19)

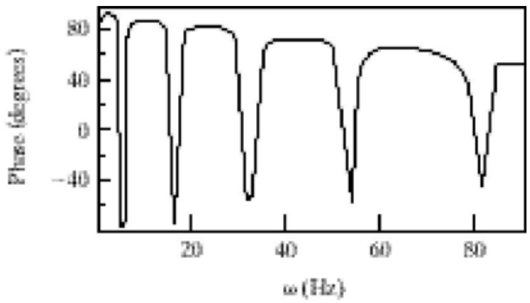
7.10 Consider the magnitude plot of Figure P7.10. How many natural frequencies does this system have, and what are their approximate values?



Solution:

The system looks to have 8 modes with approximate natural frequencies of 2, 4, 10, 15, 22, 29, 36, and 47 Hz.

7.11 Consider the experimental transfer function plot of Figure P7.11. Use the methods of Example 7.4.1 to determine ζ_i and ω_i.



Solution:

For each mode:

$$\zeta_i = \frac{\omega_{bi} - \omega_{ai}}{2\omega_i}$$

where ω_{bi} and ω_{ai} are the frequencies where the magnitude is $1/\sqrt{2}$ of the resonant magnitude. All values given in the following table are approximate.

Mode	ω_i (Hz)	$\lvert H(\omega_i)\rvert$	$\frac{\lvert H(\omega_i)\rvert}{\sqrt{2}}$	ω_{ai} (Hz)	ω_{bi} (Hz)	ζ_i
1	4.80	0.089	0.063	4.56	5.04	0.049
2	15.20	1.050	0.742	14.76	15.48	0.024
3	30.95	1.800	1.270	30.47	31.19	0.012
4	52.62	2.000	1.414	52.14	52.85	0.007
5	80.00	2.100	1.480	79.05	80.48	0.009

7.12 Consider a two-degree-of-freedom system with frequencies ω_1 = 10 rad/s, ω_2 = 15 rad/s, and damping ratios $\zeta_1 = \zeta_2$ = 0.01. With modal $s = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, calculate the transfer function of this system for an input at x_1 and a response measurement at x_2.

Solution:

Since the natural frequencies, damping ratios and mode shapes are given, the system can be expressed in modal coordinates as

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\ddot{\mathbf{r}} + \begin{bmatrix} 2(.01)10 & 0 \\ 0 & 2(.01)15 \end{bmatrix}\dot{\mathbf{r}} + \begin{bmatrix} 10^2 & 0 \\ 0 & 15^2 \end{bmatrix}\mathbf{r} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} 1 \\ 0 \end{Bmatrix} f(t) = \frac{1}{\sqrt{2}}\begin{Bmatrix} 1 \\ -1 \end{Bmatrix} f(t)$$

$$y = \frac{1}{\sqrt{2}}\{0 \quad 1\}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\mathbf{r} = \frac{1}{\sqrt{2}}\{1 \quad 1\}\mathbf{r}$$

This is the representation of the system in modal coordinates, if proportional damping is assumed. The transfer function is:

$$Y(s) = \frac{1}{\sqrt{2}}\{1 \quad 1\}R(s)$$

where

$$R(s) = \frac{1}{\sqrt{2}}\begin{Bmatrix} \dfrac{1}{s^2 + 0.2s + 100} \\ \dfrac{-1}{s^2 + 0.3s + 225} \end{Bmatrix} F(s)$$

Combining the previous two expressions yields

$$\frac{Y(s)}{F(s)} = \frac{(0.1)(s + 1250)}{(s^2 + 0.2s + 100)(s^2 + 0.3s + 225)}$$

7.13 Plot the magnitude and phase of the transfer function of Problem 7.12 and see if you can reconstruct the modal data (ω_1, ω_2, ζ_1, and ζ_2) from your plot.

Solution:

For each mode:

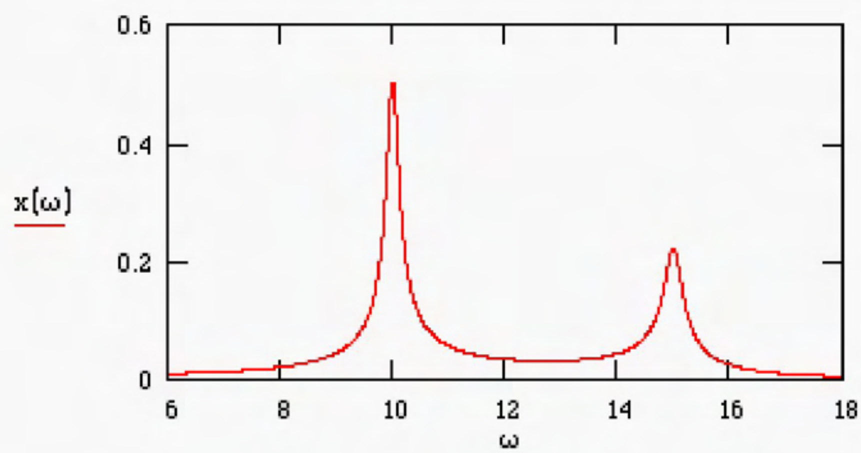
$$\zeta_i = \frac{\omega_{bi} - \omega_{ai}}{2\omega_i}$$

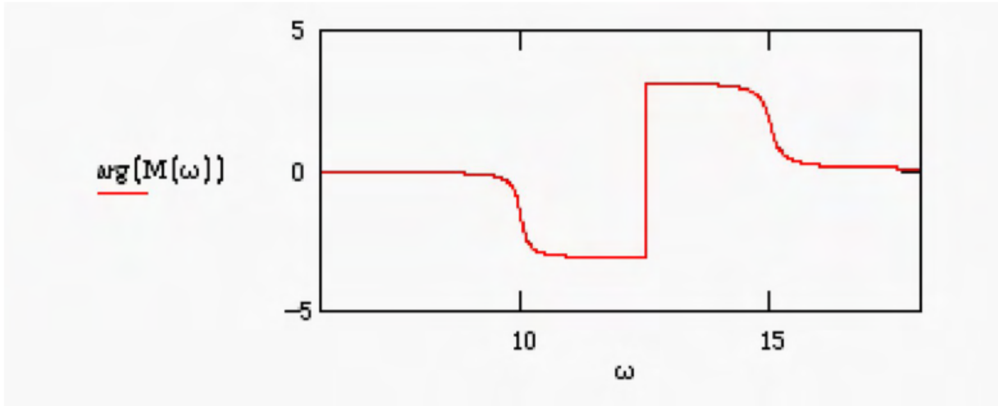
where ω_{bi} and ω_{ai} are the frequencies where the magnitude is $1/\sqrt{2}$ of the resonant magnitude. All values in the following table are approximate.

Mode	ω_i (rad/s)	$\lvert H(\omega_i)\rvert$	$\frac{\lvert H(\omega_i)\rvert}{\sqrt{2}}$	ω_{ai} (rad/s)	ω_{bi} (rad/s)	ζ_i
1	10	0.50	0.354	9.89	10.07	0.009
2	15	0.22	0.156	14.83	15.16	0.011

$$M(\omega) := \frac{(0.1)\cdot(\omega \cdot j + 1250)}{\left(-\omega^2 + 0.2\cdot\omega\cdot j + 100\right)\cdot\left(-\omega^2 + 0.3\cdot\omega\cdot j + 225\right)}$$

$$x(\omega) := |M(\omega)|$$





arg(M(ω))
5
0
−5
10
15
ω

7.14 Consider equation (7.14) for determining the damping ratio of a single mode. If the measurement in frequency varies by 1%, how much will the value of ζ change?

Solution:

$$\zeta = \frac{\omega_b - \omega_a}{2\omega_d}$$

If $\omega_d = \omega_{do}(1 \pm 0.01)$ where ω_{do} is the measured natural frequency, then the damping ratio is

$$\zeta = \frac{\omega_b - \omega_a}{2\omega_{do}}\left\{\frac{1}{1 \pm 0.01}\right\} = \zeta_o\left\{\frac{1}{1 \pm 0.01}\right\}$$

If ω_d is $0.99\,\omega_{do}$, then ζ = 1.01 ζ_o

If ω_d is $1.01\,\omega_{do}$, then ζ = 0.99 ζ_o

Thus, 1 percent changes in the measured natural frequency produce similar changes in the measured damping ratio.

7.15 Discuss the problems of using equation (7.14) if the natural frequencies of the structure are very close together.

Solution:

Equation (7.14) assumes that the response at resonance is due to a single degree of freedom system. If the natural frequencies are very close together, this assumption is not valid. This will introduce error into the damping ratio calculation since the peak response at each resonant frequency will be due to a combination of responses from each of the closely spaced modes.

7.16 Discuss the limitation of using equation (7.15) if ζ is very small. What happens if ζ is very large?

Solution: When ζ is very small (<0.01), it is difficult to determine where $R(\alpha)$ is the largest since equation (7.15) is changing very rapidly in the vicinity of resonance. When ζ is very large (>0.707), the frequency response near resonance is very flat, again making it difficult to determine the damped natural frequency. In either case, experimentally determined damping ratios will contain error since they depend on an accurate determination of the resonant frequency. Problem 7.18 contains plots that illustrate these ideas.

7.17 Consider the two-degree-of-freedom system described by

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} f_0 \sin \omega t \\ 0 \end{bmatrix}$$

and calculate the transfer function $|X/F|$ as a function of the damping parameter c.

Solution:

The equations of motion for the system are:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\ddot{\mathbf{x}} + \begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix}\dot{\mathbf{x}} + \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\mathbf{x} = \begin{Bmatrix} f_o \\ 0 \end{Bmatrix} f(t)$$

Taking the Laplace transform yields

$$\begin{bmatrix} s^2+2 & -1 \\ -1 & s^2+cs+2 \end{bmatrix} X(s) = \begin{Bmatrix} f_o \\ 0 \end{Bmatrix} F(s)$$

Inverting the matrix on the left hand side leads to an expression for $X(s)$:

$$X(s) = \frac{1}{(s^2+2)(s^2+cs+2)-1}\begin{bmatrix} s^2+cs+2 & 1 \\ 1 & s^2+2 \end{bmatrix}\begin{Bmatrix} f_o \\ 0 \end{Bmatrix} F(s)$$

Performing the multiplication leads to

$$\frac{X_1(s)}{f_o F(s)} = \frac{s^2+cs+2}{s^4+cs^3+4s^2+2cs+3}$$

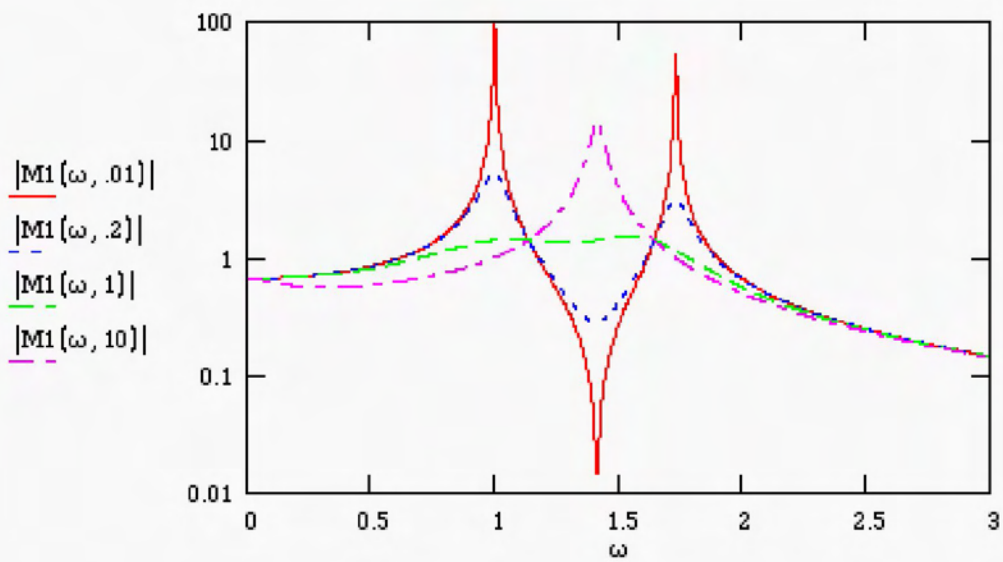
$$\frac{X_2(s)}{f_o F(s)} = \frac{1}{s^4+cs^3+4s^2+2cs+3}$$

7.18 Plot the transfer function of Problem 7.17 for the four cases: c = 0.01, c = 0.2, c = 1, and c = 10. Discuss the difficulty in using these plots to measure ζ_i and ω_i for each value of c.

Solution:

For c = 0.01, the resonant peaks are very sharp, making an accurate determination of ζ_i difficult. In the case c = 0.2, ζ_i and ω_i could be determined fairly easily using the techniques of section 7.4. Increasing c to 1.0 makes the frequency response very flat, which again makes finding ζ_i and ω_i difficult. Finally, when c = 10, it almost looks as if there is one resonant peak, which would lead to a completely erroneous result.

$$M1(\omega, c) := \frac{-\omega^2 + c\cdot\omega\cdot j + 2}{\omega^4 - c\cdot\omega^3\cdot j - 4\cdot\omega^2 + 2\cdot c\cdot\omega\cdot j + 3}$$



7.19 Use a numerical procedure to calculate the natural frequencies and damping ratios of the system of Problem 7.18. Label these on your plots from Problem 7.18 and discuss the possibility of measuring these values using the methods of Section 7.4

Solution:

For the case where $c = 0.01$

Mode	ω_i (rad/s)	$\lvert H(\omega_i)\rvert$	$\dfrac{\lvert H(\omega_i)\rvert}{\sqrt{2}}$	ω_{ai} (rad/s)	ω_{bi} (rad/s)	ζ_i
1	1.0	59	41.72	0.99	1.02	0.015
2	1.7	48	33.94	1.71	1.69	0.006

Actual values: $\omega_1 = 1.00$ $\zeta_1 = 0.003$

$\omega_2 = 1.73$ $\zeta_2 = 0.001$

The actual values are calculated directly from the equations.

For the case where $c = 0.2$

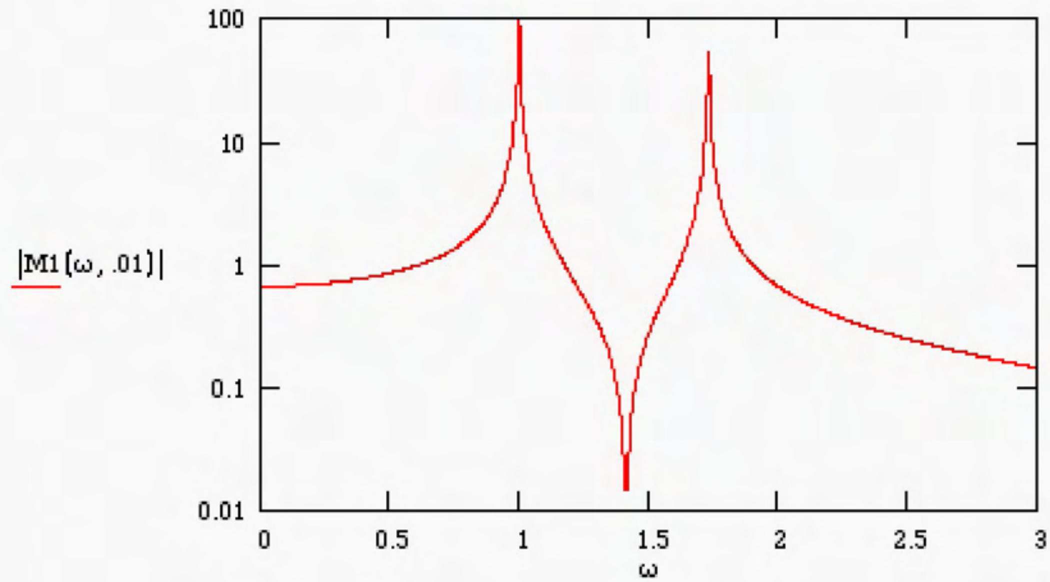
Mode	ω_i (rad/s)	$\lvert H(\omega_i)\rvert$	$\dfrac{\lvert H(\omega_i)\rvert}{\sqrt{2}}$	ω_{ai} (rad/s)	ω_{bi} (rad/s)	ζ_i
1	1.0	5.1	3.61	0.93	1.06	0.064
2	1.7	2.9	2.05	1.69	1.79	0.030

Actual values: $\omega_1 = 1.00$ $\zeta_1 = 0.050$

$\omega_2 = 1.73$ $\zeta_2 = 0.029$

For the case $c = 0.01$, there is more error in the measured parameters than for the case $c = 0.2$ due to the sharpness of the resonant peak.

$$M1(\omega, c) := \frac{-\omega^2 + c\cdot\omega\cdot j + 2}{\omega^4 - c\cdot\omega^3\cdot j - 4\cdot\omega^2 + 2\cdot c\cdot\omega\cdot j + 3}$$



Problems and Solutions Section 7.5 (7.20-7.24)

7.20 Using the definition of the mobility transfer function of Window 7.4, calculate the Re and Im parts of the frequency response function and hence verify equations (7.15) and (7.16).

Solution:

From Window 7.4:

$$\frac{sX(s)}{F(s)} = \frac{s}{ms^2 + cs + k}$$

$$\alpha(\omega) = \frac{j\omega}{(k - m\omega^2) + j\omega c}$$

$$\alpha(\omega) = \frac{j\omega\left[(k - m\omega^2) - j\omega c\right]}{(k - m\omega^2)^2 + (\omega c)^2}$$

$$\alpha(\omega) = \frac{\omega c^2 + j\omega(k - m\omega^2)}{(k - m\omega^2)^2 + (\omega c)^2}$$

The previous expression can be separated into real and imaginary parts:

$$Re\left[\alpha(\omega)\right] = \frac{\omega^2 c}{(k - \omega^2 m)^2 + (\omega c)^2} \qquad Im\left[\alpha(\omega)\right] = \frac{\omega(k - \omega^2 m)}{(k - \omega^2 m)^2 + (\omega c)^2}$$

7.21 Using equations (7.15) and (7.16), verify that the Nyquist plot of the mobility frequency response function does in fact form a circle.

Solution:

Define
$$A = \frac{\omega^2 c}{(k-\omega^2 m)^2 + (\omega c)^2} - \frac{1}{2c} = \mathrm{Re}(\alpha) - \frac{1}{2c}$$

$$B = \frac{\omega(k - \omega^2 m)}{(k-\omega^2 m)^2 + (\omega c)^2} = \mathrm{Im}(\alpha)$$

Show that

$$A^2 + B^2 = \left(\frac{1}{2c}\right)^2$$

which is a circle of radius $\frac{1}{2c}$ with center at $Re(\alpha) = \frac{1}{2c}$, $Im(\alpha) = 0$.

$$A^2 + B^2 = \left[\frac{\omega^2 c}{(k-\omega^2 m)^2 (\omega c)^2} - \frac{1}{2c}\right]^2 + \left[\frac{\omega(k-\omega^2 m)}{(k-\omega^2 m)^2 + (\omega c)^2}\right]^2$$

$$A^2 + B^2 = \frac{(\omega^2 c)^2}{\left[(k-\omega^2 m)^2 + (\omega c)^2\right]^2} + \frac{\omega^2\ (k-\omega^2 m)^2}{\left[(k-\omega^2 m)^2 + (\omega c)^2\right]^2} - \frac{\omega^2}{(k-\omega^2 m)^2 + (\omega c)^2} + \left(\frac{1}{2c}\right)$$

$$A^2 + B^2 = \frac{\omega^2}{(k-\omega^2 m)^2 + (\omega c)^2}\left[\frac{(k-\omega^2 m)^2 + (\omega c)^2}{(k-\omega^2 m)^2 + (\omega c)^2}\right] - \frac{\omega^2}{(k-\omega^2 m)^2 + (\omega c)^2} + \left(\frac{1}{2c}\right)^2$$

$$A^2 + B^2 = \left(\frac{1}{2c}\right)^2$$

Which is the equation of a circle.

7.22 Consider a single-degree-of-freedom system of mass 10 kg, stiffness 1000 N/m, and damping ratio of 0.01. Pick five values of ω between 0 and 20 rad/s and plot five points of the Nyquist circle using equations (7.15) and (7.16). Do these form a circle?

Solution:

SDOF oscillator:

$$m\ddot{x} + c\dot{x} + kx = 0$$

$$m = 10 \text{ kg} \qquad k = 1000 \text{ N/m} \quad \zeta = 0.01$$

First, calculate the damping constant *c*.

$$\omega_n^2 = \frac{k}{m} = 100$$

$$c = 2\zeta\omega_n m = 2(0.01)(10)(10) = 2 \text{ Ns/m}$$

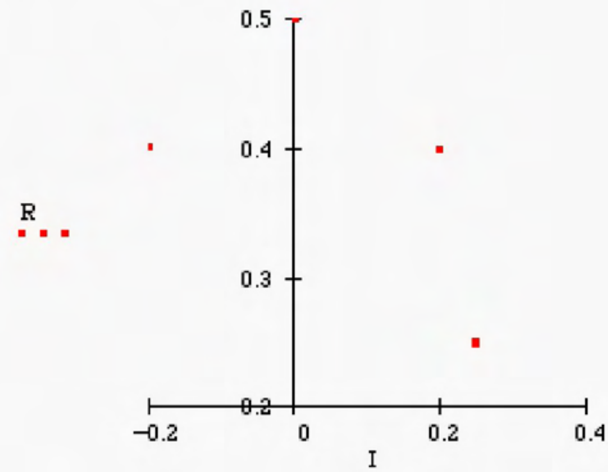
$$Re[\alpha] = \frac{2\omega^2}{(1000 - 10\omega^2)^2 + (2\omega)^2}$$

$$Im[\alpha] = \frac{\omega(1000 - 10\omega^2)}{(1000 - 10\omega^2)^2 + (2\omega)^2}$$

ω	$Re(\alpha)$	$Im(\alpha)$
9.90	0.2487	0.2500
9.95	0.3996	0.2003
10.00	0.5000	0.0000
10.05	0.4004	-0.1997
10.10	0.2512	0.2500

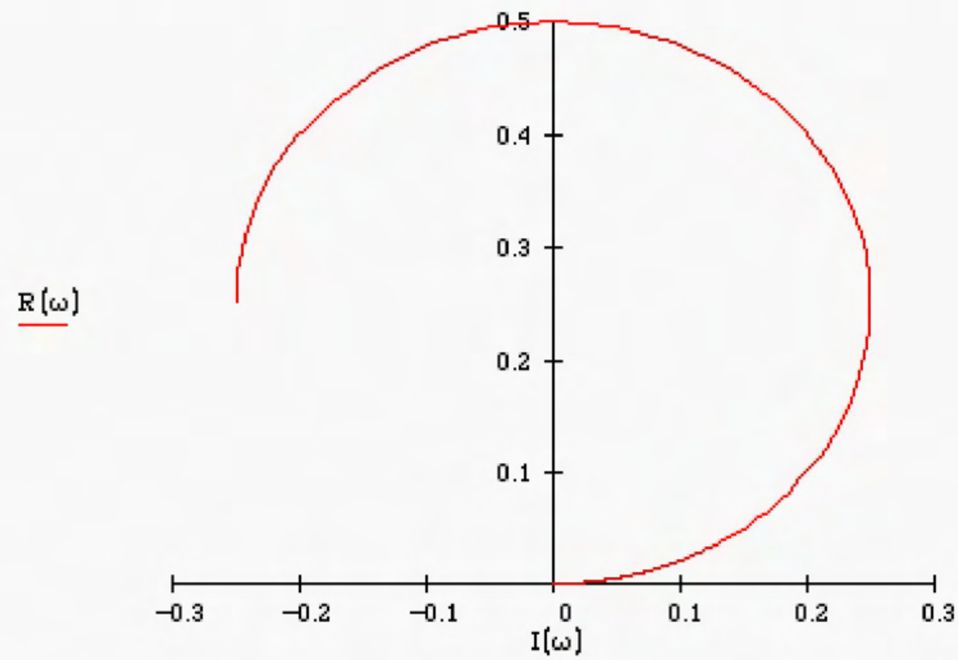
The following plot displays the 5 points listed in the table, as well as the same plot with a fine discretization of the driving frequency ω.

$$R := \begin{bmatrix} 0.2487 \\ 0.3996 \\ 0.5 \\ 0.4004 \\ 0.2512 \end{bmatrix} \qquad I := \begin{bmatrix} 0.25 \\ 0.2003 \\ 0 \\ -0.1997 \\ 0.25 \end{bmatrix}$$



$\omega := 0, 0.01 .. 10.1$

$$R(\omega) := \frac{2 \cdot \omega^2}{\left(1000 - 10 \cdot \omega^2\right)^2 + (2\ \omega)^2} \qquad I(\omega) := \frac{\omega \cdot \left(1000 - 10 \cdot \omega^2\right)}{\left(1000 - 10 \cdot \omega^2\right)^2 + (2\ \omega)^2}$$



7.23 Derive equation (7.20) for the damping ratio from equations (7.18) and (7.19). Then verify that equation (7.20) reduces to equation (7.21) at the half-power points.

Solution: Begin with equations (7.18) and (7.19)

$$\tan\left(\frac{\alpha}{2}\right)=\frac{\left(\frac{\omega_a}{\omega_3}\right)^2-1}{2\zeta_3\frac{\omega_a}{\omega_3}}$$

$$\tan\left(\frac{\alpha}{2}\right)=\frac{\left(\frac{\omega_b}{\omega_3}\right)^2-1}{2\zeta_3\frac{\omega_b}{\omega_3}}$$

Multiplying the right hand side of each expression by $\frac{\omega_3^2}{\omega_3^2}$ yields

$$\tan\left(\frac{\alpha}{2}\right)=\frac{\omega_a^2-\omega_3^2}{2\zeta_3\omega_a\omega_3}$$

$$\tan\left(\frac{\alpha}{2}\right)=\frac{\omega_3^2-\omega_b^2}{2\zeta_3\omega_b\omega_3}$$

After a suitable multiplication, these expressions are:

$$(2\zeta_3\omega_a\omega_3)\tan\left(\frac{\alpha}{2}\right)=\omega_a^2-\omega_3^2$$

$$(2\zeta_3\omega_b\omega_3)\tan\left(\frac{\alpha}{2}\right)=\omega_3^2-\omega_b^2$$

Adding the previous two equations results in:

$$2\zeta_3(\omega_a+\omega_b)\tan\left(\frac{\alpha}{2}\right)=\omega_a^2-\omega_b^2$$

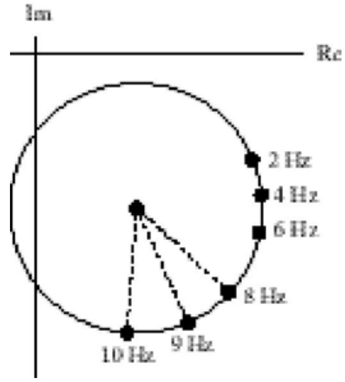
Which can be manipulated to yield equation (7.20)

$$\zeta_3=\frac{\omega_a^2-\omega_b^2}{2\omega_3\left[\omega_a\tan\left(\frac{\alpha}{2}\right)+\omega_b\tan\left(\frac{\alpha}{2}\right)\right]}$$

At the half-power points, α = 90° and $\tan\left(\frac{\alpha}{2}\right)=1$, so (7.20) reduces to:

$$\zeta_3=\frac{\omega_a-\omega_b}{2\omega_3}$$

7.24 Consider the experimental curve fit Nyquist circle of Figure P7.24. Determine the modal damping ratio for this mode



Solution:

From Figure 7.18,

$$\alpha \approx 45°$$
$$\omega_3 = 9\,\text{Hz}$$
$$\omega_b = 10\,\text{Hz}$$
$$\omega_a = 8\,\text{Hz}$$

Using (7.20)

$$\zeta_3 = \frac{10^2 - 8^2}{2(9)\left[8\tan\left(45°/2\right) + 10\tan\left(45°/2\right)\right]}$$

$$\zeta_3 = 0.27$$