Chapter 6

Problems and Solutions Section 6.2 (6.1 through 6.7)

6.1 Prove the orthogonality condition of equation (6.28).

Solution:

Calculate the integrals directly. For $n = n$, let $u = n\pi x/l$ so that $du = (n\pi/l)dx$ and the integral becomes

$$\frac{l}{n\pi}\int_0^{n\pi}\sin^2 u\,du = \frac{l}{n\pi}\left(\frac{1}{2}u - \frac{1}{4}\sin 2u\right)\Bigg|_0^{n\pi}$$

$$= \frac{l}{n\pi}\left(\frac{1}{2}n\pi - \frac{1}{4}\sin 4n\pi\right) - 0 = \frac{l}{2}$$

where the first step used a table of integrals. For $n \neq m$ let $u = \pi x/l$ so that $du = (\pi/l)dx$ and

$$\int_0^l \sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}\,dx = \frac{l}{\pi}\int_0^l \sin mu \sin nu\,du$$

which upon consulting a table of integrals is

$$\frac{l}{\pi}\left\{\frac{\sin(m-n)\pi}{2(m-n)} - \frac{\sin(n+m)\pi}{2(n+m)}\right\} = 0\,.$$

6.2 Calculate the orthogonality of the modes in Example 6.2.3.

Solution:

One needs to show that $\int_0^1 X_n(x)X_m(x)dx = 0$ for $m \neq n$, where $X_m(t) = a_n \sin \sigma_n x$. But each mode $X_n(x)$ must satisfy equation (6.14), i.e.

$$X_n'' = -\sigma_n^2 X_n \qquad (1)$$

Likewise

$$X_m'' = \sigma_m^2 X_m \qquad (2)$$

Multiply (1) by X_m and integrate from 0 to l. Then multiply (2) by $X_n(x)$ and integrate from 0 to l. This yields

$$\int_0^l X_n'' X_m dx = -\sigma_n^2 \int_0^l X_n X_m dx$$
$$\int_0^l X_m'' X_n dx = -\sigma_m^2 \int_0^l X_m X_n dx$$

Subtracting these two equations yields

$$\int_0^l \left(X_n'' X_m - X_m'' X_n\right) dx = \left(\sigma_n^2 - \sigma_m^2\right) \int_0^l X_n(x) X_m(x) dx$$

Integrate by parts on the left side to get

$$\int_0^l X_n' X_m' dx - \int_0^l X_m' X_n' dx + X_n' X_m \Big|_0^l - X_m X_n' \Big|_0^l$$
$$= X_m(l) k X_n(l) - X_n(l) k X_m(l) = 0$$

from the boundary condition given by eq. (6.50). Thus

$$\left(\sigma_n^2 - \sigma_m^2\right) \int_0^l X_n X_m dx = 0.$$

But from fig. 6.4, $\sigma_n \neq \sigma_m$ for $m \neq n$ so that

$$\int_0^l X_n X_m dx = a_n^2 \int_0^l \sin \sigma_n x \sin \sigma_m x dx = 0$$

and the modes are orthogonal.

6.3. Plot the first four modes of Example 6.2.3, for the case l = 1 m, k = 800 N/m and τ = 800 N/m.

Solution:

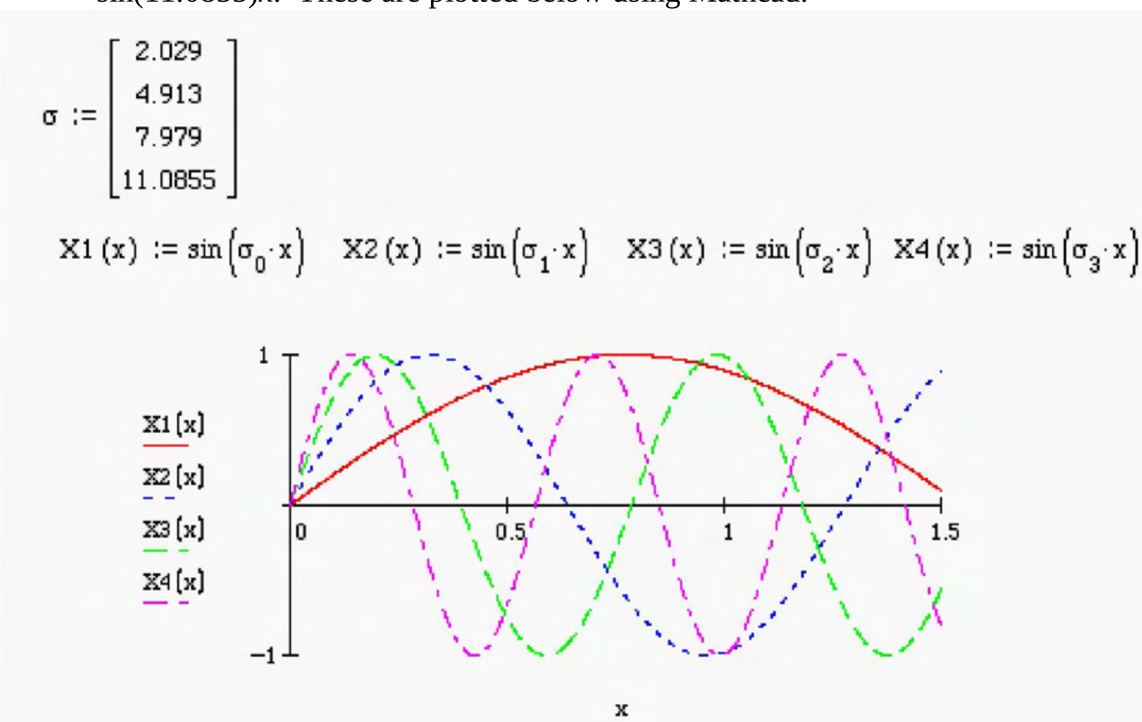
The mode shapes are given as $\sin\sigma_n x$ where σ_n satisfies eq. (6.51). To solve this numerically values of l, k and τ must be given. For example chose l = 1 m, k = 800 N/m, and τ = 800 N/m the equation (6.51) becomes

$$\tan\sigma = -\sigma$$

Solving using MATLAB for the first 4 values yields

$$\sigma_1 = 2.029,\ \sigma_2 = 4.913,\ \sigma_3 = 7.979,\ \sigma_4 = 11.0855$$

So that the mode shapes are sin(2.029)x, sin(4.913)x, sin(7.979)x and sin(11.0855)x. These are plotted below using Mathcad.



6.4 Consider a cable that has one end fixed and one end free. The free end cannot support a transverse force, so that $w_x(l,t) = 0$. Calculate the natural frequencies and mode shapes.

Solution:

The cable equation results in (6.17). The boundary conditions are

$$w(x,t) = X(x)T(t) = 0 \text{ at } x = 0 \text{ (fixed end)}$$

so that $X(0) = 0$ and

$$w_x(x,t) = X'(x)T(t) = 0 \text{ at } x = l \text{ (free end)}$$

so that $X(l) = 0$. Applying these to equation (6.17) yields

$$0 = a_1 \sin(0) + a_2 \cos(0) \text{ so that } a_2 = 0$$
$$0 = a_1 \sigma \cos \sigma(l)$$

so that cos $\sigma l = 0$ or $\sigma l = n$ for odd n and the natural frequency $\sigma_n = \frac{n\pi}{2l}$, $n = 1, 3, 5\ldots$ or $\sigma_n = \frac{2n-1\pi}{2l}$, $n = 1, 2, 3\ldots$Since $a_2 = 0$, and a_1 is arbitrary the mode shapes are

$$a_n \sin\left(\frac{(2n-1)\pi x}{2l}\right), \quad n = 1,2,3...$$

the natural frequencies are from (6.15) and (6.24):

$$\omega_n = \sqrt{\sigma_n^2 c^2} = c\sigma_n = \frac{(2n-1)\pi c}{2l} = \frac{(2n-1)\pi}{2l}\sqrt{\tau / \rho}$$

6.5 Calculate the coefficients c_n and d_n of equation (6.27) for the system of a clamped-clamped string to the initial displacement given in Figure P6.5 and an initial velocity of $w_t(x,0) = 0$.

Solution:

For the clamped-clamped string the solution is given by eq. (6.27) as

$$w(x,t) = \sum_{i=1}^{\infty}(c_n \sin\sigma_n x \sin\sigma_n ct + d_n \sin\sigma_n x \cos\sigma_n ct)$$

Series $w_t(x,0) = 0$, equation (6.33) yields that $c_n = 0$ for all n. The coefficients d_n are given by eq. (6.31) as

$$d_n = \frac{2}{l}\int_0^l \omega_0(x)\sin\frac{m\pi x}{l}dx \quad m = 1,2,...$$

From fig. 6.16 $\omega_0(x) = \begin{cases} 2x/l & 0 \le x \le l/2 \\ 2(l-x)/l & l/2 \le x \le l \end{cases}$ cm. Calculation yields

$$d_n = \frac{2}{l}\left\{\int_0^{l/2}\frac{2x}{l}\sin\frac{n\pi x}{l}dx + \int_{l/2}^{l}\frac{2}{l}(l-x)\sin\frac{n\pi x}{l}dx\right\}$$
$$= \frac{8}{\pi^2 n^2}\sin\frac{n\pi}{2} \qquad n = 1,3,5...$$

and d_n is zero for even values of n.

6.6 Plot the response of the string in Problem 6.5 for the piano string of Example 6.2.2 (l = 1.4 m, m = 110 g, τ =11.1x10^4 N) at $x = l/4$ and $x = l/2$, using 3, 5, and 10 terms in the solution.

Solution:

For the piano string of example 6.22, l = 1.4m and c = 11.89. From problem 6.5 the solution has the form

$$w(x,t) = \frac{8}{\pi^2}\left\{\sum_{m,\text{odd}=1}^{\infty} \frac{1}{m^2}\sin\frac{m\pi}{2}\sin\frac{m\pi x}{l}\cos\frac{m\pi c}{l}t\right\}$$

For 3 terms at $x = l/4 = 3.5$, this series becomes

$$w_3(3.5,t) = 0.81\left\{0.24\cos 26.68t + 0.07858\cos 80.04t - 0.02828\cos 133.40t\right\}$$

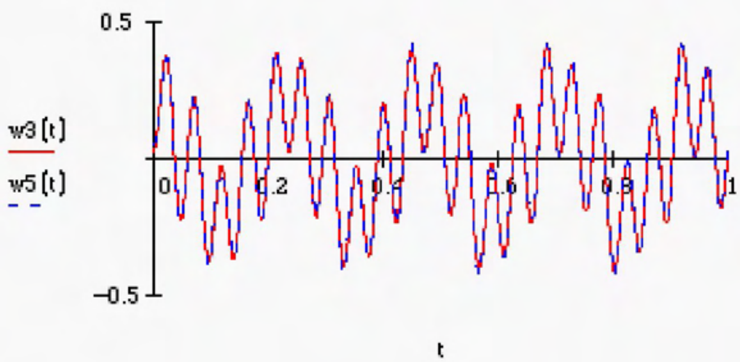
for 5 terms this becomes

$$w_5(3.5,t) = w_3 + 0.01442\cos 182t + 0.00873\cos 240.13t$$

The next terms have coefficients 0.00584, 0.00418, 0.00314, 0.00244 and 0.00195 respectively. Any of the codes can be used to easily plot these. Plot of w_3 and w_5 at $l/4$ are given below in Mathcad:

$$w3(t) := 0.81\cdot(0.24\cos(26.68\cdot t) + 0.07858\cdot\cos(80.84\cdot t) - 0.2828\cdot\cos(133.4\cdot t))$$

$$w5(t) := w3(t) + 0.01441\cdot\cos(182\cdot t) + 0.00873\cdot\cos(240.13\cdot t)$$



6.7 Consider the clamped string of Problem 6.5. Calculate the response of the string to the initial condition

$$w(x,0) = \sin\frac{3\pi x}{l} \quad w_t(x,0) = 0$$

Plot the response at $x = l/2$ and $x = l/4$, for the parameters of Example 6.2.2.

Solution:

Since $w_t = 0$ each if the coefficients c_n is zero in equation (6.33). Thus the solution is of the form

$$w(x,t) = \sum_{i=1}^{\infty} d_n \sin\frac{n\pi x}{l}\cos\frac{n\pi c}{l}t$$

as given in problem 6.5. Equation (6.31) for the initial position yields

$$d_n = \frac{2}{l}\int_0^l \sin\frac{3\pi x}{l}\sin\frac{m\pi x}{l}dx \quad m = 1,2,...$$

Because of the orthogonality all the $d_n = 0$ except d_3 and from the above integral $d_3 = 1$. Hence the solution collapses to the single term

$$w(x,t) = \sin\frac{3\pi x}{l}\sin\frac{3\pi c}{l}t$$

At $x = l/2$ this becomes

$$w\left(\frac{l}{2},t\right) = \sin\frac{3\pi}{2}\cos\frac{3\pi c}{l}t = -\cos\frac{3\pi c}{l}t$$

At $x = l/4$

$$w\left(\frac{l}{4},t\right) = \sin\frac{3\pi}{4}\cos\frac{3\pi c}{l}t = 0.707\cos\frac{3\pi c}{l}t$$

Using the values for the piano string ($l = 1.4$, $c = 1188$ m/s) $w(l/4,t)$ is simply a cosine of frequency 8000 rad/s and amplitude 0.707.

Problems and Solutions Section 6.3 (6.8 through 6.29)

6.8 Calculate the natural frequencies and mode shapes for a free-free bar. Calculate the temporal solution of the first mode.

Solution:

Following example 6.31 (with different B.C.'s), the spatial response of the bar will be

$$X(x) = a\sin\sigma x + b\cos\sigma x$$

The boundary conditions are $X'(0) = X'(l) = 0$. The expression for X' is $X'(x) = \sigma a\cos\sigma x - \sigma b\sin\sigma x$ so at 0:

$$0 = \sigma a \Rightarrow a = 0$$

at l

$$0 = -\sigma b\sin\sigma l, \quad b \neq 0$$

so that $\sigma l = n\pi$ or $\sigma = n\pi/l$ where n starts a *zero*. Hence the mode shapes are of the form

$$X_n(x) = b_n\cos\frac{n\pi x}{l} \text{ for } n = 1, 2, 3, \ldots \text{ and for } n = 0,$$

$$X_0(x) = b_0\cos\left(\frac{0\pi}{l}x\right) = b_0 \text{ a constant.}$$

The temporal solution is given by eq. (6.15) to be

$$\frac{\ddot{T}_n(t)}{c^2T_n(t)} = -\sigma^2$$

so that the temporal solution of the first mode:

$$\ddot{T}_0(t) + 0c^2T_0(t) = 0 = \ddot{T}_0(t) \quad \underline{T_0(t) = b + ct}$$

6.9 Calculate the natural frequencies and mode shapes of a clamped-clamped bar.

Solution: The calculation of the natural frequencies and mode shapes of a clamped-clamped bar is identical to that of the fixed-fixed string since the equations of motion are mathematically the same. The solution of this problem is thus given at the beginning of section 6.2, but is repeated here: Applying separation of variable to eq. (6.56) yields that the spatial variable must satisfy eq. (6.59) of example 6.3.1, i.e., $X(x) = a\sin\sigma x + b\cos\sigma x$ where a and b are constants to be determined. The clamped boundary conditions require that $X(0) = X(l) =$ or

$0 = b$ or $X = a\sin\sigma x$

$0 = a\sin\sigma l$ or $\sigma = n\pi/l$

Hence the mode shapes will be of the form

$X_n = a_n \sin\sigma_n x$

Where $\sigma_\nu = n\pi/l$. The frequencies are determined from the temporal solution and become

$$\omega_n = \sigma_n c = \frac{n\pi}{l}\sqrt{\frac{E}{\rho}}, \quad n = 1, 2, 3, \ldots$$

6.10 It is desired to design a 4.5 m, clamped-free bar such that the first natural frequency is 1878 Hz. Of what material should it be made?

Solution: First change the frequency into radians:

1878 Hz =1878x2π rad/s=11800 rad/s

The first natural frequency is given computed in Example 6.3.1, Equation (6.63) as

$$\omega_1 = \frac{2\pi}{l}\sqrt{\frac{E}{\rho}} \Rightarrow \frac{E}{\rho} = \omega_1^2 \frac{4l^2}{\pi^2} = (11800)^2 \frac{4l^2}{\pi^2}$$

$$\Rightarrow \frac{E}{\rho} = 7.143 \times 10^7$$

in Nm/kg. Examining the ratios from Table 2.1 for the values given yields that for Steel:

$$\frac{E}{\rho} = \frac{2 \times 10^{11}}{2.8 \times 10^3} = 7.143 \times 10^7 \text{ Nm/kg}$$

Thus a **steel** bar with a length 4.5 meters will have a first natural frequency of 1878 Hz. This is something like a truck chassis.

6.11 Compare the natural frequencies of a clamped-free 1-m aluminum bar to that of a 1-m bar made of steel, a carbon composite, and a piece of wood.

Solution:

For a clamped-free bar the natural frequencies are given by eq. (6.6.3) as

$$\omega_n = \frac{(2n-1)\pi}{2l}\sqrt{\frac{E}{\rho}}$$

Referring to values of r and E from table 1.2 yields (for ω_1):
Steel

$$\frac{\pi}{(2)(1)}\sqrt{\frac{2.0\times10^{11}}{7.8\times10^{3}}} = 7{,}954 \text{ rad/s (1266Hz)}$$

Aluminum

$$\frac{\pi}{(2)(1)}\sqrt{\frac{7.1\times10^{10}}{2.7\times10^{3}}} = 8{,}055 \text{ rad/s (1282 Hz)}$$

Wood

$$\frac{\pi}{(2)(1)}\sqrt{\frac{5.4\times10^{9}}{6.0\times10^{2}}} = 4{,}712 \text{ rad/s (750 Hz)}$$

Carbon composite (student must hunt for E/ρ and guess a little) from Vinson and Sierakowski's book on composites $\sqrt{E/\rho}$ = 3118 and

$$\frac{\pi}{2}(3118) = 4897 \text{ rad/s (780 Hz)}$$

6.12 Derive the boundary conditions for a clamped-free bar with a solid lumped mass, of mass M attached to free end.

Solution: At the clamped end, $x = 0$, the boundary condition is $w(0,t) = 0$ or $X(x) = 0$. At the end $x = l$ the tensile force in the bar must be equal to the inertia force of the attached mass. For an attached mass of value M, this becomes

$$EA\frac{\partial w(x,t)}{\partial x}\bigg|_{x=l} = -M\frac{\partial^2 w(x,t)}{\partial t^2}\bigg|_{x=l}$$

6.13 Calculate the mode shapes and natural frequencies of the bar of Problem 6.12. State how the lumped mass affects the natural frequencies and the mode shapes.

Solution: Via separation of variables [i.e., $w(x,t) = X(x)T(t)$], the spatial equation becomes (following example 6.3.1 for instance)

$$X(x) = a\sin\sigma x + b\cos\sigma x$$

Applying the boundary condition at $x = 0$ yields

$$X(0) = 0 = a\sin(0) + b\cos(0) \Rightarrow b = 0\; 0 = b$$

so the spatial solution reduces to $X(x) = a\sin\sigma x$. Now the second boundary condition (see 6.12) involves time deviates so that $w(x,t) = X(x)T(t)$ substituted into the boundary condition $EAW_x = -Mw_{tt}(l,t)$ becomes:

$$EAX'(l)T(t) = -MX(l)\ddot{T}(t) \Rightarrow \frac{EAX'(l)}{MX(l)} = -\frac{\ddot{T}(t)}{T(t)}$$

From equation (6.15) $\ddot{T}/T = -\sigma^2c^2$, so this boundary condition becomes

$$\frac{EA}{M}\cdot\frac{X'(l)}{X(l)} = \sigma^2c^2 \qquad (1)$$

Substitution of $X(x) = a\sin\sigma x$ and $X'(x) = a\sigma\cos\sigma x$ into (1) yields

$$\frac{EA}{M}\frac{a\sigma\cos\sigma l}{a\sin\sigma l} = \sigma^2c^2$$

or

$$\cot\sigma l = \frac{\sigma c^2 M}{EA}$$

which describes multiple values of $\sigma = \sigma_n$, $n = 1, 2, 3,\ldots$ The frequency of oscillation is related to σ_n by $\omega_n = \sigma_n c$, where $c = \sqrt{E/\rho}$. Let $\rho Al = m$ be the mass of the beam and rewrite $\cot(\sigma l)$ as

$$\cot\sigma l = \cot\left(\frac{\omega_n l}{c}\right) = \frac{\sigma(E/\rho)M}{EA} = \frac{(\omega_n l/c)}{A\rho l}\cdot M = \frac{\omega_n l}{c}\frac{M}{m}.$$

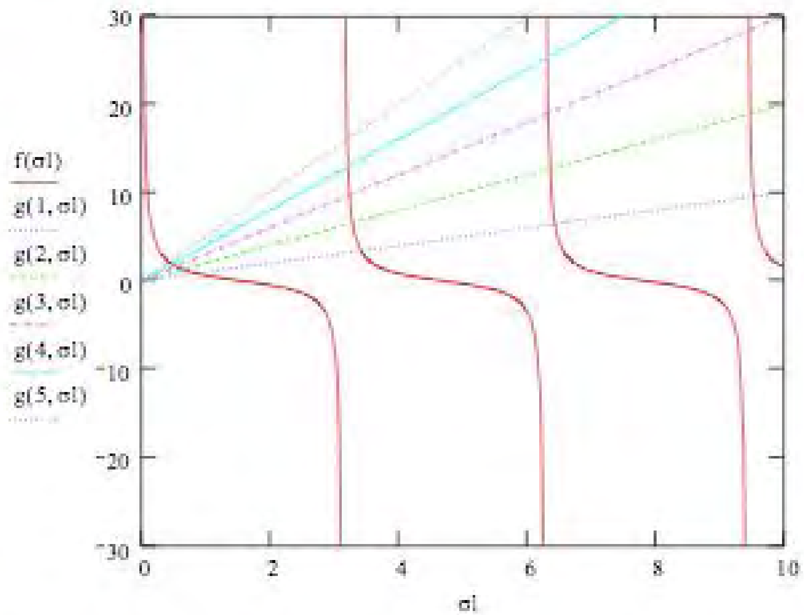
This can be rewritten as

$$\alpha\cot\alpha = \beta$$

where $\beta = m/M$ and $\alpha = \omega_n l/c$. As the mass ratio β increases (tip mass increases) the frequency increases. The mode shapes are proportional to $\sin\sigma_n x$, where σ_n is calculated numerically from $\cot(\sigma l) = (M/m)\sigma l$, similar to the calculation showing in Figure 6.4. This is illustrated in the following Mathcad session.

$f(\sigma l) := \cot(\sigma l) \qquad g(\beta, \sigma l) := \beta \cdot \sigma l$

Here, β is the end mass - to - bar mass ratio (β=M/ρAL). Hence the transcendental equation f(σl)=g(σl) is totally nondimensional which makes it possible to study the effect of end mass - to - bar mass ratio in a nondimesional basis. That is what you see in the following figure where 5 different end mass - to - bar mass ratios are investigated.



For each value of β (from 1 to 5), the intersection of f and g gives the frequency parameter σL which is directly proportional to the natural frequency of the mode of interest. Therefore, it is obvious from the figure that **increasing the end mass reduces the natural frequencies of the bar** (and looks like the first natural frequency is the most sensitive one). This makes perfect physical sense: if you add a large end mass, the structure becomes much more flexible.

The limiting behavoirs are β=0 and β=inf.. The former case (β=0) gives the natural frequencies of a clamped-free bar without tip mass (from the roots of cot(σl)=0 or just cos(σl)=0 if you like) whereas the latter case (β=inf.) gives the natural frequencies of a clamped-clamped bar (from the roots of sin(σl)=0) which is also evident from the above graph.

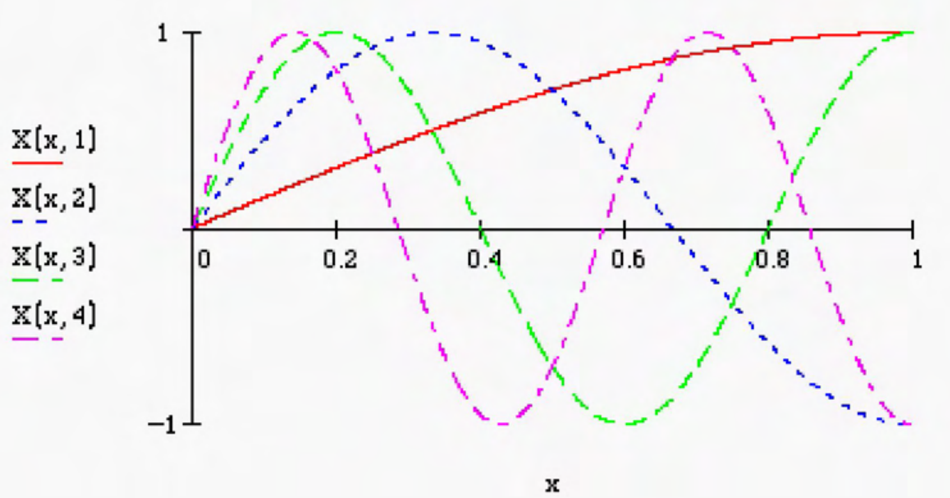
6.14 Calculate and plot the first three mode shapes of a clamed-free bar.

Solution: The second entry of Table 6.1 yields the solution

$$X_n(x) = \sin\frac{(2n-1)}{2\ell}\pi x$$

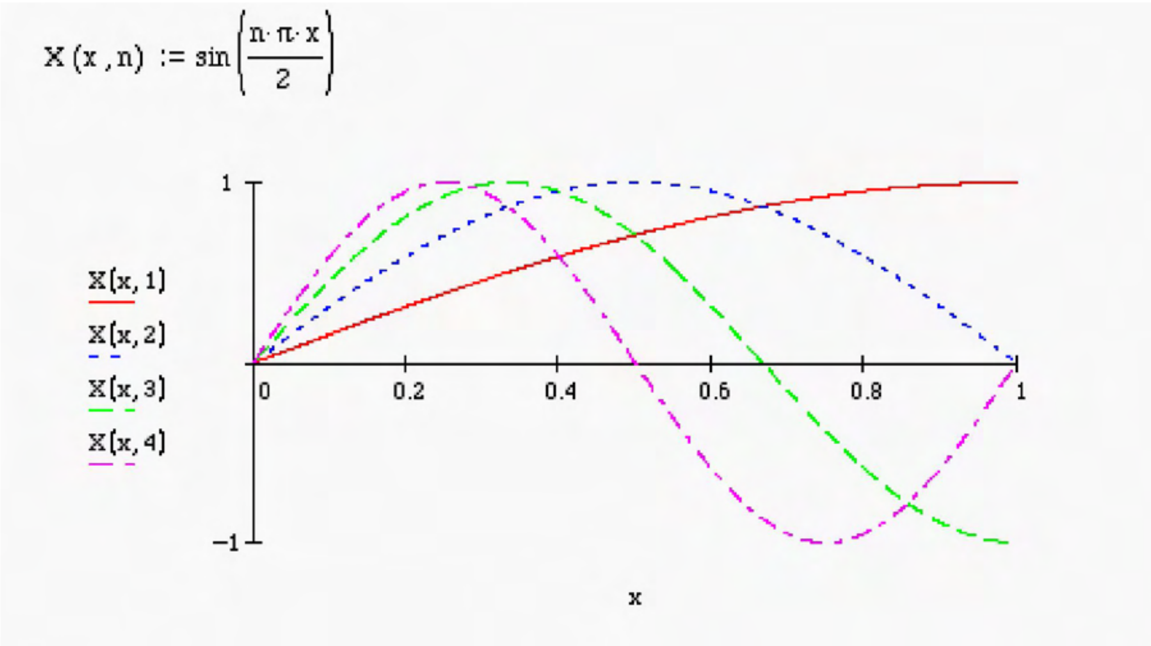
which is calculated following the procedures out lined in Example 6.3.1. The plot is given in Mathcad for the case ℓ = 1m.

$$X(x,n) := \sin\left[\frac{\left[(2n-1)\cdot\pi\cdot x\right]}{2}\right]$$



6.15 Calculate and plot the first three mode shapes of a clamed-clamped bar and compare them to the plots of Problem 6.14.

Solution: As in problem 6.14 the solution is given in table 6.1. The important item here is to notice the difference between mode shapes from the plots of $\sin\frac{(2n-1)\pi}{2l}x$ and sin $(n\pi x/l)$. In particular notice the difference at the free end.



6.16 Calculate and compare the eigenvalues of the free-free, clamped-free, and the clamped-clamed bar. Are the related? What does this state about the system's natural frequencies?

Solution:

Students can calculate these or just use the results listed in table 6.1. Note for l = 1

free-free 0, πc, $2\pi c$…

clamped-free $\frac{\pi c}{2}, \frac{3\pi c}{2}, \frac{5\pi c}{2}...$

clamped-clamped πc, $2\pi c$, $3\pi c$…

so that the free-free and clamped-clamped values are a π shift from one another with the clamped-free values falling in between: as the number of constraints increases, the frequency increases.

6.17 Consider the nonuniform bar of Figure P6.17, which changes cross-sectional area as indicated in the figure. In the figure A_1, E_1, ρ_1, and l_1 are the cross-sectional area, modulus, density and length of the first segment, respectively, and A_2, E_2, ρ_2, and l_2 are the corresponding physical parameters of the second segment. Determine the characteristic equation.

Solution: Let the subscript 1 denote the first part of the beam and 2 the second part of the beam. The bar equation must be satisfied in each part so that equation of motion is in two parts:

$$E_1 \frac{\partial^2 w_1(x,t)}{\partial x^2} = \rho_1 \frac{\partial^2 w_1(x,t)}{\partial t^2} \quad 0 < x < \ell_1$$

$$E_2 \frac{\partial^2 w_2(x,t)}{\partial x^2} = \rho_2 \frac{\partial^2 w_2(x,t)}{\partial t^2} \quad \ell_1 < x < \ell_1 + \ell_2 = \ell$$

The boundary conditions are the two from the clamped-free configuration then there are two more conditions expressing force and displacement continuity at the point where the two beams join ($x = \ell_1$). Follow the procedure of separation of variables but this time keep the constant c in the spatial equation so that we may write: $w_1(x,t) = X_1(x)T(t)$ and $w_2(x,t) = X_2(x)T(t)$ where the function of time is common to both beams. Then denoting σ^2 as the separation constant and substituting the separated forms into the equation of motion yields:

$$\frac{c_1^2 X_1''(x)}{X_1(x)} = \frac{\ddot{T}(t)}{T(t)} = -\sigma^2 \quad 0 < x < \ell_1 \text{ and } c_1 = \sqrt{\frac{E_1}{\rho_1}} \quad (1)$$

$$\frac{c_2^2 X_2''(x)}{X_2(x)} = \frac{\ddot{T}(t)}{T(t)} = -\sigma^2 \quad \ell_1 < x < \ell \text{ and } c_2 = \sqrt{\frac{E_2}{\rho_2}} \quad (2)$$

In this way the temporal equation for both parts is the same (σ does not depend on which part of the beam and will show up in the characteristic equation). Solving the two spatial equations yields:

$$(1) \Rightarrow X_1 = a_1 \sin\frac{\sigma}{c_1}x + a_2 \cos\frac{\sigma}{c_1}x \quad 0 < x < \ell_1$$

$$(2) \Rightarrow X_2 = a_3 \sin\frac{\sigma}{c_2}x + a_4 \cos\frac{\sigma}{c_2}x \quad \ell_1 < x < \ell$$

There are now 4 boundary conditions (one at each end and two in the middle) which will yield 4 equations in the 4 coefficients a_i. This set of equations must be singular yielding the characteristic equation for σ.

From the clamped end:

$$X_1(0) = 0 \Rightarrow a_1 \sin(0) + a_2 \cos(0) = 0 \quad (3)$$

From the free end:

$$X_2'(\ell) = 0 \Rightarrow \frac{\sigma}{c_2} a_3 \cos\frac{\sigma\ell}{c_2} - \frac{\sigma}{c_2} a_4 \sin\frac{\sigma\ell}{c_2} = 0 \quad (4)$$

From the middle and enforcing displacement continuity at $x = \ell_1$:

$$a_1 \sin\frac{\sigma}{c_1}\ell_1 + a_2 \cos\frac{\sigma}{c_1}\ell_1 = a_3 \sin\frac{\sigma}{c_2}\ell_1 + a_4 \cos\frac{\sigma}{c_2}\ell_1 \quad (5)$$

From the middle and enforcing force, equation (6.54) continuity at $x = \ell_1$:

$$E_1A_1X_1'(\ell_1) = E_2A_2X'(\ell_1)$$

$$\Rightarrow E_1A_1\frac{\sigma}{c_1}(a_1\cos\frac{\sigma\ell_1}{c_1} - a_2\sin\frac{\sigma\ell_1}{c_1}) = E_2A_2\frac{\sigma}{c_2}(a_3\cos\frac{\sigma\ell_1}{c_2} - \frac{\sigma}{c_2}a_4\sin\frac{\sigma\ell_1}{c_2}) \qquad (6)$$

Equations (3) through (6) are 4 equations in the 4 unknowns a_i. Writing these in matrix form as a homogeneous algebraic equation yields:

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\frac{\sigma\ell}{c_2} & -\sin\frac{\sigma\ell}{c_2} \\ \sin\frac{\sigma}{c_1}\ell_1 & \cos\frac{\sigma}{c_1}\ell_1 & -\sin\frac{\sigma}{c_2}\ell_1 & -\cos\frac{\sigma}{c_2}\ell_1 \\ \frac{E_1A_1}{c_1}\cos\frac{\sigma\ell_1}{c_1} & -\frac{E_1A_1}{c_1}\sin\frac{\sigma\ell_1}{c_1} & -\frac{E_2A_2}{c_2}\cos\frac{\sigma\ell_1}{c_2} & \frac{E_2A_2}{c_2}\sin\frac{\sigma\ell_1}{c_2} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

In order for the vector **a** to be nonzero, the determinant of the matrix coefficient must be zero (recall chapter 4). This yields the characteristic equation (computed using Mathcad):

$$E_2A_2c_1\sin\frac{\sigma\ell_1}{c_1}\left[\sin\frac{\sigma\ell}{c_2}\cos\frac{\sigma\ell_1}{c_2} - \sin\frac{\sigma\ell_1}{c_2}\cos\frac{\sigma\ell}{c_2}\right]$$

$$= E_1A_1c_2\cos\frac{\sigma\ell_1}{c_1}\left[\sin\frac{\sigma\ell_1}{c_2}\sin\frac{\sigma\ell}{c_2} + \cos\frac{\sigma\ell_1}{c_2}\cos\frac{\sigma\ell}{c_2}\right] \qquad (7)$$

$$\Rightarrow$$

$$\frac{E_2A_2c_1}{E_1A_1c_2}\tan\frac{\sigma\ell_1}{c_1}\left[\sin\frac{\sigma\ell}{c_2}\cos\frac{\sigma\ell_1}{c_2} - \cos\frac{\sigma\ell}{c_2}\sin\frac{\sigma\ell_1}{c_2}\right] = \sin\frac{\sigma\ell}{c_2}\sin\frac{\sigma\ell_1}{c_2} + \cos\frac{\sigma\ell}{c_2}\cos\frac{\sigma\ell_1}{c_2} \qquad (8)$$

Further simplifying yields

$$\frac{E_2A_2c_1}{E_1A_1c_2}\tan\frac{\sigma\ell_1}{c_1}\sin\frac{\sigma(\ell - \ell_1)}{c_2} = -\cos\frac{\sigma(\ell - \ell_1)}{c_2}$$

$$\Rightarrow \frac{E_2A_2c_1}{E_1A_1c_2}\tan\frac{\sigma\ell_1}{c_1}\tan\frac{\sigma(\ell - \ell_1)}{c_2} = -1$$

Given the parameter values, equation (9) must be solved numerically for σ, yielding the natural frequencies.

6.18 Show that the solution obtained to Problem 6.17 is consistent with that of a uniform bar.

Solution:

If the bar is the same, then $E_1 = E_2 = E$, $\rho_1 = \rho_2 = \rho$ etc. and the characteristic equation from (1) in the solution to Problem 6.17 becomes ($l = l_1$)

$$\sin\frac{\sigma\ell}{c}\left[\sin\frac{\sigma\ell}{c}\cos\frac{\sigma\ell}{c} - \sin\frac{\sigma\ell}{c}\cos\frac{\sigma\ell}{c}\right] = \cos\frac{\sigma\ell}{c}\left[\sin\frac{\sigma\ell}{c}\sin\frac{\sigma\ell}{c} + \cos\frac{\sigma\ell}{c}\cos\frac{\sigma\ell}{c}\right]$$

$$\Rightarrow \sin\frac{\sigma\ell}{c}(0) = \cos\frac{\sigma\ell}{c}\left[\sin^2\frac{\sigma\ell}{c} + \cos^2\frac{\sigma\ell}{c}\right]$$

$$\Rightarrow 0 = \cos\frac{\sigma\ell}{c}(1) \Rightarrow \frac{\sigma\ell}{c} = \frac{2n-1}{2}\pi$$

so that $\sigma_n = \omega_n = \dfrac{(2n-1)\pi}{2l}\sqrt{\dfrac{E}{\rho}}$ which according to table 6.1 entry 2 is the frequency of a clamped-free bar of length l .

6.19 Calculate the first three natural frequencies for the cable and spring system of Example 6.2.3 for $l = 1$, $k = 100$, $\tau = 100$ (SI units).

Solution:

For $l = 1$, $k = 100$ and $\tau = 100$ the frequency equation (6.51) becomes

$$\tan\sigma = -\sigma$$

Using MATLAB the first 3 solutions are

$\sigma_1 = 0$, $\sigma_2 = 2.029$, $\sigma_3 = 4.913$. But zero is not allowed because of the boundary conditions.

6.20 Calculate the first three natural frequencies of a clamped-free cable with a mass of value *m* attached to the free end. Compare these to the frequencies obtained in Problem 6.17.

Solution:

Recall example 6.1.1. The force balance at the boundary $x = l$ yields

$$\tau w_x(x,t)\big|_{x=l} = -m w_{tt}(l,t)$$

The boundary condition at $x = 3$ remains $w(0,t) = 0$. The equation of motion is (6.8) or

$$c^2 w_{xx}(x,t) = w_{tt}(x,t)$$

Again, separation of variable $w(x,t) = X(x)T(t)$ yields eq. (6.12) or

$$\frac{X''(x)}{X(x)} = \frac{\ddot{T}(t)}{c^2 T(t)} = -\sigma^2$$

The spatial equation is

$$X'' + \sigma^2 X(x) = 0$$

which has solution $X(x) = a_1 \sin \sigma x + a_2 \cos \sigma x$. Applying the boundary conditions yields $X(0) = 0$ or $a_2 = 0$. Substitution of $X(x) = a_1 \sin \sigma_2 x$ into the boundary condition at $x = l$ yields

$$[a_1 \tau\sigma \cos \sigma t]T(t) = -m\ddot{T}(t) a_1 \sin \sigma l$$

But $\ddot{T}(t)/T(t) = \sigma^2 c^2$ so this becomes

$$\tau\sigma \cos\sigma l = m\sigma^2 c^2$$

or that

$$\tan \sigma l = \frac{\tau}{m\sigma c^2} \quad (\text{or } \cot \sigma l = \frac{n\sigma}{\rho})$$

is the characteristic equation (see also table 6.1) with mode shape $\sin \sigma_n x$. A plot of their characteristic equation $\cos(\sigma l) = \frac{mc^2}{lr}\sigma l = \frac{m}{lp}(\sigma l)$ yields the value of the frequencies relative to those of problem 6.16.

6.21 Calculate the boundary conditions of a bar fixed at $x = 0$ and connected to ground through a mass and a spring as illustrated in Figure P6.21.

Solution:

A free body diagram of the boundary is shown in Figure 1.

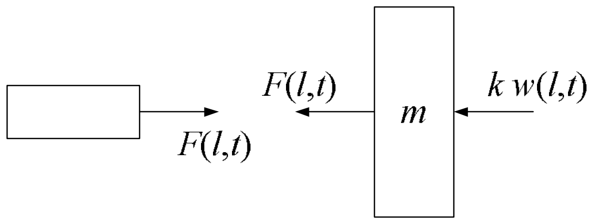


Figure 1

Consider first the end of the rod, the force is related to the axial extension of the rod though

$$F(l,t) = \sigma A\big|_{x=l} = EA\frac{\partial w(x,t)}{\partial x}\bigg|_{x=l}$$

On the other hand, applying Newton's second law to the mass yields

$$-F(l,t) - kw(x,t)\big|_{x=l} = m\frac{\partial w(x,t)}{\partial t^2}\bigg|_{x=l}$$

Hence, this yields the following boundary condition

$$m\frac{\partial w(x,t)}{\partial t^2}\bigg|_{x=l} = -EA\frac{\partial w(x,t)}{\partial x}\bigg|_{x=l} - kw(x,t)\big|_{x=l}$$

6.22 Calculate the natural frequency equation for the system of Problem 6.21.

Solution:

The boundary condition at $x = 0$ is just $w(x,t)|_{x=0} = 0$. Again from separation of variables

$$\ddot{T}(t)/T(t) = -c^2\sigma^2, \quad X(x) = a\sin\sigma x + b\cos\sigma x$$

Applying the boundary condition at 0 yields $X(0) = 0 = b$, so the spatial solution will be of the form $X(x) = a \sin \sigma x$. Substitution of the separated form $w(x,t) = X(x)T(t)$ into the boundary condition at l yields (from problem 6.21)

$$mX(l)\ddot{T}(t) = -kX(l)T(t) - EAX'(l)T(t)$$

Dividing by $T(t)$, and substitution of $\ddot{T}/T = -\sigma^2 c^2$ and $X = a \sin \sigma l$ yields

$$-EA\sigma \cos \sigma l = (-m\sigma^2 c^2 + k)\sin \sigma l \quad \text{or} \quad \tan \sigma l = -\frac{EA\sigma}{k - m\sigma^2 c^2}$$ is the frequency or characteristic equation. Note that this reduces to the values given in Table 6.1 for the special case $m = 0$ and for the case $k = 0$.

6.23 Estimate the natural frequencies of an automobile frame for vibration in its longitudinal direction (i.e., along the length of the car) by modeling the frame as a (one-dimensional) steel bar.

Solution:

Note: The fundamental frequency of an automobile is of primary importance in assuming the quality of an automobile. While an automobile certainly has numerous modes, its fundamental frequency apparently has a large correlation with the occupants perception of quality. The fundamental frequency of a Mercedes 300 series is 25 Hz. Infinity and Lexus have frequencies in the low twenties. This problem has no straightforward answer. Students should think about their own cars or that of their family. For steel $\rho = 7.8 \times 10^3$ kg/m^2, $E = 2.0 \times 10^{11}$ N/m. For a Ford Taurus $l = 4.5$ m and assume the width to be 1 meter. The frequency equation in Hertz of a free-free beam is (excluding the rigid body mode)

$$f_n = \frac{n}{2\pi}\frac{\pi}{l}\sqrt{\frac{E}{\rho}} = 562 \text{ Hz, } 1125\text{Hz}\ldots$$

where $n = 1,2,\ldots$ The frequency measured by auto engineers is from a 3 dimensional finite element model and modal test data. The frequency most felt is probably a transverse frequency.

6.24 Consider the first natural frequency of the bar of Problem 6.21 with $k = 0$ and Table 6.2, which is fixed at one end and has a lumped-mass, M, attached at the free end. Compare this to the natural frequency of the same system modeled as a single-degree-of-freedom spring-mass system given in Figure 1.21. What happens to the comparison as M becomes small and goes to zero?

Solution:

From figure 1.21, $k = EA/l$ is the stiffness of a cantilevered bar. Hence the frequency is

$$\omega_n = \sqrt{k/m} = \sqrt{\frac{EA}{lm}}$$

for the bar with tip mass m modeled as a single degree of freedom system. Now consider the first natural frequency of the distributed mass model of the same structure given in the last entry of table 6.1.

$$\omega_1 = \frac{\lambda_1 c}{l} = \frac{\lambda_1}{l}\sqrt{\frac{E}{\rho}}$$

where λ_1 satisfies $\cot\lambda_1 = \left(\frac{m}{\rho Al}\right)\lambda_1$. This last expression can be written as

$\lambda_1 \tan\lambda_1\left(\frac{\rho cl}{m}\right)$ since $\lambda_1 = \omega_1 l/c$,

$$\frac{\omega_1 l}{c}\tan\left(\frac{\omega_1 l}{c}\right) = \frac{\rho Al}{m}$$

Now for small, or negligible beam mass, c becomes very large $\left(c = \sqrt{E/\rho}\right)$ and $\omega_1 l/c$ becomes small so that $\tan\theta$ can be approximated as θ. Then this last expression becomes

$$\left(\frac{\omega_1 l}{c}\right)^2 = \frac{\rho Al}{m}, \text{ or } \omega_1 = \sqrt{\frac{EA}{lm}}$$

in agreement with the single degree of freedom values of figure 1.21. As the tip mass goes to zero, the equation for figure 1.21 does not appear to make sense. The equation for ω_1 however reduces to that of a cantilevered beam, i.e., $\omega_1 = \pi c/2l$ since the frequency equation returns to $\omega_1(l/c) = 0$.

6.25 Following the line of thought suggested in Problem 6.24, model the system of Problem 6.21 as a lumped-mass single-degree-of-freedom system and compare this frequency to the first natural frequency obtained in Problem 6.22.

Solution: Note that the system of figure P6.21 is a mass connected to two springs in parallel if the bar is modeled as spring. The stiffness of a bar is given in Chapter 1 to be

$$k_{\text{bar}} = \frac{EA}{\ell}$$

The equivalent stiffness is just the sum, so that the equation of motion is

$$m\ddot{x} + \left(\frac{EA}{\ell} + k\right)x = 0$$

Thus the natural frequency of the bar and spring of figure P6.21 modeled as a single degree of freedom system is just

$$\omega_n = \sqrt{\frac{EA}{m\ell} + \frac{k}{m}}$$

The first natural frequency of the system treated as a distributed mass systems is given by the characteristic equation given in the solution to problem 6.22. To make a comparison, chose some specific values. For a 4 m aluminum beam connected to 1000 kg mass through a 100,000 N/m spring the value is given in the following Mathcad session:

$k := 100000 \quad m := 1000 \quad \rho := 7.8 \cdot 10^3 \quad E := 2 \cdot 10^{11}$

$L := 4 \quad A := 0.2 \cdot 0.5 \quad \rho \cdot A = 780$

$c := \sqrt{\frac{E}{\rho}} \qquad \omega n := \sqrt{\frac{E \cdot A}{m \cdot L} + \frac{k}{m}}$

$\omega n = 2.236 \cdot 10^3$

$\sigma := \frac{\pi}{6} \qquad f(\sigma) := \tan(\sigma \cdot L) + \frac{E \cdot A \cdot \sigma}{\left(m \cdot \sigma^2 \cdot c^2\right) + k}$

$\text{root}(f(\sigma), \sigma) = 0.545$

$\omega 1 := c \cdot \sigma$

$\omega 1 = 2.651 \cdot 10^3$

Note for the parameter values chose the frequency of the lumped mass model is a little less then the actual value.

6.26 Calculate the response of a clamped-free bar to an initial displacement 1 cm at the free end and a zero initial velocity. Assume that ρ = 7.8 × 10^3 kg/m^3, A = 0.001 m^2, E=10^{10} N/m^2, and l = 0.5 m. Plot the response at $x = l$ and $x = l/2$ using the first three modes.

Solution:

The initial conditions are $w(x,t) = 0.01\delta(x-l)$ and $w_t(x,0) = 0$ and the boundary conditions are $w(0,t) = 0$ and $w_x(l,t) = 0$. From example 6.3.1 the mode shapes are $\sin\left(\frac{2n-1}{2l}\right)\pi x$ and the natural frequencies are

$$\omega_n = \left(\frac{2n-1}{2l}\right)\sqrt{\frac{E}{\rho}} = (2n-1)(1132.38)$$

The solution is given in example 6.3.2 as

$$w(x,t) = \sum (c_n \sin\omega_n t + d_n \cos\omega_n t)\sin\left(\frac{2n-1}{2l}\right)\pi x$$

so that the velocity is

$$w_t(x,t) = \sum_{n=1}^{\infty}(\omega_n c_n \cos\omega_n t - d_n\omega_n \sin\omega_n t)\sin\left(\frac{2n-1}{2l}\right)\pi x$$

Using $w_t(x,0) = 0$ then yields $c_n = 0$ for n = 1, 2, …, so that

$$0.01\delta(x-l) = \sum d_n \cos\omega_n t \sin\frac{2n-1}{2l}\pi x$$

Multiplying by $\sin\frac{2m-1}{2l}\pi x$ and integrating from 0 to l yields

$$0.01\int_0^l \delta(x-l)\sin\left(\frac{2m-1}{2l}\right)\pi x dx = c_m \int_0^l \sin^2\left(\frac{2m-1}{2l}\right)\pi x dx$$

using the orthogonality of sin $\sigma_n x$..

$$0.01\sin\frac{2m-1}{2}\pi = c_m\frac{l}{2},\ m = 1,2,3\ldots$$

so that $c_m = (.02)(-1)^{m+1}/l = (.004)(-1)^{m+1}$ and the solution is

$$w(x,t)=\sum_{n=1}^{\infty}(.004)(-1)^{n+1}\sin[(2n-1)(1132.28)t]\sin(2n-1)\pi x$$

For n = 3 and x = 0.5,

$$w(0.5,t)=(.004)[\sin 1132.28t-0-\sin 33968t]$$

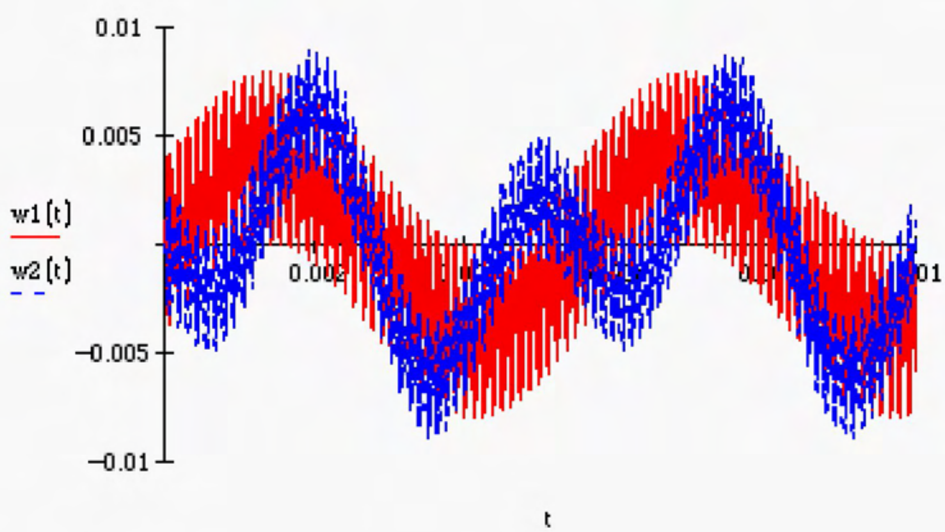
For n = 3 and $x = l/2$ = 0.25

$$w(.25,t)=(.004)[.707\sin 1132.28-\sin 2264.56t+.707\sin 339684t]$$

These are plotted below using Mathcad:

$$w1(t) := 0.004\cdot(\sin(1132.28\cdot t)-\sin(339684\cdot t))$$

$$w2(t) := 0.004\cdot(0.707\cdot\sin(1132.28\cdot t)-\sin(2264.56\cdot t)+0.707\cdot\sin(339684\cdot t))$$

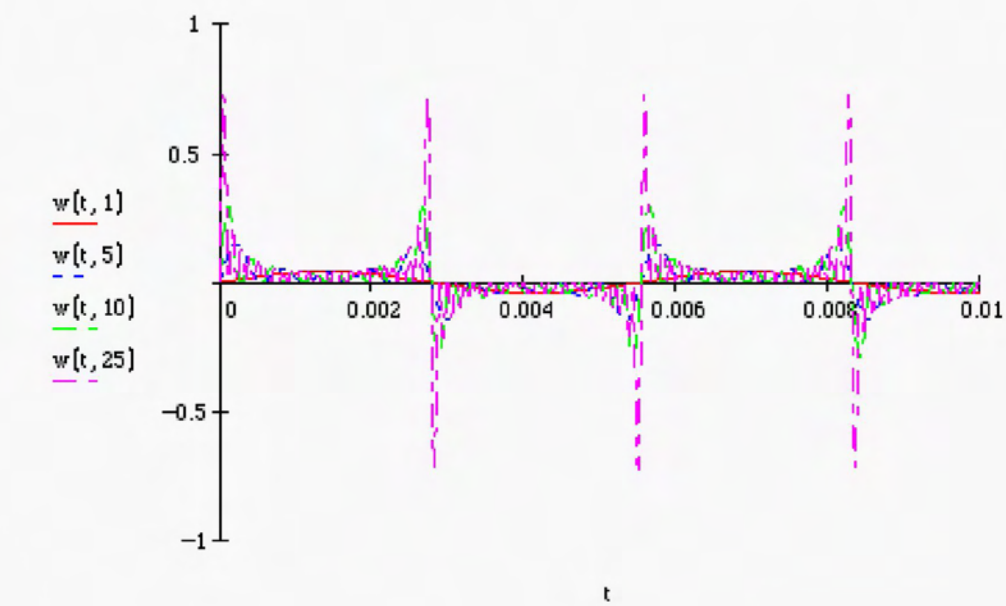


6.27 Repeat the plots of Problem 6.26 for 5 modes, 10 modes, 15 modes, and so on, to answer the question of how many modes are needed in the summation of equation (6.27) in order to yield an accurate plot of the response for this system.

Solution: The following plots in Mathcad illustrate that it takes 10 modes to capture the behavior of this series, by plotting the formula of 6.26.

$x := .5$

$$w(t,N) := \sum_{n=1}^{N} (0.04)\cdot(-1)^{n+1}\cdot\left[\sin\left[(2\cdot n-1)\cdot 1132.28\cdot t\right]\right]\cdot\sin\left[(2\cdot n-1)\cdot\pi\cdot x\right]$$



6.28 A moving bar is traveling along the x axis with constant velocity and is suddenly stopped at the end at $x = 0$, so that the initial conditions are $(x,0) = 0$ and $w(x,0) = v$. Calculate the vibration response.

Solution:

Model the bar as a free-free bar. Then from Table 6.2 the natural frequencies are $n\pi c/l$ and the mode shapes are $\cos(n\pi x/l)$. Thus the solution is of the form

$$w(x,t) = \sum_{n=1}^{\infty} (A_n \sin\omega_n t + B_n \cos\omega_n t)\cos(n\pi x/l)$$

Using the initial condition $w(x,t) = 0$ yields that $B_n = 0$ for $n = 1, 2, 3,\ldots$, i.e.

$$w(x,0) = 0 = \sum B_n \cos(n\pi x/l)$$

which is multiplied by $\cos(n\pi x/l)$ and integrated over (0,l) using orthogonality to get $B_n = 0$. Next differentiate

$$w(x,t) = \sum A_n \sin\omega_n t \cos n\pi x/l$$

to get $w_z(x,t)$, then set $t = 0$ to use the second initial condition.

$$w_t(x,0) = \sum A_n\omega_n \cos(0)\cos(n\pi x/l)$$

Modeling the initial velocity as $v\delta(x)$, multiplying by $\cos m\pi x/l$ and integrating yields

$$\int_0^l \delta(x) v\cos(n\pi x/l)dx = \omega_n\left(\frac{l}{2}\right)A_n, \quad \text{or} \quad A_n = \frac{\partial V}{l\omega_n}$$

so that

$$w(x,t) = \frac{2v}{\pi c}\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)\sin\left(\frac{n\pi ct}{l}\right)\sin\left(\frac{n\pi x}{l}\right)$$

Note that Thomson uses a form of this problem as example 3 of section 5.3, but he models the moving beam as having a clamped free rather than free-free boundary. What do you think?

6.29 Calculate the response of the clamped-clamped string of Section 6.2 to a zero initial velocity and an initial displacement of $w_0(x) = \sin(2\pi x/l)$. Plot the response at $x = l/2$.

Solution:

The clamped-clamped string has eigenfunction sin $n\pi x/l$ and solution given by equation (6.27) where the unknown coefficients c_n and d_n are given by equation (6.31) and (6.33) respectively. Since $\dot{w}_0 = 0$, equation 6.33 yields $c_n = 0$, n = 1,2,3.. with $w_0 = \sin(2\pi x/l)$,

$$d_n = \frac{2}{l}\int_0^l \sin(2\pi x/l)\sin(n\pi x/l)dx$$

which is zero for each n except n =2, in which case $d_n = 1$. Hence

$$w(x,t) = \sin(2\pi ct/l)\sin(2\pi x/l)$$

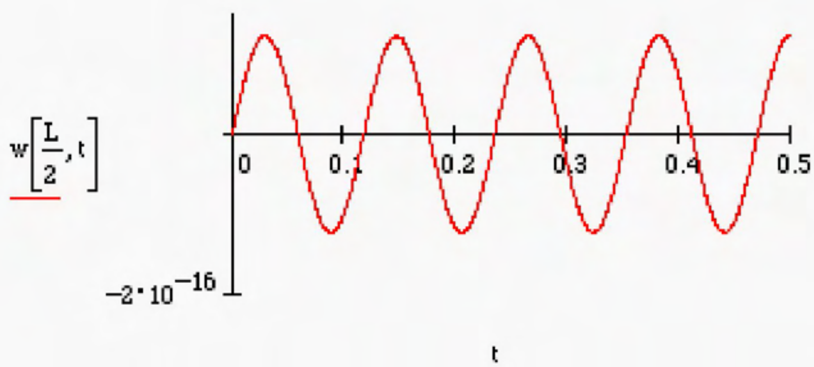
For $x = l/2$

$$w(l/2, t) = \sin(2\pi ct/l)$$

which has a well known plot given in the following Mathcad session using the values for a piano wire.

c := 11.89 L := 1.4

$$w(x,t) := \left[\sin\left[(2)\cdot\pi\cdot c\cdot\frac{t}{L}\right]\right]\cdot\sin\left(2\cdot\pi\cdot\frac{x}{L}\right)$$



Problems and Solutions Section 6.4 (6.30 through 6.39)

6.30 Calculate the first three natural frequencies of torsional vibration of a shaft of Figure 6.7 clamped at $x = 0$, if a disk of inertia $J_0 = 10$ kg m^2/rad is attached to the end of the shaft at $x = l$. Assume that $l = 0.5$ m, $J = 5$ m^4, $G = 2.5 \times 10^9$ Pa, $\rho =$ 2700 kg/m^3.

Solution: The equation of motion is $\ddot{\theta} = \dfrac{G}{\rho}\theta''$. Assume separation of variables:

$\theta = \phi(X)q(t)$ to get $\phi\ddot{q} = \dfrac{G}{\rho}\phi'' q$ or $\dfrac{\rho}{G}\dfrac{\ddot{q}}{q} = \dfrac{\phi''}{\phi} = -\sigma^2$ so that

$$\ddot{q} + \frac{G}{\rho}\sigma^2 q = 0 \text{ and } \phi'' + \sigma^2\phi = 0$$

where $\omega^2 = \dfrac{G}{\rho}\sigma^2$. The clamped-inertia boundary condition is $\theta(0,t) = 0$, and $-GJ\theta'(l,t) = J_0\ddot{\theta}(l,t)$. This yields that $\phi(0) = 0$ and

$$GJ\phi'(l)q(t) = J_0\phi(l)\ddot{q}(t) = J_0\phi(l)\frac{G}{\rho}\sigma^2 q(t)$$

or
$$J\phi'(l) = J_0\frac{\sigma^2}{\rho}\phi(l)$$

The solution of the spatial equation is of the form

$$\phi(x) = A\sin\sigma x + B\cos\sigma x$$

but the clamped boundary condition yields $B = 0$. The inertia boundary condition

$$JA\sigma\cos\sigma l = J_0\frac{\sigma^2}{\rho}A\sin\sigma l$$

yields

$$\tan\sigma l = \frac{J}{J_0}\frac{\rho l}{\sigma l} = \frac{1}{\sigma l}\left(\frac{5\text{ m}^4}{10\text{kg m}^2}\right)\quad(2700\text{kg/m}^3)(0.5\text{m})$$

So the frequency equation is

$$\tan\sigma l = \frac{675}{\sigma l}$$

Using the MATLAB function **fsolve**; this has the solutions

$$\left.\begin{aligned}\sigma_1 l &= 1.5685\\ \sigma_2 l &= 4.7054\\ \sigma_3 l &= 7.8424\end{aligned}\right\}\text{ or }\left\{\begin{aligned}\sigma_1 &= 3.1369\\ \sigma_2 &= 9.4108\\ \sigma_3 &= 15.6847\end{aligned}\right.$$

Thus
$\omega_1 = 3018.5$ rad/s $\Rightarrow$ $f_1 = 480.4$ Hz
$\omega_2 = 9055.6$ rad/s $\Rightarrow$ $f_2 = 1441.2$ Hz
$\omega_3 = 15092.6$ rad/s $\Rightarrow$ $f_3 = 2402.1$ Hz

6.31 Compare the frequencies calculated in the previous problem to the frequencies of the lumped-mass single-degree-of-freedom approximation of the same system.

Solution:

First calculate the equivalent torsional stiffness of the rod.

$$k = \frac{GJ}{l} = \frac{(2.5\times 10^9)(5)}{0.5} = 2.5\times 10^{10}$$

$$J_0\ddot{\theta} = -k\theta$$

$$J_0\ddot{\theta} + k\theta = 0$$

$$10\ddot{\theta} + 2.5\times 10^{10}\theta = 0 \quad \text{or} \quad \ddot{\theta} + 2.5\times 10^{9}\theta = 0$$

so that $\omega^2 = 2.5 \times 10^9$, $\omega = 5 \times 10^5$ rad/s or about 80,000 Hz, far from the 482 Hz of problem 6.30.

6.32 Calculate the natural frequencies and mode shapes of a shaft in torsion of shear modulus G, length l, polar inertia J, and density ρ that is free at $x = 0$ and connected to a disk of inertia J_0 at $x = l$.

Solution:

Assume zero initial conditions, i.e. $\theta(x,0) = \dot{\theta}(x,0) = 0$. From equation 6.66

$$\frac{\partial^2\theta(x,t)}{\partial t^2} = \left(\frac{G}{\rho}\right)\frac{\partial^2\theta(x,t)}{\partial x^2} \tag{1}$$

The boundary condition at $x = l$ and at $x = 0$ is

$$GJ\frac{\partial\theta(l,t)}{\partial x} = -J_0\frac{\partial^2\theta(l,t)}{\partial t^2} \qquad \frac{\partial\theta(0,t)}{\partial x} = 0$$

Using separation of variable in (1) of form $\theta(x,t) = \Theta(x)T(t)$ yields:

$$\frac{\Theta''(x)}{\Theta(x)} = \frac{1}{c^2}\frac{\ddot{T}(t)}{T(t)} = -\sigma^2 \tag{2}$$

where $c^2 = \dfrac{G}{\rho}$ and σ^2 is a separation constant. (2) can now be rewritten as 2 equations

$$\Theta''(x) + \sigma^2\Theta(x) = 0$$

$$\ddot{T}(t) + c^2\sigma^2 T(t) = 0 \rightarrow \omega = \sigma = \sigma\sqrt{\frac{G}{\rho}}$$

from the boundary condition at $x = l$

$$GJ\Theta'(l)T(t) = -J_0\Theta(l)\ddot{T}(t)$$

$$-\frac{GJ}{J_0}\frac{\Theta'(l)}{\Theta(l)} = \frac{\ddot{T}(t)}{T(t)} = -c^2\sigma^2$$

$$\Theta'(l) = \frac{J_0}{GJ}\frac{G}{\rho}\sigma^2\Theta = \frac{J_0\sigma^2}{J\rho}\Theta(l)$$

The boundary condition at $x = 0$ yields simply $\Theta'(0) = 0$. The general solution is of the form

$$\Theta(x) = a_1 \sin\sigma x + a_2 \cos\sigma x \text{ so that } \Theta'(x) = a_1\sigma\cos\sigma x - a_2\sigma\sin\sigma x$$

The boundary conditions applied to these solutions yield:

$$\Theta'(l) = a_1\sigma\cos\sigma l - a_2\sigma\sin\sigma l = \frac{J_0\sigma^2}{J\rho}[a_1\sin\sigma l + a_2\cos\sigma l]$$

$$a_1\left[\cos\sigma l - \frac{J_0\sigma^2}{J\rho}\sin\sigma l\right] = a_2\left[\sin\sigma l + \frac{J_0\sigma}{J\rho}\cos\sigma l\right]$$

$$\Theta'(0) = a_1\sigma = 0 \rightarrow a_1 = 0$$

$$a_2\left[\sin\sigma l + \frac{J_0\sigma}{J\rho}\cos\sigma l\right] = 0$$

For the non-trivial solution of this last expression, the coefficients of a_2 must vanish, which yields

$$\tan\sigma l = -\frac{J_0}{J\rho}\sigma$$

This must be solved numerically for σ (except for the rigid body case of σ = 0) and the frequency is calculated from $\omega = \sigma\sqrt{\frac{G}{\rho}}$. The mode shapes are $\Theta(x) = a_2 \cos \sigma x$. Note the solution for σ is illustrated in figure 6.4 page 479 of the text.

6.33 Consider the lumped-mass model of Figure 4.21 and the corresponding three-degree-of-freedom model of Example 4.8.1. Let $J_1 = k_1 = 0$ in this model and collapse it to a two-degree-of-freedom model. Comparing this to Example 6.4.1, it is seen that they are a lumped-mass model and a distributed mass model of the same physical device. Referring to Chapter 1 for the effects of lumped stiffness on a rod in torsion (k_2), compare the frequencies of the lumped-mass two-degree-of-freedom model with those of Example 6.4.1.

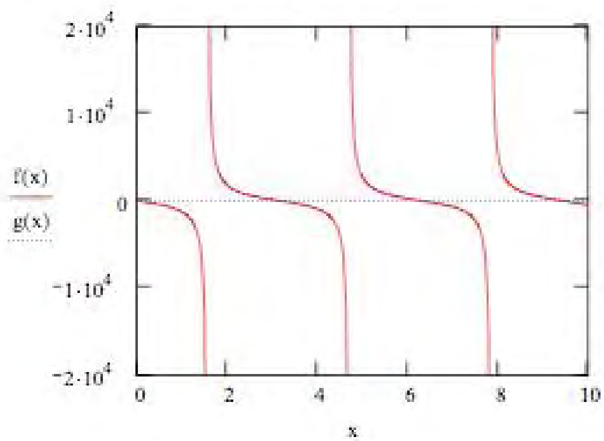
Solution: From Mathcad:

$ro := 7800 \qquad J := 5 \qquad J_1 := 10 \qquad J_2 := 10 \qquad L := 0.425 \qquad G := 80 \cdot 10^9$

$$a := \frac{ro \cdot J \cdot L}{J_1 + J_2} \qquad b := \frac{J_1 \cdot J_2}{(J_1 + J_2) \cdot ro \cdot J \cdot L} \qquad \text{from p.492}$$

$a = 828.75 \qquad b = 3.017 \times 10^{-4}$

$f(x) := (b \cdot x^2 - a) \cdot \tan(x) \qquad g(x) := x$



$x_1 := 0 \qquad x_2 := 3.138$

$$\omega_1 := x_1 \cdot \sqrt{\frac{G}{ro \cdot L^2}} \qquad \boxed{\omega_1 = 0}$$

$$\omega_2 := x_2 \cdot \sqrt{\frac{G}{ro \cdot L^2}} \qquad \omega_2 = 2.365 \times 10^4 \qquad f_2 := \frac{\omega_2}{2\pi} \qquad \boxed{f_2 = 3.763 \times 10^3}$$

2 dof model

$$k := \frac{G \cdot J}{L} \qquad k = 9.412 \times 10^{11}$$

$$M := \begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix} \qquad K := \begin{pmatrix} k & -k \\ -k & k \end{pmatrix}$$

$$M = \begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix} \qquad K = \begin{pmatrix} 9.412 \times 10^{11} & -9.412 \times 10^{11} \\ -9.412 \times 10^{11} & 9.412 \times 10^{11} \end{pmatrix}$$

$$\text{sort}\left(\text{eigenvals}\left(M^{-1} \cdot K\right)\right) = \begin{pmatrix} 1.526 \times 10^{-5} \\ 1.882 \times 10^{11} \end{pmatrix}$$

Natural frequencies: $\text{nat\_freqs} := \text{sort}\left(\sqrt{\text{eigenvals}\left(M^{-1} \cdot K\right)}\right)$

$$\frac{\text{nat\_freqs}}{2\pi} = \begin{pmatrix} 6.217 \times 10^{-4} \\ 6.905 \times 10^{4} \end{pmatrix}$$

6.34 The modulus and density of a 1-m aluminum rod are $E = 7.1 \times 10^{10}$ N/m^2, $G = 2.7 \times 10^{10}$ N/m^2, and $\rho = 2.7 \times 10^3$ kg/m^2. Compare the torsional natural frequencies with the longitudinal natural frequencies for a free-clamped rod.

Solution:

The appropriate boundary conditions are: $\theta'(0,t) = 0$ and $\theta(l,t) = 0$ for the rod and $w'(0,t) = 0 = w(l,t)$ for the bar. The separated equations are

$$\ddot{\theta} = \left(\frac{G}{\rho}\right)\theta'' \text{ and } \phi\ddot{q} = \left(\frac{G}{\rho}\right)\phi'' q$$

$$\ddot{q} + \left(\frac{G}{\rho}\right)\sigma^2 q = 0 \text{ and } \phi'' + \sigma^2 \phi = 0$$

Solutions are

$$q_n = A_n \sin\omega_n t + B_n \cos\omega_n t \quad \text{and} \quad \phi_n = C_n \sin\sigma_n x + D_n \cos\sigma_n x$$

where $\omega_n^2 = \dfrac{G}{\rho}\sigma_n^2$. But $\phi'(0) = 0$ so that $C_n = 0$. The other boundary condition yields $\phi_n(l) = D_n \cos \sigma_n l = 0$ so that

$$\sigma_n l = \frac{(2n-1)\pi}{2}, \quad n = 1, 2, \ldots$$

Thus the torsional frequencies are

$$\omega_n = \sqrt{\frac{G}{\rho}}\sigma_n$$

and the longitudinal frequencies are

$$\omega_n = \sqrt{\frac{E}{\rho}}\sigma_n$$

where

$$\sigma_n = \frac{(2n-1)\pi}{2l}$$

From the values given $\sqrt{\dfrac{G}{\rho}} = 3162$ m/s and $\sqrt{\dfrac{E}{\rho}} = 5128$ m/s. Thus the natural frequencies of the longitudinal vibration are *1.6 times larger* than the torsional vibrations.

6.35 Consider the aluminum shaft of Problem 6.32. Add a disk of inertia J_0 to the free end of the shaft. Plot the torsional natural frequencies versus increasing the tip inertia J_0 of a single-degree-of-freedom model and for the first natural frequency of the distributed-parameter model in the same plot. Are there any values of J_0 for which the single-degree–of-freedom model gives the same frequency as the full distributed model?

Solution:

Refer to problem 6.32 of the rod clamped at $x = 0$ with inertia J_0 at $x = l$. The *sdof* model of the frequency is given in example 1.5.1 as

$$\omega = \sqrt{\frac{GJ}{lJ_0}}$$

where G = torsional rigidity, J = polar moment of inertia of the rod of length l and J_0 is the disc inertia. The first natural frequency according to distributed parameter theory is given in problem 6.30 as the solution of

$$\tan \sigma / 2 = -\frac{\rho}{\sigma J_0}, \quad \omega = \sigma \frac{G}{\rho}$$

which will have a solution for a given value of J_0 equivalent to that of the *sdof* system.

6.36 Calculate the mode shapes and natural frequencies of a bar with circular cross section in torsional vibration with free-free boundary conditions. Express your answer in terms of G, l, and ρ.

Solution:

The separated equations are $\ddot{q} + \left(\frac{G}{\rho}\sigma^2\right)q = 0$ and $\phi'' + \sigma^2\phi = 0$

where $\omega_n = \sqrt{\frac{G}{\rho}}\sigma_n$. Thus

$$q_n = A_n \sin\omega_n t + B_n \cos\omega_n t \quad \text{and} \quad \phi_n = C_n \sin\sigma_n x + D_n \cos\sigma_n x$$

The boundary conditions are

$$\phi_n'(0) = 0$$
$$\phi_n'(l) = 0$$

But $\phi_n' = C_n\sigma_n \cos\sigma_n x - D_n\sigma_n \sin\sigma_n x$ so that $\phi_n'(0) = 0 \Rightarrow C_n = 0$ and the frequency equation becomes $\phi_n'(l) = 0 = -D_n\sigma_n \sin\sigma_n 0.$ This has the solution $\sigma_n l = n\pi$ or $\sigma_n = \frac{n\pi}{l}$. Hence

$$\omega_n = \sqrt{\frac{G}{\rho}}\frac{n\pi}{l} \quad \text{and} \quad \phi_n(x) = \cos\frac{n\pi x}{l}.$$

6.37 Calculate the mode shapes and natural frequencies of a bar with circular cross section in torsional vibration with fixed boundary conditions. Express you answer in terms of G, l, and ρ,

Solution: From equation 6.66

$$\frac{\partial^2\theta(x,t)}{\partial t^2} = \left(\frac{G}{\rho}\right)\frac{\partial^2\theta(x,t)}{\partial x^2}$$

Assume a solution of the form θ(x,t) = Θ(x)T(t) so that

$$\Theta(x)\ddot{T}(t) = \frac{G}{\rho}\Theta''(x)T(t)$$

Separate where σ² is the separation constant and $c^2 = \dfrac{G}{\rho}$

$$\frac{\Theta''(x)}{\Theta(x)} = \frac{1}{c^2}\frac{\ddot{T}(t)}{T(t)} = -\sigma^2$$

or $\Theta''(x) + \sigma^2\Theta(x) = 0$ and $\ddot{T}(t) + \sigma^2 c^2 T(t) = 0$ where $\omega = \sqrt{\dfrac{G}{\rho}}\sigma$. The

boundary conditions for a fixed-fixed rod are Θ(0) = 0 and Θ(l) = 0 from the solution of the spatial equations

$$\Theta(0) = a_2 = 0$$
$$\Theta(l) = a_1 \sin\sigma l = 0.$$

.

For the non-trivial solution

$$\sin\sigma \mathrm{l} = 0$$
$$\sigma = \frac{\mathrm{n}\pi}{\mathrm{l}}, \quad n = 0,1,2,..$$

natural frequency

$$\omega = \sqrt{\frac{G}{\rho}}\frac{n\pi}{l}, \quad n = 1,2,...$$

mode shape

$$\Theta(x) = a_1 \sin\frac{n\pi}{l}x, \quad n = 0,1,2,...$$

6.38 Calculate the eigenfunctions of Example 6.4.1.

Solution:

From example 6.4.1 the eigenfunctions are

$$\Theta_n(x) = a_1 \sin\sigma_n x + a_2 \cos\sigma_n x \ \text{ or } \Theta_n(x) = A_n\left(-\frac{\sigma J_1}{\rho J}\sin\sigma_n x + \cos\sigma_n x\right)$$

where σ_n are determined by equation 6.8.4.

6.39 Show that the eigenfunctions of Problem 6.38 are orthogonal.

Solution:

Orthogonality requires $\int_0^1 \theta_n(x)\theta_m(x)dx = 0, \quad m \neq n.$ From direct calculation

$$\int_0^l \left(-\frac{\sigma J_1}{\rho J}\sin\sigma_n x + \cos\sigma_n x\right)\left(-\frac{\sigma J_1}{\rho J}\sin\sigma_m x + \cos\sigma_m x\right)dx$$

$$= \left(\frac{\sigma J_1}{\rho J}\right)^2 \int_0^l \sin\sigma_m x \sin\sigma_n x dx$$

$$-\frac{\sigma J_1}{\rho J}\int_0^l \sin\sigma_n x \sin\sigma_m x dx - \frac{\sigma J_1}{\rho J}\int_0^l \sin\sigma_m x \sin\sigma_n x dx$$

$$+\int_0^l \cos\sigma_n x \cos\sigma_m x dx$$

where each integral vanishes. Also one can use the same calculation as problem 6.3 since the natural frequencies have distinct values.

Problems and Solutions Section 6.5 (6.40 through 6.47)

6.40 Calculate the natural frequencies and mode shapes of a clamped-free beam. Express your solution in terms of E, I, ρ, and l. This is called the cantilevered beam problem.

Solution:

Clamped-free boundary conditions are

$$w(0,t) = w_x(0,t) = 0 \text{ and } w_{xx}(l,t) = w_{xxx}(l,t) = 0$$

assume E, I, ρ, l constant. The equation of motion is

$$\frac{\partial^2 w}{\partial t^2} + \left(\frac{EI}{\rho A}\right)\frac{\partial^4 w}{\partial x^4} = 0$$

assume separation of variables $w(x,t) = \phi(x)q(t)$ to get

$$\left(\frac{EI}{\rho A}\right)\frac{\phi''''}{\phi} = -\frac{\ddot{q}}{q} = \omega^2$$

The spatial equation becomes

$$\phi'''' - \left(\frac{\rho A}{EI}\right)\omega^2\phi = 0$$

define $\beta^4 = \dfrac{\rho A\omega^2}{EI}$ so that $\phi'''' - \beta^4\phi = 0$ which has the solution:

$$\phi = C_1 \sin\beta x + C_2 \cos\beta x + C_3 \sinh\beta x + C_4 \cosh\beta x$$

Applying the boundary conditions

$w(0,t) = w_x(0,t) = 0$ and $w_{xx}(l,t) = w_{xxx}(l,t) = 0 \Rightarrow$

$$\phi(0) = \phi'(0) = 0 \text{ and } \phi''(l) = \phi'''(l) = 0$$

Substitution of the expression for ϕ into these yields:

$$C_2 + C_4 = 0$$

$$C_1 + C_3 = 0$$

$$-C_1 \sin\beta l - C_2 \cos\beta l + C_3 \sinh\beta l + C_4 \cosh\beta l = 0$$

$$-C_1 \cos\beta l + C_2 \sin\beta l + C_3 \cosh\beta l + C_4 \sinh\beta l = 0$$

Writing these four equations in four unknowns in matrix form yields:

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ -\sin\beta l & -\cos\beta l & \sinh\beta l & \cosh\beta l \\ -\cos\beta l & \sin\beta l & \cosh\beta l & \sinh\beta l \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = 0$$

For a nonzero solution, the determinant must be zero to that (after expansion)

$$\begin{vmatrix} -\sin\beta l - \sinh\beta l & -\cos\beta l - \cosh\beta \\ -\cos\beta l - \cosh\beta & \sin\beta l - \sinh\beta \end{vmatrix} =$$

$$(-\sin\beta l - \sinh\beta l)(\sin\beta l - \sinh\beta l) -$$
$$(-\cos\beta l - \cosh\beta l)(-\cos\beta l - \cosh\beta l) = 0$$

Thus the frequency equation is cos βl cosh βl = -1 or $\cos\beta_n l = -\dfrac{1}{\cosh\beta_n l}$ and

frequencies are $\omega_n = \sqrt{\dfrac{\beta_n^4 EI}{\rho A}}$. The mode shapes are

$$\phi_n = C_{1n}\sin\beta_n x + C_{2n}\cos\beta_n x + C_{3n}\sinh\beta_n x + C_{4n}\cosh\beta_n x$$

Using the boundary condition information that $C_4 = -C_2$ and $C_3 = -C_1$ yields

$$-C_1\sin\beta l - C_2\cos\beta l - C_1\sinh\beta l - C_2\cosh\beta l$$
$$-C_1(\sin\beta l + \sinh\beta l) = C_2(\cos\beta l + \cosh\beta l)$$

so that

$$C_1 = -C_2\left(\frac{\cos\beta l + \cosh\beta l}{\sin\beta l + \sinh\beta l}\right)$$

and the mode shapes can be expressed as:

$$\phi_n = -C_{2n}\left[-\left(\frac{\cos\beta_n l + \cosh\beta_n l}{\sin\beta_n l + \sinh\beta_n l}\right)\sin\beta_n x + \cos\beta_n x\right.$$
$$\left. + \left(\frac{\cos\beta_n l + \cosh\beta_n l}{\sin\beta_n l + \sinh\beta_n l}\right)\sinh\beta_n x - \cosh\beta_n x\right]$$

6.41 Plot the first three mode shapes calculated in Problem 6.40. Next calculate the strain mode shape [i.e., $X'(x)$], and plot these next to the displacement mode shapes $X(x)$. Where is the strain the largest?

Solution: The following Mathcad session yields the plots using the values of β taken from Table 6.4.

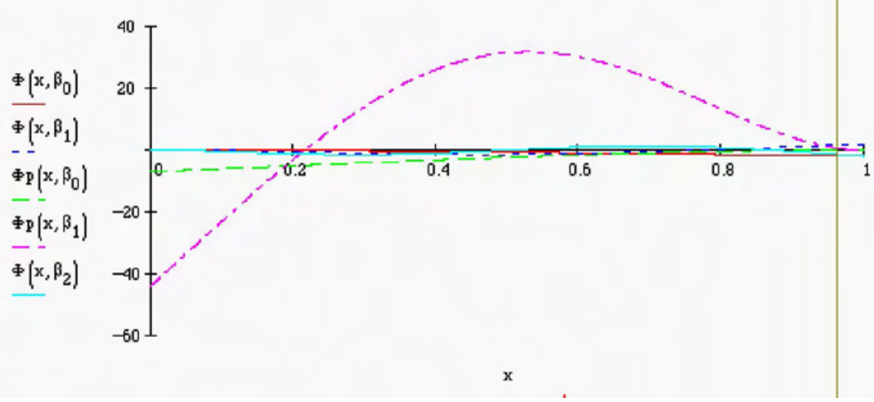
$$\beta := \begin{bmatrix} 1.87510407 \\ 4.69409113 \\ 7.85475744 \end{bmatrix}$$

$$\Phi(x,b) := \left[\left(-\frac{\cos(b)+\cosh(b)}{\sin(b)+\sinh(b)}\cdot\sin(b\cdot x)\right)+\cos(b\cdot x)+\left(\frac{\cos(b)+\cosh(b)}{\sin(b)+\sinh(b)}\right)\cdot\sinh(b\cdot x)-\cosh(b\cdot x)\right]$$

$$\left[\left(-\frac{\cos(b)+\cosh(b)}{\sin(b)+\sinh(b)}\cdot\sin(b\cdot x)\right)+\cos(b\cdot x)+\left(\frac{\cos(b)+\cosh(b)}{\sin(b)+\sinh(b)}\right)\cdot\sinh(b\cdot x)-\cosh(b\cdot x)\right]$$

$$\left[\left[\frac{-(\cos(b)+\cosh(b))}{(\sin(b)+\sinh(b))}\cdot\cos(b\cdot x)\cdot b-\sin(b\cdot x)\cdot b+\frac{(\cos(b)+\cosh(b))}{(\sin(b)+\sinh(b))}\cdot\cosh(b\cdot x)\cdot b\right]-\sinh(b\cdot x)\cdot b\right]$$

$$\Phi p(x,b) := \frac{-(-\cos(b)-\cosh(b))}{(\sin(b)+\sinh(b))}\cdot\sin(b\cdot x)\cdot b^2-\cos(b\cdot x)\cdot b^2+\frac{(\cos(b)+\cosh(b))}{(\sin(b)+\sinh(b))}\cdot\sinh(b\cdot x)\cdot b^2-\cosh(b\cdot x)\cdot b^2$$



The strain is largest at the free end.

6.42 Derive the general solution to a fourth-order ordinary differential equation with constant coefficients of equation (6.100) given by equation (6.102).

Solution:

From equation (6.100) with $\beta^4 = \rho A\omega^2 / EI$, the problem is to solve $X'''' - \beta^4 X = 0$. Following the procedure for the second order equations suggested in example 6.2.1 let $X(x) = Ae^{\lambda t}$ which yields

$$\left(\lambda^4 - \beta^4\right)Ae^{\lambda x} = 0 \text{ or } \lambda^4 = \beta^4$$

This characteristic equation in λ has 4 roots

$$\lambda = -\beta, \beta, -\beta j, \text{ and } \beta j$$

each of which corresponds to a solution, namely $A_1e^{-\beta x}$, $A_2e^{\beta x}$, $A_3e^{-\beta jx}$ and $A_4e^{\beta jx}$. The most general solution is the sum of each of these or

$$X(x) = A_1e^{-\beta x} + A_2e^{\beta x} + A_3e^{-\beta jx} + A_4e^{\beta jx} \quad \text{(a)}$$

Now recall equation (A.19), i.e., $e^{\beta x} = \cos\beta x + j\sin\beta x$, and add equations (A.21) to yield $e^{\beta jx} = \sinh\beta x + \cosh\beta x$. Substitution of these two expressions into (a) yields

$$X(x) = A\sin\beta x + B\cos\beta x + C\sinh\beta x + D\cosh\beta x$$

where *A*, *B*, *C*, and *D* are combinations of the constants A_1, A_2, A_3 and A_4 and may be complex valued.

6.43 Derive the natural frequencies and mode shapes of a pinned-pinned beam in transverse vibration. Calculate the solution for $w_0(x) = \sin 2\pi x/l$ min and $\dot{w}_0(x) = 0$.

Solution: Use $w(x,t) = \phi(x)q(t)$ in equation (6.29) with $\dot{w}(x,0) = 0$ or $\dot{q}(0) = 0$. Then the temporal solution $q = A \sin \omega t + B \cos \omega t$ with $\dot{q}(0) = 0$ yields $A = 0$. The spatial solution is $\phi = C_1 \sin \beta x + C_2 \cos \beta x + C_3 \sinh \beta x + C_4 \cosh \beta x$ where $\beta^4 = \dfrac{\rho A \omega^2}{EI}$. The boundary conditions become

$$\phi(0) = \phi''(0) = \phi(l) = \phi''(l) = 0$$

Applied to $\phi(x)$ these yield the matrix equation

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ \sin\beta l & \cos\beta l & \sinh\beta l & \cosh\beta l \\ -\sin\beta l & -\cos\beta l & \sinh\beta l & \cosh\beta l \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = 0$$

But $C_2 + C_4 = 0$ and $-C_2 + C_4 = 0$ so $C_2 = C_4 = 0$ and this reduces to

$$\begin{bmatrix} \sin\beta l & \sinh\beta \\ -\sin\beta & \sinh\beta \end{bmatrix} \begin{bmatrix} C_1 \\ C_3 \end{bmatrix} = 0$$

or $\sin \beta l \sinh \beta l + \sin \beta l \sinh \beta l = 0$,

$C_3 = -\dfrac{C_1 \sin\beta l}{\sinh\beta l}$, and $-C_1 \sin\beta l - C_1 \sin\beta l = 0$ so that the frequency equation becomes $\sin \beta l = 0$ and thus $\beta_n l = n\pi$, $n = 1,2,3,\ldots$ and $\beta_n = \dfrac{n\pi}{l}$, $n = 1,2,3,\ldots$so that $C_3 = 0$ and the frequencies are $\omega_n = \left(\dfrac{n\pi}{l}\right)^2 \sqrt{\dfrac{EI}{\rho A}}$ with mode shapes $\phi_n(x) = C_{1n} \sin \beta_n x$. The total solution is the series $w(x,t) = \sum_{n=1}^{\infty} \{\beta_n \cos\omega_n t \sin\beta_n x\}$.

Applying the second initial condition yields $w(x,0) = \sin\dfrac{2\pi x}{l} = \sum_{n=1}^{\infty} \beta_n \sin\dfrac{n\pi x}{l}$ and therefore

$$B_n = \begin{cases} 0 & n = 1 \\ & n = 3,4,\ldots \\ 1 & n = 2 \end{cases}$$

so that

$$w(x,t) = \cos \omega_2 t \sin \frac{2\pi x}{l}$$

6.44 Derive the natural frequencies and mode shapes of a fixed-fixed beam in transverse vibration.

Solution: Follow example 6.5.1 to get the solution in the 5th entry of table 6.4. The spatial equation for the transverse vibration of a beam has solution of the form (6.102)

$$X(x) = a_1 \sin \beta x + a_2 \cos \beta x + a_3 \sinh \beta x + a_4 \cosh \beta x$$

where $\beta^4 = \rho A \omega^2 / EI$. The clamped boundary conditions are given by equation (6.94) as $X(0) = X'(0) = X(l) = X'(l) = 0$. Applying these boundary conditions to the solution yields

$$X(0) = 0 = a_1(0) + a_2(1) + a_3(0) + a_4(1) \qquad (1)$$

$$X'(0) = 0 = \beta a_1(1) - \beta a_2(0) + \beta a_3(1) + \beta a_4(0) \qquad (2)$$

$$X(l) = 0 = a_1 \sin \beta l + a_2 \cos \beta l + a_3 \sinh \beta l + a_4 \cosh \beta l \qquad (3)$$

$$X'(l) = 0 = \beta a_1 \cos \beta l - \beta a_2 \sin \beta l + \beta a_3 \cosh \beta l + \beta a_4 \sinh \beta l$$

(4)

dividing (2) and (3) by $\beta \neq 0$ and writing in matrix form yields

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ \sin \beta l & \cos \beta l & \sinh \beta l & \cosh \beta l \\ \cos \beta l & -\sin \beta l & \cosh \beta l & \sinh \beta l \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The coefficient matrix must have zero determinant for a nonzero solution for the a_n. Taking the determinant yields (expanding by minors across the top row).

$$\sinh^2 \beta l - \cosh^2 \beta l - \sin \beta l \sinh \beta l + \cos \beta l \cosh \beta l +$$
$$\cos \beta l \cosh \beta l \sin \beta l \sinh \beta l - \sin^2 \beta l - \cos^2 \beta l = 0$$

which reduces to

$$-1 + 2\cos \beta l \cosh \beta l - 1 = 0 \quad \text{or} \quad \cos \beta l \cosh \beta l = 1$$

since $\sinh^2 \beta l - \cosh^2 \beta l = -1$ and $\sin^2 x + \cos^2 x = 1$. The solutions of this characteristic equation are given in table 6.4. Next from equation (1) $a_2 = -a_4$ and from equation (2) $a_1 = -a_3$ so equation (3) can be written as

$$-a_3 \sin \beta l - a_4 \cos \beta l + a_3 \sinh \beta l + a_4 \cosh \beta l = 4$$

Solving this for a_3 yields

$$a_3 = a_4 \left(\frac{\cos \beta l - \cosh \beta l}{\sinh \beta l - \sin \beta l} \right)$$

Recall also that $a_1 = -a_3$. Substitution into the solution $X(x)$ and factoring out a_4 yields

$$X(x) = a_4 \cosh \beta x - \cosh \beta x - \left(\frac{\cos \beta l - \cosh \beta l}{\sin \beta l - \sinh \beta l} \right) \left(\sinh \beta x - \sin \beta x \right)$$

in agreement with table 6.4. Note that a_4 is arbitrary as it should be.

6.45 Show that the eigenfunctions or mode shapes of Example 6.5.1 are orthogonal. Make them normal.

Solution:

The easiest way to show the orthogonality is to use the fact that the eigenvalues are not repeated and follow the solution to problem 6.2. The eigenfunctions are (table 6.4 or example 6.5).

$$X_n(x) = a_n \left\{ \cosh \beta_n x - \cos \beta_n x - \sigma_n \left(\sinh \beta_n x - \sin \beta_n x \right) \right\}$$

Note that the constant a_n is arbitrary (a constant times a mode shape is still a mode shape) and normalizing involves choosing the constant a_n so that $\int X_n X_n dx = 1$. Calculating this integral yields:

$$\begin{aligned} a_n^2 \int_0^l & \left\{ \cosh^2 \beta_n x - 2\cos \beta_n x \cosh \beta_n x + \cos^2 \beta_n x \right. \\ & - 2\sigma_n \left(\sinh \beta_n x - \sin \beta_n x \right)\left(\cosh \beta_n x - \cos \beta_n x \right) \\ & \left. + \sigma_n^2 \left(\sinh^2 \beta_n x - 2\sin \beta_n x \sinh \beta_n x + \sin^2 \beta_n x \right) \right\} dx \end{aligned}$$

so

$$\begin{aligned} 1 = a_n^2 & \left[\frac{1}{\beta_n} \left(\frac{\sinh 2\beta_n l + \sin 2\beta_n l}{4} \right) + \beta_n l \right. \\ & - \frac{1}{\beta_n} \left(\sinh \beta_n l \sin \beta_n l + \cos \beta_n l \cosh \beta_n l \right) - \frac{\sigma_n}{\beta_n} \cos^2 \beta_n l + \cosh 2\beta_n l \\ & + \sinh \beta_n l \left(\sin \beta_n l + \cos \beta_n l \right) - \cosh \beta_n l \left(\cos \beta_n - \sin \beta_n l \right) \\ & \left. + \frac{\sigma_n^2}{\beta_n} \left[\frac{\sinh^2 \beta_n l - \sin 2\beta_n l}{4} - 1 - \sin \beta_n l \sinh \beta_n l + \cosh \beta_n l \cos \beta_n l \right] \right. \end{aligned}$$

So denoting the term in [ ] as γ_n and solving for $a_n = 1/\sqrt{\gamma_n}$ yields the normalization constant.

6.46 Derive equation (6.109) from equations (6.107) and (6.108).

Solution:

Using subscript notation for the partial derivatives, equation (6.108) with $f = 0$ yields an expression for φ_x, i.e.

$$\varphi_x = (\kappa AGW_{xx} - \rho Aw_{tt}) / \kappa^2 AG \qquad \text{(a)}$$

Equation (6.107) can be differentiated once with respect to x to yield a middle term identical to the first term of equation (6.108). Substitution yields

$$EI\varphi_{xx} + \rho Aw_{tt} = \rho I\varphi_{xtt} \qquad \text{(b)}$$

Equation (a) can be differentiated twice with respect to time to get an expression for $\rho I\varphi_{xx}$ in terms of $w(x,t)$ which when substituted into (b) yields

$$EI\varphi_{xxx} + \rho Aw_{tt} = \rho Iw_{xxtt} - \left(\rho^2 I / \kappa^2 G\right)w_{tttt}$$

The first term $EI\varphi_{xxx}$ can be eliminated by differentiating (a) twice with respect to x to yield

$$EI\left(\kappa^2 AGw_{xxxx} - \rho Aw_{ttxx}\right) + \rho Aw_{tt} = \kappa^2 AGw_{xxtt} - \rho AEIw_{tttt}$$

when substituted into (c). This is an expression in $w(x,t)$ only. Rearranging terms and dividing by $\kappa^2 AG$ yields equation (6.109).

6.47 Show that if shear deformation and rotary inertia are neglected, the Timoshenko equation reduces to the Euler-Bernoulli equation and the boundary conditions for each model become the same.

Solution:

This is a bit of a discussion problem. Since ρI is the inertia of the beam in rotation about φ the term ρIw_{xxxtt} represents rotary inertia. The term $(\rho IE/\kappa^2 G)w_{tttt}$ is the shear distortion and the term $(\rho^2 I/\kappa^2 G)w_{xxtt}$ is a combination of shear distortion and rotary inertia. Removing these terms from equation (6.109) results in equation (6.92).

Problems and Solutions Section 6.6 (6.48 through 6.52)

6.48 Calculate the natural frequencies of the membrane of Example 6.6.1 for the case that one edge $x = 1$ is free.

Solution:

The equation for a square membrane is

$$w_{tt} + w_{yy} = \left(\frac{\rho}{\tau} w_{tt} \right)$$

with boundary condition given by $w(0,y) = 0$, $w_x(l,y) = 0$, $w(x,0) = 0$, $w(x,l) = 0$. Assume separation of variables $w = X(x)Y(y)q(t)$ which yields

$$\frac{X''}{X} + \frac{Y''}{Y} = \frac{1}{c^2}\frac{\ddot{q}}{q} = -\omega^2 \quad \text{where} \quad c = \sqrt{\rho / \tau}$$

Then

$$\ddot{q} + c^2\omega^2 q = 0$$

is the temporal equation and

$$\frac{X''}{X} = -\omega^2 - \frac{Y''}{Y} = -\alpha^2$$

yields

$$X'' + \alpha^2 X = 0$$
$$Y'' + \gamma^2 Y = 0$$

as the spatial equation where $\gamma^2 = \omega^2 - \alpha^2$ and $\omega^2 = \alpha^2 + \gamma^2$. The separated boundary conditions are $X(0) = 0$, $X'(l) = 0$ and $Y(0) = Y(l) = 0$. These yield

$$X = A\sin\alpha x + B\cos\alpha x$$
$$B = 0$$
$$A\cos\alpha l = 0$$
$$\alpha_n l = \frac{(2n-1)\pi}{2}$$
$$\alpha_n = \frac{(2n-1)\pi}{2l}$$

Next $Y = C \sin \gamma y + D \cos \gamma y$ with boundary conditions which yield $D = 0$ and $C \sin \gamma l = 0$. Thus

$$\gamma_m = m\pi l$$

and for $l = 1$ we get $a_n = \frac{(2n-1)\pi}{2}$, for $\gamma_m = m\pi$ $n,m = 1, 2, 3,\ldots$

$$\omega_{nm}^2 = \alpha_n^2 + \gamma_m^2 = \frac{(2n-1)^2\pi^2}{4} + m^2\pi^2 = \left[\frac{(2n-1)^2 + 4m^2}{4}\right]\pi^2$$

$$c^2\omega_{nm}^2 = c^2\left[\frac{(2n-1)^2 + 4m^2}{4}\right]\pi^2$$

So that

$$\omega_{nm} = \sqrt{(2n-1)^2 + 4m^2}\,\frac{c\pi}{2}$$

are the natural frequencies.

6.49 Repeat Example 6.6.1 for a rectangular membrane of size *a* by *b*. What is the effect of *a* and *b* on the natural frequencies?

Solution:

The solution of the rectangular membrane of size $a \times b$ is the same as given in example 6.6.1 for a unit membrane until equation 6.13.1. The boundary condition along $x = a$ becomes

$$A_1 \sin\alpha a \sin\gamma y + A_2 \sin\alpha a \cos\gamma y = 0$$

or

$$\sin\alpha a(A_1 - \sin\gamma y + A_2 \cos\gamma y) = 0$$

Thus $\sin\alpha a = 0$ and $\alpha a = n\pi$ or $\alpha = n\pi/a$, $n = 1, 2, \ldots$ Similarly, the boundary conditions along $y = b$ yields that

$$\gamma = \frac{n\pi}{b} \quad n=1,2,3,\ldots$$

Thus the natural frequency becomes

$$\omega_{nm} = \pi\sqrt{a^2n^2 + b^2m^2} \quad n, m = 1, 2, 3, \ldots$$

Note that ω_{nm} are no longer repeated, i.e., $\omega_{12} \neq \omega_{21}$, etc.

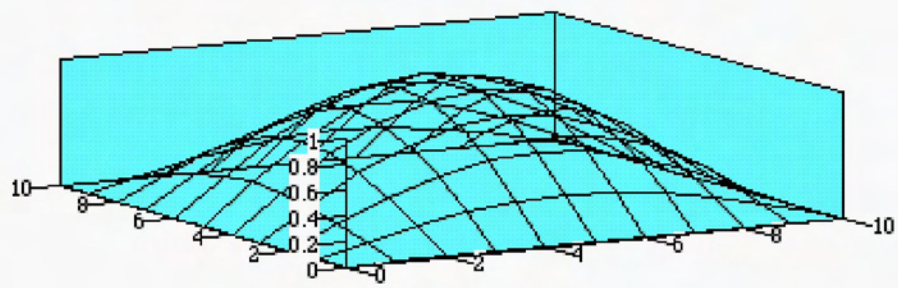
6.50 Plot the first three mode shapes of Example 6.6.1.

Solution: A three mesh routine from any of the programs can be used. Mathcad results follow for the 11, 12, 21 and 31 modes:

$N := 10 \quad i := 0..N \qquad j := 0..N \qquad x_i := i \cdot 0.1 \qquad y_j := j \cdot 0.1$

$w(x, y) := (\sin(\pi \cdot x)) \cdot \sin(\pi \cdot y)$

$M_{(i,j)} := w(x_i, y_j)$

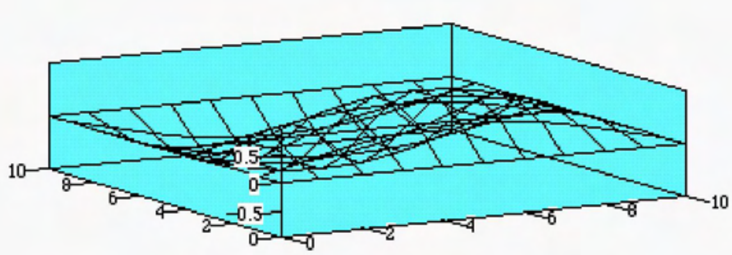


M

$N := 10 \quad i := 0..N \qquad j := 0..N \qquad x_i := i \cdot 0.1 \qquad y_j := j \cdot 0.1$

$w(x, y) := (\sin(\pi \cdot x)) \cdot \sin(2\ \pi \cdot y)$

$M_{(i,j)} := w(x_i, y_j)$

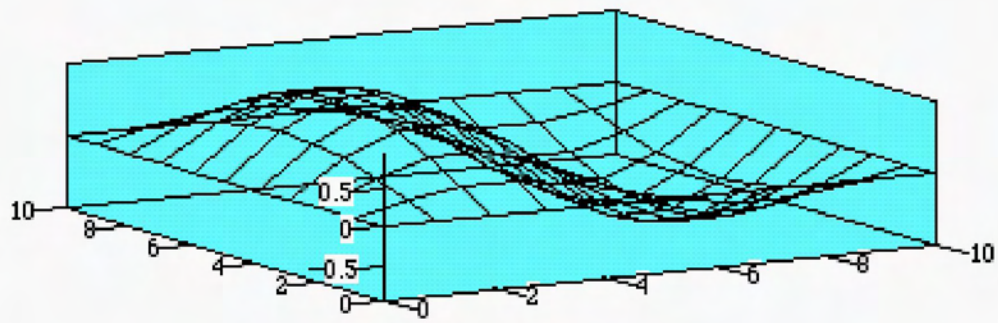


M

$N := 10 \quad i := 0..N \quad j := 0..N \quad x_i := i \cdot 0.1 \quad y_j := j \cdot 0.1$

$w(x, y) := (\sin(2\ \pi \cdot x)) \cdot \sin(\pi \cdot y)$

$M_{\langle i, j \rangle} := w(x_i, y_j)$

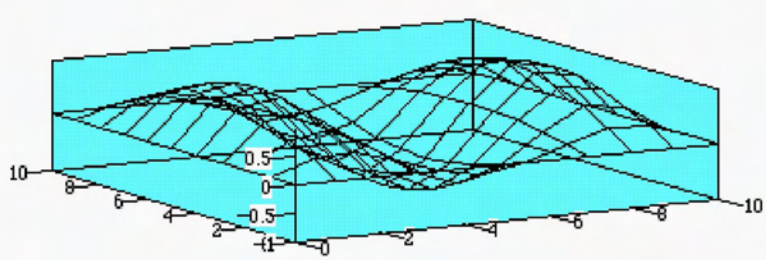


M

$N := 10 \quad i := 0..N \quad j := 0..N \quad x_i := i \cdot 0.1 \quad y_j := j \cdot 0.1$

$w(x, y) := (\sin(3\ \pi \cdot x)) \cdot \sin(\pi \cdot y)$

$M_{\langle i, j \rangle} := w(x_i, y_j)$



M

6.51 The lateral vibrations of a circular membrane are given by

$$\frac{\partial^2\omega(r,\phi,t)}{\partial r^2}+\frac{1}{r}\frac{\partial\omega(r,\phi,t)}{\partial r}+\frac{1}{r^2}\frac{\partial^2\omega(r,\phi,t)}{\partial\phi\partial r}=\frac{\rho}{\tau}\frac{\partial^2\omega(r,\phi,t)}{\partial t^2}$$

where *r* is the distance form the center point of the membrane along a radius and ϕ is the angle around the center. Calculate the natural frequencies if the membrane is clamped around its boundary at *r* = *R*.

Solution:

This is a tough problem. Assign it only if you want to introduce Bessel functions. The differential equation of a circular membrane is:

$$\frac{\partial^2 W(r,\phi)}{\partial r^2}+\frac{1}{r}\frac{\partial W(r,\phi)}{\partial r}+\frac{1}{r^2}\frac{\partial^2 W(r,\phi)}{\partial\phi^2}+\beta^2 W(r,\phi)=0$$

$$\beta^2=\left(\frac{\omega}{c}\right)^2 \quad c=\frac{T}{\rho}$$

Assume:

$$W(r,\phi)=F(r)G(\phi)$$

The differential equation separates into:

$$\frac{d^2G}{d\phi^2}+m^2G=0$$

$$\frac{d^2F}{dr^2}+\frac{1}{r}\frac{dF_1}{dr}+\left(\beta^2-\frac{m^2}{r^2}\right)F=0$$

Since the solution in ϕ must be continuous, *m* must be an integer. Therefore

$$G_m(\phi)=B_{1m}\sin m\phi+B_{2m}\cos m\phi$$

The equation in *r* is a Bessel equation and has the solution

$$F_m(r)=B_{3m}J_m(\beta r)+B_{4m}Y_m(\beta r)$$

Where $J_m(\beta r)+Y_m(\beta r)$ are the m^{th} order Bessel functions of the first and second kind, respectively. Writing the general solution $F(r)G(\phi)$ as

$$W_m(r,\phi)=A_{1m}J_m(\beta r)\sin m\phi+A_{2m}J_m(\beta r)\cos m\phi$$
$$+A_{3m}Y_m(\beta r)\sin m\phi+A_{4m}Y_m(\beta r)\cos m\phi$$

Enforcing the boundary condition

$$W_m(R,\phi) = 0 \quad m = 0,1,2,...$$

Since every interior point must be finite and $Y_m(\beta r)$ tends to infinity as r $\rightarrow$ 0, $A_{3m} = A_{4m} = 0$. At $r = R$

$$W_m(R,\phi) = A_{1m}J_m(\beta R)\sin m\phi + A_{2m}J_m(\beta R)\cos m\phi = 0$$

This can only be satisfied if

$$J_m(\beta R) = 0 \quad m = 1,2,...$$

For each m, $J_m(\beta R) = 0$ has an infinite number of solutions. Denote β_{mn} as the nth root of the mth order Bessel function of the first kind, normalized by R. Then the natural frequencies are:

$$\omega_{mn} = c\beta_{mn}$$

6.52 Discuss the orthogonality condition for Example 6.6.1.

Solution:

The eigenfuncitons of example 6.6.1 are given as

$$X_n(x)Y_n(y) = A_{nm}\sin m\pi x \sin n\pi y$$

Orthogonality in this case is generalized to two dimensions and becomes

$$\int_0^1\int_0^1 A_{nm}A_{pq}\sin m\pi x \sin n\pi y \sin p\pi y \sin q\pi y dxdy = 0 \quad mn \neq pq$$

Integrating yields

$$A_{nm}A_{pq}\int_0^1 \sin n\pi x \sin p\pi x dx \int_0^1 \sin m\pi g \sin g y dy$$
$$= A_{nm}A_{pq}\left[\frac{\sin(n-p)\pi x}{2(n-p)} - \frac{\sin(n+p)\pi x}{2(m+p)}\right]\left[\frac{\sin(m-1)\pi x}{2(m-q)} - \frac{\sin(m+p)\pi x}{2(m+p)}\right]$$

Evaluating at $x = 0$ and $x = 1$ this expression is zero. The expression is also zero provided $n = p$ and $n \neq q$ illustrating that the modes are in fact orthogonal.

Problems and Solutions Section 6.7 (6.53 through 6.63)

6.53 Calculate the response of Example 6.7.1 for l = 1 m, $E = 2.6 \times 10^{10}$ N/m^2 and ρ = 8.5×10^3 kg/m^3. Plot the response using the first three modes at $x = l/2$, $l/4$, and $3l/4$. How many modes are needed to represent accurately the response at the point $x = l/2$?

Solution:

$$w(x,t) = \sum_{n=1}^{\infty}\left(\frac{0.02}{l^2\sigma_n^2}(-1)^{n+1}\right)e^{-0.01\omega_n t}\cos\omega_{2n}t\sin\sigma_n x$$

Where

$$\sigma_n = \frac{(2n-1)\pi}{2l}$$
$$\omega_n = \sigma_n\sqrt{\frac{E}{\rho}}$$
$$\omega_{dn} = 0.9999\omega_n$$

For l = 1 m

$$E = 2.6\times 10^{10}\ \text{N/m}^2$$
$$\rho = 8.5\times 10^3\ \text{kg/m}^3$$

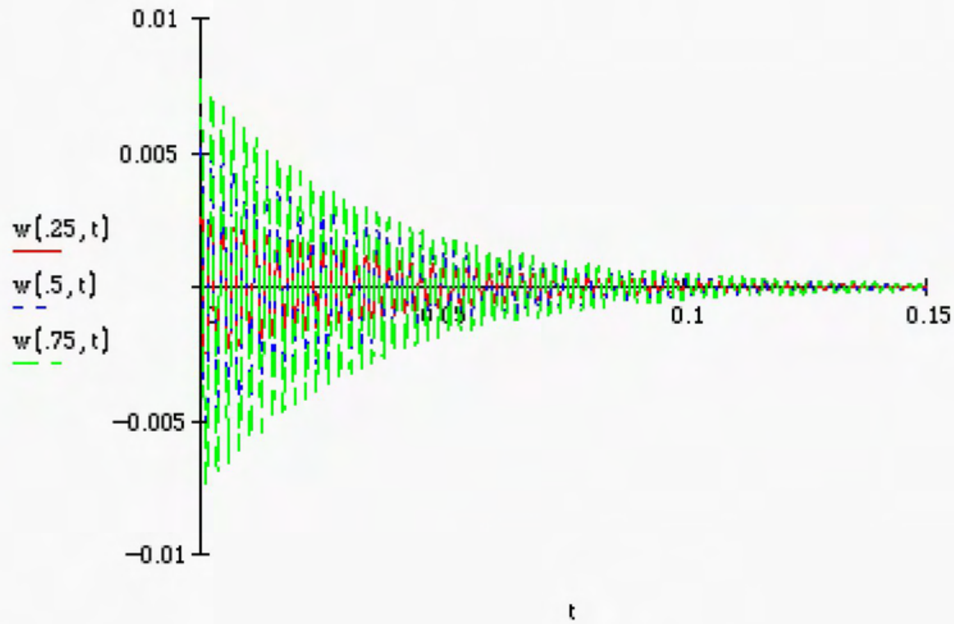
Response using first three modes at $x = \frac{l}{2}, \frac{l}{4}, \frac{3l}{4}$ plotted below.

Three modes accurately represents the response at $x = \frac{l}{2}$. The error between a three and higher mode approximation is less than 0.2%.

$$\sigma(n) := \frac{2\cdot n - 1}{2}\cdot\pi \qquad E := 2.6\cdot 10^{10} \qquad \rho := 8.5\cdot 10^{3}$$

$$\omega(n) := \sigma(n)\cdot\sqrt{\frac{E}{\rho}}$$

$$w(x,t) := \sum_{n=1}^{3} \frac{.02}{\sigma(n)^2}\,(-1)^{n+1}\cdot e^{-0.01\cdot\omega(n)\cdot t}\cdot\cos(\omega(n)\cdot t)\cdot\sin(\sigma(n)\cdot x)$$



6.54 Repeat Example 6.7.1 for a modal damping ratio of $\zeta_n = 0.01$.

Solution: Using $\zeta_n = 0.01$ and the frequency given in the example

$$\omega_{dn} = \omega_n\sqrt{1-\zeta_n^2} = 0.995\omega_n, \quad \omega_n = \frac{2n-1}{2l}\sqrt{\frac{E}{\rho}} = \sigma_n\sqrt{\frac{E}{\rho}}$$

The time response is then $T_n(t) = A_n e^{-0.1\omega_n t}\sin(\omega_{dn}t + \phi_n)$ and the total solution is:

$$w(x,t) = \sum_{n=1}^{\infty} A_n e^{-0.1\omega_n t}\sin(\omega_{dn}t + \phi_n)\sin\frac{(2n-1)}{2l}\pi x$$

The initial conditions are:

$$w(x,0) = 0.01\frac{x}{l}\text{ m} \text{ and } w_t(x,0) = 0$$

Therefore:

$$0.01\frac{x}{l} = A_n \sin\phi_n \sin\sigma_n x$$

Multiply by $\sin\sigma_m x$ and integrate over the length of the bar to get

$$0.01\frac{(-1)^{m+1}}{l\,\sigma_m^2} = A_m \sin\phi_m \frac{l}{2} \quad m = 1,2,3,\ldots$$

From the velocity initial condition

$$w_t(x,0) = 0 = \sum_{n=1}^{\infty} A_n\left[-0.1\omega_n \sin\phi_n + \omega_{dn}\cos\phi_n\right]\sin\sigma_n x$$

Again, multiply by $\sin\sigma_m x$ and integrate over the length of the bar to get

$$A_m(-0.1\omega_n \sin\phi_n + \omega_{dn}\cos\phi_n)\frac{l}{2} = 0$$

Since A_m is not zero this yields:

$$\tan\phi_n = \frac{\sin\phi_n}{\cos\phi_n} = \frac{\sqrt{1-\zeta_n^3}}{0.1} = 9.9499 \Rightarrow \phi_n = 1.4706\text{ rad } = 84.3°$$

Substitution into the equation from the displacement initial condition yields:

$$A_m = \frac{0.01}{l^2\sigma_m^2}(-1)^{m+1}\frac{1}{\sin\phi_n} = \frac{0.0201}{l^2\sigma_m^2}(-1)^{m+1}$$

The solution is then

$$w(x,t) = \sum_{n=1}^{\infty}\frac{0.01}{l^2\sigma_m^2}(-1)^{m+1}e^{-0.1\omega_n t}\sin(\omega_{dn}t + \phi_n)\sin\frac{(2n-1)}{2l}\pi x$$

6.55 Repeat Problem 6.53 for the case of Problem 6.54. Does it take more or fewer modes to accurately represent the response at $l/2$?

Solution: Use the result given in 6.54 and

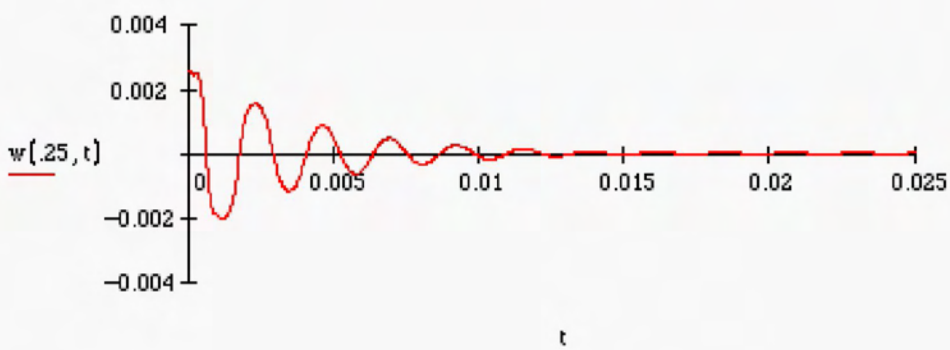
$$l = 1 \text{ m}$$
$$E = 2.6 \times 10^{10} \text{ N/m}^2$$
$$\rho = 8.5 \times 10^3 \text{ kg/m}^3$$

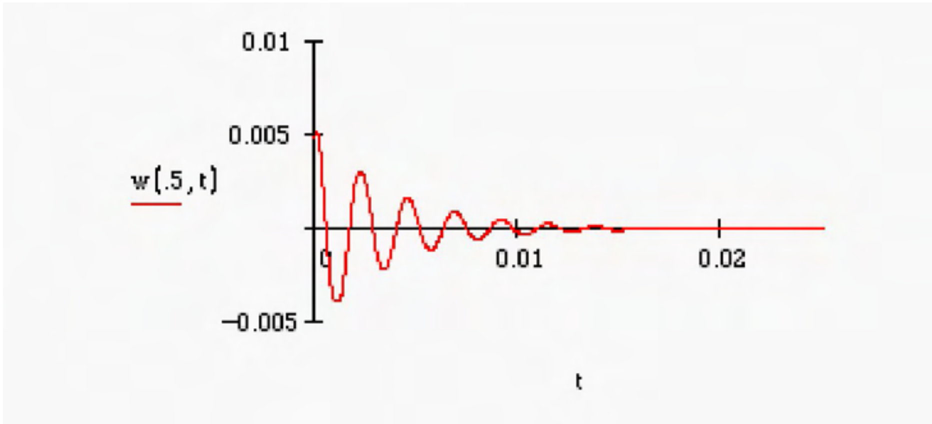
The response is plotted below at $x = \frac{l}{4}, \frac{l}{2}, \frac{3l}{4}$. An accurate representation of the response is obtained with three modes. The error between a three mode and a higher mode representation is always less than 0.2%. The results here are from Mathcad:

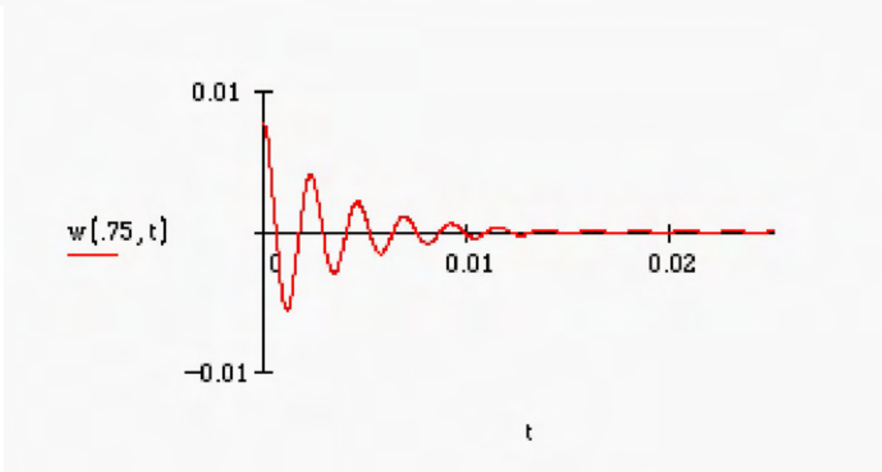
$$\sigma(n) := \frac{2 \cdot n - 1}{2} \cdot \pi \qquad E := 2.6 \cdot 10^{10} \qquad \rho := 8.5 \cdot 10^3$$

$$\omega(n) := \sigma(n) \cdot \sqrt{\frac{E}{\rho}} \qquad \omega d(n) := 0.995 \cdot \omega(n)$$

$$w(x,t) := \sum_{n=1}^{3} \frac{.0201}{\sigma(n)^2} (-1)^{n+1} \cdot e^{-0.1 \cdot \omega(n) \cdot t} \cdot \sin(\omega d(n) \cdot t + 1.4706) \cdot \sin(\sigma(n) \cdot x)$$




0.01
0.005
w(.5,t)
0
0.01
0.02
−0.005
t


0.01
w(.75,t)
0
0.01
0.02
−0.01
t

6.56 Calculate the form of modal damping ratios for the clamped string of equation (6.151) and the clamped membrane of equation (6.152).

Solution:

(a) For the string:

$$\rho w_{tt} + \gamma w_t - \tau w_{xx} = 0$$
$$\rho\phi\ddot{q} + \gamma\phi\dot{q} - \tau\phi'' q = 0$$
$$\frac{\rho}{\tau}\frac{\ddot{q}}{q} + \frac{\gamma}{\tau}\frac{\dot{q}}{q} = \frac{\phi''}{\phi} = -\sigma^2$$
$$\ddot{q} + \left(\frac{\gamma}{\rho}\right)\dot{q} + \left(\frac{\tau}{\rho}\right)\sigma^2 q = 0$$

$\phi'' + \sigma^2\phi = 0$ which has the solution $\phi = A\sin\sigma x + B\cos\sigma x$. The boundary conditions $\phi(0) = \phi(l) = 0$ yield $\sigma_n = \dfrac{n\pi}{l}, \quad n = 1,2,3,\ldots$

$$\omega_n^2 = \left(\frac{\tau}{\rho}\right)\sigma_n^2 = \frac{\tau}{\rho}\left(\frac{n\pi}{l}\right)^2$$
$$2\zeta_n\omega_n = \frac{\gamma}{\rho}$$
$$\zeta_n = \frac{\gamma}{2\rho}\sqrt{\frac{\rho}{\pi}}\left(\frac{n\pi}{l}\right)$$
$$\zeta_n = \frac{\gamma}{2\sqrt{\rho\tau}}\left(\frac{n\pi}{l}\right)$$

(b) For the membrane

$$\frac{\rho}{\tau}w_{tt} + \frac{\gamma}{\tau}w_t = w_{xx} + w_{yy}$$
$$\left(\frac{\rho}{\tau}\right)XY\ddot{q} + \left(\frac{\gamma}{\tau}\right)XY\dot{q} = X''Yq + XY''q$$
$$\left(\frac{\rho}{\tau}\right)\frac{\ddot{q}}{q} + \left(\frac{\gamma}{\tau}\right)\frac{\dot{q}}{q} = \frac{X''}{X} + \frac{Y''}{Y} = -\beta^2$$
$$\ddot{q} + \left(\frac{\gamma}{\rho}\right)\dot{q} + \left(\frac{\tau}{\rho}\right)\beta^2 q = 0$$

$\frac{X''}{X} = -\frac{Y''}{Y} - \beta^2 = -\alpha^2$ .The boundary conditions are $X(0) = X(l) = 0$ and $Y(0) = Y(l) = 0$. The two spatial solutions become

$$X'' + \alpha^2 = 0 \qquad Y'' + \gamma^2 Y = 0$$
$$X = A\sin\alpha x + B\cos\alpha x \qquad Y = C\sin\gamma x + D\cos\gamma x$$
$$B = 0 \qquad D = 0$$
$$\alpha_n + \frac{n\pi}{l} \quad n = 1,2,3,\ldots \qquad \gamma_m = \frac{n\pi}{l} \quad m = 1,2,3,\ldots$$

Thus

$$\beta_{mn}^2 = \left(n^2 + m^2\right)\left(\frac{\pi}{l}\right)^2$$

$$\omega_{mn}^2 = \frac{\tau}{\rho}\left(n^2 + m^2\right)\left(\frac{\pi}{l}\right)^2$$

$$2\zeta_m n\omega_m n = \frac{\gamma}{\rho}$$

$$\zeta_{mn} = \frac{\gamma}{2\rho\omega_{mn}} = \frac{\gamma}{2\rho}\frac{1}{\sqrt{\frac{\tau}{\rho}\left(n^2 + m^2\right)}}\frac{l}{\pi}$$

$$\zeta_{mn} = \frac{\gamma l}{2\sqrt{\rho\tau\left(n^2 + m^2\right)}}$$

6.57 Calculate the units on γ and β in equation (6.153).

Solution: The units are found from

$$\frac{\text{mg}}{\text{m}^3}\text{m}^2\frac{\text{m}}{\text{s}^2} = \gamma\frac{\text{m}}{\text{s}}$$
$$\frac{\text{kg}}{\text{s}^2}\frac{\text{s}}{\text{m}} = \gamma$$
$$\gamma = \frac{\text{kg}}{\text{m} \bullet \text{s}}$$

6.58 Assume that E, I, and ρ are constant in equations (6.153) and (6.154) and calculate the form of the modal damping ratio ζ_n.

Solution:

If E, I, and ρ are constant in equation 6.153 and 6.154. Then separation of variables works and the mode shapes become those given in table 6.4, which can be normalized so that $\int_0^l X_n X_m \, dx = \delta_{nm}$. Substitution of $w(x,t) = a_n(t)X_n(x)$ into equation (6.153) multiplying by $X_m(x)$ and integrating over x yield the mth modal equation:

$$\rho A \ddot{a}_n(t) + \gamma \dot{a}_n(t) + \beta I \left(\frac{\omega_n^2}{c^2} \right) \dot{a}_n(t) + EI \frac{\omega_n^2}{c^2} a_n(t) = 0$$

where equation (6.93) has been used to evaluate X'''' and $c^2 = EI / \rho A$. Dividing by ρA yields

$$\ddot{a}_n(t) + \left(\frac{\gamma}{\rho A} + \frac{\beta}{E} \omega_n^2 \right) \dot{a}_n(t) + \omega_n^2 a_n(t) = 0$$

which is the *sdof* form of windows 6.4. Thus the coefficients of must be and hence

$$2\zeta_n \omega_n = \frac{\gamma}{\rho A} + \frac{\beta}{E} \omega_n^2$$

and

$$\omega_n = \beta_n^2 \sqrt{\frac{EI}{\rho A}}$$

$$\zeta_n = \frac{\gamma}{2\rho A \omega_n} + \frac{\beta}{E} \omega_n$$

where β_n are given in table 6.4.

6.59 Calculate the form of the solution $w(x,t)$ for the system of Problem 6.58.

Solution:

The form of the solution of the m time equation is just

$$A_n e^{-\zeta_n \omega_n t} \sin\left(\omega_{dn} t + \phi_n\right)$$

where ζ_n and ω_n are as given in problem 6.58, $\omega_{dn} = \omega_n \sqrt{1-\zeta_n^2}$, and A_n and ϕ_n are constants determined by initial conditions. The total solution is of the form

$$w(x,t) = \sum_{n=1}^{\infty} A_n e^{-\zeta_n \omega_n t} \sin\left(\omega_{dn} t + \phi_n\right) X_n(x)$$

where $X_n(t)$ are the eigenfunctions given in table 6.4.

6.60 For a given cantilevered composite beam, the following values have been measured for bending vibration:

$E = 2.71\text{x}10^{10}$ N/m^2 $\quad\rho = 1710$ kg/m^3
$A = 0.597\text{x}10^{-3}$ m^2 $\quad l = 1$ m
$I = 1.64\text{x}10^{-9}$ m^4 $\quad\gamma = 1.75$ N s/m^2
$\beta = 20{,}500$ Ns/m^2

Calculate the solution for the beam to an initial displacement of $w_t(x,0) = 0$ and $w(x,0) = 3\sin \pi x$.

Solution:

Using the values given and the formulas for $a_n(t)$ from problem 6.58 the temporal equation becomes

$$\ddot{a}_n + \left(1.714 \times .00000075 G\omega_n^2\right)\dot{a}_n + \omega_n^2 a_n = 0$$

from problem 6.59,

$$w_t(x,t)\Big|_{t=0} = 0 = \sum A_n\left[\left(-\zeta_n\omega_n\right)\sin\phi_n + \omega_{dn}\cos\phi_n\right]X_n(x)$$

and

$$w(x,0) = 3\sin\pi x = \sum A_n \sin\phi_n X_n(x)$$

Multiplying by $X_n(x)$ and integrating yields that

$$\zeta_n\omega_n \sin\phi_n = \omega_{dn}\cos\phi_n \quad \text{or} \quad \tan\phi_n = \frac{\omega_{dn}}{\zeta_n\omega_n}$$

and $3\int_0^l \sin\pi x X_n(x)dx = A_n \sin\phi_n$ so that

$$A_n = \frac{3\int_0^l \sin\pi x X_n(x)dx}{\sin\phi_n} = \frac{3}{\sqrt{1-\zeta_n^2}}\int_0^l \sin\pi x X_n(x)dx$$

where $X_n(x)$ is given in table 6.4.

6.61 Plot the solution of Example 6.7.2 for the case $w_t(x,0) = 0$, $w(x,0)=\sin(nx/\ell)$, $\gamma=10$ Ns/m^2, $\tau = 10^4$ N, $\ell = 1$ m and $\rho = 0.01$ kg/m^3.

Solution: From equation (6.156) and the values given, $\zeta_1 = 0.159/n$ or $\zeta_n\omega_n = 500$ and $\omega_{dn} = \sqrt{1 = 0.159^2}$, so that:

$$w(x,t) = \sum_{n=1}^{\infty} A_n e^{-500t} \sin(\omega_{dn} t + \phi_n) \sin n\pi x$$

Applying the initial conditions yields

$$\int_0^l \sin n\pi x \sin m\pi x dx = \sum_{n=1}^{\infty} A_n \sin(\phi_n) \int_0^l \sin m\pi x \sin n\pi x dx$$

So that $A_n\sin\phi_n = 0$ for all n except $n = 1$, and $A_1\sin\phi_1 = 1$. So either $\phi_n = 0$ or $A_n = 0$ for n not zero. The other initial condition yields that $\phi_n = \tan^{-1}\left(\frac{-\sqrt{1-\zeta_n^2}}{\zeta_n}\right)$ so that $A_n = 0$ for n not zero. Thus the system is only excited in the first mode. Then

$$w(x,t) = A_1 e^{-500t} \sin(\omega_1\sqrt{1-\zeta_n^2}\, t + \phi_n) \sin \pi x$$
$$= -1.001 e^{-500t} \sin(3137.7t - 1.50) \sin \pi x$$

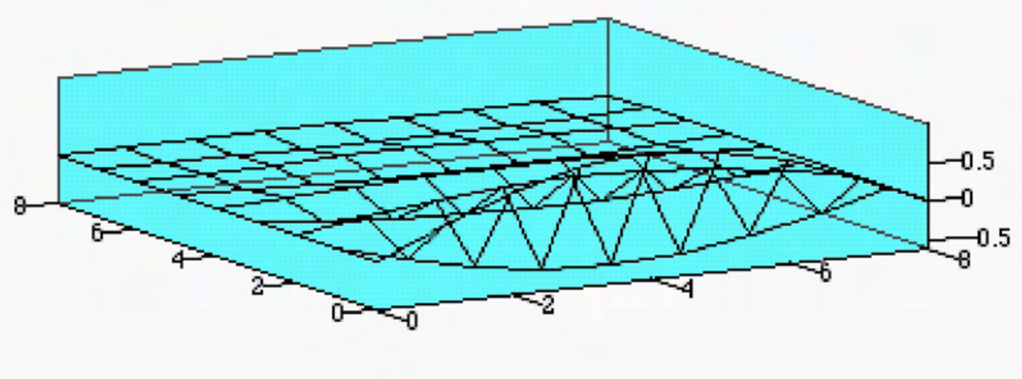
This is plotted in Mathcad below:

$N := 8 \quad i := 0..N \quad j := 0..N \quad x_i := i\cdot\frac{1}{8} \quad t_j := j\cdot 0.001$

$\zeta := 0.159 \quad \omega := \pi\cdot 10^3 \quad \phi := -1.52 \quad \omega d := \omega\cdot\sqrt{1-\zeta^2}$

$w(x,t) := -1.001\, e^{-\zeta\cdot\omega\cdot t}\cdot(\sin(\pi\cdot x))\cdot\sin(\omega d\cdot t + \phi)$

$M_{[i,j]} := w(x_i, t_j)$



M

6.62 Calculate the orthogonality condition for the system of Example 6.7.2. Then calculate the form of the temporal solution.

Solution: Problem is to fill in the details of example 6.7.2 by checking the coefficients. Equation (6.155) by performing the integration.

6.63 Calculate the form of modal damping for the longitudinal vibration of the beam of Figure 6.14 with boundary conditions specified by equation (6.157).

Solution: This is a discussion problem. The boundary condition given in equation (6.157)

$$AEw_x(0,t) = kw(0,t) + c\frac{\partial w(0,t)}{\partial t}$$
$$AEw_x(l,t) = -kw(l,t) - c\frac{\partial w(l,t)}{\partial t}$$

Do not conform readily to separation of variables and lead to time dependent boundary conditions. However one approach is to treat the damper as applied forces of the bar $cw_t(0,t)$ and $-cw_t(l,t)$. Following this approach the boundary conditions become

$$AEX'(0) = kX(0) \text{ and } AEX'(l) = -kX(l)$$

The general solution of the spatial equation of a bar has the form

$$X(x) = a\sin(\sigma x + b)$$

Where σ is the usual separation constant and a and b are constants. The first boundary condition yields that $\phi = \tan^{-1}(AE/k)$. The second boundary condition yields the characteristic equation

$$-(AE/k)\sigma_n = \tan(\sigma_n l + \phi)$$

Which can be solved for σ_n numerically. Note that σ_n are distinct so that from problem 6.39 the eigenfunctions are orthogonal, i.e. an can be calculated such that

$$X_n(x) = a_n\sin(\sigma_n x + \phi)$$

Are orthonormal. Following the procedure of example 6.8.11, the temporal solution for the forced response is

$$\ddot{T}_n(t) + \omega_n^2 = \int_0^l \left[cw_t(0,t) - cw(l,t)\right]X_r(x)dx$$
$$= \left\{\int_0^l \left[cX_n(0) - cX_n(l)\right]X_n(x)dx\right\}\dot{T}_n(t)$$

Bring the $\dot{T}_n$ term to the left side and comparing its coefficient to $2\zeta_n\omega_n$ yields

$$2\zeta_n\omega_n = \left\{\int_0^l c\left[X_n(l) - X_n(0)\right]X_n(x)\right\}dx$$

The form of the modal damping ratio is thus

$$\zeta_n - \frac{ca_n^2}{2\omega_n\sigma_n}\left[\cos\left(\sigma_n l + \phi\right) - \cos\phi\right]$$

Where a_n^2 is the normalization factor, σ_n are the eigenvalues $\omega_n^2 = c^2\sigma_n^2$ and $\tan^{-1}(AE/k)$.

Problems and Solutions Section 6.8 (6.64 through 6.68)

6.64 Calculate the response of the damped string of Example 6.8.1 to a disturbance force of $f(x,t)$ = (sin $\pi x/l$) sin10t.

Solution:

$f(x,t) = \sin\left(\frac{\pi x}{l}\right)\sin 10t.$ Assume a solution of the form:

$$w_n(x,t) = T_n(t)X_n(x)$$

where

$$X_n(x) = \sin\frac{n\pi x}{l}$$

Substitute into (6.158)

$$\left\{\rho\ddot{T}_n + \gamma\dot{T}_n - \tau\left[-\left(\frac{n\pi}{l}\right)^2\right]T_n\right\}\sin\frac{n\pi x}{l} = \sin\left(\frac{\pi x}{l}\right)\sin 10t$$

Multiply by $\sin\frac{n\pi x}{l}$ and integrate over the length of the string:

$$\left\{\rho\ddot{T}_n + \gamma\dot{T}_n + \tau\left(\frac{n\pi}{l}\right)^2 T_n\right\}\frac{l}{2} = \begin{cases} 0 & \text{for } n = 1 \\ \sin 10t & \text{for } n > 1 \end{cases}$$

Only the particular solution is of interest since we are looking for the response to the disturbance force. Therefore, dropping the subscripts:

$$\rho\ddot{T} + \gamma\dot{T} + \tau\left(\frac{\pi}{l}\right)^2 T = \sin 10t$$

$$\ddot{T} + \left(\frac{\gamma}{\rho}\right)\dot{T} + \left(\frac{c\pi}{l}\right)^2 T = \frac{\sin 10t}{\rho} \quad \text{where } c = \sqrt{\frac{\tau}{\rho}}$$

Solution is

$$T = A\sin(20t - \phi)$$

where

$$A = \frac{1}{\rho\sqrt{\left(\frac{c^2\pi^2}{l^2} - 100\right)^2 + 100\frac{\gamma^2}{\rho^2}}} = \frac{l^2}{\sqrt{\rho^2\left(c^2\pi^2 - 100l^2\right)^2 + 100\gamma^2 l^4}}$$

$$\phi = \tan^{-1}\left[\frac{10\frac{\gamma}{\rho}}{\frac{c^2\pi^2}{l^2} - 100}\right] = \tan^{-1}\left[\frac{10\gamma l^2}{\rho c^2\pi^2 - 100\rho l^2}\right]$$

$$w(x,t) = A\sin(10t - \phi)\sin\frac{\pi x}{l}$$

where A and ϕ are given above.

6.65 Consider the clamped-free bar of Example 6.3.2. The bar can be used to model a truck bed frame. If the truck hits an object (at the free end) causing an impulsive force of 100 N, calculate the resulting vibration of the frame. Note here that the truck cab is so massive compared to the bed frame that the end with the cab is modeled as clamped. This is illustrated in Figure P6.65.

Solution: Assume constant area and constant material properties. Equation of motion:

$$\rho A w_{tt} - EAw_{xx} = f(x,t) = -100\delta(x-l)\delta(t)$$

Mode shapes (eigenvalues) of a fixed-free bar are (Table 6.1)

$$X_n(x) = \sin\frac{(2n-1)\pi x}{2l}$$

Assume a solution of the form: $w_n(x,t) = X_n(x)T_n(t)$. Substitute into the equation of motion:

$$\left\{\ddot{T}_n - \left[-\left(\frac{(2n-1)\pi}{2l}\right)^2\right]c^2T_n\right\}\sin\frac{(2n-1)\pi x}{2l} = -\frac{100}{\rho A}\delta(x-l)\delta(t)dx$$

$$\left\{\ddot{T}_n + \omega_n^2 T_n\right\}\sin\frac{(2n-1)\pi x}{2l} = -\frac{100}{\rho A}\delta(x-l)\delta(t)$$

where $c^2 = \frac{E}{\rho}$ and $\omega_n = \frac{(2n-1)\pi c}{2l}$. Multiply by $\sin\frac{(2n-1)\pi x}{2l}$ and integrate over the length of the rod:

$$\ddot{T}_n + \omega_n^2 T_n = -\frac{2}{l}\int_0^l \frac{100}{\rho A}\sin\left(\frac{(2n-1)\pi x}{2l}\right)\delta(x-l)\delta(t)$$

$$= -\frac{200}{\rho A l}\sin\left(\frac{(2n-1)\pi}{2}\right)\delta(t)$$

which has the solution:

$$T_n(t) = -\frac{200}{\rho A l\omega_n}\sin\left(\frac{(2n-1)\pi}{2}\right)\sin\omega_n t$$

The total solution is:

$$w_n(x,t) = -\sum_{n=1}^{\infty}\left\{\left[\frac{400}{\rho A(2n-1)\pi c}\right]\sin\left(\frac{(2n-1)\pi}{2}\right)\right.$$
$$\left.\sin\left(\frac{(2n-1)\pi ct}{2l}\right)\sin\left(\frac{(2n-1)\pi x}{2l}\right)\right\}$$

6.66 A rotating machine sits on the second floor of a building just above a support column as indicated in Figure P6.66. Calculate the response of the column in terms of *E*, *A*, and ρ of the column modeled as a bar.

Solution: Referring to equation (6.55) for the equation of a bar and summing forces to get the effect of the applied force yields

$$\rho A w_{tt} - EA w_{xx} = \delta(x-l) F_0 \sin\omega t$$

subject to the boundary conditions $w(0,t) = w_x(0,t) = 0$. Following the method of example 6.8.1, use separation of variables where the spatial function is the clamped-free mode shapes used in example 6.3.1:

$$w(x,t) = X_n(x)T_n(t) = (a_n \sin\sigma_n x)T_n(t), \quad \sigma_n = \frac{2n-1}{2l}\pi$$

Substitution into the equation of motion yields

$$\left(\rho A \ddot{T}_n(t) + EA\sigma_n^2 T_n(t)\right) a_n \sin\sigma_n x = \delta(x-l) F_0 \sin\omega t$$

(the minus sign in front of *EA* goes away because of the second derivative of sine being negative). Next, let $a_n = 1$ (recalling that eigenvectors have arbitrary magnitude) and multiply by $\sin\sigma_n x$ and integrate over the length of the beam to get:

$$\left(\rho A \ddot{T}_n(t) + EA\sigma_n^2 T_n(t)\right)\frac{l}{2} = F_0 \sin\omega t \int_0^l \delta(x-l)\sin\sigma_n x dx$$

The integral on the right is a bit tricky as the delta function acts at the end of the interval. The details are below, however integrating yields

$$\left(\rho A \ddot{T}_n(t) + EA\sigma_n^2 T_n(t)\right)\frac{l}{2} = \frac{1}{2} F_0 \sin\omega t \frac{\sin\sigma_n l}{2} = (-1)^{n-1}\frac{F_0}{2}\sin\omega t$$

Dividing by the appropriate constants this simplifies to

$$\ddot{T}_n(t) + \frac{E}{\rho}\sigma_n^2 T_n(t) = \frac{(-1)^{n-1}F_0}{\rho A}\sin\omega t$$

This has particular solution

$$T_{np}(t) = \frac{(-1)^{n-1}}{\rho A}\left(\frac{F_0}{\omega_n^2 - \omega^2}\right)\sin\omega t \quad \text{where } \omega_n = \sqrt{\frac{E}{\rho}}\frac{(2n-1)\pi}{2l}$$

Combined with the homogenous solution, the total temporal solution is

$$T_n(t) = C_{1n}\sin\omega_n t + C_{2n}\cos\omega_n t + \left(\frac{(-1)^{n-1}}{\rho A}\frac{F_0}{\omega_n^2 - \omega^2}\right)\sin\omega t$$

So the total solution is

$$w(x,t) = \sum_{n=1}^{\infty}\left\{C_{1n}\sin\omega_n t + C_{2n}\cos\omega_n t + \frac{(-1)^{n-1}}{\rho A}\left(\frac{F_0}{\omega_n^2 - \omega^2}\right)\sin\omega t\right\}\sin\left(\frac{(2n-1)\pi x}{2l}\right)$$

The following it the evaluation of the Dirac integral used about (courtesy of Jamil Renno)
Start with the integral at hand

$$\int_0^l \delta(x-l)\sin(\sigma_m x)dx = \lim_{\tau\to 0}\left[\int_0^l d_\tau(x-l)\sin(\sigma_m x)dx\right]$$

where $d_\tau(x-l) = \begin{cases} \dfrac{1}{2\tau} & l-\tau < x < l+\tau \\ 0 & x \le l-\tau \text{ or } x \ge l+\tau \end{cases}$ is the pulse over the interval $\left[l-\tau, l+\tau\right]$.

Hence, the integral can be subdivided over two intervals

$$\int_0^l \delta(x-l)\sin(\sigma_m x)dx = \lim_{\tau\to 0}\left[\int_0^{l-\tau} d_\tau(x-l)\sin(\sigma_m x)dx + \int_{l-\tau}^{l} d_\tau(x-l)\sin(\sigma_m x)dx\right]$$

$$= \lim_{\tau\to 0}\left[\int_0^{l-\tau} 0\sin(\sigma_m x)dx + \int_{l-\tau}^{l}\frac{1}{2\tau}\sin(\sigma_m x)dx\right] = \lim_{\tau\to 0}\left[\int_{l-\tau}^{l}\frac{1}{2\tau}\sin(\sigma_m x)dx\right]$$

$$= \lim_{\tau\to 0}\left[\frac{1}{2\tau}\frac{1}{\sigma_m}\left[-\cos(\sigma_m x)\right]_{l-\tau}^{l}\right] = \lim_{\tau\to 0}\frac{\cos\left[\sigma_m(l-\tau)\right]-\cos\left[\sigma_m l\right]}{2\tau\sigma_m}$$

$$\overset{\text{L'Hopital's Rule}}{=} \lim_{\tau\to 0}\frac{\dfrac{d}{d\tau}\left\{\cos\left[\sigma_m(l-\tau)\right]-\cos\left[\sigma_m l\right]\right\}}{\dfrac{d}{d\tau}\left\{2\tau\sigma_m\right\}}$$

$$= \lim_{\tau\to 0}\frac{\left(\dfrac{-1}{\sigma_m}\right)\left(-\sin\left[\sigma_m(l-\tau)\right]\right)}{2\sigma_m} = \frac{\sin(\sigma_m l)}{2}$$

6.67 Recall Example 6.8.2, which models the vibration of a building due to a rotating machine imbalance on the second floor. Suppose that the floor is constructed so that the beam is clamped at one end and pinned at the other, and recalculate the response (recall Example 6.5.1). Compare your solution and that of Example 6.8.2, and discuss the difference.

Solution:

Clamped-pinned beam conditions yield mode shapes (eigenfunctions) of the form:

$$X_n(x) = a_n\left[\cosh\beta_n x - \cos\beta_n x - \sigma_n(\sinh\beta_n x - \sin\beta_n x)\right]$$

where $\tan\beta_n l = \tanh\beta_n l$ and

$$\sigma_n = \begin{cases} 1.0008 & \text{for } n = 1 \\ 1 & \text{for } n > 1 \end{cases}$$

Normalize the mode shape as follows:

$$\int_0^l X_n^2 dx = 1 \Rightarrow$$

$$a_n^2 \int_0^l [\cosh\beta_n x - \cos\beta_n x - \sigma_n(\sinh\beta_n x - \sin\beta_n x)]^2 dx = 1$$

From Mathematica

$$\begin{aligned} a_n^2 = 4\beta_n / \{&4\beta_n l + 2\sigma_n\cos(2\beta_n l) - 2\sigma_n\cosh(2\beta_n l) - 4\cosh(\beta_n l)\sin(\beta_n l) \\ &- 4\sigma_n^2\cosh(\beta_n l)\sin(\beta_n l) + \sin(2\beta_n l) - \sigma_n^2\sin(2\beta_n l) \\ &- 4\cos(\beta_n l)\sinh(\beta_n l) + 4\sigma_n^2\cos(\beta_n l)\sinh(\beta_n l) \\ &+ 8\sigma_n\sin(\beta_n l)\sinh(\beta_n l) + \sinh(2\beta_n l) + \sigma_n^2\sinh(2\beta_n l)\} \end{aligned}$$

The equation of motion for the system is: (constant properties)

$$\rho A w_{tt} + EI w_{xxxx} = f(x,t) = 100\sin 3t\,\delta\left(x - \frac{l}{2}\right)$$

Assume a solution of the form: $w_n(x,t) = X_n(x)T_n(t)$

$$\ddot{T}_n X_n + \frac{EI}{\rho A} T_n X_n'''' = \frac{100}{\rho A}\sin 3t\,\delta\left(x - \frac{l}{2}\right)$$

Using the mode shapes given above:

$$X_n'''' = \beta_n^4 X_n = \frac{\omega_n^2}{c^2} X_n$$

where

$$\beta_n^4 = \frac{\rho A}{EI}\omega_n^2, \quad c^2 = \frac{EI}{\rho A}$$

The equation of motion reduces to:

$$\left\{\ddot{T}_n + \omega_n^2\right\} X_n = \frac{100}{\rho A}\sin 3t\,\delta\left(x - \frac{l}{2}\right)$$

Multiply by X_n and integrate over the length of the beam:

$$\ddot{T}_n + \omega_n^2 T_n = \frac{100}{\rho A}\sin 3t \int_0^l X_n(x)\delta\left(x - \frac{l}{2}\right)dx$$

$$= \frac{100}{\rho A}\sin 3t X_n\left(\frac{l}{2}\right)$$

$$= \frac{100 a_n}{\rho A}\sin 3t\left[\cosh\frac{\beta_n l}{2} - \cos\frac{\beta_n l}{2} - \sigma_n\left(\sinh\frac{\beta_n l}{2} - \sin\frac{\beta_n l}{2}\right)\right]$$

or:

$$T_n(t) = \left[\frac{100 X_n\left(\frac{l}{2}\right)}{\rho A\left(\omega_n^2 - 9\right)}\right]\sin 3t$$

The solution is then:

$$w(x,t) = \sum_{n=1}^{\infty}\left\{a_n\left[\cosh\beta_n x - \cos\beta_n x - \sigma_n\left(\sinh\beta_n x - \sin\beta_n x\right)\right]\right.$$
$$\left.\left[\frac{100}{\rho A\left(\omega_n^2 - 9\right)}\right] X_n\left(\frac{l}{2}\right)\right\}\sin 3t$$

where a_n, ω_n, and β_n are given above. The free time response is stiffer for the clamped case as the frequencies are higher (See Table 6.4).

The comparison of the solution between the two models (one with a pinned end and one with a fixed or clamped end) had two purposes: design and modeling. From the design point of view it is important to know how to construct the floor for a minimum value of response. From the modeling point of view it is important to know how much the solution is effected by the choice of boundary conditions as part of the modeling.

Here the comparison can be made by calculating the response and then evaluating it and plotting it using a truncated solution (say 3 modes, as given in Equation 6.181) at a given point of interest (i.e. for a particular value of x). This gives an accurate comparison.

Next you can compare the differences in the details. For instance the clamped-pinned natural frequencies are lower then the clamped-clamped frequencies (just look at Table 6.4) because the clamped-clamped system is stiffer. Next, one of these sets of frequencies is going to have a natural frequency that is closer to the driving frequency, and hence produce a larger response. To make such comparisons, pick a value for the physical parameters (let omega = beta squared for instance) and check. In this case the clamped-pinned frequency is about 3.9 rad/s, which is much closer to the driving frequency of 3 rad/s then the clamped-clamped first natural frequency of 4.7 rad/s. Thus the first term in the series solution for the example will be larger then the corresponding term in the series solution for the clamped-clamped case.

6.68 Use the modal analysis procedure suggested at the end of Section 6.8 to calculate the response of a clamped free beam with a sinusoidal loading $F_0 \sin\omega t$ at its free end.

Solution:

The equation of motion is:

$$\rho A w_{tt} + EI w_{xxxx} = f(x,t) = F_0 \delta(x-l)\sin\omega t$$

Assume a solution of the form $w_n(x,t) = X_n(x) T_n(t)$

$$\ddot{T}_n X_n + \frac{EI}{\rho A} T_n X_n'''' = \frac{F_0}{\rho A}\delta(x,l)\sin\omega t$$

The mode shapes are given in Table 6.4 for a fixed-free beam:

$$X_n(x) = a_n\left[\cosh\beta_n x - \cos\beta_n x - \sigma_n\left(\sinh\beta_n x - \sin\beta_n x\right)\right]$$

Where

$$\sigma_n = \frac{\sinh\beta_n l - \sin\beta_n l}{\cosh\beta_n l + \cos\beta_n l}$$

$$\beta_n^4 = \frac{\rho A}{EI}\omega_n^2$$

And

$$\cos\beta_n l \cosh\beta_n l = -1$$

From the unforced vibration problem:

$$\ddot{T}_n X_n + \frac{EI}{\rho A} T_n X_n'''' = 0$$

$$\frac{\ddot{T}_n}{T_n} = -\left(\frac{EI}{\rho A}\right)\frac{X_n''''}{X_n} = -\omega_n^2$$

Therefore

$$X_n'''' = \frac{\rho A}{EI}\omega_n^2 X_n = \beta_n^4 X_n$$

Substitute into the equation of motion and rearrange:

$$\left\{\ddot{T}_n + \omega_n^2 T_n\right\} X_n = \frac{F_0}{\rho A}\delta(x-l)\sin\omega t$$

Normalize the mode shapes as follows:

$$\int_0^l X_n^2 dx = 1$$

$$a_n^2 \int_0^l \left[\cosh\beta_n x - \cos\beta_n x - \sigma_n\left(\sinh\beta_n x - \sin\beta_n x\right)\right]^2 dx = 1$$

$$\begin{aligned} a_n^2 = 4\beta_n / &\left\{4\beta_n l + 2\sigma_n \cos\left(2\beta_n l\right) - 2\sigma_n \cosh\left(2\beta_n l\right) - 4\cosh\left(\beta_n l\right)\sin\left(\beta_n l\right)\right. \\ &- 4\sigma_n^2 \cosh\left(\beta_n l\right) + \sin\left(2\beta_n l\right) - \sigma_n^2 \sin\left(2\beta_n l\right) \\ &- 4\cos\left(\beta_n l\right)\sinh\left(\beta_n l\right) + 4\sigma_n^2 \cos\left(\beta_n l\right)\sinh\left(\beta_n l\right) \\ &\left. + 8\sigma_n \sin\left(\beta_n l\right)\sinh\left(\beta_n l\right) + \sinh\left(2\beta_n l\right) + \sigma_n^2 \sinh\left(2\beta_n l\right)\right\} \end{aligned}$$

Multiply the equation of motion $X_n(x)$ and integrate over the length of the beam:

$$\ddot{T}_n + \omega_n^2 T_n = \frac{F_0}{\rho A}\int_0^l X_n(x)\delta(x-l)dx\sin\omega t$$

$$= \frac{F_0}{\rho A} X_n(l)\sin\omega t$$

Solving:

$$T_n(t) = \left(\frac{F_0}{\rho A}\right)\left(\frac{X_n(l)}{\omega_n^2 - \omega^2}\right)\sin\omega t$$

The total solution is:

$$\begin{aligned} w(x,t) = \sum_{n=1}^{\infty} &\left\{a_n\left[\cosh\beta_n x - \cos\beta_n x - \sigma_n\left(\sinh\beta_n x - \sin\beta_n x\right)\right]\right. \\ &\left.\left(\frac{F_0}{\rho A}\right)\left[\frac{X_n(l)}{\omega_n^2 - \omega^2}\right]\right\}\sin\omega t \end{aligned}$$

Where σ_n, w_n are given above and $\cos\beta_n l \cosh\beta_n l = -1$.