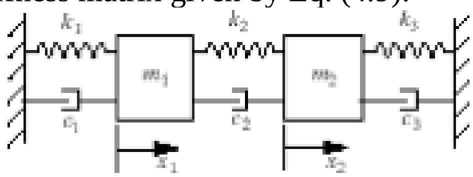
Problems and Solutions for Section 4.1 (4.1 through 4.16)

4.1 Consider the system of Figure P4.1. For $c_1 = c_2 = c_3 = 0$, derive the equation of motion and calculate the mass and stiffness matrices. Note that setting $k_3 = 0$ in your solution should result in the stiffness matrix given by Eq. (4.9).



Solution:

For mass 1:

$$m_1\ddot{x}_1 = -k_1x_1 + k_2(x_2 - x_1)$$
$$\Rightarrow m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 = 0$$

For mass 2:

$$m_2\ddot{x}_2 = -k_3x_2 - k_2(x_2 - x_1)$$
$$\Rightarrow m_2\ddot{x}_2 - k_2x_1 + (k_2 + k_3)x_2 = 0$$

So, $M\ddot{\mathbf{x}} + K\mathbf{x} = 0$

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\ddot{\mathbf{x}} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}\mathbf{x} = \mathbf{0}$$

Thus:

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \quad K = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}$$

4.2 Calculate the characteristic equation from problem 4.1 for the case

$$m_1 = 9 \text{ kg} \quad m_2 = 1 \text{ kg} \quad k_1 = 24 \text{ N/m} \quad k_2 = 3 \text{ N/m} \quad k_3 = 3 \text{ N/m}$$

and solve for the system's natural frequencies.

Solution: Characteristic equation is found from Eq. (4.9):

$$\det\left(-\omega^2 M + K\right) = 0$$

$$\begin{vmatrix} -\omega^2 m_1 + k_1 + k_2 & -k_2 \\ -k_2 & -\omega^2 m_1 + k_2 + k_3 \end{vmatrix} = \begin{vmatrix} -9\omega^2 + 27 & -3 \\ -3 & -\omega^2 + 6 \end{vmatrix} = 0$$

$$9\omega^4 - 81\omega^2 + 153 = 0$$

Solving for ω:

$$\begin{matrix} \omega_1 = \mathbf{1.642} \\ \omega_2 = \mathbf{2.511} \end{matrix} \text{ rad/s}$$

4.3 Calculate the vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ for problem 4.2.

Solution: Calculate $\mathbf{u}_1$:

$$\begin{bmatrix} (-2.697)(9) + 27 & -3 \\ -3 & -2.697 + 6 \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This yields

$$2.727u_{11} - 3u_{21} = 0$$

$$-3u_{11} + 3.303u_{21} = 0 \qquad \text{or, } u_{21} = 0.909u_{11}$$

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0.909 \end{bmatrix}$$

Calculate $\mathbf{u}_2$:

$$\begin{bmatrix} (-6.303)(9) + 27 & -3 \\ -3 & -6.303 + 6 \end{bmatrix} \begin{bmatrix} u_{12} \\ u_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-29.727u_{12} - 3u_{22} = 0$$
$$-3u_{12} = 0.303u_{22} = 0 \qquad \text{or, } u_{12} = -0.101u_{22}$$

This yields

$$\mathbf{u}_2 = \begin{bmatrix} -0.101 \\ 1 \end{bmatrix}$$

4.4 For initial conditions $\mathbf{x}(0) = [1\ \ 0]^T$ and $\dot{\mathbf{x}}(0) = [0\ 0]^T$ calculate the free response of the system of Problem 4.2. Plot the response x_1 and x_2.

Solution: Given $\mathbf{x}(0) = [1\ \ 0]^T$, $\dot{\mathbf{x}}(0) = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$, The solution is

$$\mathbf{x}(t) = A_1 \sin(\omega_1 t + \phi_1)\mathbf{u}_1 + A_2 \sin(\omega_2 t + \phi_2)\mathbf{u}_2$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} A_1 \sin(\omega_1 t + \phi_1) - 0.101A_2 \sin(\omega_2 t + \phi_2) \\ 0.909A_1 \sin(\omega_1 t + \phi_1) + A_2 \sin(\omega_2 t + \phi_2) \end{bmatrix}$$

Using initial conditions,

$$1 = A_1 \sin\phi_1 - 0.101A_2 \sin\phi_2 \qquad [1]$$
$$0 = 0.909A_1 \sin\phi_1 + A_2 \sin\phi_2 \qquad [2]$$
$$0 = 1.642A_1 \cos\phi_1 - 0.2536A_2 \cos\phi_2 \qquad [3]$$
$$0 = 6.033A_1 \cos\phi_1 + 2.511A_2 \cos\phi_2 \qquad [4]$$

From [3] and [4], $\phi_1 = \phi_2 = \pi / 2$

From [1] and [2], $A_1 = 0.916$, and $A_2 = -0.833$

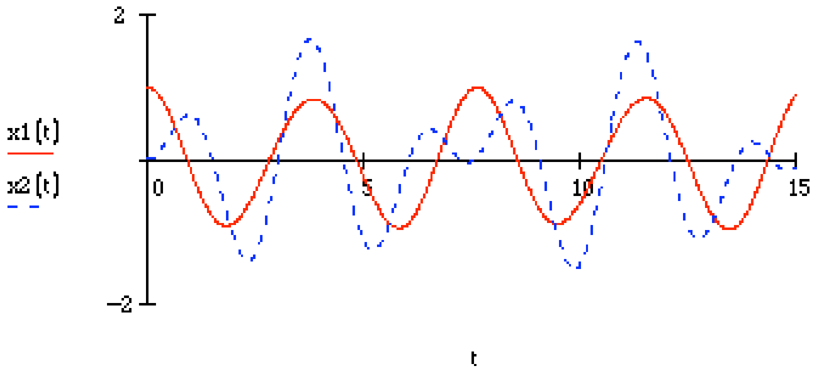
So,

$$x_1(t) = 0.916\sin(1.642t + \pi / 2) + 0.0841\sin(2.511t + \pi / 2)$$
$$x_2(t) = 0.833\sin(1.642t + \pi / 2) - 0.833\sin(2.511t + \pi / 2)$$

$$\boldsymbol{x}_1(t) = 0.916\cos 1.642t + 0.0841\cos 2.511t$$
$$\boldsymbol{x}_2(t) = 0.833(\cos 1.642t - \cos 2.511t)$$

$x1(t) := 0.916 \cdot \cos(1.642 \cdot t) + 0.0841 \cdot \cos(2.511 \cdot t)$

$x2(t) := 0.833 \cdot (\cos(1.642 \cdot t) - \cos(2.511 \cdot t))$



4.5 Calculate the response of the system of Example 4.1.7 to the initial condition $\mathbf{x}(0) = 0, \dot{\mathbf{x}}(0) = [1 \;\; 0]^T$, plot the response and compare the result to Figure 4.3.

Solution: Given: $\mathbf{x}(0) = \mathbf{0}$, $\dot{\mathbf{x}}(0) = [1 \;\; 0]^T$

From Eq. (4.27) and example 4.1.7,

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{3} A_1 \sin\left(\sqrt{2t} + \phi_1\right) - \frac{1}{3} A_2 \sin\left(2t + \phi_2\right) \\ A_1 \sin\left(\sqrt{2t} + \phi_1\right) + A_2 \sin\left(2t + \phi_2\right) \end{bmatrix}$$

Using initial conditions:

$$0 = A_1 \sin\phi_1 - A_2 \sin\phi_2 \qquad [1]$$
$$0 = A_1 \sin\phi_1 + A_2 \sin\phi_2 \qquad [2]$$
$$3 = \sqrt{2} A_1 \cos\phi_1 - 2 A_2 \cos\phi_2 \qquad [3]$$
$$0 = \sqrt{2} A_1 \cos\phi_1 + 2 A_2 \cos\phi_2 \qquad [4]$$

From [1] and [2]:

$$\phi_1 = \phi_2 = 0$$

From [3] and [4]:

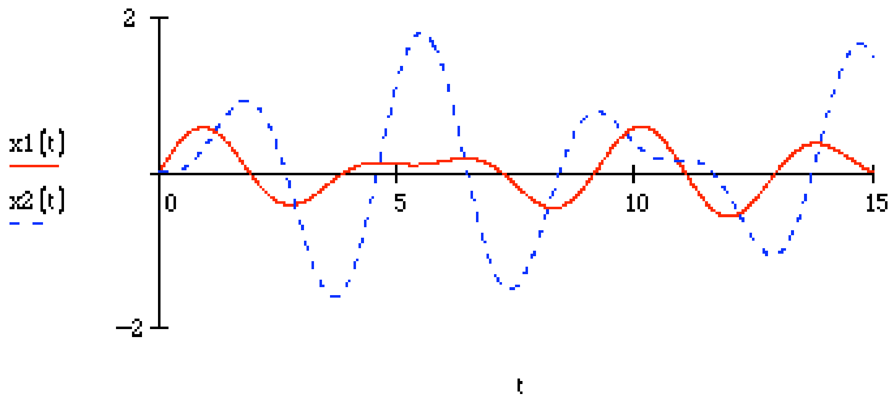
$$A_1 = \frac{3\sqrt{2}}{4}, \text{ and } A_2 = -\frac{3}{4}$$

The solution is

$$x_1(t) = 0.25\left(\sqrt{2}\sin\sqrt{2}t + \sin 2t\right)$$
$$x_2(t) = 0.75\left(\sqrt{2}\sin\sqrt{2}t - \sin 2t\right)$$

As in Fig. 4.3, the second mass has a larger displacement than the first mass.

$x1(t) := 0.25\cdot\left(\sqrt{2}\cdot\sin\left(\sqrt{2}\cdot t\right) + \sin(2\cdot t)\right)$ $\quad x2(t) := 0.75\cdot\left(\sqrt{2}\cdot\sin\left(\sqrt{2}\cdot t\right) - \sin(2\cdot t)\right)$



4.6 Repeat Problem 4.1 for the case that $k_1 = k_3 = 0$.

Solution:

The equations of motion are

$$m_1\ddot{x}_1 + k_2x_1 - k_2x_2 = 0$$
$$m_2\ddot{x}_2 - k_2x_1 + k_2x_2 = 0$$

So, $M\ddot{x} + Kx = 0$

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\ddot{x} + \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}x = 0$$
$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \text{ and } K = \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}$$

4.7 Calculate and solve the characteristic equation for Problem 4.6 with $m_1 = 9$, $m_2 = 1$, $k_2 = 10$.

Solution:

The characteristic equation is found from Eq. (4.19):

$$\det\left(-\omega^2 M + K\right) = 0$$
$$\begin{vmatrix} -9\omega^2 + 10 & -10 \\ -10 & -\omega^2 + 10 \end{vmatrix} = 9\omega^4 - 100\omega^2 = 0$$
$$\omega_{1,2}^2 = 0, 11.111$$
$$\omega_1 = 0$$
$$\omega_2 = 3.333$$

4.8 Compute the natural frequencies of the following system:

$$\begin{bmatrix} 6 & 2 \\ 2 & 4 \end{bmatrix}\ddot{\mathbf{x}}(t) + \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}\mathbf{x}(t) = 0\,.$$

Solution:

$$\det(-\omega^2 M + K) = \det\left(-\omega^2\begin{bmatrix} 6 & 2 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}\right) = 20\omega^4 - 22\omega^2 + 2 = 0,\ \omega^2 = 0.1,\ 1$$

$\underline{\omega_{1,2} = 0.316,\ 1 \text{ rad/s}}$

4.9 Calculate the solution to the problem of Example 4.1.7, to the initial conditions

$$\mathbf{x}(0) = \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix} \quad \dot{\mathbf{x}}(0) = \mathbf{0}$$

Plot the response and compare it to that of Fig. 4.3.

Solution: Given: $\mathbf{x}(0) = \begin{bmatrix} 1/3 & 1 \end{bmatrix}^T$, $\dot{\mathbf{x}}(0) = 0$

From Eq. (4.27) and example 4.1.7,

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{3}A_1 \sin\left(\sqrt{2}t + \phi_1\right) - \frac{1}{3}A_2 \sin\left(2t + \phi_2\right) \\ A_1 \sin\left(\sqrt{2}t + \phi_1\right) + A_2 \sin\left(2t + \phi_2\right) \end{bmatrix}$$

Using initial conditions:

$$1 = A_1 \sin\phi_1 - A_2 \sin\phi_2 \qquad [1]$$
$$1 = A_1 \sin\phi_1 + A_2 \sin\phi_2 \qquad [2]$$
$$0 = \sqrt{2}A_1 \cos\phi_1 - 2A_2 \cos\phi_2 \qquad [3]$$
$$0 = \sqrt{2}A_1 \cos\phi_1 + 2A_2 \cos\phi_2 \qquad [4]$$

From [3] and [4]: $\phi_1 = \phi_2 = \dfrac{\pi}{2}$

From [1] and [2]: $A_1 = 1$, and $A_2 = 0$

The solution is

$$x_1(t) = \frac{1}{3}\cos\sqrt{2}t$$
$$x_2(t) = \cos\sqrt{2}t$$

In this problem, both masses oscillate at only one frequency.

4.10 Calculate the solution to Example 4.1.7 for the initial condition

$$x(0)=\begin{bmatrix}-\frac{1}{3}\\ 1\end{bmatrix}\quad \dot{x}(0)=0$$

Solution:

Given: $\mathbf{x}(0) = [-1/3 \quad 1]^T$, $\dot{\mathbf{x}}(0)=0$

From Eq. (4.27) and example 4.1.7,

$$\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}=\begin{bmatrix}\frac{1}{3}A_1\sin\left(\sqrt{2}t+\phi_1\right)-\frac{1}{3}A_2\left(2t+\phi_2\right)\\ A_1\sin\left(\sqrt{2}t+\phi_1\right)+A_2\sin\left(2t+\phi_2\right)\end{bmatrix}$$

Using initial conditions:

$$\begin{aligned}
-1 &= A_1\sin\phi_1 - A_2\sin\phi_2 &&[1]\\
1 &= A_1\sin\phi_1 + A_2\sin\phi_2 &&[2]\\
0 &= \sqrt{2}A_1\cos\phi_1 - 2A_2\cos\phi_2 &&[3]\\
0 &= \sqrt{2}A_1\cos\phi_1 + 2A_2\cos\phi_2 &&[4]
\end{aligned}$$

From [3] and [4]

$$\phi_1=\phi_2=\frac{\pi}{2}$$

From [1] and [2]:

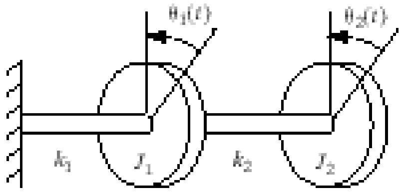
$$\begin{aligned}A_1&=0\\ A_2&=1\end{aligned}$$

The solution is

$$\begin{aligned}x_1(t)&=-\frac{1}{3}\cos 2t\\ x_2(t)&=\cos 2t\end{aligned}$$

In this problem, both masses oscillate at only one frequency (not the same frequency as in Problem 4.9, though.)

4.11 Determine the equation of motion in matrix form, then calculate the natural frequencies and mode shapes of the torsional system of Figure P4.11. Assume that the torsional stiffness values provided by the shaft are equal $\left(k_1 = k_2\right)$ and that disk 1 has three times the inertia as that of disk $2(J_1 = 3J_2)$.



Solution: Let $k = k_1 = k_2$ and $J_1 = 3J_2$. The equations of motion are

$$J_1\ddot{\theta}_1 + 2k\theta_1 - k\theta_2 = 0$$
$$J_2\ddot{\theta}_2 - k\theta_1 + k\theta_2 = 0$$

So,

$$J_2\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\ddot{\theta} + k\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\theta = 0$$

Calculate the natural frequencies:

$$\det\left(-\omega^2 J + K\right) = \begin{vmatrix} -3\omega^2 J_2 + 2k & -k \\ -k & -\omega^2 J_2 + k \end{vmatrix} = 0$$

$$\omega_1 = 0.482\sqrt{\frac{k}{J_2}}$$

$$\omega_2 = 1.198\sqrt{\frac{k}{J_2}}$$

Calculate the mode shapes: mode shape 1:

$$\begin{bmatrix} -3\left(0.2324\right)k + 2k & -k \\ -k & -\left(0.2324\right)k + k \end{bmatrix}\begin{bmatrix} u_{11} \\ u_{12} \end{bmatrix} = 0$$

$$u_{11} = 0.7676u_{12}$$

So, $\mathbf{u}_1 = \begin{bmatrix} 0.7676 \\ 1 \end{bmatrix}$

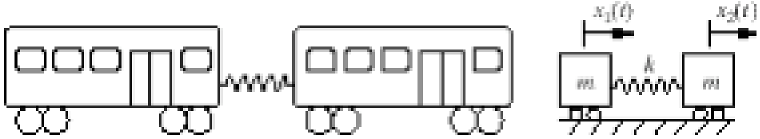
mode shape 2:

$$\begin{bmatrix} -3\left(1.434\right)k + 2k & -k \\ -k & -\left(1.434\right)k + k \end{bmatrix}\begin{bmatrix} u_{21} \\ u_{22} \end{bmatrix} = 0$$

$$u_{21} = -0.434u_{22}$$

So, $\mathbf{u}_2 = \begin{bmatrix} -0.434 \\ 1 \end{bmatrix}$

4.12 Two subway cars of Fig. P4.12 have 2000 kg mass each and are connected by a coupler. The coupler can be modeled as a spring of stiffness k = 280,000 N/m. Write the equation of motion and calculate the natural frequencies and (normalized) mode shapes.



Solution: Given: $m_1 = m_2 = m = 2000$ kg $\qquad k = 280{,}000$ N/m

The equations of motion are:

$$m\ddot{x}_1 + kx_1 - kx_2 = 0$$
$$m\ddot{x}_2 - kx_1 + kx_2 = 0$$

In matrix form this becomes:

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\ddot{x} + \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}x = 0$$
$$\begin{bmatrix} 2000 & 0 \\ 0 & 2000 \end{bmatrix}\ddot{x} + \begin{bmatrix} 280{,}000 & -280{,}000 \\ -280{,}000 & 280{,}000 \end{bmatrix}x = 0$$

Natural frequencies:

$$\det\left(-\omega^2 M + K\right) = 0$$
$$\begin{vmatrix} -2000\omega^2 + 280{,}000 & -280{,}000 \\ -280{,}000 & -2000\omega^2 + 280{,}000 \end{vmatrix} = 0$$
$$4\times10^6\omega^4 - 1.12\times10^9\omega^2 = 0$$
$$\omega^2 = 0, 280 \Rightarrow \omega_1 = 0 \text{ rad/sec and } \omega_2 = 16.73 \text{ rad/sec}$$

Mode shapes:

Mode 1, $\omega_1^2 = 0$

$$\begin{bmatrix} 280{,}000 & -280{,}000 \\ -280{,}000 & 280{,}000 \end{bmatrix}\begin{bmatrix} u_{11} \\ u_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\therefore u_{11} = u_{12}$$
$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Mode 2, $\omega_2^2 = 280$

$$\begin{bmatrix} -280{,}000 & -280{,}000 \\ -280{,}000 & -280{,}000 \end{bmatrix} \begin{bmatrix} u_{21} \\ u_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\therefore u_{21} = u_{22}$$

$$\mathbf{u}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Normalizing the mode shapes yields

$$\mathbf{u}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{u}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Note that $\mathbf{u}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is also acceptable because a mode shape times a constant (-1 in this case) is still a mode shape.

4.13 Suppose that the subway cars of Problem 4.12 are given the initial position of x_{10} = 0, x_{20} = 0.1 m and initial velocities of $v_{10} = v_{20} = 0$. Calculate the response of the cars.

Solution:

Given: $\mathbf{x}(0) = \begin{bmatrix} 0 & 0.1 \end{bmatrix}^T$, $\dot{\mathbf{x}}(0) = \mathbf{0}$

From problem 12,

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } \mathbf{u}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$
$$\omega_1 = 0 \text{ rad/s and } \omega_2 = 16.73 \text{ rad/s}$$

The solution is

$$\mathbf{x}(t) = (c_1 + c_2 t)\mathbf{u}_1 + A\sin(16.73t + \phi)\mathbf{u}_2$$
$$\Rightarrow \dot{\mathbf{x}}(0) = c_2\mathbf{u}_1 + 16.73A\cos(\phi)\mathbf{u}_2 \text{ and } \mathbf{x}(0) = c_1\mathbf{u}_1 + A\sin(\phi)\mathbf{u}_2$$

Using initial the conditions four equations in four unknowns result:

$$0 = c_1 + A\sin\phi \qquad [1]$$
$$0.1 = c_1 - A\sin\phi \qquad [2]$$
$$0 = c_2 + 16.73A\cos\phi \qquad [3]$$
$$0 = c_2 - 16.73A\cos\phi \qquad [4]$$

From [3] and [4]: $c_2 = 0$, and $\phi = \dfrac{\pi}{2}$ rad

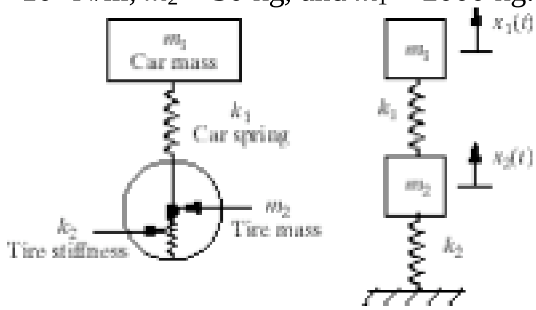
From [1] and [2]: $c_1 = 0.05$ m and $A = -0.05$ m

The solution is

$$x_1(t) = 0.05 - 0.05\cos 16.73t$$
$$x_2(t) = 0.05 + 0.05\cos 16.73t$$

Note that if $\mathbf{u}_2 = \dfrac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is chosen as the second mode shape the answer will remain the same. It might be worth presenting both solutions in class, as students are often skeptical that the two choices will yield the same result.

4.14 A slightly more sophisticated model of a vehicle suspension system is given in Figure P4.14. Write the equations of motion in matrix form. Calculate the natural frequencies for $k_1 = 10^3$ N/m, $k_2 = 10^4$ N/m, $m_2 = 50$ kg, and $m_1 = 2000$ kg.



Solution: The equations of motion are

$$2000\ddot{x}_1 + 1000x_1 - 1000x_2 = 0$$

$$50\ddot{x}_2 - 1000x_1 + 11{,}000x_2 = 0$$

In matrix form this becomes:

$$\begin{bmatrix} 2000 & 0 \\ 0 & 50 \end{bmatrix}\ddot{\mathbf{x}} + \begin{bmatrix} 1000 & -1000 \\ -1000 & 11{,}000 \end{bmatrix}\mathbf{x} = \mathbf{0}$$

Natural frequencies:

$$\det\left(-\omega^2 M + K\right) = 0$$

$$\begin{vmatrix} -2000\omega^2 + 1000 & -1000 \\ -1000 & -50\omega^2 + 11{,}000 \end{vmatrix} = 100{,}000\omega^4 - 2.205\times 10^7\omega^2 + 10^7 = 0$$

$$\omega_{1,2}^2 = 0.454, \quad 220.046 \quad \Rightarrow \omega_1 = 0.674 \text{ rad/s} \quad \text{and} \quad \omega_2 = 14.8 \text{ rad/s}$$

4.15 Examine the effect of the initial condition of the system of Figure 4.1(a) on the responses x_1 and x_2 by repeating the solution of Example 4.1.7, first for $x_{10} = 0, x_{20} = 1$ with $\dot{x}_{10} = \dot{x}_{20} = 0$ and then for $x_{10} = x_{20} = \dot{x}_{10} = 0$ and $\dot{x}_{20} = 1$. Plot the time response in each case and compare your results against Figure 4.3.

Solution: From Eq. (4.27) and example 4.1.7,

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{3}A_1 \sin\left(\sqrt{2}t + \phi_1\right) - \frac{1}{3}A_2 \sin\left(2t + \phi_2\right) \\ A_1 \sin\left(\sqrt{2}t + \phi_1\right) + A_2 \sin\left(2t + \phi_2\right) \end{bmatrix}$$

(a) $\mathbf{x}(0) = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$, $\dot{\mathbf{x}}(0) = \mathbf{0}$. Using the initial conditions:

$$0 = A_1 \sin\phi_1 - A_2 \sin\phi_2 \qquad [1]$$
$$1 = A_1 \sin\phi_1 + A_2 \sin\phi_2 \qquad [2]$$
$$0 = \sqrt{2}A_1 \cos\phi_1 - 2A_2 \cos\phi_2 \qquad [3]$$
$$0 = \sqrt{2}A_1 \cos\phi_1 + 2A_2 \cos\phi_2 \qquad [4]$$

From [3] and [4] $\qquad \phi_1 = \phi_2 = \dfrac{\pi}{2}$

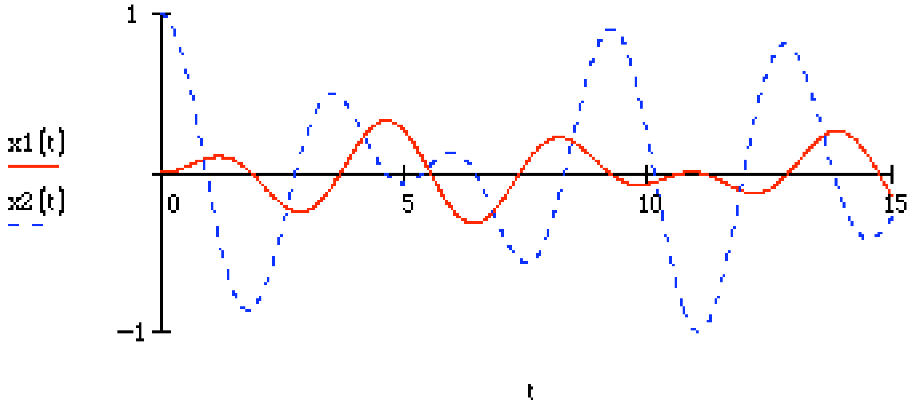
From [1] and [2] $\qquad A_1 = A_2 = \dfrac{1}{2}$

The solution is

$$x_1(t) = \frac{1}{6}\cos\sqrt{2}t - \frac{1}{6}\cos 2t$$
$$x_2(t) = \frac{1}{2}\cos\sqrt{2}t + \frac{1}{2}\cos 2t$$

This is similar to the response of Fig. 4.3

$$x1(t) := \frac{1}{6}\cdot\left(\cos\left(\sqrt{2}\cdot t\right) - \cos(2\cdot t)\right) \qquad x2(t) := \frac{1}{2}\cdot\left(\cos\left(\sqrt{2}\cdot t\right) + \cos(2\cdot t)\right)$$



(b) $\mathbf{x}(0) = 0$, $\dot{\mathbf{x}}(0) = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$. Using these initial conditions:

$$0 = A_1 \sin\phi_1 - A_2 \sin\phi_2 \qquad [1]$$

$$0 = A_1 \sin\phi_1 + A_2 \sin\phi_2 \qquad [2]$$

$$0 = \sqrt{2}A_1 \cos\phi_1 - 2A_2 \cos\phi_2 \qquad [3]$$

$$1 = \sqrt{2}A_1 \cos\phi_1 + 2A_2 \cos\phi_2 \qquad [4]$$

From [1] and [2] $\phi_1 = \phi_2 = 0$

From [3] and [4] $A_1 = \frac{\sqrt{2}}{4}$, and $A_2 = \frac{1}{4}$

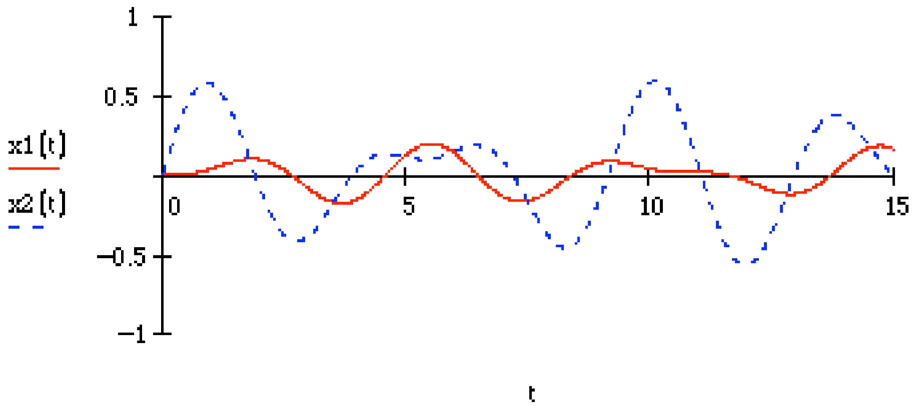
The solution is

$$x_1(t) = \frac{\sqrt{2}}{12}\sin\sqrt{2}t - \frac{1}{12}\sin 2t$$

$$x_2(t) = \frac{\sqrt{2}}{4}\sin\sqrt{2}t + \frac{1}{4}\sin 2t$$

This is also similar to the response of Fig. 4.3

$$x1(t) := \frac{\sqrt{2}}{12}\cdot\sin\left(\sqrt{2}\cdot t\right) - \frac{1}{12}\cdot\sin(2\cdot t) \qquad x2(t) := \frac{\sqrt{2}}{4}\cdot\sin\left(\sqrt{2}\cdot t\right) + \frac{1}{4}\cdot\sin(2\cdot t)$$



4.16 Refer to the system of Figure 4.1(a). Using the initial conditions of Example 4.1.7, resolve and plot x_1(t) for the cases that k_2 takes on the values 0.3, 30, and 300. In each case compare the plots of x_1 and x_2 to those obtained in Figure 4.3. What can you conclude?

Solution: Let k_2 = 0.3, 30, 300 for the example(s) in Section 4.1. Given

$$\mathbf{x}(0) = \begin{bmatrix} 1 & 0 \end{bmatrix}^T \text{ mm}, \dot{\mathbf{x}}(0) = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$$

$$m_1 = 9, m_2 = 1, k_1 = 24$$

Equation of motion becomes:

$$\begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix} \ddot{x} + \begin{bmatrix} 24 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} x = 0$$

(a) k_2 = 0.3

$$\det\left(-\omega^2 M + K\right) = \begin{vmatrix} -9\omega^2 + 24.3 & -0.3 \\ -0.3 & -\omega^2 + 0.3 \end{vmatrix} = 9\omega^4 - 27\omega^2 + 7.2 = 0$$

$$\omega^2 = 0.2598, 2.7042$$

$$\omega_1 = 0.5439$$

$$\omega_2 = 1.6444$$

Mode shapes:

Mode 1, $\omega_1^2 = 0.2958$

$$\begin{bmatrix} 21.6374 & -0.3 \\ -0.3 & 0.004159 \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$21.6374u_{11} - 0.3u_{12} = 0$$

$$u_{11} = 0.01386u_{12}$$

$$\mathbf{u}_1 = \begin{bmatrix} 0.01386 \\ 1 \end{bmatrix}$$

Mode 2, $\omega_2^2 = 2.7042$

$$\begin{bmatrix} -0.03744 & -0.3 \\ -0.3 & 2.4042 \end{bmatrix} \begin{bmatrix} u_{21} \\ u_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-0.3u_{21} = 2.4042u_{22}$$

$$u_{22} = -0.1248u_{21}$$

$$\mathbf{u}_2 = \begin{bmatrix} 1 \\ -0.1248 \end{bmatrix}$$

The solution is $x(t) = A_1 \sin(\omega_1 t + \phi_1)u_1 + A_2 \sin(\omega_2 t + \phi_2)u_2$

Using initial conditions

$$1 = A_1\left(0.01386\right)\sin\phi_1 + A_2\sin\phi_2 \qquad [1]$$
$$0 = A_1\sin\phi_1 + A_2\left(-0.1248\right)\sin\phi_2 \qquad [2]$$
$$0 = A_1\left(0.01386\right)\left(0.5439\right)\cos\phi_1 + A_2\left(1.6444\right)\cos\phi_2 \qquad [3]$$
$$0 = A_1\left(0.5439\right)\cos\phi_1 + A_2\left(1.6444\right)\left(-0.1248\right)\cos\phi_2 \qquad [4]$$

From [3] and [4],

$$\phi_1 = \phi_2 = \pi/2$$

From [1] and [2],

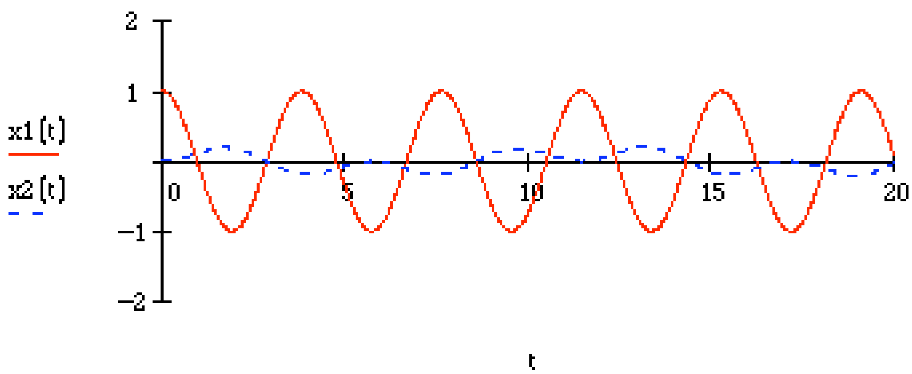
$$A_1 = 0.1246$$
$$A_2 = 0.9983$$

So,

$$x_1\left(t\right) = 0.001727\cos(0.5439t) + 0.9983\cos\left(1.6444t\right) \text{ mm}$$
$$x_2\left(t\right) = 0.1246\left[\cos\left(0.5439t\right) - \cos\left(1.6444t\right)\right] \text{ mm}$$

```
x1 (t) := 0.001727·cos (0.5439·t) + 0.9983·cos (1.644·t)

x2 (t) := 0.1246· (cos (0.5439·t) − cos (1.644·t))
```



(b) $k_2 = 30$

$$\det\left(-\omega^2 M + K\right) = \begin{vmatrix} -9\omega^2 + 54 & -30 \\ -30 & -\omega^2 + 30 \end{vmatrix} = 9\omega^4 - 32\omega^2 + 720 = 0$$

$$\omega^2 = 2.3795, 33.6205$$
$$\omega_1 = 1.5426$$
$$\omega_2 = 5.7983$$

Mode shapes:

Mode 1, $\omega_1^2 = 2.3795$

$$\begin{bmatrix} 32.5845 & -30 \\ -30 & 27.6205 \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$30u_{11} = 27.6205u_{12}$$
$$u_{11} = 0.9207u_{12}$$
$$\mathbf{u}_1 = \begin{bmatrix} 0.9207 \\ 1 \end{bmatrix}$$

Mode 2, $\omega_2^2 = 33.6205$

$$\begin{bmatrix} -248.5845 & -30 \\ -30 & -3.6205 \end{bmatrix} \begin{bmatrix} u_{21} \\ u_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$30u_{21} = -3/6205u_{22}$$
$$u_{21} = -0.1207u_{22}$$
$$\mathbf{u}_2 = \begin{bmatrix} -0.1207 \\ 1 \end{bmatrix}$$

The solution is

$$x(t) = A_1 \sin(\omega_1 t + \phi_1)u_1 + A_2 \sin(\omega_2 t + \phi_2)u_2$$

Using initial conditions,

$$1 = A_1(0.9207)\sin\phi_1 + A_2(-0.1207\sin\phi_2 \quad [1]$$
$$0 = A_1 \sin\phi_1 + A_2 \sin\phi_2 \quad [2]$$
$$0 = A_1(0.9207)(1.5426)\cos\phi_1 + A_2(-0.1207)(5.7983)\cos\phi_2 \quad [3]$$
$$0 = A_1(1.5426)\cos\phi_1 + A_2(5.7983)\cos\phi_2 \quad [4]$$

From [3] and [4]

$$\phi_1 = \phi_2 = \pi / 2$$

From [1] and [2]

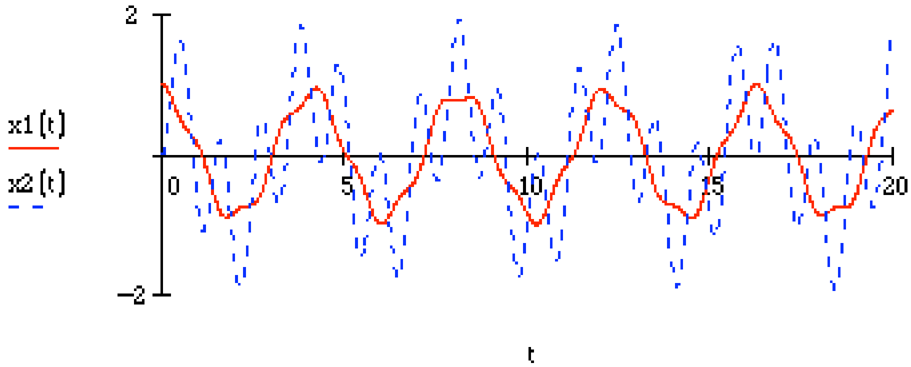
$$A_1 = 0.9602$$
$$A_2 = -0.9602$$

So,

$$x_1(t) = 0.8841\cos(1.5426t) + 0.1159\cos(5.7983t) \text{ mm}$$
$$x_2(t) = 0.9602[\cos(1.5426t) - \cos(5.7983t)] \text{ mm}$$

$x1(t) := 0.8841 \cdot \cos(1.5426 \cdot t) + 0.1159 \cdot \cos(5.7983 \cdot t)$

$x2(t) := 0.9602 \cdot (\cos(1.5426 \cdot t) - \cos(5.7983 \cdot t))$



(c) $k_2 = 300$

$$\det(-\omega^2 M + K) = \begin{vmatrix} -9\omega^2 + 324 & -300 \\ -300 & -\omega^2 + 300 \end{vmatrix} = 9\omega^4 - 3024\omega^2 + 7200 = 0$$

$$\omega^2 = 2.3981, 333.6019$$
$$\omega_1 = 1.5486$$
$$\omega_2 = 18.2648$$

Mode shapes:

Mode 1, $\omega_1^2 = 2.3981$

$$\begin{bmatrix} 302.4174 & -300 \\ -300 & 297.6019 \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$302.4174u_{11} = 300u_{12}$$

$$u_{11} = 0.9920u_{12}$$

$$\mathbf{u}_1 = \begin{bmatrix} 0.9920 \\ 1 \end{bmatrix}$$

Mode 2, $\omega_2^2 = 333.6019$

$$\begin{bmatrix} -2678.4174 & -300 \\ -300 & -33.6019 \end{bmatrix} \begin{bmatrix} u_{21} \\ u_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$300u_{21} = 33.6019u_{22}$$

$$u_{21} = -0.1120u_{22}$$

$$\mathbf{u}_2 = \begin{bmatrix} -0.1120 \\ 1 \end{bmatrix}$$

The solution is

$$x(t) = A_1 \sin(\omega_1 t + \phi_1)u_1 + A_2 \sin(\omega_2 t + \phi_2)u_2$$

Using initial conditions

$$1 = A_1(0.9920)\sin\phi_1 + A_2(-0.1120)\sin\phi_2 \quad [1]$$
$$0 = A_1 \sin\phi_1 + A_2 \sin\phi_2 \quad [2]$$
$$0 = A_1(0.9920)(1.5486)\cos\phi_1 + A_2(-0.1120)(18.2648) \quad [3]$$
$$0 = A_1(1.5486)\cos\phi_1 + A_2(18.2648)\cos\phi_2 \quad [4]$$

From [3] and [4] $\phi_1 = \phi_2 = \pi / 2$

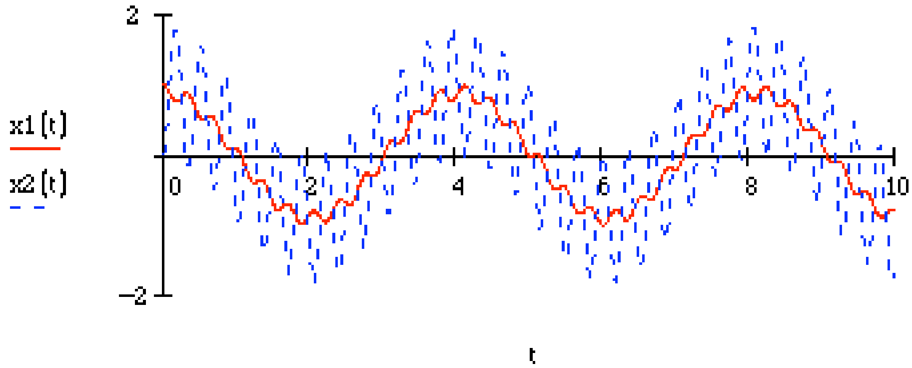
From [1] and [2], $A_1 = 0.9058$ and $A_2 = -0.9058$.

So,

$$x_1(t) = 0.8986\cos(1.5486t) + 0.1014\cos(18.2648t) \text{ mm}$$
$$x_2(t) = 0.9058\left[\cos(1.5486t) - \cos(18.2648t)\right] \text{ mm}$$

$$x1(t) := 0.8986 \cdot \cos(1.5486 \cdot t) + 0.1014 \cdot \cos(18.2648 \cdot t)$$

$$x2(t) := 0.9052 \cdot (\cos(1.5486 \cdot t) - \cos(18.2648 \cdot t))$$



As the value of k_2 increases the effect on mass 1 is small, but mass 2 oscillates similar to mass 1 with a superimposed higher frequency oscillation.

4.17 Consider the system of Figure 4.1(a) described in matrix form by Eqs. (4.11), (4.9), and (4.6). Determine the natural frequencies in terms of the parameters m_1, m_2, k_1 and k_2. How do these compare to the two single-degree-of-freedom frequencies $\omega_1 = \sqrt{k_1 / m_1}$ and $\omega_2 = \sqrt{k_2 / m_2}$?

Solution:

The equation of motion is

$$M\ddot{x} + Kx = 0$$
$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\ddot{x} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} x = 0$$

The characteristic equation is found from Eq. (4.19):

$$\det\left(-\omega^2 M + K\right) = 0$$
$$\begin{vmatrix} -m_1\omega^2 + k_1 + k_2 & -k_2 \\ -k_2 & -m_2\omega^2 + k_2 \end{vmatrix}$$
$$m_1 m_2 \omega^4 - \left(k_1 m_2 + k_2\left(m_1 + m_2\right)\right)\omega^2 + k_1 k_2 = 0$$

$$\omega_{1,2}^2 = \frac{k_1 m_2 + k_2\left(m_1 + m_2\right) \pm \sqrt{\left[k_1 m_2 + k_2\left(m_1 + m_2\right)\right]^2 - 4 m_1 m_2 k_1 k_2}}{2 m_1 m_2}$$

So,

$$\omega_{1,2} = \sqrt{\frac{k_1 m_2 + k_2\left(m_1 + m_2\right) \pm \sqrt{\left[k_1 m_2 + k_2\left(m_1 + m_2\right)\right]^2 - 4 m_1 m_2 k_1 k_2}}{2 m_1 m_2}}$$

In two-degree-of-freedom systems, each natural frequency depends on all four parameters (m_1, m_2, k_1, k_2), while a single-degree-of-freedom system's natural frequency depends only on one mass and one stiffness.

4.18 Consider the problem of Example 4.1.7 and use a trig identity to show the $x_1(t)$ experiences a beat. Plot the response to show the beat phenomena in the response.

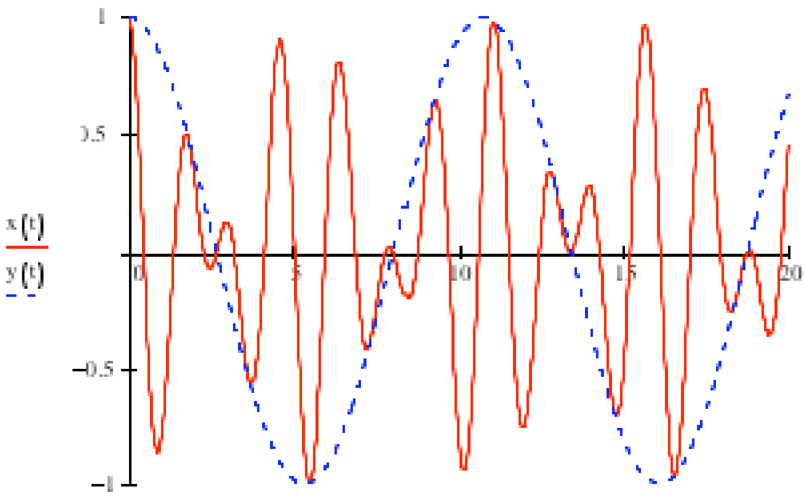
Solution Applying the trig identity of Example 2.2.2 to x_1 yields

$$x_1(t) = (\cos\sqrt{2}t + \cos 2t) = \cos(\frac{\sqrt{2}-2}{2}t)\cos(\frac{\sqrt{2}+2}{2}t) = \cos 0.586t \cos 3.414t$$

Plotting x_1 and $\cos(0.586t)$ yields the clear beat:

$x(t) := \cos(0.586 \cdot t)\ \cos(3.414 \cdot t)$

$y(t) := \cos(0.586 \cdot t)$



Problems and Solutions for Section 4.2 (4.19 through 4.33)

4.19 Calculate the square root of the matrix

$$M = \begin{bmatrix} 13 & -10 \\ -10 & 8 \end{bmatrix}$$

$$\left[\text{Hint: Let } M^{1/2} = \begin{bmatrix} a & -b \\ -b & c \end{bmatrix}; \text{ calculate } \left(M^{1/2}\right)^2 \text{ and compare to } M.\right]$$

Solution: Given:

$$M = \begin{bmatrix} 13 & -10 \\ -10 & 8 \end{bmatrix}$$

If

$$M^{1/2} = \begin{bmatrix} a & -b \\ -b & c \end{bmatrix}, \text{ then}$$

$$M = M^{1/2}M^{1/2} = \begin{bmatrix} a & -b \\ -b & c \end{bmatrix}\begin{bmatrix} a & -b \\ -b & c \end{bmatrix} = \begin{bmatrix} a^2+b^2 & -ab-bc \\ -ab-bc & b^2+c^2 \end{bmatrix} = \begin{bmatrix} 13 & -10 \\ -10 & 8 \end{bmatrix}$$

This yields the 3 nonlinear algebraic equations:

$$a^2 + b^2 = 13$$
$$ab + bc = 10$$
$$b^2 + c^2 = 8$$

There are several possible solutions but only one that makes $M^{1/2}$ positive definite which is $a = 3$, $b = c = 2$ as determined below in Mathcad. Choosing these values results in

$$M^{1/2} = \begin{bmatrix} 3 & -2 \\ -2 & 2 \end{bmatrix}$$

$a := 1 \qquad b := 1 \qquad c := 1$

given

$$a^2 + b^2 = 13$$

$$a \cdot b + b \cdot c = 10$$

$$b^2 + c^2 = 8$$

$$\text{find}(a, b, c) = \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}$$

4.20 Normalize the vectors

$$\begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \end{bmatrix}, \begin{bmatrix} -0.1 \\ 0.1 \end{bmatrix}$$

first with respect to unity (i.e., $x^T x = 1$) and then again with respect to the matrix M (i.e., $x^T Mx = 1$), where

$$M = \begin{bmatrix} 3 & -0.1 \\ -0.1 & 2 \end{bmatrix}$$

Solution:

(a) Normalize the vectors

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$\alpha_1 = \frac{1}{\sqrt{x^T x}} = \frac{1}{\sqrt{5}}$$

Normalized:

$$\mathbf{x}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 0.4472 \\ -0.8944 \end{bmatrix}$$

$$\mathbf{x}_2 = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$$

$$\alpha_2 = \frac{1}{\sqrt{x^T x}} = \frac{1}{5}$$

Normalized:

$$\mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\mathbf{x}_3 = \begin{bmatrix} -0.1 \\ 0.1 \end{bmatrix}$$

$$\alpha_3 = \frac{1}{\sqrt{x^T x}} = \frac{1}{\sqrt{0.02}}$$

Normalized:

$$\mathbf{x}_3 = \sqrt{50}\begin{bmatrix} -0.1 \\ 0.1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.7071 \\ 0.7071 \end{bmatrix}$$

(b) Mass normalize the vectors

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$\alpha_1 = \frac{1}{\sqrt{\mathbf{x}^T M\mathbf{x}}} = \frac{1}{\sqrt{11.4}}$$

Mass normalized:

$$\mathbf{x}_1 = \frac{1}{\sqrt{11.4}}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 0.2962 \\ -.5923 \end{bmatrix}$$

$$\mathbf{x}_2 = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$$

$$\alpha_2 = \frac{1}{\sqrt{\mathbf{x}^T M\mathbf{x}}} = \frac{1}{\sqrt{50}}$$

$$\mathbf{x}_2 = \frac{1}{\sqrt{50}}\begin{bmatrix} 0 \\ 5 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.7071 \end{bmatrix}$$

$$\mathbf{x}_3 = \begin{bmatrix} -0.1 \\ 0.1 \end{bmatrix}$$

$$\alpha_3 = \frac{1}{\sqrt{x^T Mx}} = \frac{1}{\sqrt{0.052}}$$

Mass normalized:

$$\mathbf{x}_3 = \frac{1}{\sqrt{0.052}}\begin{bmatrix} -0.1 \\ 0.1 \end{bmatrix} = \begin{bmatrix} -0.4385 \\ 0.4385 \end{bmatrix}$$

4.21 For the example illustrated in Figure P4.1 with $c_1 = c_2 = c_3 = 0$, calculate the matrix $\tilde{K}$.

Solution:

From Figure 4.1,

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \ddot{x} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} x = 0$$

$$\tilde{K} = M^{-1/2} K M^{-1/2} = \begin{bmatrix} m_1^{-1/2} & 0 \\ 0 & m_2^{-1/2} \end{bmatrix} \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \begin{bmatrix} m_1^{-1/2} & 0 \\ 0 & m_2^{-1/2} \end{bmatrix}$$

$$\tilde{K} = \begin{bmatrix} m_1^{-1}\left(k_1 + k_2\right) & -m_1^{-1/2} m_2^{-1/2} k_2 \\ -m_1^{-1/2} m_2^{-1/2} k_2 & m_1^{-1}\left(k_2 + k_3\right) \end{bmatrix}$$

Since $\tilde{K}^T = \tilde{K}, \tilde{K}$ is symmetric.

Using the numbers given in problem 4.2 yields

$$\tilde{K} = \begin{bmatrix} 3 & -1 \\ -1 & 6 \end{bmatrix}$$

This is obviously symmetric.

4.22 Repeat Example 4.2.5 using eight decimal places. Does $P^T P = 1$, and does $P^T \tilde{K} P = \Lambda = \text{ diag }\left[\omega_1^2 \;\; \omega_2^2\right]$ exactly?

Solution: From Example 4.2.5,

$$\tilde{K} = \begin{bmatrix} 12 & -1 \\ -1 & 3 \end{bmatrix} \Rightarrow \det\left(\tilde{K} - \lambda I\right) = \lambda^2 - 15\lambda + 35 = 0$$

$$\Rightarrow \lambda_1 = 2.89022777, \text{ and } \lambda_2 = 12.10977223$$

Calculate eigenvectors and normalize them:

$$\lambda_1 = 2.89022777$$

$$\begin{bmatrix} 9.10977223 & -1 \\ -1 & 0.10977223 \end{bmatrix}\begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow 9.10977223 v_{11} = v_{12}$$

$$\left\| v_1 \right\| = \sqrt{v_{11}^2 + v_{12}^2} = \sqrt{v_{11}^2 + \left(9.10977223\right)^2 v_{11}^2} = 1$$

$$\Rightarrow v_{11} = 0.10911677 \text{ and } \quad v_{12} = 0.99402894$$

$$\mathbf{v}_1 = \left[0.10911677 \quad 0.99402894\right]^T$$

$$\lambda_2 = 12.10977223$$

$$\Rightarrow \begin{bmatrix} -0.10977223 & -1 \\ -1 & -9.10977223 \end{bmatrix}\begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow v_{21} = 9.10977223 v_{22}$$

$$\left\| v_2 \right\| = \sqrt{v_{21}^2 + v_{22}^2} = \sqrt{\left(-9.10977223\right)^2 v_{22}^2 + v_{22}^2} = 1$$

$$v_{21} = -9.10911677, \text{ and } \quad v_{22} = -0.99402894$$

$$\mathbf{v}_2 = \left[0.99402894 \quad 0.10911677\right]^T$$

Now, $P = \left[\mathbf{v}_1 \quad \mathbf{v}_2\right] = \begin{bmatrix} 0.10911677 & -0.99402894 \\ 0.99402894 & 0.10911677 \end{bmatrix}$

Check $P^T P = I$

$$P^T P = \begin{bmatrix} 1.00000000 & 0 \\ 0 & 1.00000000 \end{bmatrix} = I \text{ (to 8 decimal places)}$$

Check $P^T \tilde{K} P = \Lambda = \text{diag}\left(\lambda_1, \lambda_2\right)$

$$\Lambda = P^T \tilde{K} P = \begin{bmatrix} 2.89022778 & 0.00000002 \\ 0.00000002 & 12.10977227 \end{bmatrix}$$

$$\mathrm{diag}\left(\lambda_1, \lambda_2\right) = \begin{bmatrix} 2.89022777 & 0 \\ 0 & 12.10977223 \end{bmatrix}$$

This is accurate to 7 decimal places.

4.23 Discuss the relationship or difference between a mode shape of equation (4.54) and an eigenvector of $\tilde{K}$.

Solution:

The relationship between a mode shape, **u**, of $M\ddot{x} + Kx = 0$ and an eigenvector, **v**, of $\tilde{K} = M^{-1/2}KM^{-1/2}$ is given by

$$\mathbf{v}_i = M^{1/2}\mathbf{u}_i \quad \text{or} \quad \mathbf{u}_i = M^{-1/2}\mathbf{v}_i$$

If **v** is normalized, then **u** is mass normalized.

This is shown by the relation

$$\mathbf{v}_i^T\mathbf{v}_i = 1 = \mathbf{u}_i^T M\mathbf{u}_i$$

4.24 Calculate the units of the elements of matrix $\tilde{K}$.

Solution:

$\tilde{K} = M^{-1/2}KM^{-1/2}$

$M^{-1/2}$ has units kg$^{-1/2}$

K has units N/m = kg/s^2

So, $\tilde{K}$ has units $\left(\text{kg}^{-1/2}\right)\left(\text{kg/s}^2\right)\left(\text{kg}^{-1/2}\right) = s^{-2}$

4.25 Calculate the spectral matrix Λ and the modal matrix P for the vehicle model of Problem 4.14, Figure P4.14.

Solution: From Problem 4.14:

$$M\ddot{\mathbf{x}} + K\mathbf{x} = \begin{bmatrix} 2000 & 0 \\ 0 & 50 \end{bmatrix}\ddot{\mathbf{x}} + \begin{bmatrix} 1000 & -1000 \\ -1000 & 11{,}000 \end{bmatrix}\mathbf{x} = \mathbf{0}$$

Calculate eigenvalues:

$$\det\left(\tilde{K} - \lambda I\right) = 0$$

$$\tilde{K} = M^{-1/2}KM^{-1/2} = \begin{bmatrix} 0.5 & -3.162 \\ -3.162 & 220 \end{bmatrix}$$

$$\begin{vmatrix} 0.5-\lambda & -3.162 \\ -3.162 & 220-\lambda \end{vmatrix} = \lambda^2 - 220.5\lambda + 100 = 0$$

$$\lambda_{1,2} = 0.454, 220.05$$

The spectral matrix is

$$\Lambda = \text{diag}\left(\lambda_1\right) = \begin{bmatrix} 0.454 & 0 \\ 0 & 220.05 \end{bmatrix}$$

Calculate eigenvectors and normalize them:

$$\lambda_1 = 0.454$$

$$\begin{bmatrix} 0.0455 & -3.162 \\ -3.162 & 219.55 \end{bmatrix}\begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = 0 \Rightarrow v_{11} = 69.426v_{12}$$

$$\left\|v_1\right\| = \sqrt{v_{11}^2 + v_{22}^2} = \sqrt{\left(69.426\right)^2 v_{12}^2 + v_{12}^2} = 69.434v_{12} = 1$$

$$\Rightarrow v_{12} = 0.0144, \text{ and } v_{11} = 0.9999$$

$$\Rightarrow \mathbf{v}_1 = \begin{bmatrix} 0.9999 \\ 0.0144 \end{bmatrix}$$

$$\lambda_2 = 220.05$$

$$\begin{bmatrix} -219.55 & -3.162 \\ -3.162 & -0.0455 \end{bmatrix}\begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = 0$$

$$v_{21} = 0.0144v_{22}$$

$$\left\|v_2\right\| = \sqrt{v_{21}^2 + v_{22}^2} = \sqrt{\left(-0.0144\right)^2 v_{22}^2 + v_{22}^2} = 1.0001v_{22} = 1$$

$$\Rightarrow v_{22} = 0.9999, \text{ and } v_{21} = -0.0144$$

$$\Rightarrow \mathbf{v}_2 = \begin{bmatrix} -0.0144 \\ 0.9999 \end{bmatrix}$$

The modal matrix is

$$P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} 0.9999 & -0.0144 \\ 0.0144 & 0.9999 \end{bmatrix}$$

4.26 Calculate the spectral matrix Λ and the modal matrix P for the subway car system of Problem 4.12, Figure P4.12.

Solution: From problem 4.12 and Figure P4.12,

$$M\ddot{\mathbf{x}} + K\mathbf{x} = \begin{bmatrix} 2000 & 0 \\ 0 & 2000 \end{bmatrix} \ddot{\mathbf{x}} + \begin{bmatrix} 280,000 & -280,000 \\ -280,000 & 280,000 \end{bmatrix} \mathbf{x} = 0$$

Calculate eigenvalues:

$$\det\left(\tilde{K} - \lambda I\right) = 0$$

$$\tilde{K} = M^{-1/2} K M^{-1/2} = \begin{bmatrix} 140 & -140 \\ -140 & 140 \end{bmatrix}$$

$$\begin{vmatrix} 140-\lambda & -140 \\ -140 & 140-\lambda \end{vmatrix} = \lambda^2 - 280\lambda = 0$$

$$\lambda_{1,2} = 0, 280$$

The spectral matrix is

$$\Lambda = \text{diag}\left(\lambda_i\right) = \begin{bmatrix} 0 & 0 \\ 0 & 280 \end{bmatrix}$$

Calculate eigenvectors and normalize them:

$$\lambda_1 = 0$$

$$\begin{bmatrix} 140 & -140 \\ -140 & 140 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = 0$$

$$v_{11} = v_{12}$$

$$\|v_1\| = \sqrt{v_{11}^2 + v_{12}^2} = \sqrt{v_{12}^2 + v_{12}^2} = 1.414 v_{12} = 1$$

$$v_{12} = 0.7071$$

$$v_{11} = 0.7071$$

$$\mathbf{v}_1 = \begin{bmatrix} 0.7071 \\ 0.7071 \end{bmatrix}$$

$$\lambda_2 = 280$$

$$\begin{bmatrix} 140 & -140 \\ -140 & 140 \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = 0 \Rightarrow v_{21} = v_{22}$$

$$\Rightarrow \|v_2\| = \sqrt{v_{21}^2 + v_{22}^2} = \sqrt{v_{22}^2 + v_{22}^2} = 14.14 v_{12} = 1 \Rightarrow v_{22} = 0.7071, v_{21} = -0.7071$$

$$\Rightarrow \mathbf{v}_2 = \begin{bmatrix} -0.7071 \\ 0.7071 \end{bmatrix}$$

The modal matrix is $P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} 0.7071 & -0.7071 \\ 0.7071 & 0.7071 \end{bmatrix}$

4.27 Calculate $\tilde{K}$ for the torsional vibration example of Problem 4.11. What are the units of $\tilde{K}$?

Solution: From Problem 4.11,

$$J\ddot{\theta} + K\theta = J_2\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\ddot{\theta} + k\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\theta = 0$$

$$\tilde{K} = J^{-1/2}KJ^{-1/2}$$

$$J^{-1/2} = J_2^{-1/2}\begin{bmatrix} 0.5774 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\tilde{K} = J_2^{-1/2}\begin{bmatrix} 0.5774 & 0 \\ 0 & 1 \end{bmatrix}k\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}J_2^{-1/2}\begin{bmatrix} 0.5774 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\tilde{K} = \frac{k}{J_2}\begin{bmatrix} 0.6667 & -0.5774 \\ -0.5774 & 1 \end{bmatrix}$$

The units of $\tilde{K}$ are

$$\left(\frac{kg \cdot m^2}{\text{rad}}\right)^{-1/2}\left(\frac{N \cdot m}{\text{rad}}\right)\left(\frac{kg \cdot m^2}{\text{rad}}\right)^{-1/2} = s^{-2}$$

4.28 Consider the system in the Figure P4.28 for the case where $m_1 = 1$ kg, $m_2 = 4$ kg, $k_1 = 240$ N/m and k_2=300 N/m. Write the equations of motion in vector form and compute each of the following

a) the natural frequencies
b) the mode shapes
c) the eigenvalues
d) the eigenvectors
e) show that the mode shapes are not orthogonal
f) show that the eigenvectors are orthogonal
g) show that the mode shapes and eigenvectors are related by $M^{-\frac{1}{2}}$
h) write the equations of motion in modal coordinates

Note the purpose of this problem is to help you see the difference between these various quantities.

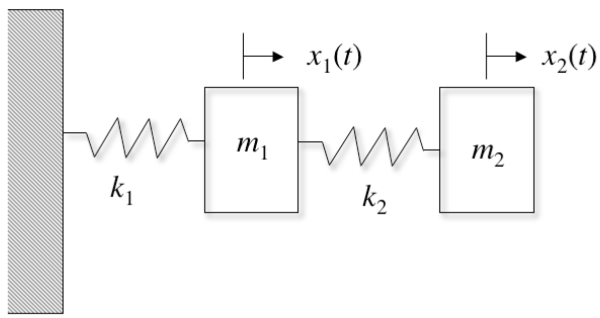


Figure P1.28 A two-degree of freedom system

Solution From a free body diagram, the equations of motion in vector form are

$$\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}\ddot{\mathbf{x}} + \begin{bmatrix} 540 & -300 \\ -300 & 300 \end{bmatrix}\mathbf{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The natural frequencies can be calculated in two ways. The first is using the determinant following example 4.1.5:

a) $$\det(-\omega^2 M + K) = 0 \Rightarrow \underline{\omega_1 = 5.5509, \omega_2 = 24.1700 \text{ rad/s}}$$

The second approach is to compute the eigenvalues of the matrix $\tilde{K} = M^{-\frac{1}{2}}KM^{-\frac{1}{2}}$ following example 4.4.4, which yields the same answers. The mode shapes are calculate following the procedures of example 4.1.6 or numerically using `eig(K,M)` in Matlab

b) $$\mathbf{u}_1 = \begin{bmatrix} 0.5076 \\ 0.8616 \end{bmatrix}, \ \mathbf{u}_2 = \begin{bmatrix} 0.9893 \\ -0.1457 \end{bmatrix}$$

The eigenvectors are vectors that satisfy $\tilde{K}\mathbf{v} = \lambda\mathbf{v}$, where λ are the eigenvalues. These can be computed following example 4.2.2, or using `[V,Dv]=eig(Kt)` in Matlab. The eigenvalues and eigenvectors are

c) $$\lambda_1 = 30.8120, \quad \lambda_2 = 584.1880,$$

d) $$\mathbf{v}_1 = \begin{bmatrix} 0.2826 \\ 0.9592 \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} -0.9592 \\ 0.2826 \end{bmatrix}$$

To show that the mode shapes are not orthogonal, show that $\mathbf{u}_1^T\mathbf{u}_2 \neq 0$:

e) $$\mathbf{u}_1^T\mathbf{u}_2 = (0.5076)(0.9893) + (0.8616)(-0.1457) = 0.3767 \neq 0$$

To show that the eigenvectors are orthogonal, compute the inner product to show that $\mathbf{v}_1^T\mathbf{v}_2 = 0$:

f) $$\mathbf{v}_1^T\mathbf{v}_2 = (0.2826)(-0.9592) + (0.9592)(0.2826) = 0$$

To solve the next part merely compute $M^{-½}\mathbf{v}_2$ and show that it is equal to $\mathbf{u}_2$ (see the discussion at the top of page 262.

g) $$M^{-½}\mathbf{v}_2 = \begin{bmatrix} 0.9592 \\ -0.1413 \end{bmatrix}, \text{ normalize to get } \begin{bmatrix} -0.9893 \\ 0.1457 \end{bmatrix} = -\mathbf{u}_2$$

Likewise, $M^{-½}\mathbf{v}_1 = \mathbf{u}_1$. Note that if you use Matlab you'll automatically get normalized vectors. But the product $M^{-½}\mathbf{v}_2$ will not be normalized, so it must be normalized before comparing it to $\mathbf{u}_2$.

h) We can write down the modal equations, just as soon as we know the eigenvalues (squares of the frequencies). They are:

$$\ddot{r}_1(t) + 30.812 r_1(t) = 0$$
$$\ddot{r}_2(t) + 583.189 r_2(t) = 0$$

4.29 Consider the following system:

$$\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}\ddot{\mathbf{x}} + \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}\mathbf{x} = \mathbf{0}$$

where M is in kg and K is in N/m. (a) Calculate the eigenvalues of the system. (b) Calculate the eigenvectors and normalize them.

Solution: Given:

$$M\ddot{\mathbf{x}} + K\mathbf{x} = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}\ddot{\mathbf{x}} + \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}\mathbf{x} = \mathbf{0}$$

Calculate eigenvalues:

$$\det\left(\tilde{K} - \lambda I\right) = 0$$

$$\tilde{K} = M^{-1/2}KM^{-1/2} = \begin{bmatrix} 3 & -0.5 \\ -0.5 & 0.25 \end{bmatrix}$$

$$\begin{vmatrix} 3-\lambda & -0.5 \\ -0.5 & 0.25-\lambda \end{vmatrix} = \lambda^2 - 3.25\lambda + 0.5 = 0$$

$$\lambda_{1,2} = 0.162, 3.088$$

The spectral matrix is

$$\Lambda = \text{diag}\left(\lambda_i\right) = \begin{bmatrix} 0.162 & 0 \\ 0 & 3.088 \end{bmatrix}$$

Calculate eigenvectors and normalize them:

$$\lambda_1 = 0.162$$

$$\begin{bmatrix} 2.838 & -0.5 \\ -0.5 & 0.088 \end{bmatrix}\begin{bmatrix} v_{11} \\ v_{21} \end{bmatrix} = 0 \Rightarrow v_{11} = 1.762 v_{21}$$

$$\left\| v_1 \right\| = \sqrt{v_{11}^2 + v_{21}^2} = \sqrt{\left(0.1762\right)^2 v_{21}^2 + v_{21}^2} = 1.015 v_{21} = 1$$

$$v_{21} = 0.9848 \text{ and } v_{11} = 0.1735 \Rightarrow \mathbf{v}_1 = \begin{bmatrix} 0.1735 \\ 0.9848 \end{bmatrix}$$

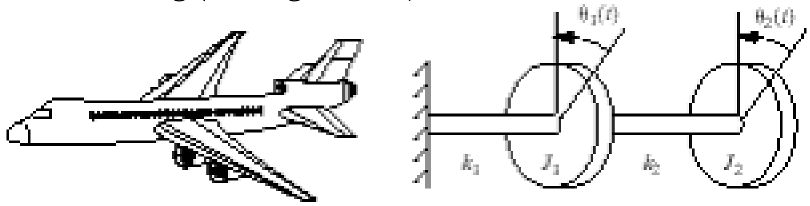
$$\lambda_2 = 3.088$$

$$\begin{bmatrix} -0.088 & -0.5 \\ -0.5 & -2.838 \end{bmatrix}\begin{bmatrix} v_{12} \\ v_{22} \end{bmatrix} = 0 \Rightarrow v_{12} = 1.762 v_{22}$$

$$\left\| v_2 \right\| = \sqrt{v_{12}^2 + v_{22}^2} = \sqrt{\left(-5.676\right)^2 v_{22}^2 + v_{22}^2} = 5.764 v_{22} = 1$$

$$\Rightarrow v_{22} = 0.1735 \text{ and } v_{12} = -0.9848 \Rightarrow \mathbf{v}_2 = \begin{bmatrix} -0.9848 \\ 0.1735 \end{bmatrix}$$

4.30 The torsional vibration of the wing of an airplane is modeled in Figure P4.30. Write the equation of motion in matrix form and calculate the natural frequencies in terms of the rotational inertia and stiffness of the wing (See Figure 1.22).



Airplane wing with engines

Wing modeled as two shafts and two disks for torsional vibration

Solution: From Figure 1.22,

$$k_1 = \frac{GJ_p}{l_1} \text{ and } k_2 = \frac{GJ_p}{l_2}$$

Equation of motion:

$$\begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix}\ddot{\theta} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\theta = 0$$

$$\begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix}\ddot{\theta} + \begin{bmatrix} GJ_p\left(\frac{1}{l_1} + \frac{1}{l_2}\right) & \frac{-GJ_p}{l_2} \\ \frac{-GJ_p}{l_2} & \frac{GJ_p}{l_2} \end{bmatrix}\theta = 0$$

Natural frequencies:

$$\tilde{K} = M^{-1/2}KM^{-1/2} = \begin{bmatrix} \frac{GJ_p}{J_1}\left(\frac{1}{l_1} + \frac{1}{l_2}\right) & \frac{-GJ_p}{l_2\sqrt{J_1J_2}} \\ \frac{-GJ_p}{l_2\sqrt{J_1J_2}} & \frac{GJ_p}{J_2l_2} \end{bmatrix}$$

$$\det\left(\tilde{K} - \lambda I\right) = \begin{bmatrix} \frac{GJ_p}{J_1}\left(\frac{1}{l_1} + \frac{1}{l_2}\right) - \lambda & \frac{-GJ_p}{l_2\sqrt{J_1J_2}} \\ \frac{-GJ_p}{l_2\sqrt{J_1J_2}} & \frac{GJ_p}{J_2l_2} - \lambda \end{bmatrix}$$

Solving for λ yields

$$\lambda_{1,2} = \frac{GJ_p}{2}\left[\frac{1}{J_1}\left(\frac{1}{l_1} + \frac{1}{l_2}\right) + \frac{1}{J_2l_2}\right] \pm \frac{GJ_p}{2}\sqrt{\left[\frac{1}{J_1}\left(\frac{1}{l_1} + \frac{1}{l_2}\right) + \frac{1}{J_2l_2}\right]^2 - \frac{4}{J_1J_2l_1l_2}}$$

The natural frequencies are

$$\omega_1 = \sqrt{\lambda_1} \text{ and } \omega_2 = \sqrt{\lambda_2}$$

4.31 Calculate the value of the scalar a such that $\mathbf{x}_1 = [a \quad -1 \quad 1]^T$ and $\mathbf{x}_2 = [1 \quad 0 \quad 1]^T$ are orthogonal.

Solution: To be orthogonal, $\mathbf{x}_1^T \mathbf{x}_2 = 0$

So, $\mathbf{x}_1^T \mathbf{x}_2 = \begin{bmatrix} a & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = a + 1 = 0$. Therefore, a = -1.

4.32 Normalize the vectors of Problem 4.31. Are they still orthogonal?

Solution: From Problem 4.31, with a = -1,

$$\mathbf{x}_1 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \text{ and } \mathbf{x}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Normalize $\mathbf{x}_1$:

$$(\alpha x_1)^T (\alpha x_1) = 1$$

$$a^2 \begin{bmatrix} -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = 3\alpha^2 = 1$$

$$\alpha = 0.5774$$

So, $\mathbf{x}_1 = 0.5774 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$

Normalize $\mathbf{x}_2$:

$$(\alpha x_2)^T (\alpha x_2) = 1$$

$$a^2 \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 2\alpha^2 = 1$$

$$\alpha = 0.7071$$

So, $\mathbf{x}_2 = 0.7071 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

Check orthogonality:

$$\mathbf{x}_1^T \mathbf{x}_2 = (0.5774)(0.7071) \begin{bmatrix} -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 0 \text{ Still orthogonal}$$

4.33 Which of the following vectors are normal? Orthogonal?

$$\mathbf{x}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} \quad \mathbf{x}_2 = \begin{bmatrix} 0.1 \\ 0.2 \\ 0.3 \end{bmatrix} \quad \mathbf{x}_3 = \begin{bmatrix} 0.3 \\ 0.4 \\ 0.3 \end{bmatrix}$$

Solution:

Check vectors to see if they are normal:

$$\|x_1\| = \sqrt{1/2 + 0 + 1/2} = \sqrt{1} = 1 \qquad \text{Normal}$$

$$\|\mathrm{x}_2\| = \sqrt{.1^2 + .2^2 + .3^2} = \sqrt{.14} = 0.3742 \qquad \text{Not normal}$$

$$\|\mathrm{x}_3\| = \sqrt{.3^2 + .4^2 + .3^2} = \sqrt{.34} = 0.5831 \qquad \text{Not normal}$$

Check vectors to see if they are orthogonal:

$$\mathbf{x}_1^T\mathbf{x}_2 = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} .1 \\ .2 \\ .3 \end{bmatrix} = .2828 \qquad \text{Not orthogonal}$$

$$\mathbf{x}_2^T\mathbf{x}_3 = \begin{bmatrix} .1 & .2 & .3 \end{bmatrix} \begin{bmatrix} .3 \\ .4 \\ .3 \end{bmatrix} = 0.2 \qquad \text{Not orthogonal}$$

$$\mathbf{x}_3^T\mathbf{x}_1 = \begin{bmatrix} .3 & .4 & .3 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} = 0.4243 \qquad \text{Not orthogonal}$$

$\therefore$ Only $\mathbf{x}_1$ is normal, and none are orthogonal.

Problems and Solutions for Section 4.3 (4.34 through 4.43)

4.34 Solve Problem 4.11 by modal analysis for the case where the rods have equal stiffness (i.e., $k_1 = k_2$), $J_1 = 3J_2$, and the initial conditions are $\mathbf{x}(0) = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$ and $\dot{\mathbf{x}}(0) = \mathbf{0}$.

Solution: From Problem 4.11 and Figure P4.11, with $k = k_1 = k_2$ and $J_1 = 3J_2$:

$$J_2\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\ddot{\theta} + k\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\theta = 0$$

Calculate eigenvalues and eigenvectors:

$$J^{-1/2} = J_2^{-1/2}\begin{bmatrix} \frac{1}{\sqrt{3}} & 0 \\ 0 & 1 \end{bmatrix}$$

$$\tilde{K} = J^{-1/2}KJ^{-1/2} = \frac{k}{J_2}\begin{bmatrix} \frac{2}{3} & \frac{-1}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} & 1 \end{bmatrix} \Rightarrow \det\left(\tilde{K} - \lambda I\right) = \lambda^2 - \frac{5k}{3J_2}\lambda + \frac{k^2}{3J_2^2} = 0$$

$$\lambda_1 = \frac{\left(5 - \sqrt{13}\right)k}{6J_2} \Rightarrow \quad \omega_1 = \sqrt{\lambda_1}, \text{ and } \quad \frac{\left(5 + \sqrt{13}\right)k}{6J_2} \quad \Rightarrow \omega_2 = \sqrt{\lambda_2}$$

$$\lambda_1 = \frac{\left(5 - \sqrt{13}\right)k}{6J_2} \Rightarrow \begin{bmatrix} \frac{\left(5 + \sqrt{13}\right)k}{6J_2} & \frac{-k}{\sqrt{3J_2}} \\ \frac{-k}{\sqrt{3J_2}} & \frac{\left(5 + \sqrt{13}\right)k}{6J_2} \end{bmatrix}\begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = 0$$

$$\Rightarrow v_{11} = 1.3205 v_{12} \Rightarrow \mathbf{v}_1 = \begin{bmatrix} 0.7992 \\ 0.6011 \end{bmatrix}$$

$$\lambda_2 = \frac{\left(5+\sqrt{13}\right)k}{6J_2} \Rightarrow \begin{bmatrix} \frac{\left(-1-\sqrt{13}\right)k}{6J_2} & \frac{-k}{\sqrt{3J_2}} \\ \frac{-k}{\sqrt{3J_2}} & \frac{\left(1-\sqrt{13}\right)k}{6J_2} \end{bmatrix} \begin{bmatrix} v2_{11} \\ v_{22} \end{bmatrix} = 0$$

$$\Rightarrow v_{21} = -0.7522 v_{22} \Rightarrow \mathbf{v}_2 = \begin{bmatrix} -0.6011 \\ 0.7992 \end{bmatrix}$$

Now, $P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} 0.7992 & -0.6011 \\ 0.6011 & 0.7992 \end{bmatrix}$

Calculate S and S^{-1}:

$$S = J^{-1/2} P = \frac{1}{\sqrt{J_2}} \begin{bmatrix} 0.4614 & -0.3470 \\ 0.6011 & 0.7992 \end{bmatrix}$$

$$S^{-1} = P^T J^{1/2} = J_2^{1/2} \begin{bmatrix} 1.3842 & 0.6011 \\ -1.0411 & 0.7992 \end{bmatrix}$$

Modal initial conditions:

$$\mathbf{r}(0) = S^{-1}\theta(0) = S^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = J_2^{1/2} \begin{bmatrix} 0.6011 \\ 0.7992 \end{bmatrix}$$

$$\dot{\mathbf{r}}(0) = S^{-1}\dot{\theta}(0) = 0$$

Modal solution:

$$r_1(t) = \frac{\sqrt{\omega_1^2 r_{10}^2 + \dot{r}_{10}^2}}{\omega_1} \sin\left[\omega_1 t + \tan^{-1}\frac{\omega_1 r_{10}}{\dot{r}_{10}}\right]$$

$$r_2(t) = \frac{\sqrt{\omega_2^2 r_{20}^2 + \dot{r}_{20}^2}}{\omega_2} \sin\left[\omega_2 t + \tan^{-1}\frac{\omega_2 r_{10}}{\dot{r}_{20}}\right]$$

$$r_1(t) = 0.6011 J_2^{1/2} \sin\left[\omega_1 t + \frac{\pi}{2}\right] = 0.6011 J_2^{1/2} \cos\omega_1 t$$

$$r_2(t) = 0.7992 J_2^{1/2} \sin\left[\omega_2 t + \frac{\pi}{2}\right] = 0.6011 J_2^{1/2} \cos\omega_2 t$$

$$\mathbf{r}(t) = \begin{bmatrix} 0.6011 J_2^{1/2} \cos\omega_1 t \\ 0.7992 J_2^{1/2} \cos\omega_2 t \end{bmatrix}$$

Convert to physical coordinates:

$$\theta(t) = S\mathbf{r}(t) = J_2^{1/2}\begin{bmatrix} 0.4614 & -0.3470 \\ 0.6011 & 0.7992 \end{bmatrix}\begin{bmatrix} 0.6011 J_2^{1/2} \cos\omega_1 t \\ 0.7992 J_2^{1/2} \cos\omega_2 t \end{bmatrix}$$

$$\theta(t) = \begin{bmatrix} 0.2774\cos\omega_1 t - 0.2774\cos\omega_2 t \\ 0.3613\cos\omega_1 t + 0.6387\cos\omega_2 t \end{bmatrix}$$

where $\omega_1 = 0.4821\sqrt{\frac{k}{J_2}}$ and $\omega_2 = 1.1976\sqrt{\frac{k}{J_2}}$,

4.35 Consider the system of Example 4.3.1. Calculate a value of $\mathbf{x}(0)$ and $\dot{\mathbf{x}}(0)$ such that both masses of the system oscillate with a single frequency of 2 rad/s.

Solution:

From Example 4.3.1,

$$S = \frac{1}{\sqrt{2}}\begin{bmatrix} 1/3 & 1/3 \\ 1 & -1 \end{bmatrix}$$

$$S^{-1} = \frac{1}{\sqrt{2}}\begin{bmatrix} 3 & 1 \\ 3 & -1 \end{bmatrix}$$

From Equations (4.67) and (4.68),

$$r_1(t) = \frac{\sqrt{\omega_1^2 r_{10}^2 + r_{10}^2}}{\omega_1}\sin\left[\omega_1 t + \tan^{-1}\frac{\omega_1 r_{10}}{\dot{r}_{10}}\right]$$

$$r_2(t) = \frac{\sqrt{\omega_2^2 r_{20}^2 + r_{20}^2}}{\omega_2}\sin\left[\omega_2 t + \tan^{-1}\frac{\omega_2 r_{20}}{\dot{r}_{20}}\right]$$

Choose $\mathbf{x}(0)$ and $\dot{\mathbf{x}}(0)$ so that $r_1(t) = 0$. This will cause the frequency $\sqrt{2}$ to drop out. For $r_1(t) = 0$, its coefficient must be zero.

$$\frac{\sqrt{\omega_1^2 r_{10}^2 + r_{10}^2}}{\omega_1} = 0 \quad \text{or} \quad \omega_1^2 r_{10}^2 + r_{10}^2 = 0$$

Choose $r_{10} = \dot{r}_{20} = 0$.

Let $r_{20} = 3/\sqrt{2}$ and $\dot{r}_{20} = 0$ as calculated in Example 4.3.1.

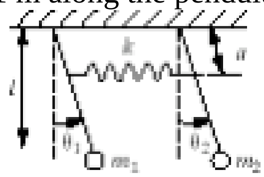
So, $\mathbf{r}(0) = \begin{bmatrix} 0 & 3/\sqrt{2} \end{bmatrix}^T$ and $\dot{\mathbf{r}}(0) = \mathbf{0}$.

Solve for $\mathbf{x}(0)$ and $\dot{\mathbf{x}}(0)$:

$$\mathbf{x}(0) = S\mathbf{r}(0) = \frac{1}{\sqrt{12}} \begin{bmatrix} 1/3 & 1/3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 3/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 0.5 \\ -1.5 \end{bmatrix}$$

$$\dot{\mathbf{x}}(0) = S\dot{\mathbf{r}}(0) = \mathbf{0}$$

4.36 Consider the system of Figure P4.36 consisting of two pendulums coupled by a spring. Determine the natural frequency and mode shapes. Plot the mode shapes as well as the solution to an initial condition consisting of the first mode shape for k = 20 N/m, l = 0.5 m and $m_1 = m_2$ = 10 kg, a = 0.1 m along the pendulum.



Solution: Given:

$$k = 20 \text{ N/m} \quad m_1 = m_2 = 10 \text{ kg}$$
$$a = 0.1 \text{ m} \qquad l = 0.5 \text{ m}$$

For gravity use $g = 9.81$ m/s^2. For a mass on a pendulum, the inertia is: $I = ml^2$

Calculate mass and stiffness matrices (for small θ). The equations of motion are:

$$\begin{matrix} I_1\ddot{\theta}_1 = ka^2\left(\theta_2 - \theta_1\right) - m_1gl\theta_1 \\ I_2\ddot{\theta}_2 = -ka^2\left(\theta_2 - \theta_1\right) - m_2gl\theta_2 \end{matrix} \Rightarrow ml^2\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} mgl + ka^2 & -ka^2 \\ -ka^2 & mgl + ka^2 \end{bmatrix}\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Substitution of the given values yields:

$$\begin{bmatrix} 2.5 & 0 \\ 0 & 2.5 \end{bmatrix}\ddot{\theta} + \begin{bmatrix} 49.05 & -0.2 \\ -0.2 & 49.05 \end{bmatrix}\theta = 0$$

Natural frequencies:

$$\tilde{K} = M^{-1/2}KM^{-1/2} = \begin{bmatrix} 19.7 & -0.08 \\ -0.08 & 19.7 \end{bmatrix}$$

$$\Rightarrow \lambda_1 = 19.54 \text{ and } \lambda_2 = 19.7 \Rightarrow \omega_1 = 4.42 \text{ rad/s and } \omega_2 = 4.438 \text{ rad/s}$$

Eigenvectors:

$$\lambda_1 = 19.54$$

$$\begin{bmatrix} 0.08 & -0.08 \\ -0.08 & 0.08 \end{bmatrix}\begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \quad \mathbf{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda_2 = 19.7$$

$$\begin{bmatrix} -0.08 & -0.08 \\ -0.08 & -0.08 \end{bmatrix}\begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \quad \mathbf{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

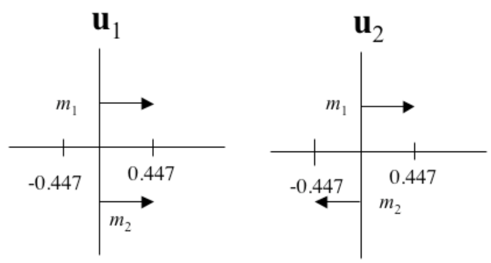
Now, $P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

Mode shapes:

$$\mathbf{u}_1 = M^{-1/2}\mathbf{v}_1 = \begin{bmatrix} 0.4472 \\ 0.4472 \end{bmatrix}$$

$$\mathbf{u}_2 = M^{-1/2}\mathbf{v}_2 = \begin{bmatrix} 0.4472 \\ -0.4472 \end{bmatrix}$$

A plot of the mode shapes is simply



This shows the first mode vibrates in phase and in the second mode the masses vibrate out of phase.

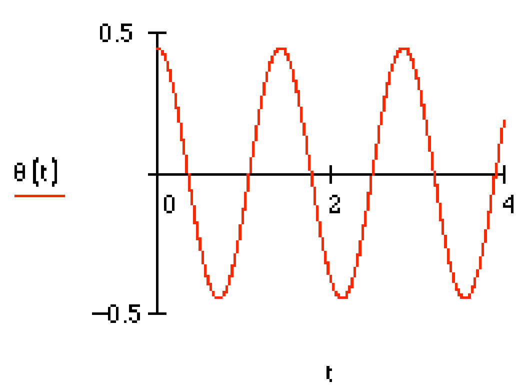
$$\theta(0) = \begin{bmatrix} 0.4472 \\ 0.4472 \end{bmatrix} \qquad \dot{\theta}(0) = 0,\ \ S = M^{-1/2}P = \begin{bmatrix} 0.4472 & 0.4472 \\ 0.4472 & -0.4472 \end{bmatrix}$$

$$S^{-1} = P^T M^{1/2} = \begin{bmatrix} 1.118 & 1.118 \\ 1.118 & -1.118 \end{bmatrix},\ \ \mathbf{r}(0) = S^{-1}\theta(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \dot{\mathbf{r}}(0) = 0$$

From Eq. (4.67) and (4.68): $r_1(t) = \sin\left(4.42t + \frac{\pi}{2}\right) = \cos 4.45t,\ \ r_2(t) = 0$

Convert to physical coordinates: $\theta(t) = S\mathbf{r}(t) = \begin{bmatrix} 0.4472\cos 4.42t \\ 0.4472\cos 4.42t \end{bmatrix}$ rad

$$\theta(t) := 0.4472 \cdot \cos(4.429 \cdot t)$$



4.37 Resolve Example 4.3.2 with m_2 changed to 10 kg. Plot the response and compare the plots to those of Figure 4.6.

Solution: From examples 4.3.2 and 4.2.5, with m_2 = 10 kg,

$$M\ddot{\mathbf{x}} + K\mathbf{x} = \begin{bmatrix} 1 & 0 \\ 0 & 10 \end{bmatrix}\ddot{\mathbf{x}} + \begin{bmatrix} 12 & -2 \\ -2 & 12 \end{bmatrix}\mathbf{x} = 0$$

Calculate eigenvalues and eigenvectors:

$$M^{-1/2} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\sqrt{10}} \end{bmatrix}$$

$$\tilde{K} = M^{-1/2}KM^{-1/2} = \begin{bmatrix} 12 & -0.6325 \\ -0.6325 & 1.2 \end{bmatrix}$$

$$\det(\tilde{K} - \lambda I) = \lambda^2 - 13.2\lambda + 14 = 0$$

$$\lambda_1 = 1.163 \quad \omega_1 = 1.078 \text{ rad/s}$$

$$\lambda_2 = 12.04 \quad \omega_2 = 3.469 \text{ rad/s}$$

$$P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} 0.0583 & -0.9983 \\ 0.9983 & 0.0583 \end{bmatrix}$$

Calculate S and S^{-1}:

$$S = M^{-1/2}P = \begin{bmatrix} 0.0583 & -0.9983 \\ 0.9983 & 0.0583 \end{bmatrix}$$

$$S^{-1} = P^T M^{1/2} = \begin{bmatrix} 0.0583 & 3.1569 \\ -0.9983 & 0.1842 \end{bmatrix}$$

Modal initial conditions:

$$\mathbf{r}(0) = S^{-1}\mathbf{x}(0) = S^{-1}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3.2152 \\ -0.8141 \end{bmatrix}$$

$$\dot{\mathbf{r}}(0) = S^{-1}\dot{\mathbf{x}}(0) = 0$$

Modal solution (from Eqs. (4.67) and (4.68):

$$r_1(t) = 3.2152\sin\left[1.078t + \frac{\pi}{2}\right] = 3.2152\cos 1.078t$$

$$r_2(t) = -0.8141\cos 3.469t$$

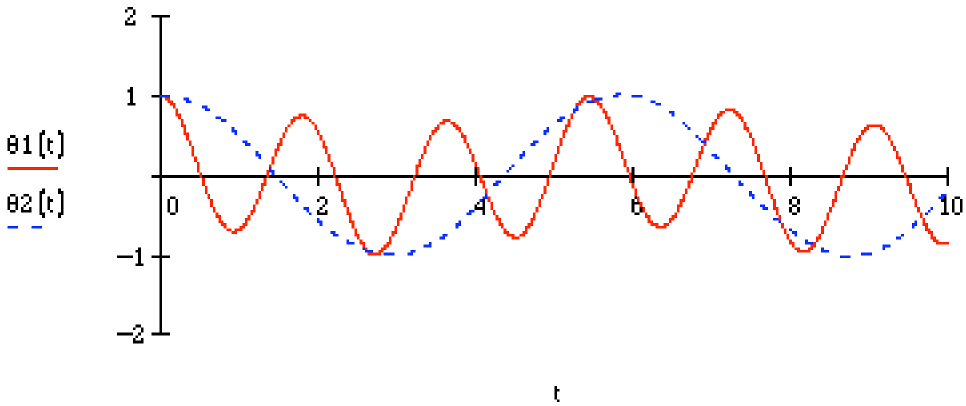
Covert to physical coordinates:

$$\mathbf{x}(t) = S\mathbf{r}(t) = \begin{bmatrix} 0.0583 & -0.9983 \\ 0.3157 & 0.0184 \end{bmatrix}\begin{bmatrix} 3.2152\cos 1.078t \\ -0.8141\cos 3.469t \end{bmatrix}$$

$$\mathbf{x}(t) = \begin{bmatrix} 0.1873\cos 1.078t + 0.8127\cos 3.469t \\ 1.015\cos 1.078t - 0.0150\cos 3.469t \end{bmatrix}$$

$$\theta 1(t) := 0.1873 \cdot \cos(1.078 \cdot t) + 0.8127 \cdot \cos(3.469 \cdot t)$$

$$\theta 2(t) := 1.015 \cdot \cos(1.078 \cdot t) - 0.0150 \cdot \cos(3.469 \cdot t)$$



These figures are similar to those of Figure 4.6, except the responses are reversed (θ_2 looks like x_2 in Figure 4.6, and θ_1 looks like x_1 in Figure 4.6)

4.38 Use modal analysis to calculate the solution of Problem 4.29 for the initial conditions

$$\mathbf{x}(0) = \begin{bmatrix} 0 & 1 \end{bmatrix}^T (\text{mm}) \text{ and } \dot{\mathbf{x}}(0) = \begin{bmatrix} 0 & 0 \end{bmatrix}^T \ (\text{mm/s})$$

Solution: From Problem 4.29,

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$$
$$\omega_1 = \sqrt{\lambda_1} = 0.4024 \text{ rad/s}$$
$$\omega_2 = \sqrt{\lambda_2} = 1.7573 \text{ rad/s}$$
$$\text{P} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} 0.1735 & -0.9848 \\ 0.9848 & 0.1735 \end{bmatrix}$$

Calculate S and S^{-1}:

$$S = M^{-1/2}P = \begin{bmatrix} 0.1735 & -0.9848 \\ 0.4924 & 0.0868 \end{bmatrix}$$
$$S^{-1} = P^T M^{1/2} = \begin{bmatrix} 0.1735 & 1.9697 \\ -0.9848 & 0.3470 \end{bmatrix}$$

Modal initial conditions:

$$\mathbf{r}(0) = S^{-1}\mathbf{x}(0) = S^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.9697 \\ 0.3470 \end{bmatrix}$$
$$\mathbf{r}(0) = S^{-1}\dot{\mathbf{x}}(0) = 0$$

Modal solution (from Eqs. (4.67) and (4.68):

$$r_1(t) = 1.9697\cos 0.4024t$$
$$r_2(t) = -.3470\cos 1.7573t$$

Convert to physical coordinates:

$$\mathbf{x}(t) = S\mathbf{r}(t) = \begin{bmatrix} 0.1735 & -0.9848 \\ 0.4924 & 0.0868 \end{bmatrix}\begin{bmatrix} 1.9697\cos 0.4024t \\ 0.3470\cos 1.7573t \end{bmatrix}$$
$$\Rightarrow \mathbf{x}(t) = \begin{bmatrix} 0.3417\cos 0.4024t - 0.3417\cos 1.7573t \\ 0.9699\cos 0.4024t + 0.0301\cos 1.7573t \end{bmatrix}\text{mm}$$

4.39 For the matrices

$$M^{-1/2} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & 4 \end{bmatrix} \text{ and } P = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

calculate $M^{-1/2}P$, $\left(M^{-1/2}P\right)^T$, and $P^T M^{-1/2}$ and hence verify that the computations in Eq. (4.70) make sense.

Solution:

Given

$$M^{-1/2} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & 4 \end{bmatrix} \text{ and } P = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

Now

$$M^{-1/2}P = \begin{bmatrix} 0.5 & 0.5 \\ -2\sqrt{2} & -2\sqrt{2} \end{bmatrix}$$

So

$$\left(M^{-1/2}P\right)^T = \begin{bmatrix} 0.5 & -2\sqrt{2} \\ 0.5 & -2\sqrt{2} \end{bmatrix}$$

$$P^T M^{-1/2} = \begin{bmatrix} 0.5 & -2\sqrt{2} \\ 0.5 & -2\sqrt{2} \end{bmatrix}$$

Thus, $\left(M^{-1/2}P\right)^T = P^T M^{-1/2}$ [Equation (4.71)]

4.40 Consider the 2-degree-of-freedom system defined by:

$$M = \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{and} \quad K = \begin{bmatrix} 27 & -3 \\ -3 & 3 \end{bmatrix}.$$

Calculate the response of the system to the initial condition

$$\mathbf{x}_0 = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix} \quad \dot{\mathbf{x}}_0 = \mathbf{0}$$

What is unique about your solution compared to the solution of Example 4.3.1.

Solution: Following the calculations made for this system in Example 4.3.1,

$$\omega_1 = \sqrt{\lambda_1} = 1.414 \text{ rad/s}, \quad \omega_2 = \sqrt{\lambda_2} = 2 \text{ rad/s}$$

$$P = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \Rightarrow S = M^{-1/2} P = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ 1 & -1 \end{bmatrix} \text{ and } S^{-1} = P^T M^{1/2} = \frac{1}{\sqrt{2}} \begin{bmatrix} 3 & 1 \\ 3 & -1 \end{bmatrix}$$

Next compute the modal initial conditions

$$\mathbf{r}(0) = S^{-1}\mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \dot{\mathbf{r}}(0) = S^{-1}\dot{\mathbf{x}}(0) = \mathbf{0}$$

Modal solution (from Eqs. (4.67) and (4.68)):

$$\mathbf{r}(t) = \begin{bmatrix} \cos 1.414t \\ 0 \end{bmatrix}$$

Note that the second coordinate modal coordinate has zero initial conditions and is hence not vibrating. Convert this solution back into physical coordinates:

$$\mathbf{x}(t) = S\mathbf{r}(t) = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \cos 1.414t \\ 0 \end{bmatrix}$$

$$\Rightarrow \mathbf{x}(t) = \begin{bmatrix} 0.236 \cos 1.414t \\ 0.707 \cos 1.414t \end{bmatrix}$$

The unique feature about the solution is that both masses are vibrating at only one frequency. That is the frequency of the first mode shape. This is because the system is excited with a position vector equal to the first mode of vibration.

4.41 Consider the 2-degree-of-freedom system defined by:

$$M = \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{and} \quad K = \begin{bmatrix} 27 & -3 \\ -3 & 3 \end{bmatrix}.$$

Calculate the response of the system to the initial condition

$$\mathbf{x}_0 = \mathbf{0}, \quad \text{and} \quad \dot{\mathbf{x}}_0 = \frac{1}{\sqrt{2}}\begin{bmatrix} \frac{1}{3} \\ -1 \end{bmatrix}$$

What is unique about your solution compared to the solution of Example 4.3.1 and to Problem 4.40, if you also worked that?

Solution: From example 4.3.1,

$$\omega_1 = \sqrt{\lambda_1} = 1.414 \text{ rad/s}, \; \omega_2 = \sqrt{\lambda_2} = 2 \text{ rad/s}, \; P = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\Rightarrow S = M^{-1/2}P = \frac{1}{\sqrt{2}}\begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ 1 & -1 \end{bmatrix}, \text{ and } S^{-1} = P^T M^{1/2} = \frac{1}{\sqrt{2}}\begin{bmatrix} 3 & 1 \\ 3 & -1 \end{bmatrix}$$

Modal initial conditions:

$$\mathbf{r}(0) = S^{-1}\mathbf{x}(0) = \mathbf{0}, \text{ and } \dot{\mathbf{r}}(0) = S^{-1}\dot{\mathbf{x}}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Modal solution (from Eqs. (4.67) and (4.68)):

$$\mathbf{r}(t) = \begin{bmatrix} 0 \\ \frac{1}{\omega_2}\cos 2t \end{bmatrix} = \begin{bmatrix} 0 \\ 0.5\cos 2t \end{bmatrix}$$

Convert to physical coordinates:

$$\mathbf{x}(t) = S\mathbf{r}(t) = \frac{1}{\sqrt{2}}\begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 0.5\cos 2t \end{bmatrix} = \begin{bmatrix} 0.118\cos 2t \\ -0.354\cos 2t \end{bmatrix}$$

Compared to Example 4.3.1, only the second mode is excited, because the initial velocity is proportional to the second mode shape, and the displacement is zero. Compared to the previous problem, here it is the second mode rather then the first mode that is excited.

4.42 Consider the system of Problem 4.1. Let k_1 = 10,000 N/m, k_2 = 15,000 N/m, and k_3 = 10,000 N/m. Assume that both masses are 100 kg. Solve for the free response of this system using modal analysis and the initial conditions

$$\mathbf{x}(0) = \begin{bmatrix} 1 & 0 \end{bmatrix}^T \quad \dot{\mathbf{x}}(0) = \mathbf{0}$$

Solution: Given:

$$k_1 = 10{,}000 \text{ N/m} \quad m_1 = m_2 = 100 \text{ kg}$$
$$k_2 = 15{,}000 \text{ N/m} \quad \mathbf{x}(0) = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$$
$$k_3 = 10{,}000 \text{ N/m} \quad \dot{\mathbf{x}}(0) = \mathbf{0}$$

Equation of motion:

$$M\ddot{\mathbf{x}} + K\mathbf{x} = \mathbf{0}$$
$$\begin{bmatrix} 100 & 0 \\ 0 & 100 \end{bmatrix}\ddot{\mathbf{x}} + \begin{bmatrix} 25{,}000 & -15{,}000 \\ -15{,}000 & 25{,}000 \end{bmatrix}\mathbf{x} = \mathbf{0}$$

Calculate eigenvalues and eigenvectors:

$$M^{-1/2} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}$$
$$\tilde{K} = M^{-1/2}KM^{-1/2} = \begin{bmatrix} 250 & -150 \\ -150 & 250 \end{bmatrix}$$
$$\det\left(\tilde{K} - \lambda I\right) = \lambda^2 - 500\lambda + 40{,}000 = 0$$

$$\lambda_1 = 100 \quad \omega_1 = 10 \text{ rad/s}$$
$$\lambda_2 = 400 \quad \omega_2 = 20 \text{ rad/s}$$

$\lambda_1 = 100$

$$\begin{bmatrix} 150 & -150 \\ -150 & 150 \end{bmatrix}\begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\mathbf{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\lambda_2 = 400$

$$\begin{bmatrix} -150 & -150 \\ -150 & -150 \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\mathbf{v}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Now, $P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

Calculate S and S^{-1}:

$$S = M^{-1/2} P = \frac{1}{\sqrt{2}} \begin{bmatrix} 0.1 & 0.1 \\ 0.1 & -0.1 \end{bmatrix}$$

$$S^{-1} = P^T M^{1/2} = \frac{1}{\sqrt{2}} \begin{bmatrix} 10 & 10 \\ 10 & -10 \end{bmatrix}$$

Modal initial conditions:

$$\mathbf{r}(0) = S^{-1}\mathbf{x}(0) = \frac{1}{\sqrt{2}} \begin{bmatrix} 10 \\ 10 \end{bmatrix}$$

$$\dot{\mathbf{r}}(0) = S^{-1}\dot{\mathbf{x}}(0) = \mathbf{0}$$

Modal solutions:

$$r_1(t) = \frac{\sqrt{\omega_1^2 r_{10}^2 + \dot{r}_{10}^2}}{\omega_1} \sin\left[\omega_1 t + \tan^{-1} \frac{\omega_1 r_{10}}{\dot{r}_{10}}\right]$$

$$r_2(t) = \frac{\sqrt{\omega_2^2 r_{20}^2 + \dot{r}_{20}^2}}{\omega_2} \sin\left[\omega_2 t + \tan^{-1} \frac{\omega_2 r_{20}}{\dot{r}_{20}}\right]$$

So

$$r_1(t) = 7.071 \sin(10t + \pi / 2) = 7.071 \cos 10t$$

$$r_2(t) = 7.071 \sin(20t + \pi / 2) = 7.071 \cos 20t$$

$$\mathbf{r}(t) = \begin{bmatrix} 7.071 \cos 10t \\ 7.071 \cos 20t \end{bmatrix}$$

Convert to physical coordinates:

$$\mathbf{x}(t) = S\mathbf{r}(t) = \frac{1}{\sqrt{2}}\begin{bmatrix} 0.1 & 0.1 \\ 0.1 & 0.1 \end{bmatrix}\begin{bmatrix} 7.071\cos 10t \\ 7.7071\cos 20t \end{bmatrix}$$

$$\mathbf{x}(t) = \begin{bmatrix} 0.5(\cos 10t + \cos 20t) \\ 0.5(\cos 10t - \cos 20t) \end{bmatrix}$$

4.43 Consider the model of a vehicle given in Problem 4.14 and illustrated in Figure P4.14. Suppose that the tire hits a bump which corresponds to an initial condition of

$$\mathbf{x}(0) = \begin{bmatrix} 0 \\ 0.01 \end{bmatrix} \quad \dot{\mathbf{x}}(0) = \mathbf{0}$$

Use modal analysis to calculate the response of the car $x_1(t)$. Plot the response for three cycles.

Solution: From Problem 4.14,

$$M\ddot{\mathbf{x}} + K\mathbf{x} = \begin{bmatrix} 2000 & 0 \\ 0 & 50 \end{bmatrix}\ddot{\mathbf{x}} + \begin{bmatrix} 1000 & -1000 \\ -1000 & 11{,}000 \end{bmatrix}\mathbf{x} = \mathbf{0}$$

Calculate the eigenvalues and eigenvectors:

$$M^{-1/2} = \begin{bmatrix} 0.0224 & 0 \\ 0 & 0.1414 \end{bmatrix},\ \tilde{K} = M^{-1/2}KM^{-1/2} = \begin{bmatrix} 0.5 & -3.1623 \\ -3.1623 & 0.1414 \end{bmatrix}$$

$$\Rightarrow \det\left(\tilde{K} - \lambda I\right) = \lambda^2 - 220.05\lambda + 100 = 0 \Rightarrow \begin{matrix} \lambda_1 = 0.4545 & \omega_1 = 0.6741 \text{ rad/s} \\ \lambda_2 = 220.05 & \omega_2 = 14.834 \text{ rad/s} \end{matrix}$$

$$P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} 0.9999 & -0.0144 \\ 0.0144 & 0.9999 \end{bmatrix}$$

Calculate S and S^{-1}:

$$S = M^{-1/2}P = \begin{bmatrix} 0.0224 & -0.003 \\ 0.0020 & 0.1414 \end{bmatrix},\ S^{-1} = P^T M^{1/2} = \begin{bmatrix} 44.7167 & 0.1018 \\ -0.6441 & 7.0703 \end{bmatrix}$$

Modal initial conditions:

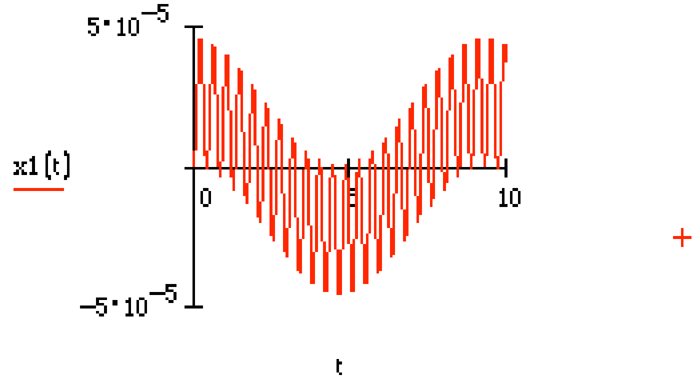
$$\mathbf{r}(0) = S^{-1}\mathbf{x}(0) = S^{-1}\begin{bmatrix} 0 \\ 0.01 \end{bmatrix} = \begin{bmatrix} 0.001018 \\ 0.07070 \end{bmatrix},\ \dot{\mathbf{r}}(0) = S^{-1}\dot{\mathbf{x}}(0) = \mathbf{0}$$

Modal solution (from equations (4.67) and (4.68)): $\mathbf{r}(t) = \begin{bmatrix} 0.001018\cos 0.6741t \\ 0.07070\cos 14.834t \end{bmatrix}$

Convert to physical coordinates:

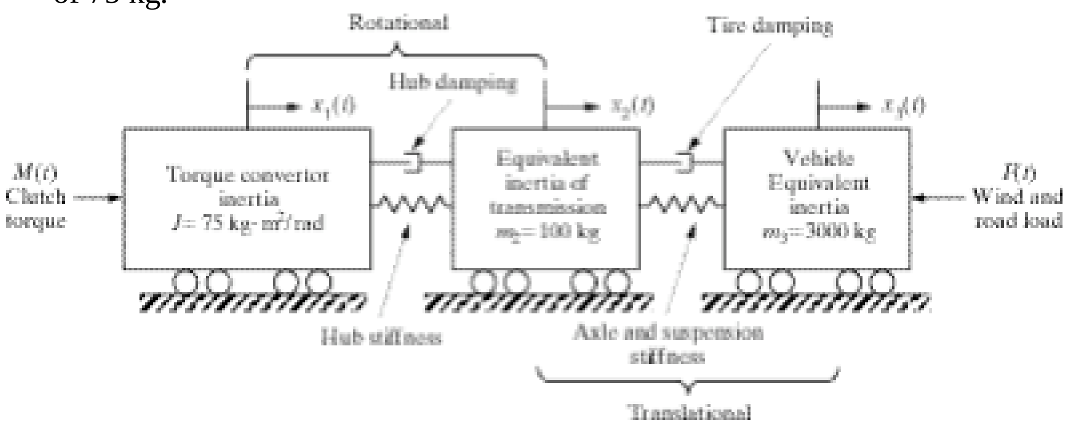
$$\mathbf{x}(t) = S\mathbf{r}(t) = \begin{bmatrix} 0.0224 & -0.0003 \\ 0.0020 & 0.1414 \end{bmatrix} \begin{bmatrix} 0.001018\cos 0.6741t \\ 0.07070\cos 14.834t \end{bmatrix} = \begin{bmatrix} 2.277\times 10^{-5}\cos 0.6741t - 2.277\times 10^{-5}\cos 14.834t \\ 2.074\times 10^{-6}\cos 0.6741t + 9.998\times 10^{-3}\cos 14.834t \end{bmatrix}$$

$$x1(t) := 2.277\cdot 10^{-5}\cdot\cos(0.674\cdot t) - 2.277\cdot 10^{-5}\cdot\cos(14.834\cdot t)$$



Problems and Solutions for Section 4.4 (4.44 through 4.55)

4.44 A vibration model of the drive train of a vehicle is illustrated as the three-degree-of-freedom system of Figure P4.44. Calculate the undamped free response [i.e. $M(t) = F(t) = 0$, $c_1 = c_2 = 0$] for the initial condition $\mathbf{x}(0) = \mathbf{0}$, $\dot{\mathbf{x}}(0) = [0 \quad 0 \quad 1]^T$. Assume that the hub stiffness is 10,000 N/m and that the axle/suspension is 20,000 N/m. Assume the rotational element J is modeled as a translational mass of 75 kg.



Solution: Let k_1 = hub stiffness and k_2 = axle and suspension stiffness.
The equation of motion is

$$\begin{bmatrix} 75 & 0 & 0 \\ 0 & 100 & 0 \\ 0 & 0 & 3000 \end{bmatrix}\ddot{\mathbf{x}} + 10{,}000\begin{bmatrix} 1 & -1 & 0 \\ -1 & 3 & -2 \\ 0 & -2 & 2 \end{bmatrix}\mathbf{x} = \mathbf{0}$$

$$\mathbf{x}(0) = \mathbf{0} \text{ and } \dot{\mathbf{x}}(0) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T \text{ m/s}$$

Calculate eigenvalues and eigenvectors:

$$M^{-1/2} = \begin{bmatrix} 0.1155 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.0183 \end{bmatrix}$$

$$\tilde{K} = M^{-1/2}KM^{-1/2} = \begin{bmatrix} 133.33 & -115.47 & 0 \\ -115.47 & 300 & -36.515 \\ 0 & -36.515 & 6.6667 \end{bmatrix}$$

$$\det\left(\tilde{K} - \lambda I\right) = \lambda^3 - 440\lambda^2 + 28{,}222\lambda = 0$$

$$\lambda_1 = 0 \qquad \omega_1 = 0 \text{ rad/s}$$

$$\lambda_2 = 77.951 \quad \omega_2 = 8.8290 \text{ rad/s}$$

$$\lambda_3 = 362.05 \quad \omega_3 = 19.028 \text{ rad/s}$$

$$\mathbf{v}_1 = \begin{bmatrix} 0.1537 \\ 0.1775 \\ 0.9721 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -0.8803 \\ -0.4222 \\ 0.2163 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0.4488 \\ -0.8890 \\ 0.0913 \end{bmatrix}$$

Use the mode summation method to find the solution.
Transform the initial conditions:

$$\mathbf{q}(0) = M^{-1/2}\mathbf{x}(0) = \mathbf{0}, \quad \dot{\mathbf{q}}(0) = M^{1/2}\dot{\mathbf{x}}(0) = \begin{bmatrix} 0 & 0 & 54.7723 \end{bmatrix}^T$$

The solution is given by:

$$\mathbf{q}(t) = (c_1 + c_4 t)\mathbf{v}_1 + c_2 \sin(\omega_2 t + \phi_2)\mathbf{v}_2 + c_3 \sin(\omega_3 t + \phi_3)\mathbf{v}_3$$

where

$$\phi_i = \tan^{-1}\left(\frac{\omega_i \mathbf{v}_i^T \mathbf{q}(0)}{\mathbf{v}_i^T \mathbf{q}(0)}\right) \qquad i = 2,3$$

$$c_i = \frac{\mathbf{v}_i^T \dot{\mathbf{q}}(0)}{\omega_i \cos\phi} \qquad i = 2,3$$

Thus,

$$\phi_2 = \phi_3 = 0, c_2 - 1.3417, \text{ and } c_3 = 0.2629$$

So,

$$\mathbf{q}(0) = c_1\mathbf{v}_1 + \sum_{i=2}^{3} c_i \sin\phi_i \mathbf{v}_i$$

$$\dot{\mathbf{q}}(0) = c_4\mathbf{v}_1 + \sum_{i=2}^{3} \omega_i c_i \cos\phi_i \mathbf{v}_i$$

Premultiply by v_1^T;

$$\mathbf{v}_1^T \mathbf{q}(0) = 0 = c_1$$

$$\mathbf{v}_1^T \dot{\mathbf{q}}(0) = 53.2414 = c_4$$

So,

$$\mathbf{q}(t) = 53.2414t\mathbf{v}_1 + 1.3417\sin(8.8290t)\mathbf{v}_2 + 0.2629\sin(19.028t)\mathbf{v}_3$$

Change to **q**(t):

$$\mathbf{x}(t) = M^{-1/2}\mathbf{q}(t)$$

$$\mathbf{x}(t) = 0.9449t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -0.1364 \\ -0.05665 \\ 0.005298 \end{bmatrix}\sin 8.8290t + \begin{bmatrix} 0.01363 \\ -0.02337 \\ 0.0004385 \end{bmatrix}\sin 19.028t \text{ m}$$

4.45 Calculate the natural frequencies and normalized mode shapes of

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \ddot{\mathbf{x}} + \begin{bmatrix} 4 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \mathbf{x} = \mathbf{0}$$

Solution: Given the indicated mass and stiffness matrix, calculate eigenvalues:

$$M^{-1/2} = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.7071 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \tilde{K} = M^{-1/2} K M^{-1/2} = \begin{bmatrix} 1 & -0.3536 & 0 \\ -0.3536 & 1 & -0.7071 \\ 0 & -0.7071 & 1 \end{bmatrix}$$

$$\det\left(\tilde{K} - \lambda I\right) = \lambda^3 - 3\lambda^2 + 2.375\lambda - 0.375 = 0$$

$$\lambda_1 = 0.2094, \quad \lambda_2 = 1, \quad \lambda_3 = 1.7906$$

The natural frequencies are:

$$\omega_1 = 0.4576 \text{ rad/s}$$
$$\omega_2 = 1 \text{ rad/s}$$
$$\omega_3 = 1.3381 \text{ rad/s}$$

The corresponding eigenvectors are:

$$\mathbf{v}_1 = \begin{bmatrix} -0.3162 \\ -0.7071 \\ -0.6325 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 0.8944 \\ 0 \\ -0.4472 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 0.3162 \\ -0.7071 \\ 0.6325 \end{bmatrix}$$

The relationship between eigenvectors and mode shapes is

$$\mathbf{u} = M^{-1/2}\mathbf{v}$$

The mode shapes are:

$$\mathbf{u}_1 = \begin{bmatrix} -0.1581 \\ -0.5 \\ -0.6325 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 0.4472 \\ 0 \\ -0.4472 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 0.1581 \\ -0.5 \\ 0.6325 \end{bmatrix}$$

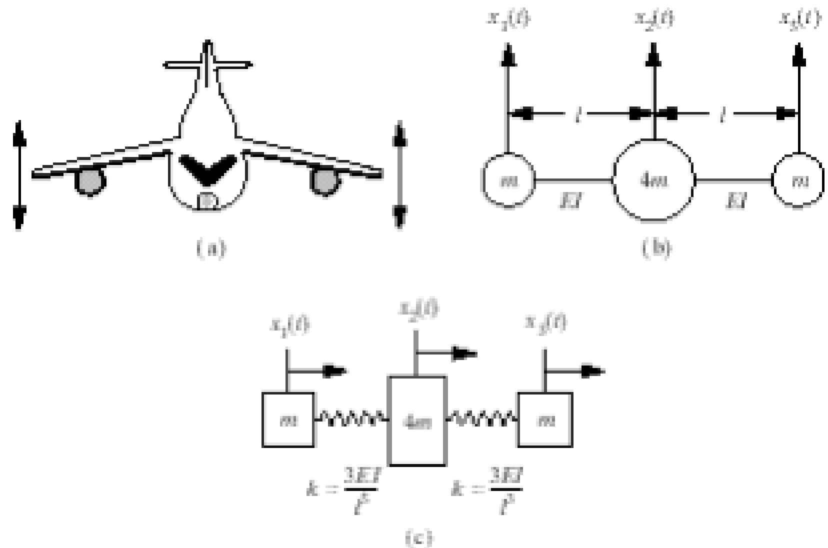
The normalized mode shapes are

$$\hat{\mathbf{u}}_1 = \frac{\mathbf{u}_1}{\sqrt{\mathbf{u}_1^T \mathbf{u}_1}} = \begin{bmatrix} 0.192 \\ 0.609 \\ 0.77 \end{bmatrix}, \quad \hat{\mathbf{u}}_2 = \begin{bmatrix} 0.707 \\ 0 \\ -0.707 \end{bmatrix}, \quad \hat{\mathbf{u}}_3 = \begin{bmatrix} 0.192 \\ -0.609 \\ 0.77 \end{bmatrix}.$$

4.46 The vibration is the vertical direction of an airplane and its wings can be modeled as a three-degree-of-freedom system with one mass corresponding to the right wing, one mass for the left wing, and one mass for the fuselage. The stiffness connecting the three masses corresponds to that of the wing and is a function of the modulus E of the wing. The equation of motion is

$$m\begin{bmatrix}1 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\ddot{x}_1\\ \ddot{x}_2\\ \ddot{x}_3\end{bmatrix} + \frac{EI}{l^3}\begin{bmatrix}3 & -3 & 0\\ -3 & 6 & -3\\ 0 & -3 & 3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

The model is given in Figure P4.46. Calculate the natural frequencies and mode shapes. Plot the mode shapes and interpret them according to the airplane's deflection.



Solution: Given the equation of motion indicated above, the mass-normalized stiffness matrix is calculated to be

$$M^{-1/2} = \frac{1}{\sqrt{m}}\begin{bmatrix}1 & 0 & 0\\ 0 & 0.5 & 0\\ 0 & 0 & 1\end{bmatrix}, \quad \tilde{K} = M^{-1/2}KM^{-1/2} = \frac{EI}{m\ell^3}\begin{bmatrix}3 & -1.5 & 0\\ -1.5 & 1.5 & -1.5\\ 0 & -1.5 & 3\end{bmatrix}$$

Computing the matrix eigenvalue by factoring out the constant $\frac{EI}{m\ell^3}$ yields

$$\det(\tilde{K} - \lambda I) = 0 \Rightarrow \lambda_1 = 0, \quad \lambda_2 = 3\frac{EI}{m\ell^3}, \quad \lambda_3 = 4.5\frac{EI}{m\ell^3}$$

and eigenvectors:

$$\mathbf{v}_1 = \begin{bmatrix}0.4082\\ 0.8165\\ 0.4082\end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix}-0.7071\\ 0\\ 0.7071\end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix}0.5774\\ -0.5774\\ 0.5774\end{bmatrix}$$

The natural frequencies are $\omega_1 = 0$, $\omega_2 = 1.7321\sqrt{\dfrac{EI}{m\ell^3}}$ rad/s, and $\omega_3 = 2.1213\sqrt{\dfrac{EI}{m\ell^3}}$ rad/s.

The relationship between the mode shapes and eigenvectors **u** is just $\mathbf{u} = M^{-1/2}\mathbf{v}$. The fist mode shape is the rigid body mode. The second mode shape corresponds to one wing up and one down the third mode shape corresponds to the wings moving up and down together with the body moving opposite. Normalizing the mode shapes yields (calculations in Mathcad):

$$Mh := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad K := \begin{bmatrix} 3 & -3 & 0 \\ -3 & 6 & -3 \\ 0 & -3 & 3 \end{bmatrix} \quad Kh := Mh^{-1}\cdot K\cdot Mh^{-1}$$

$$Kh = \begin{bmatrix} 3 & -1.5 & 0 \\ -1.5 & 1.5 & -1.5 \\ 0 & -1.5 & 3 \end{bmatrix}$$

$$\lambda a := \text{eigenvals}(Kh) \qquad \lambda a = \begin{bmatrix} 4.5 \\ 3 \\ 0 \end{bmatrix}$$

$$v1 := \text{eigenvec}(Kh, \lambda a_2) \quad v1 = \begin{bmatrix} 0.408 \\ 0.816 \\ 0.408 \end{bmatrix} \qquad v2 := \text{eigenvec}(Kh, \lambda a_1) \quad v2 = \begin{bmatrix} -0.707 \\ 0 \\ 0.707 \end{bmatrix}$$

$$v3 := \text{eigenvec}(Kh, \lambda a_0) \qquad v3 = \begin{bmatrix} 0.577 \\ -0.577 \\ 0.577 \end{bmatrix}$$

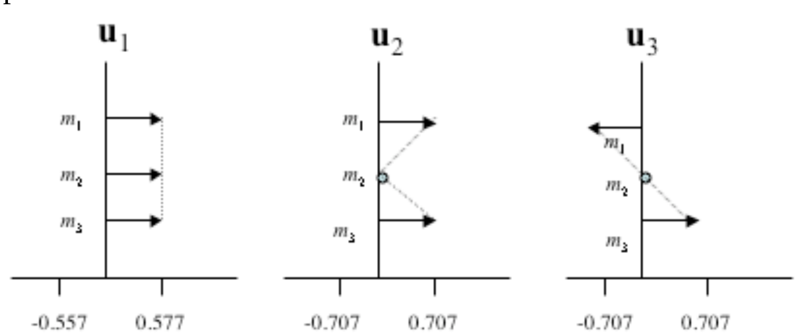
$$\omega := \begin{bmatrix} \sqrt{\lambda a_2} \\ \sqrt{\lambda a_1} \\ \sqrt{\lambda a_0} \end{bmatrix} \quad \omega = \begin{bmatrix} 2.967\cdot 10^{-8} \\ 1.732 \\ 2.121 \end{bmatrix} \qquad u1 := Mh^{-1}\cdot v1 \quad u1 = \begin{bmatrix} 0.408 \\ 0.408 \\ 0.408 \end{bmatrix}$$

$$u1n := \frac{u1}{|u1|} \quad u1n = \begin{bmatrix} 0.577 \\ 0.577 \\ 0.577 \end{bmatrix} \qquad u2 := Mh^{-1}\cdot v2 \quad u2 = \begin{bmatrix} -0.707 \\ 0 \\ 0.707 \end{bmatrix}$$

$$u3 := Mh^{-1}\cdot v3 \quad u3 = \begin{bmatrix} 0.577 \\ -0.289 \\ 0.577 \end{bmatrix} \qquad u2n := \frac{u2}{|u2|} \quad u2n = \begin{bmatrix} -0.707 \\ 0 \\ 0.707 \end{bmatrix}$$

$$u3n := \frac{u3}{|u3|} \quad u3n = \begin{bmatrix} 0.667 \\ -0.333 \\ 0.667 \end{bmatrix}$$

These are plotted:



4.47 Solve for the free response of the system of Problem 4.46. Where $E = 6.9 \times 10^9$ N/m^2, l = 2 m, m = 3000 kg, and $I = 5.2 \times 10^{-6}$m^4. Let the initial displacement correspond to a gust of wind that causes an initial condition of $\dot{\mathbf{x}}(0) = \mathbf{0}$, $\mathbf{x}(0) = [0.2 \quad 0 \quad 0]^T$ m. Discuss your solution.

Solution: From problem 4.43 and the given data

$$\begin{bmatrix} 3000 & 0 & 0 \\ 0 & 12{,}000 & 0 \\ 0 & 0 & 3{,}000 \end{bmatrix} \ddot{\mathbf{x}} + \begin{bmatrix} 1.346 & -1.346 & 0 \\ -1.346 & 2.691 & -1.346 \\ 0 & -1.346 & 1.346 \end{bmatrix} \times 10^4 \mathbf{x} = \mathbf{0}$$

$$\mathbf{x}(0) = \begin{bmatrix} 0.2 & 0 & 0 \end{bmatrix}^T m$$

$$\dot{\mathbf{x}}(0) = \mathbf{0}$$

Convert to **q**:

$$I\ddot{\mathbf{q}} + \begin{bmatrix} 4.485 & -2.242 & 0 \\ -2.242 & 2.242 & -2.242 \\ 0 & -2.242 & 4.485 \end{bmatrix} \mathbf{q} = \mathbf{0}$$

Calculate eigenvalues and eigenvectors:

$$\det\left(\tilde{K} - \lambda I\right) = 0 \Rightarrow$$

$$\begin{aligned} \lambda_2 &= 0 & \omega_1 &= 0 \text{ rad/s} \\ \lambda_2 &= 4.485 & \omega_2 &= 2.118 \text{ rad/s} \\ \lambda_3 &= 6.727 & \omega_3 &= 2.594 \text{ rad/s} \end{aligned}$$

$$\mathbf{v}_1 = \begin{bmatrix} 0.4082 \\ 0.8165 \\ 0.4082 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} -0.7071 \\ 0 \\ 0.7071 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 0.5774 \\ -0.5774 \\ 0.5774 \end{bmatrix}$$

The solution is given by

$$\mathbf{q}(t) = \left(c_1 + c_4 t\right)\mathbf{v}_1 + c_2 \sin\left(\omega_2 t + \phi_2\right)\mathbf{v}_2 + c_3 \sin\left(\omega_3 t + \phi_3\right)\mathbf{v}_3$$

where

$$\phi_i = \tan^{-1}\left(\frac{\omega_i \mathbf{v}_i^T \mathbf{q}(0)}{\mathbf{v}_i^T \dot{\mathbf{q}}(0)}\right) \qquad i = 2,3$$

$$c_i = \frac{\mathbf{v}_i^T \mathbf{q}(0)}{\sin\phi_i} \quad i = 2,3$$

Thus, $\phi_2 = \phi_3 = \frac{\pi}{2}, c_2 = -7.7459,$ and $c_3 = 6.3251$

So,

$$\mathbf{q}(0) = c_1 \mathbf{v}_1 + \sum_{i=2}^{3} c_i \sin\phi_i \mathbf{v}_i$$

$$\dot{\mathbf{q}}(0) = c_4 \mathbf{v}_i + \sum_{i=2}^{3} \omega_i c_i \cos\phi_i \mathbf{v}_i$$

Premultiply by $\mathbf{v}_i^T$:

$$\mathbf{v}_i^T \mathbf{q}(0) = 4.4716 = c_1$$

$$\mathbf{v}_i^T \dot{\mathbf{q}}(0) = 0 = c_4$$

So, $\mathbf{q}(t) = 4.4716\mathbf{v}_1 - 7.7459\cos(2.118t)\mathbf{v}_2 + 6.3251\cos(2.594t)\mathbf{v}_3$

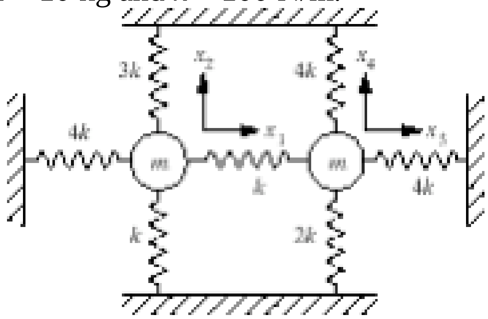
Convert to physical coordinates:

$$\mathbf{x}(t) = M^{-1/2}\mathbf{q}(t) \Rightarrow$$

$$\mathbf{x}(t) = \begin{bmatrix} 0.0333 \\ 0.0333 \\ 0.0333 \end{bmatrix} + \begin{bmatrix} 0.1 \\ 0 \\ -0.1 \end{bmatrix} \cos 2.118t + \begin{bmatrix} 0.0667 \\ -0.0333 \\ 0.0667 \end{bmatrix} \cos 2.594t \text{ m}$$

The first term is a rigid body mode, which represents (in this case) a fixed displacement around which the three masses oscillate. Mode two has the highest amplitude (0.1 m).

4.48 Consider the two-mass system of Figure P4.48. This system is free to move in the $x_1 - x_2$ plane. Hence each mass has two degrees of freedom. Derive the linear equations of motion, write them in matrix form, and calculate the eigenvalues and eigenvectors for m = 10 kg and k = 100 N/m.



Solution: Given: $m = 10\,\text{kg}, k = 100$ N/m

Mass 1

$$x_1 - \text{direction:}\quad m\ddot{x}_1 = -4kx_1 + k\left(x_3 - x_1\right) = -5kx_1 + kx_3$$

$$x_2 - \text{direction:}\quad m\ddot{x}_2 = -3kx_2 - kx_2 = -4kx_2$$

Mass 2

$$x_3 - \text{direction:}\quad m\ddot{x}_3 = -4kx_3 - k\left(x_3 - x_1\right) = -kx_1 - 5kx_3$$

$$x_4 - \text{direction:}\quad m\ddot{x}_4 = -4kx_4 - 2kx_4 = -6kx_4$$

In matrix form with the values given:

$$\begin{bmatrix} 10 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 10 \end{bmatrix} \ddot{\mathbf{x}} + \begin{bmatrix} 500 & 0 & -100 & 0 \\ 0 & 400 & 0 & 0 \\ -100 & 0 & 500 & 0 \\ 0 & 0 & 0 & 600 \end{bmatrix} \mathbf{x} = \mathbf{0}$$

$$\tilde{K} = M^{-1/2} K M^{-1/2} = \begin{bmatrix} 50 & 0 & -10 & 0 \\ 0 & 40 & 0 & 0 \\ -10 & 0 & 50 & 0 \\ 0 & 0 & 0 & 60 \end{bmatrix}$$

$$\det\left(\tilde{K} - \lambda I\right) = \lambda^4 - 200\lambda^3 + 14{,}800\lambda^2 - 480{,}000\lambda + 5{,}760{,}000 = 0$$

$$\Rightarrow \lambda_1 = 40,\quad \lambda_2 = 40,\quad \lambda_3 = 60,\quad \lambda_4 = 60$$

The corresponding eigenvectors are found from solving $(\tilde{K} - \lambda_i)\mathbf{v}_i = 0$ for each value of the index and normalizing:

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 0.7071 \\ 0 \\ 0.7071 \\ 0 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 0.7071 \\ 0 \\ -0.7071 \\ 0 \end{bmatrix} \quad \mathbf{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

These are not unique.

4.49 Consider again the system discussed in Problem 4.48. Use modal analysis to calculate the solution if the mass on the left is raised along the x_2 direction exactly 0.01 m and let go.

Solution: From Problem 4.48:

$$\begin{bmatrix} 10 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 10 \end{bmatrix} \ddot{\mathbf{x}} + \begin{bmatrix} 500 & 0 & -100 & 0 \\ 0 & 400 & 0 & 0 \\ -100 & 0 & 500 & 0 \\ 0 & 0 & 0 & 600 \end{bmatrix} \mathbf{x} = \mathbf{0}$$

$$M^{-1/2} = 0.3162 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\tilde{K} = M^{-1/2} K M^{-1/2} = \begin{bmatrix} 50 & 0 & -10 & 0 \\ 0 & 40 & 0 & 0 \\ -10 & 0 & 50 & 0 \\ 0 & 0 & 0 & 60 \end{bmatrix}$$

$$\lambda_1 = 40 \quad \omega_1 = 6.3246 \text{ rad/s}$$
$$\lambda_2 = 40 \quad \omega_2 = 6.3246 \text{ rad/s}$$
$$\lambda_3 = 60 \quad \omega_3 = 7.7460 \text{ rad/s}$$
$$\lambda_4 = 60 \quad \omega_4 = 7.7460 \text{ rad/s}$$

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 0.7071 \\ 0 \\ 0.7071 \\ 0 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 0.7071 \\ 0 \\ -0.7071 \\ 0 \end{bmatrix} \quad \mathbf{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Also, $\mathbf{x}(0)=[0 \quad 0.01 \quad 0 \quad 0]^T$ m and $\dot{\mathbf{x}}(0) = \mathbf{0}$

Use the mode summation method to find the solution.
Transform the initial conditions:

$$\mathbf{q}(0) = M^{1/2}\mathbf{x}(0) = \begin{bmatrix} 0 & 0.003162 & 0 & 0 \end{bmatrix}^T$$
$$\dot{\mathbf{q}}(0) = M^{1/2}\dot{\mathbf{x}}(0) = \mathbf{0}$$

The solution is given by Eq. (4.103),

$$\mathbf{x}(t) = \sum_{i=1}^{4} d_i \sin(\omega_i t + \phi_i)\mathbf{u}_i$$

where

$$\phi_i = \tan^{-1}\left(\frac{\omega_i \mathbf{v}_i^T \mathbf{q}(0)}{\mathbf{v}_i^T \dot{\mathbf{q}}(0)}\right) \quad i = 1,2,3,4 \quad \text{(Eq.(4.97))}$$
$$\mathrm{d_i} = \frac{\mathbf{v}_i^T \mathbf{q}(0)}{\sin\phi_i} \qquad i = 1,2,3,4 \quad \left(\text{Eq. } (4.98)\right)$$
$$\mathbf{u}_\mathrm{i} = M^{-1/2}\mathbf{v}_i$$

Substituting known values yields

$$\phi_1 = \phi_2 = \phi_3 = \phi_4 = \frac{\pi}{2} \text{ rad}$$

$$\mathrm{d}_1 = 0.003162$$
$$d_2 = d_3 = d_4 = 0$$

$$\mathbf{u}_1 = \begin{bmatrix} 0 \\ 0.3162 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{u}_2 = \begin{bmatrix} 0.2236 \\ 0 \\ 0.2236 \\ 0 \end{bmatrix} \quad \mathbf{u}_3 = \begin{bmatrix} 0.2236 \\ 0 \\ -0.2236 \\ 0 \end{bmatrix} \quad \mathbf{u}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.3162 \end{bmatrix}$$

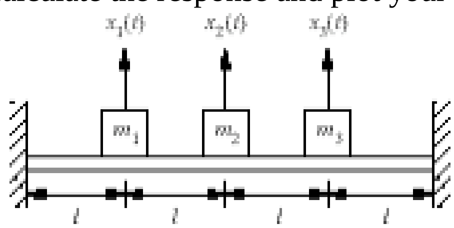
The solution is

$$\mathbf{x}(t) = \begin{bmatrix} 0 \\ 0.001\cos 6.3246t \\ 0 \\ 0 \end{bmatrix}$$

4.50 The vibration of a floor in a building containing heavy machine parts is modeled in Figure P4.50. Each mass is assumed to be evenly spaced and significantly larger than the mass of the floor. The equation of motion then becomes $\left(m_1 = m_2 = m_3 = m\right)$.

$$m I \ddot{\mathbf{x}} + \frac{EI}{l^3}\begin{bmatrix} \frac{9}{64} & \frac{1}{6} & \frac{13}{192} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \\ \frac{13}{192} & \frac{1}{6} & \frac{9}{64} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{0}$$

Calculate the natural frequencies and mode shapes. Assume that in placing box m_2 on the floor (slowly) the resulting vibration is calculated by assuming that the initial displacement at m_2 is 0.05 m. If l = 2 m, m = 200 kg, $E = 0.6 \times 10^9$ N/m^2, $I = 4.17 \times 10^{-5}$ m^4. Calculate the response and plot your results.



Solution:

The equations of motion can be written as

$$m\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + \frac{EI}{l^3}\begin{bmatrix} \frac{9}{64} & \frac{1}{6} & \frac{13}{192} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \\ \frac{13}{192} & \frac{1}{6} & \frac{9}{64} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_2 \end{bmatrix} = 0$$

or $mI\ddot{x} + Kx = 0$ where I is the 3x3 identity matrix.

The natural frequencies of the system are obtained using the characteristic equation

$$\left|K - \omega^2 M\right| = 0$$

Using the given mass and stiffness matrices yields the following characteristic equation

$$m^3\omega^6 - \frac{59EI\,m^3}{96l^3}\omega^4 + \frac{41(EI)^2\,m}{768l^6}\omega^2 - \frac{7(EI)^3}{6912l^9} = 0$$

Substituting for *E*, *I*, *m*, and *l* yields the following answers for the natural frequency

$$\omega_1 = \pm\sqrt{\frac{\left(13-\sqrt{137}\right)EI}{ml^3}}\ ,\ \omega_2 = \pm\sqrt{\frac{7EI}{96ml^3}}\ ,\ \omega_3 = \pm\sqrt{\frac{\left(13+\sqrt{137}\right)EI}{48ml^3}}$$

The plus minus sign shown above will cause the exponential terms to change to trigonometric terms using Euler's formula. Hence, the natural frequencies of the system are 0.65 rad/sec, 1.068 rad/sec and 2.837 rad/sec.

Let the mode shapes of the system be $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$. The mode shapes should satisfy the following equation

$$\left[K - \omega_1^2 M\right]\begin{Bmatrix} u_{i1} \\ u_{i2} \\ u_{i3} \end{Bmatrix} = 0, i = 1,2,3$$

Notice that the system above does not have a unique solution for $\mathbf{u}_1$ since $\left[K - \omega_1^2 M\right]$ had to be singular in order to solve for the natural frequency ω. Solving the above equation yields the following relations

$$\frac{u_{i2}}{u_{i3}} = \frac{1}{3}\frac{96m\omega_i^2 l^3 - 7EI}{13m\omega_i^2 l^3 + EI}, i = 1,2,3$$

and $u_{i1} = u_{i3}, i = 1,3$ but for the second mode shape this is different $u_{21} = u_{23}$

Substituting the values given yields

$$\frac{u_{12}}{u_{13}} = \frac{1}{3}\frac{96m\omega_1^2 l^3 - 7EI}{13m\omega_1^2 l^3 + EI} = -1.088$$

$$\frac{u_{22}}{u_{23}} = \frac{1}{3}\frac{96m\omega_2^2 l^3 - 7EI}{13m\omega_2^2 l^3 + EI} = 0$$

$$\frac{u_{32}}{u_{33}} = \frac{1}{3}\frac{96m\omega_3^2 l^3 - 7EI}{13m\omega_3^2 l^3 + EI} = 1.838$$

If we let $u_{i3} = 1, i = 1,2,3$, then

$$u_1 = \begin{Bmatrix} 1 \\ -1.088 \\ 1 \end{Bmatrix}, u_2 = \begin{Bmatrix} -1 \\ 0 \\ 1 \end{Bmatrix}, u_3 = \begin{Bmatrix} 1 \\ 1.838 \\ 1 \end{Bmatrix}$$

These mode shapes can be normalized to yield

$$u_1 = \begin{Bmatrix} 0.5604 \\ -0.6098 \\ 0.5604 \end{Bmatrix}, u_2 = \begin{Bmatrix} -0.7071 \\ 0 \\ 0.7071 \end{Bmatrix}, u_3 = \begin{Bmatrix} 0.4312 \\ 0.7926 \\ 0.4312 \end{Bmatrix}$$

This solution is the same if obtained using MATLAB

$$u_1 = \begin{Bmatrix} -0.5604 \\ 0.6098 \\ -0.5604 \end{Bmatrix},\ u_2 = \begin{Bmatrix} -0.7071 \\ 0.0000 \\ 0.7071 \end{Bmatrix},\ u_3 = \begin{Bmatrix} 0.4312 \\ 0.7926 \\ 0.4312 \end{Bmatrix}$$

The second box, m_2, is placed slowly on the floor; hence, the initial velocity can be safely assumed zero. The initial displacement at m_2 is given to be 0.05 m.

Hence, the initial conditions in vector form are given as

$$x(0) = \begin{Bmatrix} 0 \\ -0.05 \\ 0 \end{Bmatrix} \text{ and } \dot{x}(0) = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

The equations of motion given by $M\ddot{\mathbf{x}}(t) + K\mathbf{x}(t) = \mathbf{0}$ can be transformed into the modal coordinates by applying the following transformation

$\mathbf{x}(t) = S\mathbf{r}(t) = M^{-\frac{1}{2}}P\mathbf{r}(t)$ where P is the basis formed by the mode shapes of the system, given by

$$P = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}$$

Hence, the transformation S is given by

$$S = \begin{bmatrix} -0.04 & -0.05 & 0.03 \\ 0.043 & 0 & 0.056 \\ -0.04 & 0.05 & 0.03 \end{bmatrix}$$

The initial conditions will be also transformed

$$\mathbf{r}(0) = S^{-1}\mathbf{x}(0) = \begin{Bmatrix} -0.431 \\ 0 \\ -0.56 \end{Bmatrix}$$

Hence, the modal equations are

with the above initial conditions.

The solution will then be

$$\mathbf{r}(t) = \begin{Bmatrix} 0.431\cos(0.65t) \\ 0 \\ 0.56\cos(2.837t) \end{Bmatrix}$$

The solution can then be determined by

$$\mathbf{x}(t) = \begin{Bmatrix} 0.0172\cos(0.65t) - 0.0168\cos(2.837t) \\ -0.0185\cos(0.65t) - 0.0313\cos(2.837t) \\ 0.0172\cos(0.65t) - 0.0168\cos(2.837t) \end{Bmatrix}$$

The equations of motion can be also be solved using MATLAB to yield the following response.

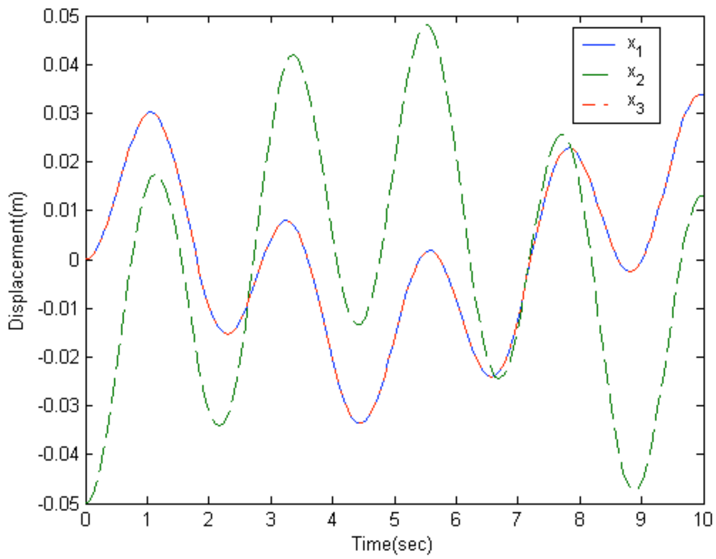


Figure 1 Numerical response due to initial deflection at m_2

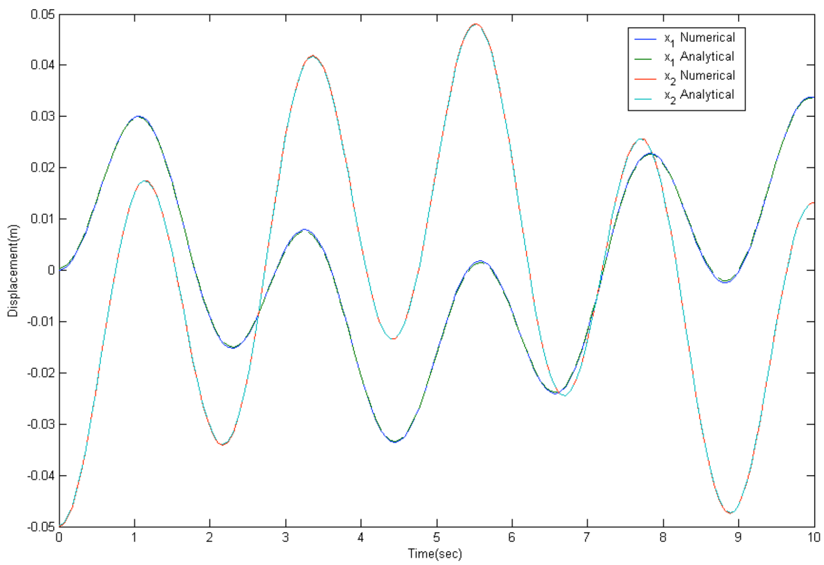


Figure 2 Numerical vs. Analytical Response (shown for x_1 and x_2 only)

The MATLAB code is attached below

```
% Set the values of the physical parameters
%
***********************************************************************
*

% Declare global variables to be used in the differential equation file
global M K
```

```
% Define the mass of the each box
m=200;

% Define the distance l
l=2;

% Define the area moment of inertia
I=4.17*10^-5;

% Define the modulus of elasticity
E=0.6*10^9;

% Define the flexural rigidity
EI=E*I;

% Define the system matrices
%
*************************************************************************
*

% Define the mass matrix
M=m*eye(3,3);

% Define the stiffness matrix
K=EI/l^3*[9/64 1/6 13/192;1/6 1/3 1/6;13/192 1/6 9/64];

% Solve the eigen value problem
[u,lambda]=eig(M\K);

% Simulate the response of the system to the given initial conditions
% The states are arranges as: [x1;x2;x3;x1_dot;x2_dot;x3_dot]
[t,xn]=ode45('sys4p47',[0 10],[0 ; -0.05 ; 0 ; 0 ; 0 ; 0]);

% Plot the results
plot(t,xn(:,1),t,xn(:,2),'--',t,xn(:,3),'-.');
set(gcf,'Color','White');
xlabel('Time(sec)');
ylabel('Displacement(m)');
legend('x_1','x_2','x_3');

% Analytical solution

for i=1:length(t)
   xa(:,i)=[0.0172*cos(0.65*t(i))-0.0168*cos(2.837*t(i));
         -0.0185*cos(0.65*t(i))-0.0313*cos(2.837*t(i)) ;
```

```
        0.0172*cos(0.65*t(i))-0.0168*cos(2.837*t(i))];
end;

% Camparison
figure;
plot(t,xn(:,1),t,xa(1,:),'--',t,xn(:,2),t,xa(2,:),'--');
set(gcf,'Color','White');
xlabel('Time(sec)');
ylabel('Displacement(m)');
legend('x_1 Numerical','x_1 Analytical','x_2 Numerical','x_2 Analytical');
```

4.51 Recalculate the solution to Problem 4.50 for the case that m_2 is increased in mass to 2000 kg. Compare your results to those of Problem 4.50. Do you think it makes a difference where the heavy mass is placed?

Solution: Given the data indicated the equation of motion becomes:

$$\begin{bmatrix} 200 & 0 & 0 \\ 0 & 2000 & 0 \\ 0 & 0 & 200 \end{bmatrix}\ddot{\mathbf{x}} + 3.197\times 10^{-4}\begin{bmatrix} 9/64 & 1/6 & 13/192 \\ 1/6 & 1/3 & 1/6 \\ 13/192 & 1/6 & 9/64 \end{bmatrix}\mathbf{x} = \mathbf{0}$$

$$\mathbf{x}(0) = [0 \quad 0.05 \quad 0]^T, \dot{\mathbf{x}}(0) = \mathbf{0}$$

Calculate eigenvalues and eigenvectors:

$$M^{-1/2} = \begin{bmatrix} 0.07071 & 0 & 0 \\ 0 & 0.02246 & 0 \\ 0 & 0 & 0.07071 \end{bmatrix}$$

$$\tilde{K} = M^{-1/2}KM^{-1/2} = \begin{bmatrix} 2.2482 & 0.8246 & 1.0825 \\ 0.8246 & 0.5329 & 0.8246 \\ 1.0825 & 0.8246 & 2.2482 \end{bmatrix}\times 10^{-7}$$

$$\det\left(\tilde{K} - \lambda I\right) = \lambda^3 - 9.8255\times 10^{-7}\lambda^2 + 1.3645\times 10^{-14}\lambda - 4.1382\times 10^{-22} = 0$$

$$\lambda_1 = 4.3142\times 10^{-9} \quad \omega_1 = 2.0771\times 10^{-5} \text{ rad/s}$$

$$\lambda_2 = 1.1657\times 10^{-7} \quad \omega_2 = 3.4143\times 10^{-4} \text{ rad/s}$$

$$\lambda_3 = 8.2283\times 10^{-7} \quad \omega_3 = 9.0710\times 10^{-4} \text{ rad/s}$$

$$\mathbf{v}_1 = \begin{bmatrix} 0.2443 \\ -0.9384 \\ 0.2443 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 0.7071 \\ 0 \\ -0.7071 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 0.6636 \\ 0.3455 \\ 0.6636 \end{bmatrix}$$

Use the mode summation method to find the solution. Transform the initial conditions:

$$\mathbf{q}(0) = M^{1/2}\mathbf{x}(0) = \begin{bmatrix} 0 & 2.2361 & 0 \end{bmatrix}^T$$
$$\dot{\mathbf{q}}(0) = M^{1/2}\dot{\mathbf{x}}(0) = \mathbf{0}$$

The solution is given by Eq. (4.103),

$$\mathbf{x}(t) = \sum_{i=1}^{4} d_i \sin(\omega_i t + \phi_i)\mathbf{u}_i$$

where

$$\phi_i = \tan^{-1}\left(\frac{\omega_i \mathbf{v}_i^T \mathbf{q}(0)}{\mathbf{v}_i^T \dot{\mathbf{q}}(0)}\right) \quad i = 1,2,3 \quad (\text{Eq. } (4.97))$$
$$d_\text{i} = \frac{\mathbf{v}_i^T \mathbf{q}(0)}{\sin\phi_i} \qquad i = 1,2,3 \quad (\text{Eq. } (4.98))$$
$$\mathbf{u}_i = M^{-1/2}\mathbf{v}_i$$

Substituting known values yields

$$\phi_1 = \phi_2 = \phi_3 = \frac{\pi}{2} \text{ rad}$$
$$d_1 = -2.0984$$
$$d_2 = 0$$
$$d_3 = 0.7726$$

$$\mathbf{u}_1 = \begin{bmatrix} 0.0178 \\ -0.02098 \\ 0.01728 \end{bmatrix} \quad \mathbf{u}_2 = \begin{bmatrix} 0.05 \\ 0 \\ -0.05 \end{bmatrix} \quad \mathbf{u}_3 = \begin{bmatrix} 0.04692 \\ 0.007728 \\ 0.04692 \end{bmatrix}$$

The solution is

$$\mathbf{x}(t) = \begin{bmatrix} -0.03625 \\ 0.04403 \\ -0.03625 \end{bmatrix} \cos\left(9.7044 \times 10^{-5} t\right) + \begin{bmatrix} 0.03625 \\ 0.005969 \\ 0.0325 \end{bmatrix} \cos\left(6.1395 \times 10^{-4} t\right) \text{ m}$$

The results are very similar to Problem 50. The responses of mass 1 and 3 are the same for both problems, except the amplitudes and frequencies are changed due to the increase in mass 2. There would have been a greater change if the heavy mass was placed at mass 1 or 3.

4.52 Repeat Problem 4.46 for the case that the airplane body is 10 m instead of 4 m as indicated in the figure. What effect does this have on the response, and which design (4m or 10 m) do you think is better as to vibration?

Solution: Given:

$$m\begin{bmatrix} 1 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 1 \end{bmatrix}\ddot{\mathbf{x}} + \frac{EI}{l^3}\begin{bmatrix} 3 & -3 & - \\ -3 & 6 & -3 \\ 0 & -3 & 3 \end{bmatrix}\mathbf{x} = \mathbf{0}$$

Calculate eigenvalues and eigenvectors:

$$M^{-1/2} = m^{-1/2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.3612 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\tilde{K} = M^{-1/2}KM^{-1/2} = \frac{EI}{ml^3}\begin{bmatrix} 3 & -0.9487 & 0 \\ -0.9487 & 0.6 & -0.9487 \\ 0 & -0.9487 & 3 \end{bmatrix}$$

Again choose the parameters so that the coefficient is 1 and compute the eigenvalues:

$$\det\left(\tilde{K} - \lambda I\right) = \lambda^3 - 6.6\lambda^2 + 10.8\lambda = 0$$

$$\lambda_1 = 0$$

$$\lambda_2 = 3$$

$$\lambda_3 = 3.6$$

$$\mathbf{v}_1 = \begin{bmatrix} -0.2887 \\ -0.9129 \\ -0.2887 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 0.7071 \\ 0 \\ -0.7071 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 0.6455 \\ -0.4082 \\ 0.6455 \end{bmatrix}$$

The natural frequencies are

$$\omega_1 = 0 \text{ rad/s}$$

$$\omega_2 = 1.7321 \text{ rad/s}$$

$$\omega_3 = 1.8974 \text{ rad/s}$$

The relationship between eigenvectors and mode shapes is

$$\mathbf{u} = M^{-1/2}\mathbf{v}$$

$$\mathbf{u}_1 = m^{-1/2}\begin{bmatrix} -0.2887 \\ -0.2887 \\ -0.2887 \end{bmatrix} \quad \mathbf{u}_2 = m^{-1/2}\begin{bmatrix} 0.7071 \\ 0 \\ -0.7071 \end{bmatrix} \quad \mathbf{u}_3 = \begin{bmatrix} 0.6455 \\ -0.1291 \\ 0.6455 \end{bmatrix}$$

It appears that the mode shapes contain less "amplitude" for the wing masses. This seems to be a better design from a vibration standpoint.

4.53 Often in the design of a car, certain parts cannot be reduced in mass. For example, consider the drive train model illustrated in Figure P4.44. The mass of the torque converter and transmission are relatively the same from car to car. However, the mass of the car could change as much as 1000 kg (e.g., a two-seater sports car versus a family sedan). With this in mind, resolve Problem 4.44 for the case that the vehicle inertia is reduced to 2000 kg. Which case has the smallest amplitude of vibration?

Solution: Let k_1 = hub stiffness and k_2 = axle and suspension stiffness. From Problem 4.44, the equation of motion becomes

$$\begin{bmatrix} 75 & 0 & 0 \\ 0 & 100 & 0 \\ 0 & 0 & 2000 \end{bmatrix} \ddot{\mathbf{x}} + 10{,}000 \begin{bmatrix} 1 & -1 & 0 \\ -1 & 3 & -2 \\ 0 & -2 & 2 \end{bmatrix} \mathbf{x} = \mathbf{0}$$

$$\mathbf{x}(0) = \mathbf{0} \text{ and } \dot{\mathbf{x}}(0) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T \text{ m/s.}$$

Calculate eigenvalues and eigenvectors.

$$M^{-1/2} = \begin{bmatrix} 0.1155 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.0224 \end{bmatrix}$$

$$\tilde{K} = M^{-1/2} K M^{-1/2} = \begin{bmatrix} 133.33 & -115.47 & 0 \\ -115.47 & 300 & -44.721 \\ 0 & -44.721 & 10 \end{bmatrix}$$

$$\det\left(\tilde{K} - \lambda I\right) = \lambda^3 - 443.33\lambda^2 + 29{,}000\lambda = 0$$

$$\lambda_1 = 0 \qquad \omega_1 = 0 \text{ rad/s}$$

$$\lambda_2 = 70.765 \quad \omega_2 = 8.9311 \text{ rad/s}$$

$$\lambda_3 = 363.57 \quad \omega_3 = 19.067 \text{ rad/s}$$

$$\mathbf{v}_1 = \begin{bmatrix} -0.1857 \\ -0.2144 \\ -0.9589 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 0.8758 \\ 0.4063 \\ -0.2065 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 0.4455 \\ -0.8882 \\ 0.1123 \end{bmatrix}$$

Use the mode summation method to find the solution. Transform the initial conditions:

$$\mathbf{q}(0) = M^{1/2}\mathbf{x}(0) = 0$$

$$\dot{\mathbf{q}}(0) = M^{1/2}\dot{\mathbf{x}}(0) = \begin{bmatrix} 0 & 0 & 44.7214 \end{bmatrix}^T$$

The solution is given by

$$\mathbf{q}(t) = (c_1 + c_4 t)\mathbf{v}_1 + c_2 \sin(\omega_2 t + \phi_2)\mathbf{v}_2 + c_3 \sin(\omega_3 t + \phi_3)\mathbf{v}_3$$

where

$$\phi_i = \tan^{-1}\left(\frac{\omega_i \mathbf{v}_i^T \mathbf{q}(0)}{\mathbf{v}_i^T \dot{\mathbf{q}}(0)}\right), \quad i = 2,3$$

$$c_i = \frac{\mathbf{v}_i^T \mathbf{q}(0)}{\omega_i \cos\phi_i}, \quad i = 2,3$$

Thus $\phi_2 = \phi_3 = 0$, $c_2 = -1.3042$ and $c_3 = 0.2635$. Next apply the initial conditions:

$$\mathbf{q}(0) = c_1 \mathbf{v}_1 + \sum_{i=2}^{3} c_i \sin\phi_i \mathbf{v}_i \text{ and } \dot{\mathbf{q}}(0) = c_4 \mathbf{v}_1 + \sum_{i=2}^{3} c_i \sin\phi_i \mathbf{v}_i$$

Pre multiply each of these by $\mathbf{v}_1^T$ to get:

$$c_1 = 0 = \mathbf{v}_1^T \mathbf{q}(0) \text{ and } c_4 = -42.8845 = \mathbf{v}_1^T \dot{\mathbf{q}}(0)$$

So

$$\mathbf{q}(t) = -42.8845 t\mathbf{v}_1 - 1.3042 \sin(8.9311t)\mathbf{v}_2 + 0.2635 \sin(19.067t)\mathbf{v}_3$$

Next convert back to the physical coordinates by

$$\mathbf{x}(t) = M^{-1/2}\mathbf{q}(t)$$

$$= 0.9195t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -0.1319 \\ -0.05299 \\ 0.007596 \end{bmatrix} \sin 8.9311t + \begin{bmatrix} 0.01355 \\ -0.02340 \\ 0.0006620 \end{bmatrix} \sin 19.067t \text{ m}$$

Comparing this solution to problem 4.44, the car will vibrate at a slightly higher amplitude when the mass is reduced to 2000 kg.

4.54 Use *mode summation method* to compute the analytical solution for the response of the 2-degree-of-freedom system of Figure P4.28 with the values where $m_1 = 1$ kg, $m_2 = 4$ kg, $k_1 = 240$ N/m and $k_2 = 300$ N/m, to the initial conditions of

$$\mathbf{x}_0 = \begin{bmatrix} 0 \\ 0.01 \end{bmatrix}, \quad \dot{\mathbf{x}}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Solution: Following the development of equations (4.97) through (4.103) for the mode summation for the free response and using the values of computed in problem 1, compute the initial conditions for the "**q**" coordinate system:

$$M^{1/2} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \Rightarrow \mathbf{q}(0) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 0.01 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.02 \end{bmatrix}, \mathbf{q}(0) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

From equation (4.97):

$$\phi_1 = \tan^{-1}\left(\frac{x}{0}\right) = \phi_2 = \tan^{-1}\left(\frac{x}{0}\right) = \frac{\pi}{2}$$

From equation (4.98):

$$d_1 = \frac{\mathbf{v}_1^T\mathbf{q}(0)}{\sin\left(\pi/2\right)} = \mathbf{v}_1^T\mathbf{q}(0), d_2 = \frac{\mathbf{v}_2^T\mathbf{q}(0)}{\sin\left(\pi/2\right)} = \mathbf{v}_2^T\mathbf{q}(0)$$

Next compute $\mathbf{q}(t)$ from (4.92) and multiply by $M^{1/2}$ to get $\mathbf{x}(t)$ or use (4.103) directly to get

$$\mathbf{q}(t) = d_1\cos(\omega_1 t)\mathbf{v}_1 + d_2\cos(\omega_2 t)\mathbf{v}_2 = \cos(\omega_1 t)\mathbf{v}_1^T\mathbf{q}(0)\mathbf{v}_1 + \cos(\omega_2 t)\mathbf{v}_1^T\mathbf{q}(0)\mathbf{v}_1$$

$$= \cos(5.551t)\begin{bmatrix} 0.0054 \\ 0.0184 \end{bmatrix} + \cos(24.170t)\begin{bmatrix} -0.0054 \\ 0.0016 \end{bmatrix}$$

Note that as a check, substitute $t = 0$ in this last line to recover the correct initial condition $\mathbf{q}(0)$. Next transform the solution back to the physical coordinates

$$\mathbf{x}(t) = M^{-1/2}\mathbf{q}(t) = \cos(5.551t)\begin{bmatrix} 0.0054 \\ 0.0092 \end{bmatrix} + \cos(24.170t)\begin{bmatrix} -0.0054 \\ 0.0008 \end{bmatrix} \text{ m}$$

4.55 For a zero value of an eigenvalue and hence frequency, what is the corresponding time response? Or asked another way, the form of the modal solution for a non-zero frequency is $A\sin(\omega_n t + \phi)$, what is the form of the modal solution that corresponds to a zero frequency? Evaluate the constants of integration if the modal initial conditions are: $r_1(0) = 0.1$, and $\dot{r}_1(0) = 0.01$.

Solution: A zero eigenvalue corresponds to the modal equation:

$$\ddot{r}_1(t) = 0 \Rightarrow r_1(t) = a + bt$$

Applying the given initial conditions:

$$r_1(0) = a + b(0) = 0.1 \Rightarrow a = 0.1$$

$$\dot{r}_1(0) = b = 0.01$$

$$\Rightarrow r_1(t) = 0.1 + 0.01t$$

Problems and Solutions for Section 4.5 (4.56 through 4.66)

4.56 Consider the example of the automobile drive train system discussed in Problem 4.44. Add 10% modal damping to each coordinate, calculate and plot the system response.

Solution: Let k_1 = hub stiffness and k_2 = axle and suspension stiffness. From Problem 4.44, the equation of motion with damping is

$$\begin{bmatrix} 75 & 0 & 0 \\ 0 & 100 & 0 \\ 0 & 0 & 3000 \end{bmatrix}\ddot{\mathbf{x}} + 10{,}000\begin{bmatrix} 1 & -1 & 0 \\ -1 & 3 & -2 \\ 0 & -2 & 2 \end{bmatrix}\mathbf{x} = \mathbf{0}$$

$$\mathbf{x}(0) = \mathbf{0} \text{ and } \dot{\mathbf{x}}(0) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T \text{ m/s}$$

Other calculations from Problem 4.44 yield:

$$\lambda_1 = 0 \qquad \omega_1 = 0 \text{ rad/s}$$

$$\lambda_2 = 77.951 \quad \omega_2 = 8.8290 \text{ rad/s}$$

$$\lambda_3 = 362.05 \quad \omega_3 = 19.028 \text{ rad/s}$$

$$\mathbf{v}_1 = \begin{bmatrix} 0.1537 \\ 0.1775 \\ 0.9721 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} -0.8803 \\ -0.4222 \\ 0.2163 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 0.4488 \\ -0.8890 \\ 0.0913 \end{bmatrix}$$

Use the summation method to find the solution. Transform the initial conditions:

$$\mathbf{q}(0) = M^{1/2}\mathbf{x}(0) = \mathbf{0}$$

$$\dot{\mathbf{q}}(0) = M^{1/2}\dot{\mathbf{x}}(0) = \begin{bmatrix} 0 & 0 & 54.7723 \end{bmatrix}^T$$

Also, $\zeta_1 = \zeta_2 = \zeta_3 = 0.1$.

$$\omega_{d2} = 8.7848 \text{ rad/s}$$

$$\omega_{d3} = 18.932 \text{ rad/s}$$

The solution is given by

$$\mathbf{q}(t) = (c_1 + c_2 t)\mathbf{v}_1 + \sum_{2}^{3} d_i e^{-\zeta_i \omega_i t} \sin(\omega_{di} t + \phi_i)\mathbf{v}_i$$

where $\phi_i = \tan^{-1}\left(\dfrac{\omega_{di}\mathbf{v}_i^T\mathbf{q}(0)}{\mathbf{v}_i^T\dot{\mathbf{q}}(0) + \zeta_i\omega_i\mathbf{v}_i^T\mathbf{q}(0)}\right) \qquad i = 2,3$ Eq. (4.114)

$$d_i = \frac{\mathbf{v}_i^T\dot{\mathbf{q}}(0)}{\omega_{di}\cos\phi_i - \zeta_i\omega_i\sin\phi_i} \qquad i = 2,3$$

Thus,

$$\phi_2 = \phi_3 = 0$$

$$d_2 = 1.3485$$

$$d_3 = 0.2642$$

Now,

$$\mathbf{q}(0) = c_1\mathbf{v}_1 + \sum_{i=2}^{3} d_i \sin\phi_i \mathbf{v}_i$$

$$\dot{\mathbf{q}}(0) = c_2\mathbf{v}_1 + \sum_{i=2}^{3}\left[-\zeta_i\omega_i d_i \sin\phi_i + \omega_{di} d_i \cos\phi_i\right]\mathbf{v}_i$$

Pre-multiply by $\mathbf{v}_1^T$:

$$\mathbf{v}_1^T\mathbf{q}(0) = 0 = c_1$$

$$\mathbf{v}_1^T\dot{\mathbf{q}}(0) = 53.2414 = c_2$$

So,

$$\mathbf{q}(t) = 53.2414\mathbf{v}_1 - 1.3485e^{-0.8829t}\sin(8.7848t)\mathbf{v}_2 + 0.2648te^{-1.9028t}\sin(18.932t)\mathbf{v}_3$$

The solution is given by

$$\mathbf{x}(t) = M^{-1/2}\mathbf{q}(t)$$

$$\mathbf{x}(t) = 0.9449t\begin{bmatrix}1\\1\\1\end{bmatrix} - \begin{bmatrix}-0.1371\\-0.05693\\0.005325\end{bmatrix}e^{-0.8829t}\sin(8.7848t) + \begin{bmatrix}0.01369\\-0.002349\\0.0004407\end{bmatrix}e^{-1.9028t}\sin(18.932t)\ \text{m}$$

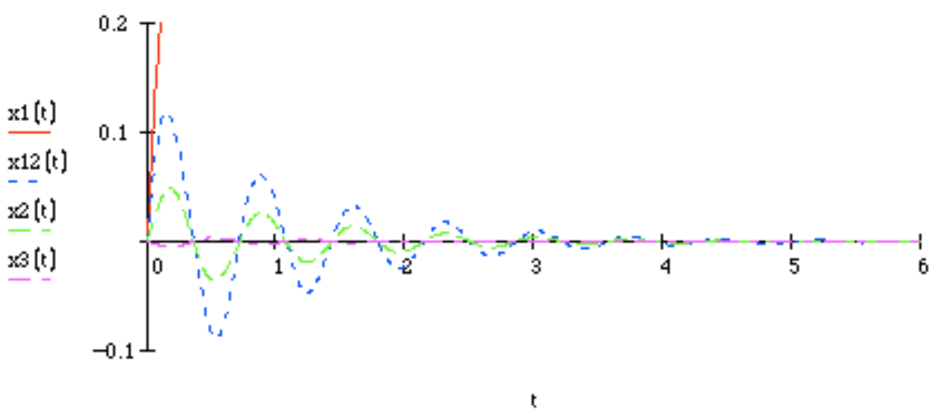
The following Mathcad session illustrates the solution without the rigid body mode (except for x_1 which shows both with and without the rigid mode)

$$x1(t) := 0.9449\cdot t + 0.1371\cdot e^{-.8829\cdot t}\cdot\sin(8.7848\cdot t) + 0.01369\cdot e^{-1.9028\cdot t}\cdot\sin(18.932\cdot t)$$

$$x12(t) := \left(0.1371\cdot e^{-.8829\cdot t}\cdot\sin(8.7848\cdot t) + 0.01369\cdot e^{-1.9028\cdot t}\cdot\sin(18.932\cdot t)\right)$$

$$x2(t) := 0.05693\cdot e^{-.8829\cdot t}\cdot\sin(8.7848\cdot t) - 0.002349\cdot e^{-1.9028\cdot t}\cdot\sin(18.932\cdot t)$$

$$x3(t) := -0.005325\cdot e^{-.8829\cdot t}\cdot\sin(8.7848\cdot t) + 0.000447\cdot e^{-1.9028\cdot t}\cdot\sin(18.932\cdot t)$$



The read solid line is the first mode with the rigid body mode included.

4.57 Consider the model of an airplane discussed in problem 4.47, Figure P4.46. (a) Resolve the problem assuming that the damping provided by the wing rotation is $\zeta_i = 0.01$ in each mode and recalculate the response. (b) If the aircraft is in flight, the damping forces may increase dramatically to $\zeta_i = 0.1$. Recalculate the response and compare it to the more lightly damped case of part (a).

Solution:

From Problem 4.47, with damping

$$\begin{bmatrix} 3000 & 0 & 0 \\ 0 & 12,000 & 0 \\ 0 & 0 & 3,000 \end{bmatrix}\ddot{\mathbf{x}} + C\dot{\mathbf{x}} + \begin{bmatrix} 13455 & -13455 & 0 \\ -13,455 & 26910 & -13,455 \\ 0 & -13,455 & 13,455 \end{bmatrix}\mathbf{x} = \mathbf{0}$$

$$\mathbf{x}(0) = \begin{bmatrix} 0.02 & 0 & 0 \end{bmatrix}^T \text{ m}$$

$$\dot{\mathbf{x}}(0) = \mathbf{0}$$

$$\lambda_1 = 0 \qquad \omega_1 = 0 \text{ rad/s}$$

$$\lambda_2 = 4.485 \quad \omega_2 = 2.118 \text{ rad/s}$$

$$\lambda_3 = 6.727 \quad \omega_3 = 2.594 \text{ rad/s}$$

$$\mathbf{v}_1 = \begin{bmatrix} -0.4082 \\ -0.8165 \\ -0.4082 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 0.7071 \\ 0 \\ -0.7071 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 0.5774 \\ -0.5774 \\ 0.5774 \end{bmatrix}$$

The solution is given by

$$\mathbf{q}(t) = (c_1 + c_2 t)\mathbf{v}_1 + \sum_{i=2}^{3} d_i e^{-\zeta_i \omega_i t} \sin(\omega_{di} t + \phi_i)\mathbf{v}_i$$

where

$$\phi_i = \tan^{-1}\left(\frac{\omega_{di}\mathbf{v}_i^T\mathbf{q}(0)}{\mathbf{v}_i^T\dot{\mathbf{q}}(0) + \zeta_i\omega_i\mathbf{v}_i^T\mathbf{q}(0)}\right) \qquad i = 2,3$$

$$d_i = \frac{\mathbf{v}_i^T\mathbf{q}(0)}{\sin\phi_i} \qquad i = 2,3$$

(Eq. (4.114))

Now,

$$\mathbf{q}(0) = c_1\mathbf{v}_1 + \sum_{i=2}^{3} d_i \sin\phi_i \mathbf{v}_i$$

$$\dot{\mathbf{q}}(0) = c_2\mathbf{v}_1 + \sum_{i=2}^{3}\left[-\zeta_i\omega_i d_i \sin\phi_i + \omega_{di} d_i \cos\phi_i\right]\mathbf{v}_i$$

Premultiply by $\mathbf{v}_1^T$:

$$\mathbf{v}_1^T \mathbf{q}(0) = 4.4721 = c_1$$
$$\mathbf{v}_1^T \dot{\mathbf{q}}(0) = 0 = c_2$$

(a) $\zeta_1 = \zeta_2 = \zeta_3 = 0.01$

$$\omega_{d2} = 2.1177 \text{ rad/s}, \quad \omega_{d3} = 2.593 \text{ rad/s}$$
$$\phi_2 = -1.5808 \text{ rad}, \quad \phi_3 = 1.5608 \text{ rad}$$
$$d_2 = 7.7464, \quad d_3 = 6.3249$$

Mode shapes:

$$\mathbf{u}_i = M^{-1/2}\mathbf{v}_i$$
$$\mathbf{u}_1 = \begin{bmatrix} -0.007454 \\ -0.007454 \\ -0.007454 \end{bmatrix} \quad \mathbf{u}_2 = \begin{bmatrix} 0.01291 \\ 0 \\ -0.01291 \end{bmatrix} \quad \mathbf{u}_3 = \begin{bmatrix} 0.01054 \\ -0.005270 \\ 0.01054 \end{bmatrix}$$

The solution is given by

$$\mathbf{x}(t) = (c_1 + c_2 t)\mathbf{u}_1 + \sum_{i=2}^{3} d_i e^{-\zeta_i \omega_i t} \sin(\omega_{di} t + \phi_i)\mathbf{u}_i$$

$$x(t) = 0.0333\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0.100 \\ 0 \\ 0.100 \end{bmatrix} e^{-0.0212t} \sin(2.1178t - 1.5808)$$
$$+ \begin{bmatrix} 0.0667 \\ -0.0333 \\ 0.0677 \end{bmatrix} e^{-0.0259t} \sin(2.5937t + 1.5608)$$

b) $\zeta_1 = \zeta_2 = \zeta_3 = 0.1$

Same thing as part (a), but now the following values are obtained

$$\omega_{d2} = 2.1072 \text{ rad/sec} \quad \omega_{d3} = 2.5807 \text{ rad/sec}$$
$$\phi_2 = -1.6710 \text{ rad} \quad \phi_3 = 1.4706 \text{ rad}$$
$$d_2 = 7.7850 \quad d_3 = 6.3564$$

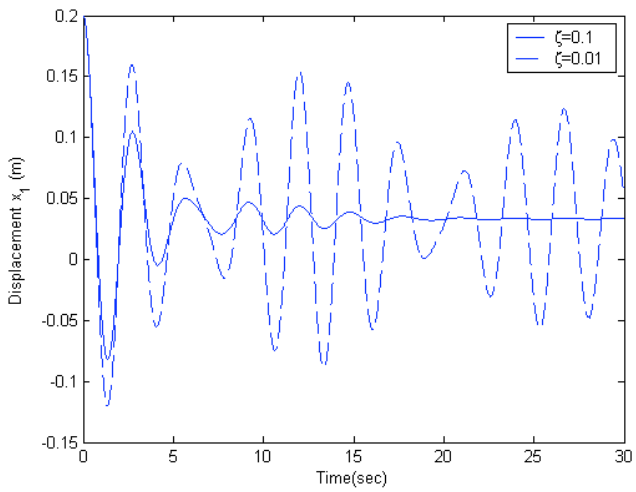
Notice that the rigid mode is not effected by changing the damping ratio, and hence

$c = 4.4721$

Consequently, the solution becomes

$$x(t) = 0.0333\begin{bmatrix}1\\1\\1\end{bmatrix} + \begin{bmatrix}-0.1005\\0\\0.1005\end{bmatrix}e^{-0.2118t}\sin(2.1072t - 1.6710) + \begin{bmatrix}0.0670\\-0.0335\\0.0670\end{bmatrix}e^{-0.2594t}\sin(2.5807t + 1.4706)$$

Below is the plot of the displacement of the left wing



4.58 Repeat the floor vibration problem of Problem 4.50 using modal damping ratios of

$$\zeta_1 = 0.01 \quad \zeta_2 = 0.1 \quad \zeta_3 = 0.2$$

Solution: The equation of motion will be of the form:

$$200\ddot{\mathbf{x}} + C\dot{\mathbf{x}} + 3.197\times10^{-4}\begin{bmatrix} 9/64 & 1/6 & 13/192 \\ 1/6 & 1/3 & 1/6 \\ 13/192 & 1/6 & 9/64 \end{bmatrix}\mathbf{x} = \mathbf{0}$$

$$\mathbf{x}(0) = \begin{bmatrix} 0 & 0.05 & 0 \end{bmatrix}^T \text{ m and } \dot{\mathbf{x}}(0) = \mathbf{0}.$$

$$M^{-1/2} = 0.7071$$

$$\tilde{K} = M^{-1/2}KM^{-1/2} = \begin{bmatrix} 2.2482 & 2.6645 & 1.0825 \\ 2.6645 & 5.3291 & 2.6645 \\ 1.0825 & 2.6645 & 2.2482 \end{bmatrix}\times10^{-7}$$

$$\det\left(\tilde{K} - \lambda I\right) = \lambda^3 - 9.8255\times10^{-7}\lambda^2 + 1.3645\times10^{-13}\lambda - 4.1382\times10^{-21} = 0$$

$$\lambda_1 = 4.3142\times10^{-8} \quad \omega_1 = 2.0771\times10^{-4} \text{ rad/s}$$

$$\lambda_2 = 1.1657\times10^{-7} \quad \omega_2 = 3.34143\times10^{-4} \text{ rad/s}$$

$$\lambda_3 = 8.2283\times10^{-7} \quad \omega_3 = 9.0710\times10^{-4} \text{ rad/s}$$

$$\mathbf{v}_1 = \begin{bmatrix} 0.5604 \\ -0.6098 \\ 0.5604 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} -0.7071 \\ 0 \\ 0.7071 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 0.4312 \\ 0.7926 \\ 0.4312 \end{bmatrix}$$

Use the mode summation method to find the solution. First transform the initial conditions:

$$\mathbf{q}(0) = M^{1/2}\mathbf{x}(0) = \begin{bmatrix} 0 & 0.7071 & 0 \end{bmatrix}^T$$

$$\dot{\mathbf{q}}(0) = M^{1/2}\dot{\mathbf{x}}(0) = \mathbf{0}$$

The solution is given by Eq. (4.115):

$$\mathbf{x}(t) = \sum_{i=1}^{3} d_i e^{-\zeta_i\omega_i t}\sin\left(\omega_{di}t + \phi_i\right)\mathbf{u}_i$$

where $$\phi_i = \tan^{-1}\left(\frac{\omega_{di}\mathbf{v}_i^T\mathbf{q}(0)}{\mathbf{v}_i^T\dot{\mathbf{q}}(0) + \zeta_i\omega_i\mathbf{v}_i^T\mathbf{q}(0)}\right) \quad i = 1,2,3$$

$$d_i = \frac{\mathbf{v}_i^T \mathbf{q}'(0)}{\sin\phi_i} \qquad i = 1,2,3, \quad \mathbf{u}_i = M^{-1/2}\mathbf{v}_i$$

$$\zeta_1 = 0.01, \quad \zeta_2 = 0.1, \qquad \zeta_3 = 0.2$$

Substituting

$$\omega_{d1} = 2.0770 \times 10^{-4} \text{ rad/s}, \ \omega_{\mathrm{d2}} = 3.3972 \times 10^{-4} \text{ rad/s} \ \ \omega_{\mathrm{d3}} = 8.8877 \times 10^{-4} \text{ rad/s}$$

$$\phi_1 = 1.5808 \text{ rad}, \quad \phi_2 = 1.6710 \text{ rad}, \quad \phi_3 = 1.3694 \text{ rad}$$

$$d_1 = 0.4312, \quad d_2 = 0, \qquad d_3 = 0.5720$$

The mode shapes are

$$\mathbf{u}_1 = \begin{bmatrix} 0.03963 \\ -0.04312 \\ 0.03963 \end{bmatrix} \quad \mathbf{u}_2 = \begin{bmatrix} -0.05 \\ 0 \\ 0.05 \end{bmatrix} \quad \mathbf{u}_3 = \begin{bmatrix} 0.03049 \\ 0.05604 \\ 0.03049 \end{bmatrix}$$

The solution is

$$\mathbf{x}(t) = \begin{bmatrix} 0.01709 \\ -0.01859 \\ 0.01709 \end{bmatrix} e^{-2.0771\times10^{-4}t} \sin\left(2.0770 \times 10^{-4} t - 1.5808\right)$$

$$+ \begin{bmatrix} 0.01744 \\ 0.03206 \\ 0.01744 \end{bmatrix} e^{-2.0771\times10^{-4}t} \sin\left(8.8877 \times 10^{-4} t + 1.3694\right) \text{m}$$

4.59 Repeat Problem 4.58 with constant modal damping of ζ_1, ζ_2, ζ_3 = 0.1 and compare this with the solution of Problem 4.58.

Solution: Use the equations of motion and initial conditions from Problem 4.58. The mode shapes, natural frequencies and transformed initial conditions remain the same. However the constants of integration are effected by the damping ratio so the solution

$$\mathbf{x}(t) = \sum_{i=1}^{3} d_i e^{-\zeta_i \omega_i t} \sin(\omega_{di} t + \phi_i)\mathbf{u}_i$$

has new constants determined by $\phi_i = \tan^{-1}\left(\dfrac{\omega_{di}\mathbf{v}_i^T\mathbf{q}(0)}{\mathbf{v}_i^T\dot{\mathbf{q}}(0) + \zeta_i\omega_i\mathbf{v}_i^T\mathbf{q}(0)}\right) \quad i = 1,2,3$

$$d_i = \frac{\mathbf{v}_i^T\mathbf{q}'(0)}{\sin\phi_i} \qquad i = 1,2,3$$

$$\mathbf{u}_i = M^{-1/2}\mathbf{v}_i$$

$$\zeta_1 = \zeta_2 = \zeta_3 = 0.1$$

Substituting yields

$$\omega_{d1} = 2.0667\times10^{-4} \text{ rad/s}, \quad \omega_{\mathrm{d2}} = 3.3972\times10^{-4} \text{ rad/s}, \quad \omega_{\mathrm{d3}} = 9.0255\times10^{-4} \text{ rad/s}$$

$$\phi_1 = -1.6710 \text{ rad}, \quad \phi_2 = -1.6710 \text{ rad}, \quad \phi_3 = 1.4706 \text{ rad}$$

$$d_1 = 0.4334, \quad d_2 = 0.0, \quad d_3 = 0.5633$$

Mode shapes:

$$\mathbf{u}_1 = \begin{bmatrix} 0.03963 \\ -0.04312 \\ 0.03963 \end{bmatrix} \quad \mathbf{u}_2 = \begin{bmatrix} -0.05 \\ 0 \\ 0.05 \end{bmatrix} \quad \mathbf{u}_3 = \begin{bmatrix} 0.03049 \\ 0.05604 \\ 0.03049 \end{bmatrix}$$

The solution is

$$\mathbf{x}(t) = \begin{bmatrix} 0.01717 \\ -0.01869 \\ 0.01717 \end{bmatrix} e^{-2.0771\times10^{-4}t}\sin(2.0667\times10^{-4}t - 1.6710)$$
$$+ \begin{bmatrix} 0.01717 \\ 0.03157 \\ 0.01717 \end{bmatrix} e^{-9.0710\times10^{-4}t}\sin(9.0255\times10^{-4}t + 1.4706)\text{m}$$

The primary difference between problems 4.58 and 4.59 is the settling time; the responses in Problem 4.59 decay faster than those of Problem 4.58.

4.60 Consider the damped system of Figure P4.1. Determine the damping matrix and use the formula of Eq. (4.119) to determine values of the damping coefficient c_I for which this system would be proportionally damped.

Solution:

From Fig. 4.29,

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\ddot{\mathbf{x}} + \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 + c_3 \end{bmatrix}\dot{\mathbf{x}} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}\mathbf{x} = \mathbf{0}$$

From Eq. (4.119)

$$C = \alpha M + \beta K$$

$$\begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 + c_3 \end{bmatrix} = \begin{bmatrix} \alpha m_1 + \beta(k_1 + k_2) & -\beta k_2 \\ -\beta k_2 & \alpha m_1 + \beta(k_2 + k_3) \end{bmatrix}$$

To be proportionally damped,

$$c_2 = \beta k_2$$
$$c_1 = \alpha m_1 + \beta k_1$$
$$c_3 = \alpha m_2 + \beta k_3$$

Alternately, compute $KM^{-1}C$ symbolically and show that the condition for symmetry:

$$\begin{bmatrix} k1 + k2 & -k2 \\ -k2 & k2 + k3 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{m1} & 0 \\ 0 & \frac{1}{m2} \end{bmatrix} \cdot \begin{bmatrix} c1 + c2 & -c2 \\ -c2 & c2 + c3 \end{bmatrix}$$

$$\begin{bmatrix} \frac{(m2 \cdot k1 \cdot c1 + m2 \cdot k1 \cdot c2 + m2 \cdot k2 \cdot c1 + m2 \cdot k2 \cdot c2 + k2 \cdot c2 \cdot m1)}{(m1 \cdot m2)} & \frac{-(m2 \cdot k1 \cdot c2 + m2 \cdot k2 \cdot c2 + k2 \cdot c2 \cdot m1 + k2 \cdot m1 \cdot c3)}{(m1 \cdot m2)} \\ \frac{-(m2 \cdot k2 \cdot c1 + m2 \cdot k2 \cdot c2 + k2 \cdot c2 \cdot m1 + c2 \cdot m1 \cdot k3)}{(m1 \cdot m2)} & \frac{(m2 \cdot k2 \cdot c2 + k2 \cdot c2 \cdot m1 + k2 \cdot m1 \cdot c3 + c2 \cdot m1 \cdot k3 + m1 \cdot k3 \cdot c3)}{(m1 \cdot m2)} \end{bmatrix}$$

Requiring the off diagonal elements to be equal enforces symmetry. This requires

$$m_1 k_2 c_3 = m_2 k_2 c_1 + (m_2 k_1 - m_1 k_3) c_2$$

4.61 Let $k_3 = 0$ in Problem 4.60. Also let $m_1 = 1, m_2 = 4, k_1 = 2, k_2 = 1$ and calculate c_1, c_2 and c_3 such that $\zeta_1 = 0.01$ and $\zeta_2 = 0.1$.

Solution:

From Figure P4.1 the equation of motion is,

$$\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \ddot{\mathbf{x}} + \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 + c_3 \end{bmatrix} \dot{\mathbf{x}} + \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} \mathbf{x} = \mathbf{0}$$

Calculate natural frequencies:

$$\tilde{K} = M^{-1/2} K M^{-1/2} = \begin{bmatrix} 3 & -0.5 \\ -0.5 & 0.25 \end{bmatrix}$$

$$\det\left(\tilde{K} - \lambda I\right) = \lambda^2 - 3.25\lambda + 0.5 = 0$$

$$\lambda_1 = 0.1619 \quad \omega_1 = 0.4024 \text{ rad/s}$$

$$\lambda_2 = 3.0881 \quad \omega_2 = 1.7573 \text{ rad/s}$$

From Eq. (4.124)

$$\zeta_i = \frac{\alpha}{2\omega_i} + \frac{\beta\omega}{2}$$

So, $0.01 = \dfrac{\alpha}{2\left(0.4024\right)} + \dfrac{\beta\left(0.4024\right)}{2}$

and $0.1 = \dfrac{\alpha}{2\left(1.7573\right)} + \dfrac{\beta\left(1.7573\right)}{2}$

Solving for α and β yields

$$\alpha = -0.01096$$

$$\beta = 0.1174$$

From Eq. (4.119),

$$C = \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 + c_3 \end{bmatrix} = \alpha M + \beta K = \begin{bmatrix} 0.3411 & -0.1174 \\ -0.1174 & 0.07354 \end{bmatrix}$$

Thus,

$$c_1 = 0.2238$$

$$c_2 = 0.1174$$

$$c_3 = -0.04382$$

Since negative damping is not usually possible, this design would not work.

4.62 Calculate the constants α and β for the two-degree-of-freedom system of Problem 4.29 such that the system has modal damping of $\zeta_1 = \zeta_2 = 0.3$.

Solution:

From Problem 4.29 with proportional damping added,

$$\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}\ddot{\mathbf{x}} + \left(\alpha M + \beta K\right)\dot{\mathbf{x}} + \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}\mathbf{x} = \mathbf{0}$$

Calculate natural frequencies:

$$\tilde{K} = M^{-1/2}KM^{-1/2} = \begin{bmatrix} 3 & -0.5 \\ -0.5 & 0.25 \end{bmatrix}$$
$$\det\left(\tilde{K} - \lambda I\right) = \lambda^2 - 3.25\lambda + 0.5 = 0$$

$$\lambda_1 = 0.1619 \quad \omega_1 = 0.4024 \text{ rad/s}$$
$$\lambda_2 = 3.0881 \quad \omega_2 = 1.7573 \text{ rad/s}$$

From Eq. (4.124)

$$\zeta_i = \frac{\alpha}{2\omega_i} + \frac{\beta\omega_i}{2}$$

So, $0.3 = \dfrac{\alpha}{2(0.4024)} + \dfrac{\beta(0.4024)}{2}$

and $0.3 = \dfrac{\alpha}{2(1,7573)} + \dfrac{\beta(1.7573)}{2}$

Solving for α and β yields

$$\alpha = 0.1966$$
$$\beta = 0.2778$$

4.63 Equation (4.124) represents *n* equations in only two unknowns and hence cannot be used to specify all the modal damping ratios for a system with $n > 2$. If the floor vibration system of Problem 4.51 has measured damping of $\zeta_1 = 0.01$ and $\zeta_2 = 0.05$, determine ζ_3.

Solution:

From Problem 4.51

$$\det\left(\tilde{K} - \lambda I\right) = \lambda^3 - 9.8255\times10^{-7}\lambda^2 + 1.3645\times10^{-14}\lambda - 4.1382\times10^{-22} = 0$$
$$\lambda_1 = 4.3142\times10^{-9} \quad \omega_1 = 2.0771\times10^{-5} \text{ rad/s}$$
$$\lambda_2 = 1.1657\times10^{-7} \quad \omega_2 = 3.4143\times10^{-4} \text{ rad/s}$$
$$\lambda_3 = 8.2283\times10^{-7} \quad \omega_3 = 9.0710\times10^{-4} \text{ rad/s}$$

Eq. (4.124)

$$\zeta_i = \frac{\alpha}{2\omega_i} + \frac{\beta\omega_i}{2}$$

Since the problem contains three modes only, and since the first and second modal damping ratios are give as $\zeta_1 = 0.01$ and $\zeta_2 = 0.05$ then the following linear system can be set up

$$\frac{\alpha}{2\left(2.0771\times10^{-5}\right)} + \frac{\beta\left(2.0771\times10^{-5}\right)}{2} = 0.01$$
$$\frac{\alpha}{2\left(3.4143\times10^{-4}\right)} + \frac{\beta\left(3.4143\times10^{-4}\right)}{2} = 0.05$$

which can be solve to yield $\alpha = 2.9\times10^{-7}$ and $\beta = 290.397$. Hence, the modal damping of the third mode can be obtained using 4.124

$$\zeta_3 = \frac{\alpha}{2\omega_3} + \frac{\beta\omega_3}{2} = 0.132$$

4.64 Does the following system decouple? If so, calculate the mode shapes and write the equation in decoupled form.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\ddot{\mathbf{x}} + \begin{bmatrix} 5 & -3 \\ -3 & 3 \end{bmatrix}\dot{\mathbf{x}} + \begin{bmatrix} 5 & -1 \\ -1 & 1 \end{bmatrix}\mathbf{x} = \mathbf{0}$$

Solution:

The system will decouple if

$$C = \alpha M + \beta K$$
$$\begin{bmatrix} 5 & -3 \\ -3 & 3 \end{bmatrix} = \begin{bmatrix} \alpha + 5\beta & -\beta \\ -\beta & \alpha + \beta \end{bmatrix}$$

Clearly the off-diagonal terms require

$$\beta = 3$$

Therefore, the diagonal terms require

$$5 = \alpha + 15$$
$$3 = \alpha + 3$$

These yield different values of α, so the system does not decouple. An easier approach is to compute $CM^{-1}K$ to see if it is symmetric:

$$CM^{-1}K = \begin{bmatrix} 5 & -3 \\ -3 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 5 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 9 & -2 \\ -12 & 6 \end{bmatrix}$$

Since this is not symmetric, the system cannot be decoupled.

4.65 Calculate the damping matrix for the system of Problem 4.63. What are the units of the elements of the damping matrix?

Solution:

From Problem 4.58,

$$\alpha = -8.8925 \times 10^{-7}$$
$$\beta = 3.0052 \times 10^{2}$$

From Problem 4.48

$$M = \begin{bmatrix} 200 & 0 & 0 \\ 0 & 2000 & 0 \\ 0 & 0 & 200 \end{bmatrix}$$

$$K = 3.197 \times 10^{-4} \begin{bmatrix} 9/64 & 1/6 & 13/192 \\ 1/6 & 1/3 & 1/6 \\ 13/192 & 1/6 & 9/64 \end{bmatrix}$$

So,

$$C = \alpha M + \beta K$$

$$C = \begin{bmatrix} 0.01334 & 0.01602 & 0.006506 \\ 0.01602 & 0.03025 & 0.01602 \\ 0.006506 & 0.01602 & 0.01334 \end{bmatrix}$$

The units are kg/s

4.66 Show that if the damping matrix satisfies $C = \alpha M + \beta K$, then the matrix $CM^{-1}K$ is symmetric and hence that $CM^{-1}K = KM^{-1}C$.

Solution: Compute the product $CM^{-1}K$ where C has the form: $C = \alpha M + \beta K$.

$$CM^{-1} = (\alpha M + \beta K)M^{-1} = \alpha I + \beta KM^{-1} \Rightarrow CM^{-1}K = \alpha K + \beta KM^{-1}K$$
$$KM^{-1}C = KM^{-1}(\alpha M + \beta K) = \alpha K + \beta KM^{-1}K$$
$$\Rightarrow KM^{-1}C = CM^{-1}K$$

Problems and Solutions for Section 4.6 (4.67 through 4.76)

4.67 Calculate the response of the system of Figure 4.16 discussed in Example 4.6.1 if $F_1(t) = \delta(t)$ and the initial conditions are set to zero. This might correspond to a two-degree-of-freedom model of a car hitting a bump.

Solution: From example 4.6.1, with $F_1(t) = \delta(t)$, the modal equations are

$$\ddot{r}_1 + 0.2\dot{r}_1 + 2r_1 = 0.7071\delta(t)$$

$$\ddot{r}_2 + 0.4\dot{r}_2 + 4r_2 = 0.7071\delta(t)$$

Also from the example,

$$\omega_{n1} = \sqrt{2} \text{ rad/s} \qquad \zeta_1 = 0.07071 \qquad \omega_{d1} = 1.4106 \text{ rad/s}$$

$$\omega_{n2} = 2 \text{ rad/s} \qquad \zeta_2 = 0.1 \qquad \omega_{d2} = 1.9899 \text{ rad/s}$$

The solution to an impulse is given by equations (3.7) and (3.8):

$$r_i(t) = \frac{\hat{F}}{m_i\omega_{di}} e^{-\zeta_i\omega_{ni}t} \sin\omega_{di}t$$

This yields

$$\mathbf{r}(t) = \begin{bmatrix} 0.5012e^{-0.1t}\sin 1.4106t \\ 0.3553e^{-0.2t}\sin 1.9899t \end{bmatrix}$$

The solution in physical coordinates is

$$\mathbf{x}(t) = M^{-1/2}P\mathbf{r}(t) = \begin{bmatrix} .2357 & -.2357 \\ .7071 & .7071 \end{bmatrix} \begin{bmatrix} 0.167e^{-0.1t}\sin 1.4106t \\ -0.118e^{-.02t}\sin 1.9899t \end{bmatrix}$$

$$\mathbf{x}(t) = \begin{bmatrix} 0.0394e^{-0.1t}\sin 1.4106t + 0.0279e^{-0.2t}\sin 1.9899t \\ 0.118e^{-0.1t}\sin 1.4106t - 0.0834e^{-0.2t}\sin 1.9899t \end{bmatrix}$$

4.68 For an undamped two-degree-of-freedom system, show that resonance occurs at one or both of the system's natural frequencies.

Solution:

Undamped two-degree-of-freedom system:

$$M\ddot{\mathbf{x}} + K\mathbf{x} = \mathbf{F}(t)$$

Let $\mathbf{F}(t) = \begin{bmatrix} F_1(t) \\ 0 \end{bmatrix}$

Note: placing F_1 on mass 1 is one way to do this. A second force could be placed on mass 2 with or without F_1.

Proceeding through modal analysis,

$$I\ddot{\mathbf{r}} + \Lambda\mathbf{r} = P^T M^{-1/2}\mathbf{F}(t)$$

Or,

$$\begin{aligned} \ddot{r}_1 + \omega_1^2 r_1 &= b_1 F_1(t) \\ \ddot{r}_2 + \omega_2^2 r_2 &= b_2 F_1(t) \end{aligned}$$

where b_1 and b_2 are constants from the matrix $P^T M^{-1/2}$.

If $F_1(t) = a \cos \omega t$ and $\omega = \omega_1$ then the solution for r_1 is (from Section 2.1),

$$r_1(t) = \frac{\dot{r}_{10}}{\omega_1}\sin\omega_1 t + r_{10}\cos\omega_1 t + \frac{b_1 a}{2\omega_1} t \sin\omega_1 t$$

The solution for r_2 is

$$r_2(t) = \frac{\dot{r}_{20}}{\omega_2}\sin\omega_2 t + \left(r_{20} - \frac{b_2 a}{\omega_2^2 - \omega_1^2} \right)\cos\omega_2 t + \frac{b_2 a}{\omega_2^2 - \omega_1^2} t \sin\omega_1 t$$

If the initial conditions are zero,

$$r_1(t) = \frac{b_1 a}{2\omega_1} t \sin \omega_1 t$$

$$r_2(t) = \frac{b_2 a}{\omega_2^2 - \omega_1^2} \left(\cos \omega_1 t - \cos \omega_2 t \right)$$

Converting to physical coordinates $X(t) = M^{-1/2} P r(t)$ yields

$$x_1(t) = c_1 r_1(t) + c_2 r_2(t)$$
$$x_2(t) = c_3 r_1(t) + c_4 r_2(t)$$

where c_i is a constant from $M^{-1/2}P$.

So, if the driving force contains just one natural frequency, both masses will be excited at resonance. The driving force could contain the other natural frequency ($\omega = \omega_{n2}$), which would cause r_1 and r_2 to be

$$r_1(t) = \frac{b_1 a}{\omega_1^2 - \omega_2^2} \left(\cos \omega_2 t - \cos \omega_1 t \right)$$

$$r_2(t) = \frac{b_2 a}{2\omega_2} t \sin \omega_2 t$$

and

$$x_1(t) = c_1 r_1(t) + c_2 r_2(t)$$
$$x_2(t) = c_3 r_1(t) + c_4 r_2(t)$$

so both masses still oscillate at resonance.

Also, if $F_1(t) = a_1 \cos \omega_1 t + a_2 \cos \omega_2 t$ where $\omega_1 = \omega_{n1}$ and $\omega_2 = \omega_{n2}$, then both r_1 and r_2 would be at resonance, so $x_1(t)$ and $x_2(t)$ would also be at resonance.

4.69 Use modal analysis to calculate the response of the drive train system of Problem 4.44 to a unit impulse on the car body (i.e., and location q_3). Use the modal damping of Problem 4.56. Calculate the solution in terms of physical coordinates, and after subtracting the rigid-body modes, compare the responses of each part.

Solution:

Let k_1 = hub stiffness and k_2 = axle and suspension stiffness.

From Problems 41 and 51,

$$\begin{bmatrix} 75 & 0 & 0 \\ 0 & 100 & 0 \\ 0 & 0 & 3000 \end{bmatrix}\ddot{\mathbf{q}} + 10{,}000\begin{bmatrix} 1 & -1 & 0 \\ -1 & 3 & -2 \\ 0 & -2 & 2 \end{bmatrix}\mathbf{q} = \mathbf{0}$$

$$M^{-1/2} = \begin{bmatrix} .1155 & 0 & 0 \\ 0 & .1 & 0 \\ 0 & 0 & .0183 \end{bmatrix}$$

$$P = \begin{bmatrix} .1537 & -.8803 & .4488 \\ .1775 & -.4222 & -.88910 \\ .9721 & .2163 & .0913 \end{bmatrix}$$

$$\lambda_1 = 0 \qquad \omega_{n1} = 0 \text{ rad/s}$$
$$\lambda_2 = 77.951 \qquad \omega_{n2} = 8.8290 \text{ rad/s}$$
$$\lambda_3 = 362.05 \qquad \omega_{n3} = 19.028 \text{ rad/s}$$

The initial conditions are **0**.

Also

$$\zeta_1 = \zeta_2 = \zeta_3 = .1$$
$$\omega_{d1} = 8.7848 \text{ rad/s}$$
$$\omega_{d2} = 18.932 \text{ rad/s}$$

From equation (4.129):

$$\ddot{\mathbf{r}} + \text{diag}(2\zeta_i\omega_{ni})\dot{\mathbf{r}} + \Lambda\mathbf{r} = P^T M^{-1/2}\mathbf{F}(t)$$

Modal force vector:

$$P^T M^{-1/2}\mathbf{F}(t) = \begin{bmatrix} .01775 \\ .003949 \\ .001668 \end{bmatrix}\delta(t)$$

The modal equations are

$$\begin{aligned}
\ddot{r}_1 &= .01775\delta(t) \\
\ddot{r}_2 + 1.7658\dot{r}_2 + 77.951 r_2 &= .003949\delta(t) \\
\ddot{r}_3 + 3.8055\dot{r}_3 + 362.05 r_3 &= .001668\delta(t)
\end{aligned}$$

The solution for r_1 is

$$r_1(t) = .01775t$$

The solutions for r_2 and r_3 are given by equations 3.7 and (3.8)

$$r_i(t) = \frac{\hat{F}}{m_i \omega_{di}} e^{-\zeta_i \omega_i t} \sin \omega_{di} t$$

This yields

$$\begin{aligned}
r_2(t) &= 4.4949 \times 10^{-4} e^{-.8829t} \sin 8.7848t \\
r_3(t) &= 8.8083 \times 10^{-5} e^{-1.9028t} \sin 18.932t
\end{aligned}$$

The solution in physical coordinates is

$$\mathbf{q}(t) = M^{-1/2} P\mathbf{r}(t)$$

$$\mathbf{q}(t) = 3.1496 \times 10^{-4} t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -4.5691 \times 10^{-5} \\ -1.8978 \times 10^{-5} \\ 1.7749 \times 10^{-6} \end{bmatrix} e^{-.8829t} \sin 8.7848t$$
$$+ \begin{bmatrix} 4.5647 \times 10^{-6} \\ -7.8301 \times 10^{-6} \\ 1.4689 \times 10^{-7} \end{bmatrix} e^{-1.9028t} \sin 18.932t \text{ m}$$

The magnitude of the components is much smaller than that in problem 51, but they do oscillate at the same frequencies.

4.70 Consider the machine tool of Figure 4.28. Resolve Ex. 4.8.3 if the floor mass *m* = 1000 kg, is subject to a force of 10 sin*t* (in Newtons). Calculate the response. How much does this floor vibration affect the machine's toolhead?

Solution:

From example 4.8.3, with $F_3(t)$ = 10 sin*t* N and m_3 = 1000 kg.

$$\left(10^3\right)\begin{bmatrix} .4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\ddot{\mathbf{x}} + \left(10^4\right)\begin{bmatrix} 30 & -30 & 0 \\ -30 & 38 & -8 \\ 0 & -8 & 88 \end{bmatrix}\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 10\sin t \end{bmatrix}$$

Calculating the eigenvalues and eigenvectors yields

$$\begin{aligned} \lambda_1 &= 29.980 & \omega_1 &= 5.4761 \text{ rad/s} \\ \lambda_2 &= 868.2743 & \omega_2 &= 29.4665 \text{ rad/s} \\ \lambda_3 &= 921.7378 & \omega_3 &= 30.3601 \text{ rad/s} \end{aligned}$$

And

$$P = \begin{bmatrix} -.4215 & .4989 & .7573 \\ -.9048 & -.1759 & -.3877 \\ -.0602 & -.8486 & .5255 \end{bmatrix}$$

Modal force vector:

$$P^T M^{-1/2}\mathbf{F}(t) = \begin{bmatrix} -.01904 \\ -.2684 \\ .1662 \end{bmatrix}\sin t$$

Undamped modal equations:

$$\begin{aligned} \ddot{r}_1 + 29.9880 r_1 &= -.01904\sin t \\ \ddot{r}_2 + 868.2743 r_2 &= -.2684\sin t \\ \ddot{r}_3 + 921.7378 r_3 &= .1662\sin t \end{aligned}$$

Inserting the damping terms,

$$\zeta_1 = .1 \qquad 2\zeta_1\omega_1 = 1.0952$$
$$\zeta_2 = .01 \qquad 2\zeta_2\omega_2 = .5893$$
$$\zeta_3 = .05 \qquad 2\zeta_3\omega_3 = 3.0360$$
$$\ddot{r}_1 + 1.0952\dot{r}_1 + 29.9880 r_1 = -.01904 \sin t$$
$$\ddot{r}_2 + .5893\dot{r}_2 + 868.2734 r_2 = -.2684 \sin t$$
$$\ddot{r}_3 + 3.0360\dot{r}_3 + 921.7378 r_3 = .1662 \sin t$$

The damped natural frequencies are

$$\omega_{d1} = \omega_{n1}\sqrt{1-\zeta_1^2} = 5.4487 \text{ rad/s}$$
$$\omega_{d2} = \omega_{n2}\sqrt{1-\zeta_2^2} = 29.4650 \text{ rad/s}$$
$$\omega_{d3} = \omega_{n3}\sqrt{1-\zeta_3^2} = 30.3222 \text{ rad/s}$$

The general solution is

$$r_i(t) = A_i e^{-\zeta_i \omega_{ni} t} \sin(\omega_{di} t - \theta_i) + A_{0i} \sin(\omega t - \phi_i)$$

where

$$A_{0i} = \frac{f_{0i}}{\sqrt{\left(\omega_{ni}^2 - \omega^2\right)^2 + \left(2\zeta_i \omega_{ni}\omega\right)^2}} \quad \text{and} \quad \phi_i = \tan^{-1}\left(\frac{2\zeta_i\omega_{ni}\omega}{\omega_{ni}^2 - \omega}\right)$$

Inserting values,

$$A_{01} = -6.5643 \times 10^{-4} \text{ m} \qquad \phi_1 = 3.7764 \times 10^{-2} \text{ rad}$$
$$A_{02} = -3.0943 \times 10^{-4} \text{ m} \qquad \phi_2 = 6.7952 \times 10^{-4} \text{ rad}$$
$$A_{03} = 1.8049 \times 10^{-4} \text{ m} \qquad \phi_3 = 3.2974 \times 10^{-3} \text{ rad}$$

So,

$$r_1(t) = A_1 e^{-.5476t} \sin(5.4487t - \theta_1) - 6.543 \times 10^{-4} \sin(t - 3.7764 \times 10^{-2})$$
$$r_2(t) = A_2 e^{-.2947t} \sin(29.4650t - \theta_2) - 3.0943 \times 10^{-4} \sin(t - 6.7952 \times 10^{-4})$$
$$r_3(t) = A_3 e^{-1.5180t} \sin(30.3222t - \theta_3) + 1.8049 \times 10^{-4} \sin(t - 3.2974 \times 10^{-3})$$

With zero initial conditions:

$$A_1 = 1.2047\times10^{-4} \text{ m} \qquad \theta_1 = .2072 \text{ rad}$$
$$A_2 = 1.0502\times10^{-5} \text{ m} \qquad \theta_2 = .02002 \text{ rad}$$
$$A_3 = -5.9524\times10^{-6} \text{ m} \qquad \theta_3 = .1002 \text{ rad}$$

Now,

$$r_1(t) = 1.2047\times10^{-4} e^{-.5476t} \sin(5.4487t - .2027) - 6.543\times10^{-4} \sin(t - 3.7764\times10^{-2})$$
$$r_2(t) = 1.0502\times10^{-5} e^{-.2947t} \sin(29.4650t - .02002) - 3.0943\times10^{-4} \sin(t - 6.7952\times10^{-4})$$
$$r_3(t) = -5.9524\times10^{-6} e^{-1.5180t} \sin(30.3222t - .1002) + 1.8049\times10^{-4} \sin(t - 3.2974\times10^{-3})$$

Convert to physical coordinates:

$$\mathbf{x}(t) = M^{-1/2}P\mathbf{r}(t) = \begin{bmatrix} -.02108 & .02494 & .03786 \\ -.02023 & -.003993 & -.008670 \\ -.001904 & -.02684 & .01662 \end{bmatrix}\mathbf{r}(t)$$

Therefore

$$x_1(t) = -.02108r_1 + .02494r_2 + .03786r_3$$
$$x_2(t) = -.02023r_1 - .003933r_2 - .008670r_3$$
$$x_3(t) = -.001904r_1 - .02684r_2 + .01662r_3$$

4.71 Consider the airplane of Figure P4.46 with damping as described in Problem 4.57 with $\zeta_1 = 0.1$. Suppose that the airplane hits a gust of wind, which applies an impulse of $3\delta(t)$ at the end of the left wing and $\delta(t)$ at the end of the right wing. Calculate the resulting vibration of the cabin [$x_2(t)$].

Solution: From Problems 4.46 and 4.57

$$M^{-1/2} = \begin{bmatrix} .01826 & 0 & 0 \\ 0 & .009129 & 0 \\ 0 & 0 & .01826 \end{bmatrix}$$

$$P = \begin{bmatrix} 0.4082 & -0.7071 & 0.5774 \\ 0.8165 & 0 & -0.5774 \\ 0.4082 & 0.7071 & 0.5774 \end{bmatrix}$$

$$\lambda_1 = 0 \qquad \omega_{\text{n1}} = 0 \text{ rad/s}$$
$$\lambda_2 = 4.485 \qquad \omega_{\text{n2}} = 2.118 \text{ rad/s}$$
$$\lambda_3 = 6.727 \qquad \omega_{\text{n3}} = 2.594 \text{ rad/s}$$

Also:

$$\zeta_1 = \zeta_2 = \zeta_3 = 0.1$$

$$\mathbf{F}(t) = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}\delta(t)$$

$$\omega_{d1} = 0 \text{ rad/s}, \ \ \omega_{d2} = 2.1072 \text{ rad/s}, \ \omega_{d3} = 2.5807 \text{ rad/s}$$

From equation (4.129):

$$\ddot{\mathbf{r}} + \text{diag}(2\zeta_i\omega_{ni})\dot{\mathbf{r}} + \Lambda\mathbf{r} = P^T M^{-1/2}\mathbf{F}(t)$$

Modal force vector:

$$P^T M^{-1/2}\mathbf{F}(t) = \begin{bmatrix} -0.0298 \\ 0.0258 \\ 0.0422 \end{bmatrix}\delta(t)$$

The modal equations are

$$\ddot{r}_1 = -0.02981\delta(t)$$
$$\ddot{r}_2 + 0.424\dot{r}_2 + 4.485r_2 = 0.0258\delta(t)$$
$$\ddot{r}_3 + 0.519\dot{r}_3 + 6.727r_3 = 0.0422\delta(t)$$

The solution for r_1 is

$$r_1(t) = -0.02981t$$

The solutions for r_2 and r_3 are given by equations (3.7) and (3.8)

$$r_i(t) = \frac{\hat{F}}{m_i\omega_{di}} e^{-\zeta_i\omega_i t} \sin\omega_{di}t$$

This yields

$$r_2(t) = 1.2253\times10^{-2} e^{-0.212t} \sin 2.107t$$

$$r_3(t) = 1.6338\times10^{-2} e^{-0.259t} \sin 2.581t$$

The solution in physical coordinates is

$$\mathbf{x}(t) = M^{-1/2} P\mathbf{r}(t)$$

For x_2:

$$x_2(t) = 2.221\times10^{-4} t + 8.06\times10^{-5} e^{-0.259t} \sin 2.581t$$

4.72 Consider again the airplane of Figure P4.46 with the modal damping model of Problem 4.57 (ζ_i = 0.1). Suppose that this is a propeller-driven airplane with an internal combustion engine mounted in the nose. At a cruising speed the engine mounts transmit an applied force to the cabin mass (4m at x_2) which is harmonic of the form 50 sin 10*t*. Calculate the effect of this harmonic disturbance at the nose and on the wind tips after subtracting out the translational or rigid motion.

Solution: From Problems 4.47 and 4.57

$$M^{-1/2} = \begin{bmatrix} .01826 & 0 & 0 \\ 0 & .009129 & 0 \\ 0 & 0 & .01826 \end{bmatrix}, \quad P = \begin{bmatrix} -.4082 & .7071 & .5774 \\ -.8165 & 0 & -.5774 \\ -.4082 & -.7071 & .5774 \end{bmatrix}$$

$$\lambda_1 = 0 \qquad \omega_{n1} = 0 \text{ rad/s}$$
$$\lambda_2 = 17.94 \qquad \omega_{n2} = 4.2356 \text{ rad/s}$$
$$\lambda_3 = 26.91 \qquad \omega_{n3} = 5.1875 \text{ rad/s}$$

Also,

$$\zeta_1 = \zeta_2 = \zeta_3 = 0.1, \Rightarrow \omega_{d1} = 0 \text{ rad/s}, \ \ \omega_{d2} = 4.2143 \text{ rad/s}, \ \ \omega_{d3} = 5.1615 \text{ rad/s}$$

$$\mathbf{F}(t) = \begin{bmatrix} 0 \\ 50\sin 10t \\ 0 \end{bmatrix}$$

The initial conditions are **0**. From equation (4.129):

$$\ddot{\mathbf{r}} + \text{diag}(2\zeta_i\omega_{ni})\dot{\mathbf{r}} + \Lambda\mathbf{r} = P^T M^{-1/2}\mathbf{F}(t)$$

Modal force vector:

$$P^T M^{-1/2}\mathbf{F}(t) = \begin{bmatrix} -.3727 \\ 0 \\ -.2635 \end{bmatrix} \sin 10t$$

The modal equations are

$$\ddot{r}_1 = -.3727\sin 10t$$
$$\ddot{r}_2 + .8471\dot{r}_2 + 17.94 r_2 = 0$$
$$\ddot{r}_3 + 1.0375\dot{r}_3 + 26.91 r_3 = -.2635\sin 10t$$

The solutions are

$$
\begin{aligned}
r_1(t) &= .003727\sin 10 \\
r_2(t) &= 0 \\
r_3(t) &= -.006915e^{-.5188t}\sin(5.1615t + .0726) + .003569\sin(10t + .141)
\end{aligned}
$$

The solutions in physical coordinates is

$$\mathbf{x}(t) = M^{-1/2}P\mathbf{r}(t)$$

The wing tips are x_1 and x_3, so

$$
\begin{aligned}
x_1(t) = x_3(t) = {} & 2.7780\times 10^{-5}\sin 10t - 7.2891\times 10^{-5}e^{-.5188t}\sin(5.1615t + .0726) \\
& + 3.7621\times 10^{-5}\sin(10t + .141)
\end{aligned}
$$

4.73 Consider the automobile model of Problem 4.14 illustrated in Figure P4.14. Add modal damping to this model of $\zeta_1 = 0.01$ and $\zeta_2 = 0.2$ and calculate the response of the body [$x_2(t)$] to a harmonic input at the second mass of 10 sin3t N.

Solution: From problem 4.14

$$M = \begin{bmatrix} 2000 & 0 \\ 0 & 50 \end{bmatrix}, \quad K = \begin{bmatrix} 1000 & -1000 \\ -1000 & 11000 \end{bmatrix}, \quad P = \begin{bmatrix} .9999 & -.1044 \\ .1044 & .9999 \end{bmatrix}$$

$\lambda_1 = 0.4545 \qquad \omega_1 = 0.6741$ rad/s, and $\lambda_2 = 220.05 \qquad \omega_2 = 14.834$ rad/s

Also,

$$\zeta_1 = .01, \quad \zeta_2 = 0.2, \quad \omega_{d1} = 0.6741 \text{ rad/s}, \quad \omega_{d2} = 14.534 \text{ rad/s}$$

$$\mathbf{F}(t) = \begin{bmatrix} 0 \\ 10\sin 3t \end{bmatrix}$$

The initial conditions are all **0**. From equation (4.129):

$$\ddot{\mathbf{r}} + \text{diag}(2\zeta_i\omega_{ni})\dot{\mathbf{r}} + \Lambda\mathbf{r} = P^T M^{-1/2}\mathbf{F}(t)$$

Modal force vector:

$$P^T M^{-1/2}\mathbf{F}(t) = \begin{bmatrix} 0.02036 \\ 1.4141 \end{bmatrix}\sin 3t$$

The modal equations are

$$\ddot{r}_1 + 0.01348\dot{r}_1 + 0.454 r_1 = 0.02036\sin 3t$$

$$\ddot{r}_2 + 5.9336\dot{r}_2 + 220.046 r_2 = 1.4141\sin 3t$$

The solutions are

$$r_1(t) = -0.1088e^{-0.006741t}\sin(0.6741t + 1.0914\times 10^{-4}) + .002445\sin(3t - .004857)$$

$$r_2(t) = -0.07500e^{-2.9668t}\sin(14.534t + 1.3087) + .07586\sin(3t + 1.26947)$$

The solutions in physical coordinates is

$$\mathbf{x}(t) = M^{-1/2}P\mathbf{r}(t)$$

The response of the body is

$$\begin{aligned} x_1(t) = &-.002433e^{-0.006741t}\sin(.6471t - 1.0914\times 10^{-4}) \\ &+ 5.4665\times 10^{-5}\sin(3t - .004857) \\ &+ 2.4153\times 10^{-5}e^{-2.9668t}\sin(14.534t - 1.3087) \\ &- 2.4430\times 10^{-5}\sin(3t + 1.2694) \end{aligned}$$

4.74 Determine the *modal equations* for the following system and comment on whether or not the system will experience resonance.

$$\ddot{\mathbf{x}} + \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\sin(0.618t)$$

Solution: Here $M = I$ so that the eigenvectors and mode shapes are the same. Computing the natural frequencies from $\det(-\omega^2 I + K) = 0$ yields:

$$\omega_1 = 0.618 \text{ rad/s} \ \text{ and } \ \omega_2 = 1.681 \text{ rad/s}$$

Next solve for the mode shapes and normalize them to get

$$P = \begin{bmatrix} 0.526 & -0.851 \\ 0.851 & 0.526 \end{bmatrix}, \quad \text{so that } P^T \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.526 \\ -0.851 \end{bmatrix}$$

The modal equations then become:

$$\ddot{r}_1 + (0.618)^2 r_1 = \ddot{r}_1 + 0.3819 r_1 = 0.526\sin(0.618t)$$

$$\ddot{r}_2 + (1.618)^2 r_2 = \ddot{r}_2 + 2.6179 r_2 = -0.851\sin(0.618t)$$

The driving frequency is equal to the natural frequency of mode one so the system exhibits resonance.

4.75 Consider the following system and compute the solution using the mode summation method.

$$M = \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix}, \ K = \begin{bmatrix} 27 & -3 \\ -3 & 3 \end{bmatrix}, \ \mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ \dot{\mathbf{x}}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solution: From Example 4.2.4

$$M^{\frac{1}{2}} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}, M^{-\frac{1}{2}} = \begin{bmatrix} 1/3 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } V = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}. \text{ Also } \omega_1 = \sqrt{2}, \omega_2 = 2 \text{ rad/s}$$

$$\text{Appropriate IC are } \mathbf{q}_0 = M^{\frac{1}{2}}\mathbf{x}_0 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \ \dot{\mathbf{q}}_0 = M^{\frac{1}{2}}\mathbf{v}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\phi_i = \tan^{-1}\frac{\omega_i \mathbf{v}_i^T \mathbf{q}(0)}{\mathbf{v}_i^T \dot{\mathbf{q}}(0)} = \tan^{-1}\frac{\omega_i \mathbf{v}_i^T \mathbf{q}(0)}{0} \Rightarrow \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} \frac{\pi}{2} \\ \frac{\pi}{2} \end{bmatrix}$$

$$d_i = \frac{\mathbf{v}_i^T \mathbf{q}(0)}{\sin\phi_i} \Rightarrow \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 3\sqrt{2}/2 \\ 3\sqrt{2}/2 \end{bmatrix}$$

$$\begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} = \frac{3\sqrt{2}}{2}\sin\left(\sqrt{2}t + \frac{\pi}{2}\right)\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{3\sqrt{2}}{2}\sin\left(2t + \frac{\pi}{2}\right)\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} = \frac{3}{2}\cos\left(\sqrt{2}t\right)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{3}{2}\cos\left(2t\right)\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\mathbf{x}(t) = M^{-1/2}\mathbf{q}(t) = \frac{3}{2}\cos\left(\sqrt{2}t\right)\begin{bmatrix} 1/3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{3}{2}\cos\left(2t\right)\begin{bmatrix} 1/3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\mathbf{x}(t) = \frac{3}{2}\cos\left(\sqrt{2}t\right)\begin{bmatrix} 1/3 \\ 1 \end{bmatrix} + \frac{3}{2}\cos\left(2t\right)\begin{bmatrix} 1/3 \\ -1 \end{bmatrix}$$

$$\mathbf{x}(t) = \begin{bmatrix} \frac{1}{2}\cos\left(\sqrt{2}t\right) + \frac{1}{2}\cos(2t) \\ \frac{3}{2}\cos\left(\sqrt{2}t\right) - \frac{3}{2}\cos\left(2t\right) \end{bmatrix}$$

Problems and Solutions for Section 4.7 (4.76 through 4.79)

4.76 Use Lagrange's equation to derive the equations of motion of the lathe of Fig. 4.21 for the undamped case.

Solution: Let the generalized coordinates be θ_1, θ_2 and θ_3.

The kinetic energy is

$$T = \frac{1}{2}J_1\dot{\theta}_1^2 + \frac{1}{2}J_2\dot{\theta}_2^2 + \frac{1}{2}J_3\dot{\theta}_3^2$$

The potential energy is

$$U = \frac{1}{2}k_1\left(\theta_2 - \theta_2\right)^2 + \frac{1}{2}k_2\left(\theta_3 - \theta_2\right)^2$$

There is a nonconservative moment *M(t)* on inertia 3. The Lagrangian is

$$L = T - U = \frac{1}{2}J_1\dot{\theta}_1^2 + \frac{1}{2}J_2\dot{\theta}_2^2 + \frac{1}{2}J_3\dot{\theta}_3^2 - \frac{1}{2}k_1\left(\theta_2 - \theta_1\right)^2 - \frac{1}{2}k_2\left(\theta_3 - \theta_2\right)^2$$

Calculate the derivatives from Eq. (4.136):

$$\frac{\partial L}{\partial \dot{\theta}_1} = J_1\dot{\theta}_1 \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) = J_1\ddot{\theta}_1$$

$$\frac{\partial L}{\partial \dot{\theta}_2} = J_2\dot{\theta}_2 \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) = J_2\ddot{\theta}_2$$

$$\frac{\partial L}{\partial \dot{\theta}_3} = J_3\dot{\theta}_3 \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_3}\right) = J_3\ddot{\theta}_3$$

$$\frac{\partial L}{\partial \theta_1} = -k_1\theta_1 + k_1\theta_2$$

$$\frac{\partial L}{\partial \theta_2} = -k_1\theta_1 - \left(k_1 + k_2\right)\theta_2 + k_2\theta_3$$

$$\frac{\partial L}{\partial \theta_3} = -k_2\theta_2 - k_2\theta_3$$

Using Eq. (4.136) yields

$$J_1\ddot{\theta}_1 + k_1\theta_1 - k_2\theta_2 = 0$$

$$J_2\ddot{\theta}_2 - k_1\theta_1 + \left(k_1 + k_2\right)\theta_2 - k_2\theta_3 = 0$$

$$J_3\ddot{\theta}_3 - k_2\theta_2 + k_2\theta_3 = M\left(t\right)$$

In matrix form this yields

$$\begin{bmatrix} J_1 & 0 & 0 \\ 0 & J_2 & 0 \\ 0 & 0 & J_3 \end{bmatrix} \ddot{\theta} + \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \theta = \begin{bmatrix} 0 \\ 0 \\ M(t) \end{bmatrix}$$

4.77 Use Lagrange's equations to rederive the equations of motion for the automobile of Example 4.8.2 illustrated in Figure 4.25 for the case $c_1 = c_2 = 0$.

Solution: Let the generalized coordinates be *x* and θ.
The kinetic energy is

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}J\dot{\theta}^2$$

The potential energy is (ignoring gravity)

$$U = \frac{1}{2}k_1\left(x - l_1\theta\right)^2 + \frac{1}{2}k_2\left(x + l_2\theta\right)^2$$

The Lagrangian is

$$L = T - U = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}J\dot{\theta}^2 - \frac{1}{2}k_1\left(x - l_1\theta\right)^2 - \frac{1}{2}k_2\left(x + l_2\theta\right)^2$$

Calculate the derivatives from Eq. (4.136):

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x}$$

$$\frac{\partial L}{\partial \dot{\theta}} = J\dot{\theta} \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = J\ddot{\theta}$$

$$\frac{\partial L}{\partial x} = -\left(k_1 + k_2\right)x + \left(k_1 l_1 - k_1 l_2\right)\theta$$

$$\frac{\partial L}{\partial \theta} = \left(k_1 l_1 - k_2 l_2\right)x - \left(k_1 l_2^2\right)\theta$$

Using Eq. (4.136) yields

$$m\ddot{x} + \left(k_1 + k_2\right)x + \left(k_1 l_1 - k_2 l_2\right)\theta = 0$$

$$J\ddot{\theta} + \left(k_1 l_2 - k_1 l_1\right)x - \left(k_1 l_1^2 + k_2 l_2^2\right)\theta = 0$$

In matrix form this yields

$$\begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix}\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & k_2 l_2 - k_1 l_1 \\ k_2 l_2 - k_1 l_1 & k_1 l_1^2 + k_2 l_2^2 \end{bmatrix}\begin{bmatrix} x \\ \theta \end{bmatrix} = \mathbf{0}$$

4.78 Use Lagrange's equations to rederive the equations of motion for the building model presented in Fig. 4.9 of Ex. 4.4.3 for the undamped case.

Solution:

Let the generalized coordinates be x_1, x_2, x_3 and x_4.
The kinetic energy is

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}m_3\dot{x}_3^2 + \frac{1}{2}m_4\dot{x}_4^2$$

The potential energy is (ignoring gravity)

$$U = \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2\left(x_2 - x_1\right)^2 + \frac{1}{2}k_3\left(x_3 - x_2\right)^2 + \frac{1}{2}k_4\left(x_4 - x_3\right)^2$$

The Lagrangian is

$$\begin{aligned} L = T - U = {} & \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 + \frac{1}{2}m\dot{x}_3^2 + \frac{1}{2}m\dot{x}_4^2 \\ & - \frac{1}{2}k_1x_1^2 - \frac{1}{2}k_2\left(x_2 - x_1\right)^2 \frac{1}{2}k_3\left(x_3 - x_2\right)^2 - \frac{1}{2}k_4\left(x_4 - x_3\right)^2 \end{aligned}$$

Calculate the derivatives from Eq. (4.136):

$$\frac{\partial L}{\partial \dot{x}_1} = m_1\dot{x}_1 \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) = m_1\ddot{x}_1$$

$$\frac{\partial L}{\partial \dot{x}_1} = m_2\dot{x}_2 \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) = m_2\ddot{x}_2$$

$$\frac{\partial L}{\partial \dot{x}_1} = m_3\dot{x}_3 \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) = m_3\ddot{x}_3$$

$$\frac{\partial L}{\partial \dot{x}_1} = m_4\dot{x}_4 \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) = m_4\ddot{x}_4$$

$$\frac{\partial L}{\partial x_1} = -\left(k_1 + k_2\right)x_1 + k_2 x_2$$
$$\frac{\partial L}{\partial x_2} = k_2 x_1 - \left(k_2 + k_3\right)x_2 + k_3 x_3$$
$$\frac{\partial L}{\partial x_3} = k_2 x_2 - \left(k_2 + k_4\right)x_3 - k_4 x_4$$
$$\frac{\partial L}{\partial x_4} = k_4 x_3 - k_4 x_4$$

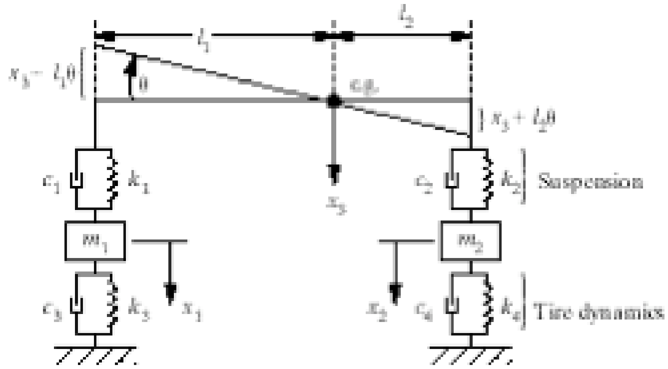
Using Eq. (4.136) yields

$$m_1\ddot{x}_1 + \left(k_1 + k_2\right)x_1 - k_2 x_2 = 0$$
$$m_2\ddot{x}_2 - k_2 x_1 + \left(k_2 + k_3\right)x_2 - k_3 x_3 = 0$$
$$m_3\ddot{x}_3 - k_3 x_2 + \left(k_3 + k_4\right)x_3 - k_4 x_4 = 0$$
$$m_4\ddot{x}_4 - k_4 x_3 + k_4 x_4 = 0$$

In matrix form this yields

$$\begin{bmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \\ 0 & 0 & m_3 & 0 \\ 0 & 0 & 0 & m_4 \end{bmatrix}\ddot{\mathbf{x}} + \begin{bmatrix} k_1 + k_2 & -k_2 & 0 & 0 \\ -k_2 & k_2 + k_3 & -k_3 & 0 \\ 0 & -k_3 & k_3 + k_4 & -k_4 \\ 0 & 0 & -k_4 & k_4 \end{bmatrix}\mathbf{x} = \mathbf{0}$$

4.79 Consider again the model of the vibration of an automobile of Fig. 4.25. In this case include the tire dynamics as indicated in Fig. P4.79. Derive the equations of motion using Lagrange formulation for the undamped case. Let m_3 denote the mass of the car acting at c.g.



Solution:

Let the generalized coordinates be x_1, x_2, x_3 and θ. The kinetic energy is

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}m_3\dot{x}_3^2 + \frac{1}{2}J\dot{\theta}^2$$

The potential energy is (ignoring gravity)

$$U = \frac{1}{2}k_1\left(x_3 - \ell_1\theta - x_1\right)^2 + \frac{1}{2}k_2(x_3 - \ell_2\theta - x_2)^2 + \frac{1}{2}k_3x_1^2 + \frac{1}{2}k_4x_2^2$$

The Lagrangian is thus:

$$L = T - U = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}m_3\dot{x}_3^2 + \frac{1}{2}J\dot{\theta}^2 - \frac{1}{2}k_1\left(x_3 - l_1\theta - x_1\right)^2$$

$$- \frac{1}{2}k_2\left(x_3 + l_2\theta - x_2\right)^2 - \frac{1}{2}k_3x_1^2 - \frac{1}{2}k_4x_2^2$$

Calculate the derivatives indicated in Eq. (4.146):

$$\frac{\partial L}{\partial \dot{x}_1} = m_1\dot{x}_1 \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) = m_1\ddot{x}_1$$

$$\frac{\partial L}{\partial \dot{x}_2} = m_2\dot{x}_2 \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) = m_1\ddot{x}_2$$

$$\frac{\partial L}{\partial \dot{x}_3} = m_3\dot{x}_3 \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_3}\right) = m_3\ddot{x}_3$$

$$\frac{\partial L}{\partial \theta} = J\dot{\theta} \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{\theta}}\right) = J\ddot{\theta}$$

$$\frac{\partial L}{\partial x_1} = -\left(k_1 + k_3\right)x_1 + k_1x_3 - k_1l_1\theta$$
$$\frac{\partial L}{\partial x_2} = -\left(k_2 + k_4\right)x_2 + k_2x_3 - k_2l_2\theta$$
$$\frac{\partial L}{\partial x_3} = k_1x_1 + k_2x_2 - \left(k_1 + k_2\right)x_3 + \left(k_1l_1 + k_2l_2\right)\theta$$
$$\frac{\partial L}{\partial \theta} = -k_1l_1x_1 - k_2l_2x_2 + \left(k_1l_1 + k_2l_2\right)x_3 - \left(k_1l_1^2 + k_2l_2^2\right)\theta$$

Using Eq. (4.146) yields

$$m_1\ddot{x}_1 + \left(k_3 + k_1\right)x_1 - k_1x_3 + k_1l_1\theta = 0$$
$$m_2\ddot{x}_2 + \left(k_4 + k_2\right)x_2 - k_2x_3 - k_2l_2\theta = 0$$
$$m_3\ddot{x}_3 - k_1x_1 - k_2x_2 + \left(k_1 + k_2\right)x_3 - \left(k_1l_1 - k_2l_2\right)\theta = 0$$
$$J\ddot{\theta} + k_1l_1x_1 - k_2l_2x_2 - \left(k_1l_1 - k_2l_2\right)x_3 + \left(k_1l_1^2 + k_2l_2^2\right) = 0$$

in matrix form

$$\begin{bmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \\ 0 & 0 & m_3 & 0 \\ 0 & 0 & 0 & J \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} \left(k_3 + k_1\right) & 0 & -k_1 & k_1l_1 \\ 0 & \left(k_4 + k_2\right) & -k_2 & k_2l_2 \\ -k_1 & -k_2 & \left(k_1 + k_2\right) & -\left(k_2l_2 + k_1l_1\right) \\ k_1l_1 & k_2l_2 & -\left(k_2l_2 + k_1l_1\right) & \left(k_1l_1^2 + k_2l_2^2\right) \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ \theta \end{Bmatrix} = 0$$

Problems and Solutions for Section 4.9 (4.80 through 4.90)

4.80 Consider the mass matrix

$$M = \begin{bmatrix} 10 & -1 \\ -1 & 1 \end{bmatrix}$$

and calculate M^{-1}, $M^{-1/2}$, and the Cholesky factor of M. Show that

$$LL^T = M$$

$$M^{-1/2}M^{-1/2} = I$$

$$M^{1/2}M^{1/2} = M$$

Solution: Given

$$M = \begin{bmatrix} 10 & -1 \\ -1 & 1 \end{bmatrix}$$

The matrix, P, of eigenvectors is

$$P = \begin{bmatrix} -0.1091 & -0.9940 \\ -0.9940 & 0.1091 \end{bmatrix}$$

The eigenvalues of M are

$$\lambda_1 = 0.8902$$

$$\lambda_2 = 10.1098$$

From Equation

$$M^{-1} = P\text{diag}\left[\frac{1}{\lambda_1}, \frac{1}{\lambda_2}\right]P^T, \quad M^{-1} = \begin{bmatrix} 0.1111 & 0.1111 \\ 0.1111 & 1.1111 \end{bmatrix}$$

From Equation

$$M^{-1/2} = Vdiag\left[\lambda_1^{-1/2}, \lambda_2^{-1/2}\right]V^T$$

$$M^{-1/2} = \begin{bmatrix} 0.3234 & 0.0808 \\ 0.0808 & 1.0510 \end{bmatrix}$$

The following Mathcad session computes the Cholesky decomposition.

$$M := \begin{bmatrix} 10 & -1 \\ -1 & 1 \end{bmatrix} \qquad M^{-1} = \begin{bmatrix} 0.11111 & 0.11111 \\ 0.11111 & 1.11111 \end{bmatrix}$$

$L := \text{cholesky}(M)$

$$L = \begin{bmatrix} 3.16228 & 0 \\ -0.31623 & 0.94868 \end{bmatrix} \qquad L \cdot L^T = \begin{bmatrix} 10 & -1 \\ -1 & 1 \end{bmatrix} \qquad L^{-1} \cdot M \cdot \left(L^T\right)^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

4.81 Consider the matrix and vector

$$A = \begin{bmatrix} 1 & -\varepsilon \\ -\varepsilon & \varepsilon \end{bmatrix} \qquad b = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$$

use a code to solve $Ax = b$ for $\varepsilon = 0.1, 0.01, 0.001, 10^{-6}$, and 1.

Solution:

The equation is

$$\begin{bmatrix} 1 & -\varepsilon \\ -\varepsilon & \varepsilon \end{bmatrix} \mathbf{x} = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$$

The following Mathcad session illustrates the effect of ε on the solution, a entire integer difference. Note that no solution exists for the case ε = 1.

$$b := \begin{bmatrix} 10 \\ 10 \end{bmatrix} \qquad A(\varepsilon) := \begin{bmatrix} 1 & -\varepsilon \\ -\varepsilon & \varepsilon \end{bmatrix} \qquad x(\varepsilon) := A(\varepsilon)^{-1} \cdot b$$

$$x(0.1) = \begin{bmatrix} 22.222 \\ 122.222 \end{bmatrix} \qquad x(0.01) = \begin{bmatrix} 20.202 \\ 1.02 \cdot 10^{3} \end{bmatrix} \qquad x(0.001) = \begin{bmatrix} 20.02 \\ 1.002 \cdot 10^{4} \end{bmatrix}$$

$$x\left(10^{-6}\right) = \begin{bmatrix} 20 \\ 1 \cdot 10^{7} \end{bmatrix}$$

So the solution to this problem is very sensitive, and ill conditioned, because of the inverse.

4.82 Calculate the natural frequencies and mode shapes of the system of Example 4.8.3. Use the undamped equation and the form given by equation (4.161).

Solution:

The following MATLAB program will calculate the natural frequencies and mode shapes for Example 4.8.3 using Equation (4.161).

```
m=[0.4 0 0;0 2 0;0 0 8]*1e3;
k=[30 –30 0;-30 38 –8;0 –8 88]1e4;
[u, d]=eig(k,m);
w=sqrt (d);
```

The matrix `d` contains the square of the natural frequencies, and the matrix `u` contains the corresponding mode shapes.

4.83 Compute the natural frequencies and mode shapes of the undamped version of the system of Example 4.8.3 using the formulation of equation (4.164) and (4.168). Compare your answers.

Solution:

The following MATLAB program will calculate the natural frequencies and mode shapes for Example 4.8.3 using Equation (4.161).

```
m=[0.4 0 0;0 2 0;0 0 8]*1e3;
k=[30 –30 0;-30 38 –8;0 –8 88]1e4;
mi=inv(m);
kt=mi*k;
[u, d]=eig(k,m);
w=sqrt (d);
```

The number of floating point operations needed is 439.
The matrix `d` contains the square of the natural frequencies, and the matrix `u` contains the corresponding mode shapes.

The following MATLAB program will calculate the natural frequencies and mode shapes for Example 4.8.3 using Equation (4.168).

```
m=[0.4 0 0;0 2 0;0 0 8]*1e3;
k=[30 –30 0;-30 38 –8;0 –8 88]1e4;
msi=inv(sqrt(m));
kt=msi*k*msi;
[p, d]=eig(kt);
w=sqrt (d);
u=msi*p;
```

The number of floating point operations needed is 461.
The matrix `d` contains the square of the natural frequencies, and the matrix `u` contains the corresponding mode shapes.

The method of Equation (4.161) is faster.

4.84 Use a code to solve for the modal information of Example 4.1.5.

Solution: See Toolbox or use the following Mathcad code:

$$\omega := 1$$

Given

$$\left[\omega^4 - \left(6\cdot\omega^2\right)\right] + 8 = 0$$

$$\text{Find}(\omega) = 1.414$$

$$\omega := 2$$

Given

$$\left[\omega^4 - \left(6\cdot\omega^2\right)\right] + 8 = 0$$

$$\text{Find}(\omega) = 2$$

4.85 Write a program to perform the normalization of Example 4.4.2 (i.e., calculate α such that the vector $\alpha\mathbf{v}_1$ is normal).

Solution:

The following MATLAB program will perform the normalization of Example 4.4.2.

```
x=[.4450 .8019 1];
mag=sqrt(sum(x.^2));
xnorm=x/mag;
```

The variable `mag` is the same as α, and `xnorm` is the normalized vector. The original vector `x` can be any length.

4.86 Use a code to calculate the natural frequencies and mode shapes obtained for the system of Example 4.2.5 and Figure 4.4.

Solution: See Toolbox or use the following Mathcad code:

$$M := \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \qquad K := \begin{bmatrix} 12 & -2 \\ -2 & 12 \end{bmatrix} \qquad Mr := \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$Kd := Mr^{-1} \cdot K \cdot Mr^{-1} \qquad Kd = \begin{bmatrix} 12 & -1 \\ -1 & 3 \end{bmatrix}$$

$$\lambda := \text{eigenvals}(Kd) \qquad \lambda = \begin{bmatrix} 12.11 \\ 2.89 \end{bmatrix}$$

$$\omega 1 := \sqrt{\lambda_1} \qquad \omega 2 := \sqrt{\lambda_0} \qquad \omega 1 = 1.7 \qquad \omega 2 = 3.48$$

$$v1 := \text{eigenvec}(Kd, \lambda_1) \qquad v2 := \text{eigenvec}(Kd, \lambda_0)$$

$$v1 = \begin{bmatrix} 0.109 \\ 0.994 \end{bmatrix} \qquad v2 = \begin{bmatrix} -0.994 \\ 0.109 \end{bmatrix} \qquad v1^T \cdot v2 = 0$$

$$v1^T \; v1 = 1 \qquad v2^T \cdot v2 = 1$$

$$P := \text{augment}(v1, v2) \qquad P^T \cdot Kd \cdot P = \begin{bmatrix} 2.89 & 0 \\ 0 & 12.11 \end{bmatrix} \qquad P^T \cdot P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

4.87 Following the modal analysis solution of Window 4.4, write a program to compute the time response of the system of Example 4.3.2.

Solution: The following MATLAB program will compute and plot the time response of the system of Example 4.3.2.

```
   t=(0:.1:10)’;

   m=[1 0;0 4];
   k=[12 –2;-2 12];
n=max(size(m));

   x0=[1 1]’;
xd0=[0 0]’;

   msi=inv(sqrtm(m));
   kt=msi*k*msi;

   [p, w]=eig(kt);
   for i=1: n-1
    for j=1: n-I
        if w(j,j)>w(j+1,j+1)
             dummy=w(j,j);
             w(j,j)=w(j+1,j+1);
             w(j+1,j+1)=dummy;
             dummy=p(:,j);
             p(:,j)=p(:,j+1);
             p(:,j+1)=dummy;
        end
    end
   end
   pt=p’;
   s=msi*p;
   si=pt*sqrtm(m);

   r0=si*x0;
   rd0=si*xd0;
   r=[];
   for i=1: n,
    wi=sqrt(w(i,i));
    rcol=(swrt((wi*r0(i))^2+rd0(i)^2/wi)*…
        sin(wi*t+atan2(wi*r0(i),rd0(i)));
    r(:,i)=rcol;
   end
   x=s*r;
   plot(t,x);
   end
```

4.88 Use a code to solve the damped vibration problem of Example 4.6.1 by calculating the natural frequencies, damping ratios, and mode shapes.

Solution: See Toolbox or use the following Mathcad code (all will do this)

$$M := \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix} \qquad K := \begin{bmatrix} 27 & -3 \\ -3 & 3 \end{bmatrix} \qquad Mr := \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \qquad C := \begin{bmatrix} 2.7 & -.3 \\ -.3 & 0.3 \end{bmatrix}$$

$$Kd := Mr^{-1} \cdot K \cdot Mr^{-1} \qquad Kd = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} \qquad Cd := Mr^{-1} \cdot \left(C \cdot Mr^{-1}\right) \qquad Cd = \begin{bmatrix} 0.3 & -0.1 \\ -0.1 & 0.3 \end{bmatrix}$$

$$\lambda := \text{eigenvals}(Kd) \qquad \lambda = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$\omega 1 := \sqrt{\lambda_1} \qquad \omega 2 := \sqrt{\lambda_0} \qquad \omega 1 = 1.414 \qquad \omega 2 = 2$$

$$v1 := \text{eigenvec}\left(Kd, \lambda_1\right) \qquad v2 := \text{eigenvec}\left(Kd, \lambda_0\right)$$

$$v1 = \begin{bmatrix} 0.707 \\ 0.707 \end{bmatrix} \qquad v2 = \begin{bmatrix} -0.707 \\ 0.707 \end{bmatrix} \qquad v1^T \cdot v2 = 0$$

$$v1^T \; v1 = 1 \qquad v2^T \cdot v2 = 1$$

$$P := \text{augment}(v1, v2) \qquad P^T \cdot Kd \cdot P = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \qquad P^T \cdot P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 0.707 & -0.707 \\ 0.707 & 0.707 \end{bmatrix} \qquad Cz := P^T \cdot Cd \cdot P \qquad Cz = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.4 \end{bmatrix}$$

$$\zeta 1 := \frac{Cz_{0,0}}{2 \cdot \omega 1} \qquad \zeta 1 = 0.071 \qquad \zeta 2 := \frac{Cz_{1,1}}{2 \cdot \omega 2} \qquad \zeta 2 = 0.1$$

$$\omega d1 := \omega 1 \cdot \sqrt{1 - \zeta 1^2} \qquad \omega d2 := \omega 2 \cdot \sqrt{1 - \zeta 2^2} \qquad \omega d2 = 1.99$$

$$\omega d1 = 1.411$$

4.89 Consider the vibration of the airplane of Problems 4.46 and 4.47 as given in Figure P4.46. The mass and stiffness matrices are given as

$$M = m\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad K = \frac{EI}{l^3}\begin{bmatrix} 3 & -3 & 0 \\ -3 & 6 & -3 \\ 0 & -3 & 3 \end{bmatrix}$$

where m = 3000 kg, l = 2 m, $I = 5.2 \times 10^{-6}$ m^4, $E = 6.9 \times 10^9$ N/m^2, and the damping matrix C is taken to be $C = (0.002)K$. Calculate the natural frequencies, normalized mode shapes, and damping ratios.

Solution: Use the Toolbox or use a code directly such as the following Mathcad session:

$E := 6.9\cdot 10^9 \qquad I := 5.2\cdot 10^{-6} \qquad m := 3000 \qquad L := 2$

$$M := m\cdot\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad K := \frac{E\cdot I}{L^3}\cdot\begin{bmatrix} 3 & -3 & 0 \\ -3 & 6 & -3 \\ 0 & -3 & 3 \end{bmatrix} \qquad C := 0.002\cdot K$$

$$M = \begin{bmatrix} 3\cdot 10^3 & 0 & 0 \\ 0 & 1.2\cdot 10^4 & 0 \\ 0 & 0 & 3\cdot 10^3 \end{bmatrix} \qquad K = \begin{bmatrix} 1.346\cdot 10^4 & -1.346\cdot 10^4 & 0 \\ -1.346\cdot 10^4 & 2.691\cdot 10^4 & -1.346\cdot 10^4 \\ 0 & -1.346\cdot 10^4 & 1.346\cdot 10^4 \end{bmatrix}$$

$i := 0, 1 .. 2 \qquad j := 0, 1 .. 2$

$Mr_{i,j} := \sqrt{M_{i,j}}$

$$Kh := Mr^{-1}\cdot K\cdot Mr^{-1} \qquad Kh = \begin{bmatrix} 4.485 & -2.242 & 0 \\ -2.242 & 2.242 & -2.242 \\ 0 & -2.242 & 4.485 \end{bmatrix} \qquad Ch := Mr^{-1}\cdot\left(C\cdot Mr^{-1}\right)$$

$$\text{eigenvals}(Kh) = \begin{bmatrix} 6.727 \\ 4.485 \\ 0 \end{bmatrix} \qquad \lambda_1 := 0 \qquad \omega_1 := \sqrt{\lambda_1}$$

$$\lambda_2 := 4.485 \quad \omega_2 := \sqrt{\lambda_2} \qquad \omega_2 = 2.118$$

$$\lambda_3 := 6.727 \quad \omega_3 := \sqrt{\lambda_3} \qquad \omega_3 = 2.594$$

$$v_1 := \text{eigenvec}(Kh, \lambda_1) \qquad v_1 = \begin{bmatrix} 0.408 \\ 0.816 \\ 0.408 \end{bmatrix}$$

$$v_2 := \text{eigenvec}(Kh, \lambda_2) \qquad v_2 = \begin{bmatrix} -0.707 \\ 0 \\ 0.707 \end{bmatrix} \quad v_3 := \text{eigenvec}(Kh, \lambda_3) \qquad v_3 = \begin{bmatrix} 0.577 \\ -0.577 \\ 0.577 \end{bmatrix}$$

$$P1 := \text{augment}(v_1, v_2) \quad P := \text{augment}(P1, v_3) \qquad Ac := P^T \cdot Ch \cdot P$$

$$Ac = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 8.97 \cdot 10^{-3} & 0 \\ 0 & 0 & 0.013 \end{bmatrix} \qquad \zeta 2 := \frac{Ac_{1,1}}{2 \cdot \omega_2} \quad \zeta 2 = 2.118 \cdot 10^{-3}$$

$$\zeta 3 := \frac{Ac_{2,2}}{2 \cdot \omega_3} \qquad \zeta 3 = 2.594 \cdot 10^{-3}$$

The normalized mode shapes are

$$u1 := Mr^{-1} \cdot v_1 \qquad u1n := \frac{u1}{|u1|} \qquad u1n = \begin{bmatrix} 0.577 \\ 0.577 \\ 0.577 \end{bmatrix}$$

$$u2 := Mr^{-1} \cdot v_2 \qquad u2n := \frac{u2}{|u2|} \qquad u2n = \begin{bmatrix} -0.707 \\ 0 \\ 0.707 \end{bmatrix}$$

$$u3 := Mr^{-1} \cdot v_3 \qquad u3n := \frac{u3}{|u3|} \qquad u3n = \begin{bmatrix} 0.667 \\ -0.333 \\ 0.667 \end{bmatrix}$$

4.90 Consider the proportionally damped, dynamically coupled system given by

$$M = \begin{bmatrix} 9 & -1 \\ -1 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} \quad K = \begin{bmatrix} 49 & -2 \\ -2 & 2 \end{bmatrix}$$

and calculate the mode shapes, natural frequencies, and damping ratios.

Solution: Use the Toolbox or any of the codes. A Mathcad solution is shown:

$$M := \begin{bmatrix} 9 & -1 \\ -1 & 1 \end{bmatrix} \qquad K := \begin{bmatrix} 49 & -2 \\ -2 & 2 \end{bmatrix} \qquad C := \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}$$

$$L := \mathrm{cholesky}(M)$$

$$Kh := L^{-1} \cdot K \cdot \left(L^{T}\right)^{-1} \quad Kh = \begin{bmatrix} 5.444 & 1.218 \\ 1.218 & 2.431 \end{bmatrix} \qquad Ch := L^{-1} \cdot \left[C \cdot \left(L^{T}\right)^{-1}\right]$$

$$Ch = \begin{bmatrix} 0.333 & -0.236 \\ -0.236 & 0.917 \end{bmatrix} \qquad \mathrm{eigenvals}(Kh) = \begin{bmatrix} 5.875 \\ 2 \end{bmatrix}$$

$$\lambda_1 := 2 \quad \omega_1 := \sqrt{\lambda_1} \quad \omega_1 = 1.414 \quad \lambda_2 := 5.875 \quad \omega_2 := \sqrt{\lambda_2} \quad \omega_2 = 2.424$$

$$v_1 := \mathrm{eigenvec}(Kh, \lambda_1) \quad v_1 = \begin{bmatrix} -0.333 \\ 0.943 \end{bmatrix} \quad v_2 := \mathrm{eigenvec}(Kh, \lambda_2) \quad v_2 = \begin{bmatrix} 0.943 \\ 0.333 \end{bmatrix}$$

$$P := \mathrm{augment}(v_1, v_2) \qquad \Lambda c := P^{T} \cdot Ch \cdot P \qquad \Lambda c = \begin{bmatrix} 1 & 0 \\ 0 & 0.25 \end{bmatrix}$$

$$\zeta 1 := \frac{\Lambda c_{0,0}}{2 \cdot \omega_1} \quad \zeta 1 = 0.354 \quad \zeta 2 := \frac{\Lambda c_{1,1}}{2 \cdot \omega_2} \quad \zeta 2 = 0.052$$

Computing the mode shapes from the eigenvectors yields:

$$M := \begin{pmatrix} 9 & -1 \\ -1 & 1 \end{pmatrix} \qquad R := \text{cholesky}(M)$$

$$R^{-1} = \begin{pmatrix} 0.333 & 0 \\ 0.118 & 1.061 \end{pmatrix}$$

$$u1 := R^{-1} \cdot \begin{pmatrix} -0.333 \\ 0.943 \end{pmatrix} \quad u1 = \begin{pmatrix} -0.111 \\ 0.961 \end{pmatrix} \qquad u2 := R^{-1} \cdot \begin{pmatrix} 0.934 \\ 0.333 \end{pmatrix} \quad u2 = \begin{pmatrix} 0.311 \\ 0.463 \end{pmatrix}$$

Problems and Solutions Section 4.10 (4.91 through 4.98)

4.91* Solve the system of Example 1.7.3 for the vertical suspension system of a car with m = 1361 kg, k = 2.668 x 10^5 N/m, and c = 3.81 x 10^4 kg/s subject to the initial conditions of $x(0)$ = 0 and $v(0)$ = 0.01 m/s^2.

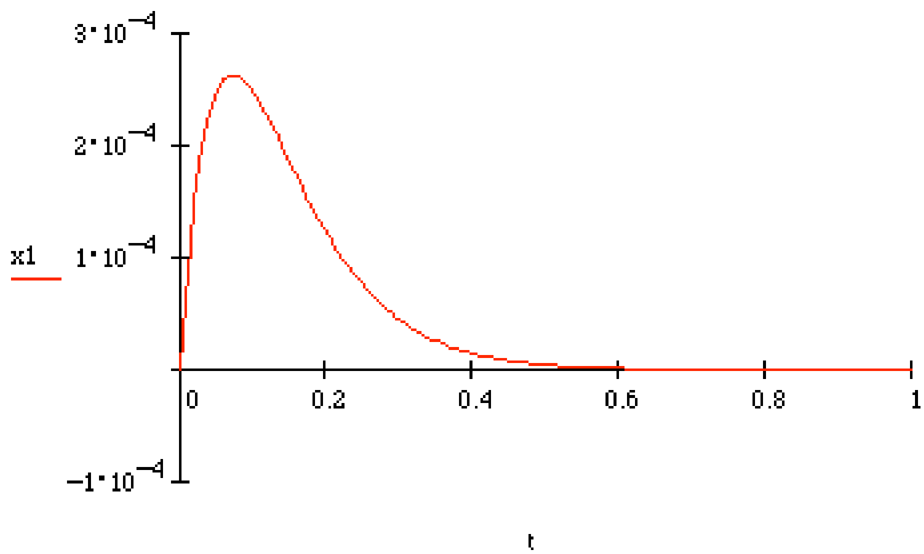
Solution: Use a Runge Kutta routine such as the one given in Mathcad here or use the toolbox:

$m := 1361 \qquad k := 2.668\cdot 10^5 \qquad c := 3.81\cdot 10^4$

$$X := \begin{bmatrix} 0 \\ 0.01 \end{bmatrix} \qquad A := \begin{bmatrix} 0 & 1 \\ \frac{-k}{m} & \frac{-c}{m} \end{bmatrix} \qquad D(t,X) := A\cdot X$$

$Z := \text{rkfixed}(X, 0, 20, 3000, D)$

$t := Z^{\langle 0 \rangle} \qquad x1 := Z^{\langle 1 \rangle}$



4.92* Solve for the time response of Example 4.4.3 (i.e., the four-story building of Figure 4.9). Compare the solutions obtained with using a modal analysis approach to a solution obtained by numerical integration.

Solution: The following code provides the numerical solution.

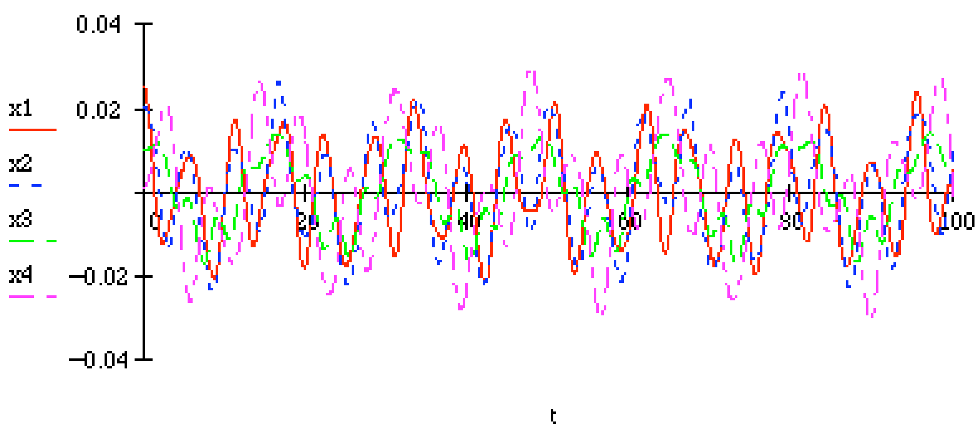
$$I := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad O := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad M := 4000 \cdot I$$

$$K := \begin{bmatrix} 10000 & -5000 & 0 & 0 \\ -5000 & 10000 & -5000 & 0 \\ 0 & -5000 & 10000 & -5000 \\ 0 & 0 & -5000 & 5000 \end{bmatrix} \quad C := O \quad X := \begin{bmatrix} 0.025 \\ 0.02 \\ 0.01 \\ 0.001 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A := \text{augment}\left(\text{stack}\left(O, -M^{-1} \cdot K\right), \text{stack}\left(I, -M^{-1} C\right)\right)$$

$$D(t, X) := A \cdot X \qquad Z := \text{rkfixed}(X, 0, 200, 3000, D)$$

$$t := Z^{\langle 0 \rangle} \quad x1 := Z^{\langle 1 \rangle} \quad x2 := Z^{\langle 2 \rangle} \quad x3 := Z^{\langle 3 \rangle} \quad x4 := Z^{\langle 4 \rangle}$$



which compares very well with the plots given in Figure 4.11 obtained by plotting the modal equations. One could also plot the modal response and numerical response on the same graph to see a more rigorous comparison.

4.93* Reproduce the plots of Figure 4.13 for the two-degree of freedom system of Example 4.5.1 using a code.

Solution: Use any of the codes. The trick here is to construct the damping matrix from the given modal information by first creating it in modal form and then transforming it back to physical coordinates as indicated in the following Mathcad session:

$$M := \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix} \qquad P := \frac{1}{\sqrt{2}} \cdot \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad K := \begin{bmatrix} 27 & -3 \\ -3 & 3 \end{bmatrix} \qquad O := \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad I := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$Mr := \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \qquad \omega 1 := \sqrt{2} \qquad \omega 2 := 2 \qquad \zeta 1 := 0.05 \qquad \zeta 2 := 0.1$$

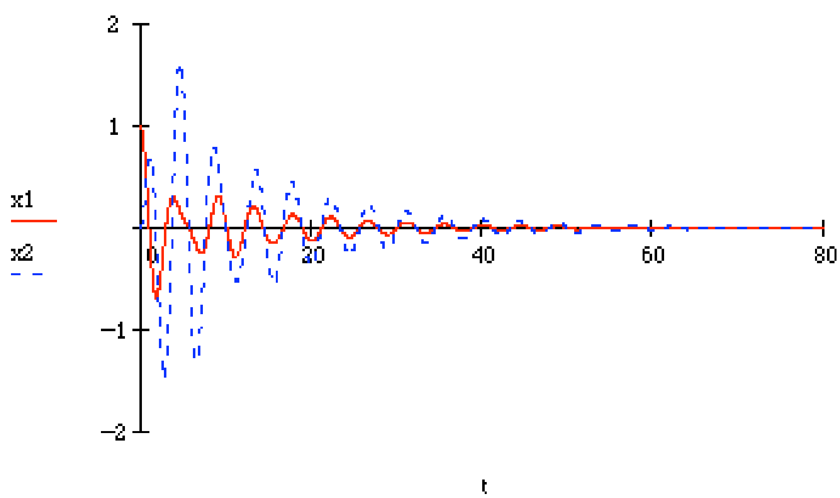
$$Ac := \begin{bmatrix} 2 \cdot \zeta 1 \cdot \omega 1 & 0 \\ 0 & 2 \cdot \zeta 2 \cdot \omega 2 \end{bmatrix} \qquad C := Mr \cdot P \cdot Ac \cdot P^T \cdot Mr$$

$$A := \mathrm{augment}\left(\mathrm{stack}\left(O, -M^{-1} \cdot K\right), \mathrm{stack}\left(I, -M^{-1}\, C\right)\right) \qquad X := \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$D(t, X) := A \cdot X$$

$$Z := \mathrm{rkfixed}(X, 0, 80, 4000, D)$$

$$t := Z^{\langle 0 \rangle} \qquad x1 := Z^{\langle 1 \rangle} \qquad x2 := Z^{\langle 2 \rangle}$$



4.94*. Consider example 4.8.3 and a) using the damping ratios given, compute a damping matrix in physical coordinates, b) use numerical integration to compute the response and plot it, and c) use the numerical code to design the system so that all 3 physical coordinates die out within 5 seconds (i.e., change the damping matrix until the desired response results).

Solution: A Mathcad solution is presented. The damping matrix is found, as in the previous problem, by keeping track of the various transformations. Using the notation of the text, the damping matrix is constructed from:

$$C = M^{1/2} P \begin{bmatrix} 2\zeta_1\omega_1 & 0 & 0 \\ 0 & 2\zeta_2\omega_2 & 0 \\ 0 & 0 & 2\zeta_3\omega_3 \end{bmatrix} P^T M^{1/2} = \begin{bmatrix} 1.062\times10^3 & -679.3 & 187.0 \\ -679.3 & 2.785\times10^3 & 617.8 \\ 187.0 & 617.8 & 2.041\times10^3 \end{bmatrix}$$

as computed using the code that follows. With this form of the matrix the damping ratios are adjusted until the desired criteria are met:

$$I := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad O := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad M := \begin{bmatrix} 0.4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix}\cdot 10^3$$

$$\omega1 := 5.3872 \quad \omega2 := 10.6755 \quad \omega3 := 30.1166$$

$$K := \begin{bmatrix} 30 & -30 & 0 \\ -30 & 38 & -8 \\ 0 & -8 & 88 \end{bmatrix}\cdot 10^4 \quad Mr := \begin{bmatrix} \sqrt{0.4\cdot 10^3} & 0 & 0 \\ 0 & \sqrt{2\cdot 10^3} & 0 \\ 0 & 0 & \sqrt{8\cdot 10^3} \end{bmatrix}$$

$$\zeta1 := 0.2 \quad \zeta2 := 0.05 \quad \zeta3 := 0.05$$

$$P := \begin{bmatrix} -0.4116 & -0.1021 & 0.9056 \\ -0.8848 & -0.1935 & -0.4239 \\ -0.2185 & 0.9758 & 0.0106 \end{bmatrix} \quad Ac := \begin{bmatrix} 2\cdot\omega1\cdot\zeta1 & 0 & 0 \\ 0 & 2\cdot\omega2\cdot\zeta2 & 0 \\ 0 & 0 & 2\cdot\omega3\cdot\zeta3 \end{bmatrix}$$

$$C := Mr\cdot P\cdot Ac\cdot P^T\cdot Mr \qquad A := \text{augment}\left(\text{stack}\left(O, -M^{-1}\cdot K\right), \text{stack}\left(I, -M^{-1}\, C\right)\right)$$

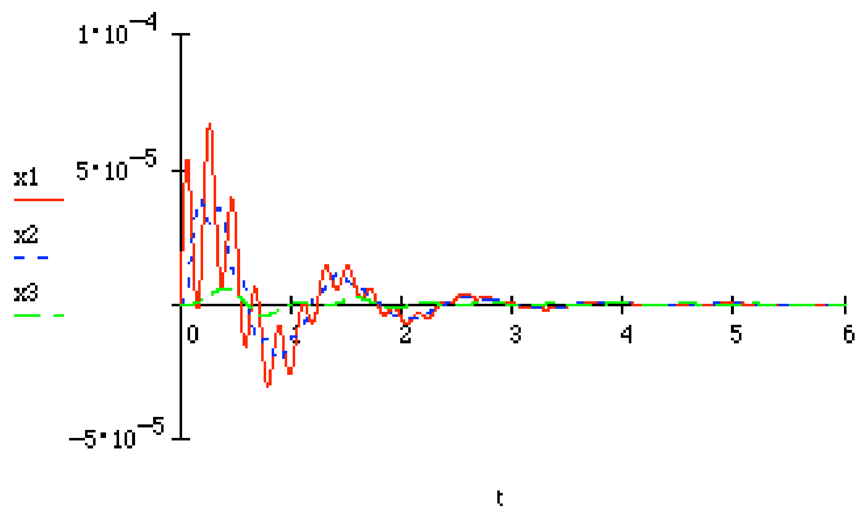
In changing the damping ratios it is best to start with the rubber component which is the first mode-damping ratio. Doubling it nails the first two coordinates but does not affect the third coordinate enough. Hence the second mode-damping ratio must be changed (doubled here) to attack this mode. This could be accomplished by adding a viscoelastic strip as described in Chapter 5 to the metal. Thus the ratios given in the code above do the trick as the following plots show. Note also how much the damping matrix changes.

$$X := \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad B := \begin{bmatrix} 1000 \\ 0 \\ 0 \end{bmatrix} \quad f := M^{-1} \cdot B \quad C = \begin{bmatrix} 1.138 \cdot 10^3 & -313.286 & 208.132 \\ -313.286 & 4.536 \cdot 10^3 & 805.984 \\ 208.132 & 805.984 & 8.958 \cdot 10^3 \end{bmatrix}$$

$$D(t, X) := A \cdot X + \begin{bmatrix} 0 \\ 0 \\ 0 \\ f_0 \\ f_1 \\ f_2 \end{bmatrix} \cdot \left(\Phi(t) - \Phi(t - 0.001) \right)$$

kfixed (X , 0 , 15 , 4000 , D)

$t := Z^{\langle 0 \rangle}$ $\quad x1 := Z^{\langle 1 \rangle}$ $\quad x2 := Z^{\langle 2 \rangle}$ $\quad x3 := Z^{\langle 3 \rangle}$



4.95*. Compute and plot the time response of the system (Newtons):

$$\begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 3 & -0.5 \\ -0.5 & 0.5 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\sin(4t)$$

subject to the initial conditions:

$$\mathbf{x}_0 = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} \text{ m}, \quad \mathbf{v}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ m/s}$$

Solution: The following Mathcad session illustrates the numerical solution of this problem using a Runge Kutta solver.

$$I := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad O := \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad M := \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix} \quad K := \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} \quad X := \begin{bmatrix} 0 \\ 0.1 \\ 1 \\ 0 \end{bmatrix}$$

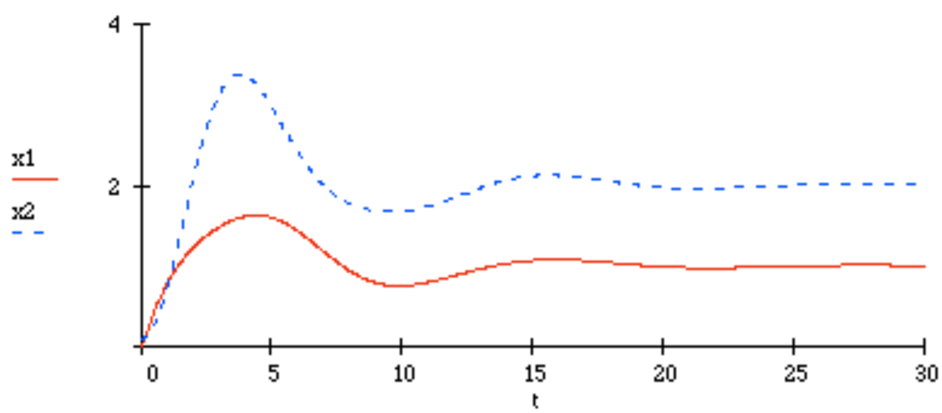
$$C := \begin{bmatrix} 3 & -0.5 \\ -0.5 & 0.5 \end{bmatrix}$$

$$A := \text{augment}\left(\text{stack}\left(O, -M^{-1}\cdot K\right), \text{stack}\left(I, -M^{-1}\ C\right)\right)$$

$$D(t, X) := A\cdot X + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{5} \\ 1 \end{bmatrix}$$

$$Z := \text{rkfixed}(X, 0, 100, 3000, D)$$

$$t := Z^{\langle 0 \rangle} \qquad x1 := Z^{\langle 1 \rangle} \qquad x2 := Z^{\langle 2 \rangle}$$



4.96* Consider the following system excited by a pulse of duration 0.1 s (in Newtons):

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 0.3 & -0.05 \\ -0.05 & 0.05 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}[\Phi(t-1) - \Phi(t-3)]$$

and subject to the initial conditions:

$$\mathbf{x}_0 = \begin{bmatrix} 0 \\ -0.1 \end{bmatrix} \text{ m}, \quad \mathbf{v}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ m/s}$$

Compute and plot the response of the system. Here Φ indicates the Heaviside Step Function introduced in Section 3.2.

Solution: The following Mathcad solution (see example4.10.3 for the other codes) gives the solution:

$$I := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad O := \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad M := \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \quad K := \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} \quad X := \begin{bmatrix} 0 \\ -0.1 \\ 0 \\ 0 \end{bmatrix}$$

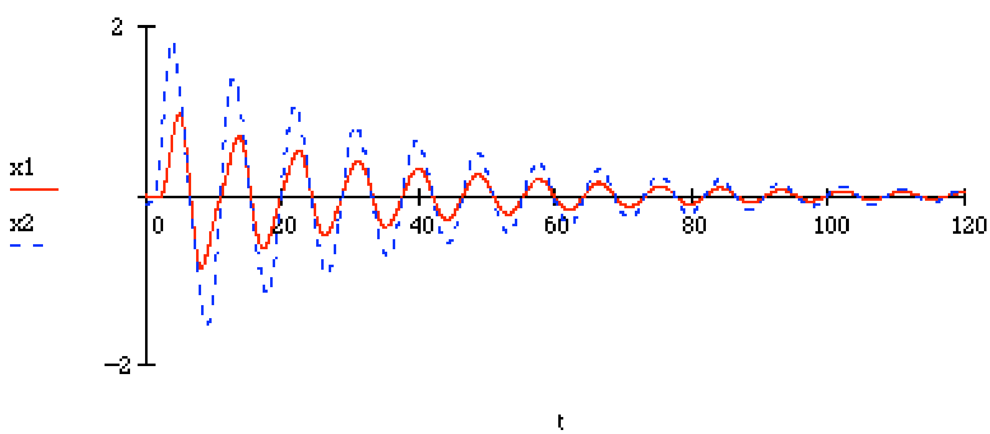
$$C := \begin{bmatrix} 0.33 & -0.05 \\ -0.05 & 0.05 \end{bmatrix} \quad B := \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad f := M^{-1} \cdot B$$

$$A := \text{augment}\left(\text{stack}\left(O, -M^{-1}\cdot K\right), \text{stack}\left(I, -M^{-1}\,C\right)\right)$$

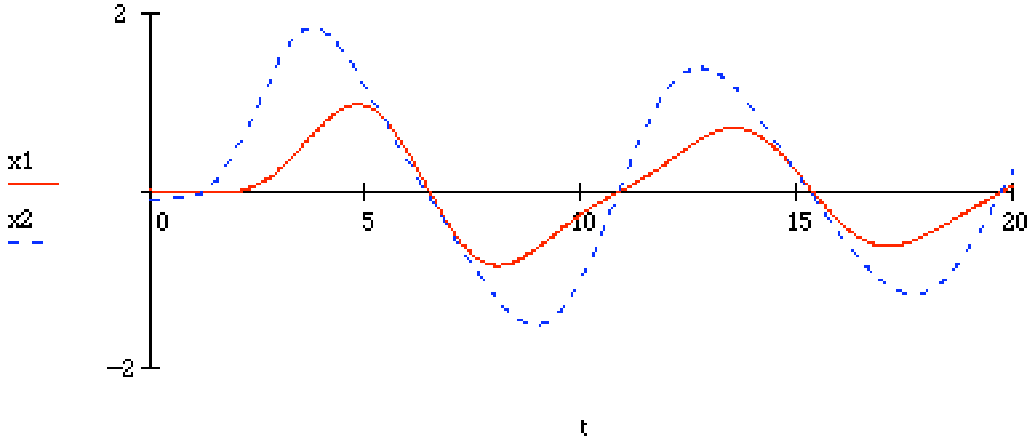
$$D(t, X) := A\cdot X + \begin{bmatrix} 0 \\ 0 \\ f_0 \\ f_1 \end{bmatrix} \cdot \left(\Phi(t-1) - \Phi(t-3)\right)$$

$$Z := \text{rkfixed}(X, 0, 120, 3000, D)$$

$$t := Z^{\langle 0 \rangle} \quad x1 := Z^{\langle 1 \rangle} \quad x2 := Z^{\langle 2 \rangle}$$



It is also interesting to examine the first 20 seconds more closely to see the effect of the impact:



Note that the impact has much more of an effect on the response than does the initial condition.

4.97.* Compute and plot the time response of the system (Newtons):

$$\begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 3 & -0.5 \\ -0.5 & 0.5 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} 30 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\sin(4t)$$

subject to the initial conditions:

$$\mathbf{x}_0 = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} \text{m}, \quad \mathbf{v}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{m/s}$$

Solution: Following the codes of Example 4.10.2 yields the solution directly.

$$I := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad O := \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad M := \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix} \quad K := \begin{bmatrix} 30 & -1 \\ -1 & 1 \end{bmatrix} \quad X := \begin{bmatrix} 0 \\ 0.1 \\ 1 \\ 0 \end{bmatrix}$$

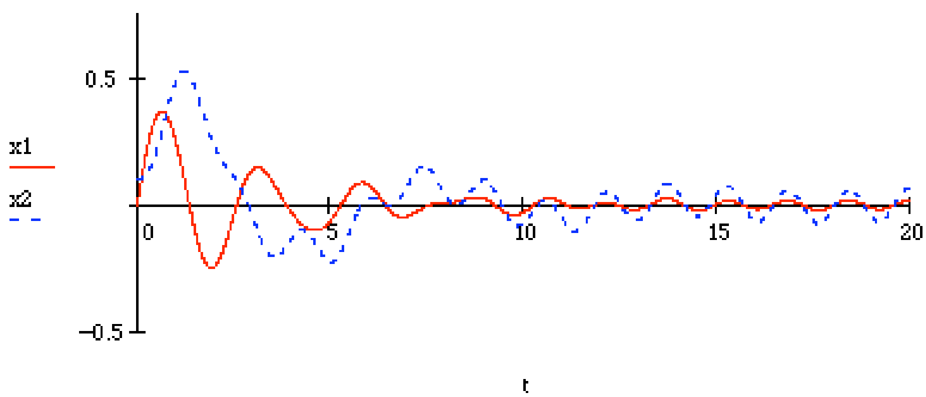
$$C := \begin{bmatrix} 3 & -0.5 \\ -0.5 & 0.5 \end{bmatrix} \quad B := \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad f := M^{-1} \cdot B$$

$$A := \text{augment}\left(\text{stack}\left(O, -M^{-1} \cdot K\right), \text{stack}\left(I, -M^{-1}\, C\right)\right)$$

$$D(t, X) := A \cdot X + \begin{bmatrix} 0 \\ 0 \\ f_0 \\ f_1 \end{bmatrix} \cdot \sin(4 \cdot t)$$

$$Z := \text{rkfixed}(X, 0, 20, 3000, D)$$

$$t := Z^{\langle 0 \rangle} \quad x1 := Z^{\langle 1 \rangle} \quad x2 := Z^{\langle 2 \rangle}$$



4.98.* Compute and plot the time response of the system (Newtons):

$$\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2.5 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \\ \ddot{x}_4 \end{bmatrix} + \begin{bmatrix} 4 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} + \begin{bmatrix} 500 & -100 & 0 & 0 \\ -100 & 200 & -100 & 0 \\ 0 & -100 & 200 & -100 \\ 0 & 0 & -100 & 100 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \sin(4t)$$

subject to the initial conditions:

$$\mathbf{x}_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.01 \end{bmatrix} \text{ m}, \quad \mathbf{v}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ m/s}$$

Solution: Again follow Example 4.10.2 for the various codes. Mathcad is given.

$$I := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad O := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad M := \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2.5 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

$$K := \begin{bmatrix} 500 & -100 & 0 & 0 \\ -100 & 200 & -100 & 0 \\ 0 & -100 & 200 & -100 \\ 0 & 0 & -100 & 100 \end{bmatrix} \quad C := \begin{bmatrix} 4 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \quad X := \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.01 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$B := \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad f := M^{-1} \cdot B$$

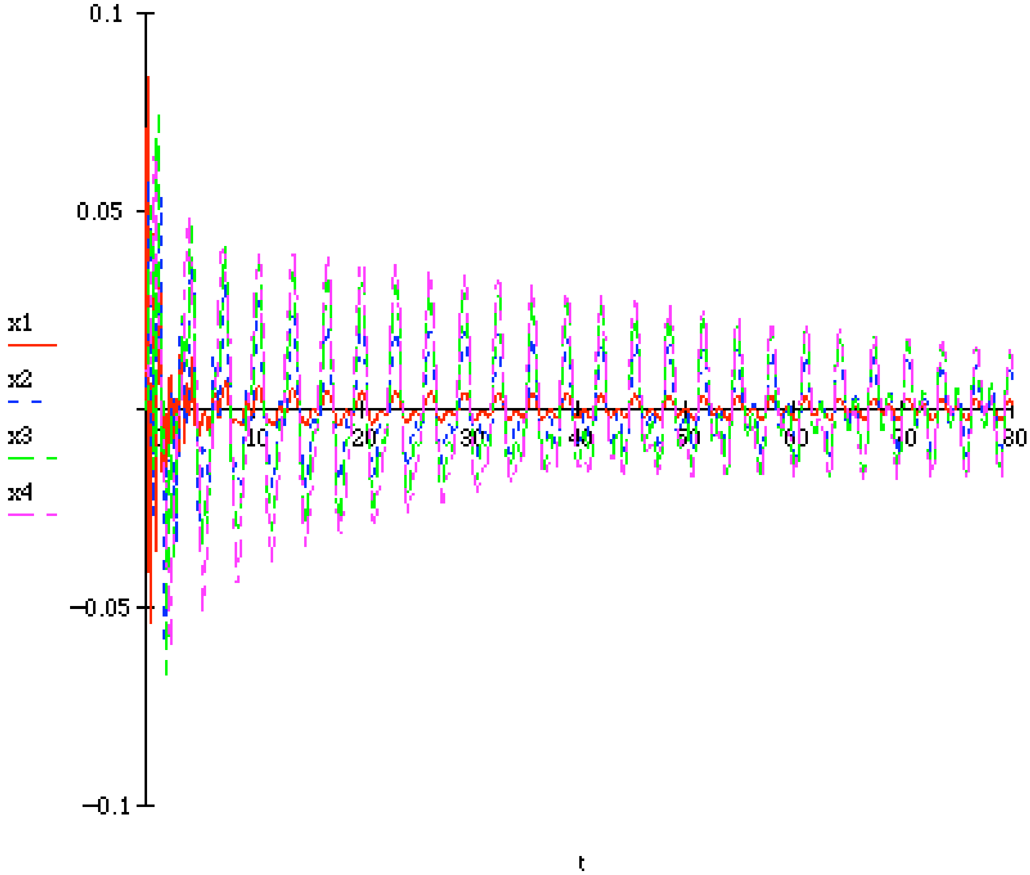
$$A := \text{augment}\left(\text{stack}\left(O, -M^{-1} \cdot K\right), \text{stack}\left(I, -M^{-1} C\right)\right)$$

$$D(t, X) := A \cdot X + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ f_0 \\ f_1 \\ f_2 \\ f_3 \end{bmatrix} \cdot \sin(4 \cdot t)$$

$$Z := \text{rkfixed}(X, 0, 200, 3000, D)$$

$$x2 := Z^{\langle 2 \rangle} \quad x3 := Z^{\langle 3 \rangle} \quad x4 := Z^{\langle 4 \rangle}$$

$$t := Z^{\langle 0 \rangle} \quad x1 := Z^{\langle 1 \rangle}$$


0.1
0.05
−0.05
−0.1
x1
x2
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x4
10
20
30
40
50
60
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80
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