Chapter Three Solutions

Problem and Solutions for Section 3.1 (3.1 through 3.14)

3.1 Calculate the solution to

$$\ddot{x} + 2\dot{x} + 2x = \delta(t - \pi)$$
$$x(0) = 1 \quad \dot{x}(0) = 0$$

and plot the response.

Solution: Given: $\ddot{x} + 2\dot{x} + 2x = \delta(t - \pi) \quad x(0) = 1, \quad \dot{x}(0) = 0$

$$\omega_n = \sqrt{\frac{k}{m}} = 1.414 \text{ rad/s}, \quad \zeta = \frac{c}{2\sqrt{km}} = 0.7071, \quad \omega_d = \omega_n\sqrt{1-\zeta^2} = 1 \text{ rad/s}$$

Total Solution: $x(t) = x_h(t) + x_p(t)$

Homogeneous: From Equation (1.36)

$$x_h(t) = Ae^{-\zeta\omega_n t}\sin(\omega_d t + \phi)$$

$$A = \sqrt{\frac{(v_0 + \zeta\omega_n x_0)^2 + (x_0\omega_d)^2}{\omega_d^2}}, \quad \phi = \tan^{-1}\left[\frac{x_0\omega_d}{v_0 + \zeta\omega_n x_0}\right] = .785 \text{ rad}$$

$$\Rightarrow x_h(t) = 1.414e^{-t}\sin(t + .785)$$

Particular: From Equation. (3.9)

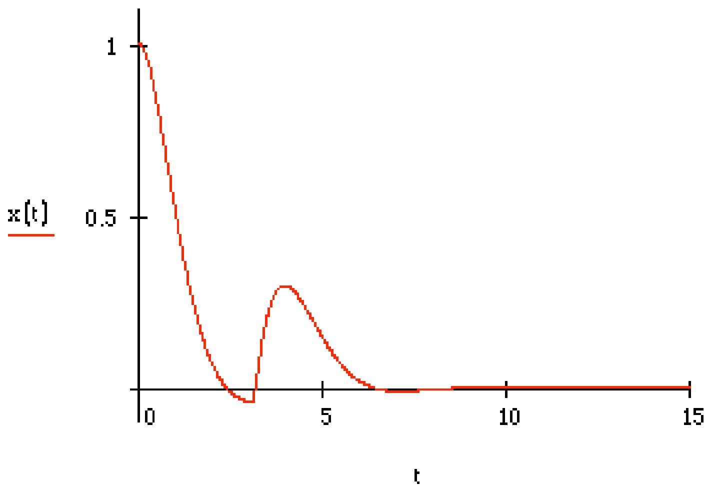
$$x_p(t) = \frac{1}{m\omega_d}e^{-\zeta\omega_n(t-\tau)}\sin\omega_d(t-\tau) = \frac{1}{(1)(1)}e^{-(t-\pi)}\sin(t-\pi)$$

But, $\sin(-t) = -\sin t$ So, $x_p(t) = -e^{-(t-\pi)}\sin t \quad \Rightarrow$

$$x(t) = 1.414e^{-t}\sin(t + 0.785) \qquad 0 < t < \pi$$
$$x(t) = 1.414e^{-t}\sin(t + 0.785) - e^{-(t-\pi)}\sin t \qquad t > \pi$$

This is plotted below using the Heaviside function.

$$x(t) := 1.414 \cdot e^{-t} \cdot \sin(t + 0.785) - e^{-(t-\pi)} \cdot \sin(t) \cdot \Phi(t - \pi)$$



3.2 Calculate the solution to

$$\ddot{x} + 2\dot{x} + 3x = \sin t + \delta(t - \pi)$$
$$x(0) = 0 \quad \dot{x}(0) = 1$$

and plot the response.

Solution: Given: $\ddot{x} + 2\dot{x} + 3x = \sin t + \delta(t - \pi), \quad x(0) = 0, \quad \dot{x}(0) = 0$

$$\omega_n = \sqrt{\frac{k}{m}} = 1.732 \text{ rad/s}, \ \zeta = \frac{c}{2\sqrt{km}} = 0.5774, \ \omega_d = \omega_n\sqrt{1-\zeta^2} = 1.414 \text{ rad/s}$$

Total Solution:

$$x(t) = x_h + x_{p1} \qquad 0 < t < \pi$$
$$x(t) = x_h + x_{p1} + x_{p2} \qquad t > \pi$$

Homogeneous: Eq. (1.36)

$$x_h(t) = Ae^{-\zeta\omega_n t}\sin(\omega_d t + \phi) = Ae^{-t}\sin(1.414t + \phi)$$

Particular: #1 (Chapter 2)

$$x_{p1}(t) = X\sin(\omega t - \theta), \text{ where } \omega = 1 \text{ rad/s . Note that } f_0 = \frac{F_0}{m} = 1$$

$$\Rightarrow X = \frac{f_0}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}} = 0.3536, \text{ and } \theta = \tan^{-1}\left[\frac{2\zeta\omega_n\omega}{\omega_n^2 - \omega^2}\right] = 0.785 \text{ rad}$$

$$\Rightarrow x_{p1}(t) = 0.3536\sin(t - 0.7854)$$

Particular: #2 Equation 3.9

$$x_{p2}(t) = \frac{1}{m\omega_d}e^{-\zeta\omega_n(t-\pi)}\sin\omega_d(t - \tau) = \frac{1}{(1)(1.414)}e^{-(t-\pi)}\sin 1.414(t - \pi)$$

$$\Rightarrow x_{p2}(t) = 0.7071e^{-(t-\pi)}\sin 1.414(t - \pi)$$

The total solution for 0< t<π becomes:

$$x(t) = Ae^{-t}\sin(1.414t + \phi) + 0.3536\sin(t - 0.7854)$$
$$\dot{x}(t) = -Ae^{-t}\sin(1.414t + \phi) + 1.414Ae^{-t}\cos(1.414t + \phi) + 0.3536\cos(t - 0.7854)$$
$$x(0) = 0 = A\sin\phi - 0.25 \Rightarrow A = \frac{0.25}{\sin\phi}$$
$$\dot{x}(0) = 1 = -A\sin\phi + 1.414A\cos\phi + 0.25 \Rightarrow 0.75 = 0.25 - 1.414(0.25)\frac{1}{\tan\phi}$$
$$\Rightarrow \phi = 0.34 \text{ and } A = 0.75$$

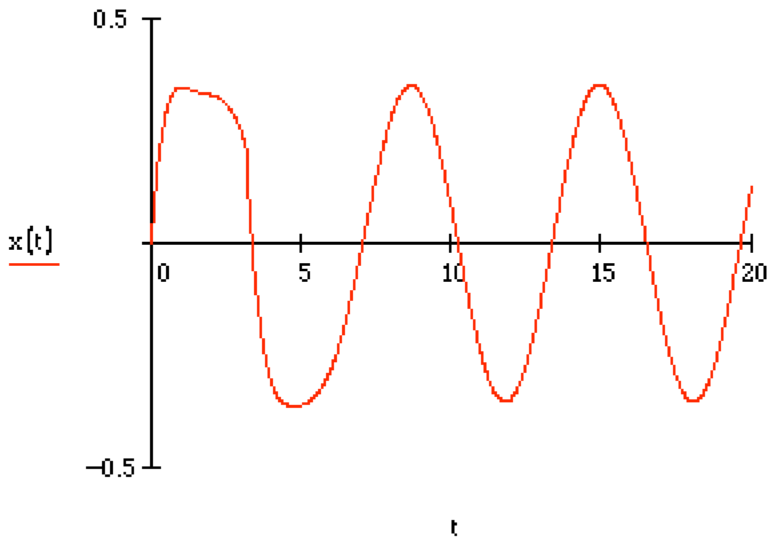
Thus for the first time interval, the response is

$$x(t) = 0.75e^{-t}\sin(1.414t + 0.34) + 0.3536\sin(t - 0.7854) \qquad 0 < t < \pi$$

Next consider the application of the impulse at t = π:

$$x(t) = x_h + x_{p1} + x_{p2}$$
$$x(t) = -0.433e^{-t}\sin(1.414t + 0.6155) + 0.3536\sin(t - 0.7854) - 0.7071e^{-(t-\pi)}\sin(1.414t - \pi) \quad t > \pi$$

The response is plotted in the following (from Mathcad):



3.3 Calculate the impulse response function for a critically damped system.

Solution:

The change in the velocity from an impulse is $v_0 = \frac{\hat{F}}{m}$, while $x_0 = 0$. So for a critically damped system, we have from Eqs. 1.45 and 1.46 with $x_0 = 0$:

$$x(t) = v_0 t e^{-\omega_n t}$$
$$\Rightarrow x(t) = \frac{\hat{F}}{m} t e^{-\omega_n t}$$

3.4 Calculate the impulse response of an overdamped system.

Solution:

The change in velocity for an impulse $v_0 = \frac{\hat{F}}{m}$, while $x_0 = 0$. So, for an overdamped system, we have from Eqs. 1.41, 1.42 and 1.43:

$$x(t) = e^{-\zeta\omega_n t}\left[\frac{-v_0}{2\omega_n\sqrt{\zeta^2-1}}e^{-\omega(_n\sqrt{\zeta^2-1})t} + \frac{v_0}{2\omega_n\sqrt{\zeta^2-1}}e^{-\omega(_n\sqrt{\zeta^2-1})t}\right]$$

$$x(t) = \frac{\hat{F}}{2m\omega_n\sqrt{\zeta^2-1}}e^{-\zeta\omega_n t}\left[e^{-\omega(_n\sqrt{\zeta^2-1})t} - e^{-\omega(_n\sqrt{\zeta^2-1})t}\right]$$

3.5 Derive equation (3.6) from equations (1.36) and (1.38).

Solution:

Equation 1.36: $x(t) = Ae^{-\zeta\omega_n t}\sin(\omega_d t + \phi)$

Equation 1.38: $A = \sqrt{\frac{(v_0 + \zeta\omega_n x_0)^2 + (x_0\omega_d)^2}{\omega_d^2}}$, $\phi = \tan^{-1}\left[\frac{x_0\omega_d}{v_0 + \zeta\omega_n x_0}\right]$

Since $x_0 = 0$ and $v_0 = \frac{\hat{F}}{m}$, Equation 1.38 becomes

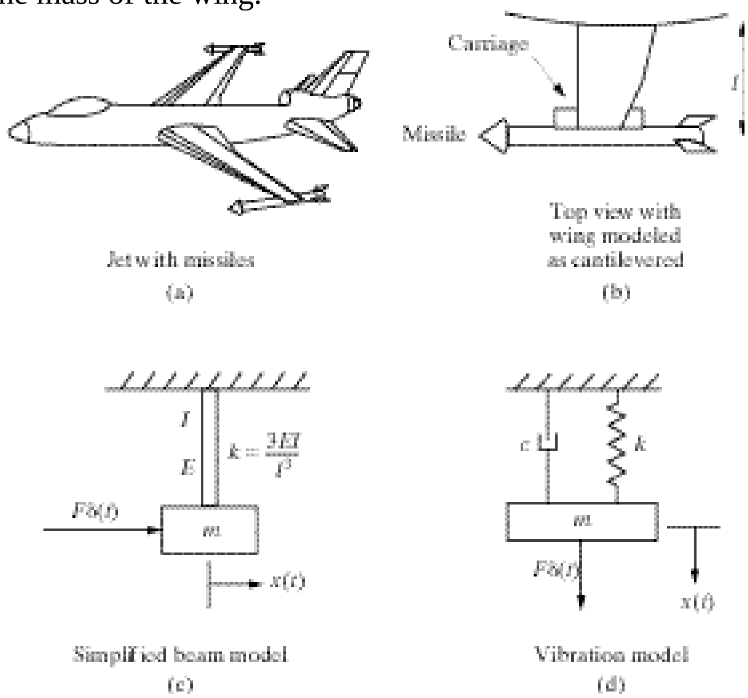
$$A = \frac{v_0}{\omega_d} = \frac{\hat{F}}{m\omega_d}$$

$$\phi = \tan^{-1}(0) = 0$$

So Equation 1.36 becomes

$$x(t) = \frac{\hat{F}}{m\omega_d}e^{-\zeta\omega_n t}\sin(\omega_d t)$$ which is Equation 3.6

3.6 Consider a simple model of an airplane wing given in Figure P3.6. The wing is approximated as vibrating back and forth in its plane, massless compared to the missile carriage system (of mass *m*). The modulus and the moment of inertia of the wing are approximated by *E* and *I*, respectively, and *l* is the length of the wing. The wing is modeled as a simple cantilever for the purpose of estimating the vibration resulting from the release of the missile, which is approximated by the impulse funciton $F\delta(t)$. Calculate the response and plot your results for the case of an aluminum wing 2 m long with $m = 1000$ kg, $\zeta = 0.01$, and $I = 0.5\ m^4$. Model *F* as 1000 N lasting over 10^{-2}s.
Modeling of wing vibration resulting from the release of a missile. (a) system of interest; (b) simplification of the detail of interest; (c) crude model of the wing: a cantilevered beam section (recall Figure 1.24); (d) vibration model used to calculate the response neglecting the mass of the wing.



Solution: Given:

$$m = 1000 \text{ kg} \quad \zeta = 0.01$$
$$l = 4 \text{ m} \quad I = 0.5 \text{ m}^4$$
$$F = 1000 \text{ N} \quad \Delta t = 10^{-2} \text{ s}$$

From Table 1.2, the modulus of Aluminum is $E = 7.1\times10^{10}$ N/m^2

The stiffness is

$$k = \frac{3EI}{\ell^3} = \frac{3\left(7.1\times10^{10}\right)\left(0.5\right)}{4^3} = 1.664\times10^9 \text{ N/m}$$

$$\omega_n = \sqrt{\frac{k}{m}} = 1.29\times10^3 \text{ rad/s (205.4 Hz)}$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 1.29\times10^3$$

Solution (Eq. 3.6):

$$x(t) = \frac{(F\Delta t)e^{-\zeta\omega_n t}}{m\omega_d}\sin\omega_d t = 7.753\times10^{-6}e^{-12.9t}\sin(1290t)\ \text{m}$$

The following m-file

```
t=(0:0.0001:0.5);
F=1000;dt=0.01;m=1000;zeta=0.01;E=7.1*10^10;I=0.5;L=4;
wn=sqrt((3*I*E/L^3)/m);
wd=wn*sqrt(1-zeta^2);
x=(F*dt/(m*wd))*exp(-zeta*wn*t).*sin(wd*t);
plot(t,x)
```

The solution worked out in Mathcad is given in the following:

$E := 7.1\cdot 10^{10}$

$m := 1000$ $\quad L := 4$ $\quad I := 0.5$ $\quad F := 1000$ $\quad \Delta t := 10^{-2}$ sec

$\zeta := 0.01$

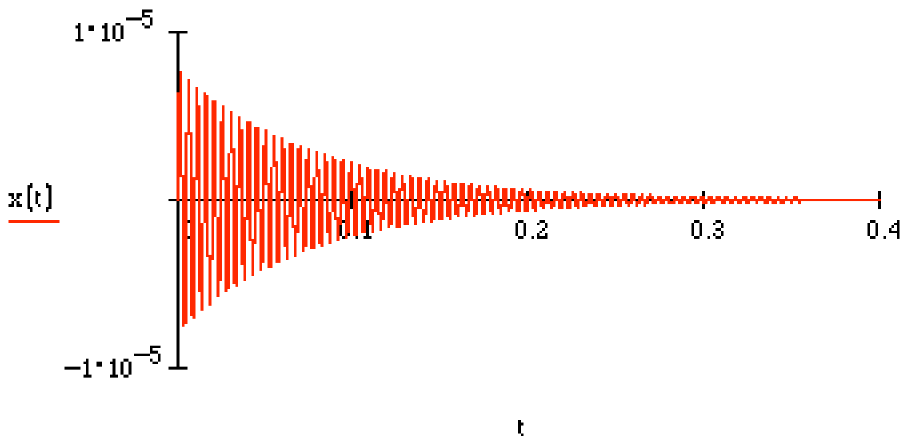
$k := \dfrac{3\cdot I\cdot E}{L^3}$ $\quad \omega n := \sqrt{\dfrac{k}{m}}$ $\quad k = 1.664\cdot 10^{9}$ $\quad \omega n = 1.29\cdot 10^{3}$

$fn := \dfrac{\omega n}{2\cdot \pi}$ $\quad fn = 205.308$ Hz $\quad \omega d := \omega n\cdot\sqrt{1-\zeta^2}$ $\quad \omega d = 1.29\cdot 10^{3}$

$\zeta\cdot\omega n = 12.9$ $\quad \dfrac{F}{m\cdot\omega d}\cdot(\Delta t) = 7.752\cdot 10^{-6}$

$t := 0, 0.00001 .. 0.4$

$x(t) := \dfrac{F}{m\cdot\omega d}\cdot\left(\Delta t\cdot e^{-\zeta\cdot\omega n\cdot t}\right)\cdot\sin(\omega d\cdot t)$ meters



3.7 A cam in a large machine can be modeled as applying a 10,000 N-force over an interval of 0.005 s. This can strike a valve that is modeled as having physical parameters: m = 10 kg, c = 18 N•s/m, and stiffness k = 9000 N/m. The cam strikes the valve once every 1 s. Calculate the vibration response, $x(t)$, of the valve once it has been impacted by the cam. The valve is considered to be closed if the distance between its rest position and its actual position is less than 0.0001 m. Is the valve closed the very next time it is hit by the cam?

Solution: Given:

$$F = 10{,}000 \text{ N} \qquad \Delta t = 0.005 \text{ s}$$

$$m = 10 \text{ kg} \qquad c = 18 \text{ N}\cdot\text{s/m} \qquad k = 9000 \text{ N/m}$$

$$\omega_n = \sqrt{\frac{k}{m}} = 30 \text{ rad/s} \quad \zeta = \frac{c}{2\sqrt{km}} = 0.03 \quad \omega_d = \omega_n\sqrt{1-\zeta^2} = 29.99 \text{ rad/s}$$

Solution Eq. (3.6):

$$x(t) = \frac{(F\Delta t)e^{-\zeta\omega_n t}}{m\omega_d}\sin\omega_d t$$

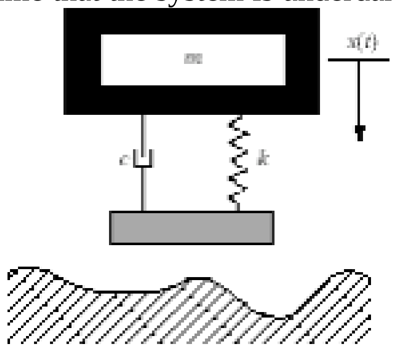
$$x(t) = \frac{(10{,}000)(0.005)e^{-(0.03)(30)t}}{(10)(29.99)}\sin(29.99t)$$

$$x(t) = 0.1667e^{-0.9t}\sin(29.99t)\text{m}$$

At t=1 s: $x(1) = 0.1667e^{-0.9}\sin(29.99) = -.06707 \text{ m}$

Since $|x(1)| = 0.06707 > 0.0001,$ the valve is not closed.

3.8 The vibration packages dropped from a height of h meters can be approximated by considering Figure P3.8 and modeling the point of contact as an impulse applied to the system at the time of contact. Calculate the vibration of the mass m after the system falls and hits the ground. Assume that the system is underdamped.



Solution: When the system hits the ground, it responds as if an impulse force acted on it.

From Equation (3.6): $x(t) = \dfrac{\hat{F}e^{-\zeta\omega_n t}}{m\omega_d}\sin\omega_d t \qquad$ where $\dfrac{\hat{F}}{m} = v_0$

Calculate v_0:

For falling mass: $\qquad x = \dfrac{1}{2}at^2$

So, $v_0 = gt^*$, where t^* is the time of impact from height h

$$h = \frac{1}{2}gt^{*2} \Rightarrow t^* = \sqrt{\frac{2h}{g}}$$

$$v_0 = \sqrt{2gh}$$

Let $t = 0$ when the end of the spring hits the ground

The response is $\qquad x(t) = \dfrac{\sqrt{2gh}}{\omega_d}e^{-\zeta\omega_n t}\sin\omega_d t$

Where ω_n, ω_d, and ζ are calculated from m, c, k. Of course the problem could be solved as a free response problem with $x_0 = 0$, $v_0 = \sqrt{2gh}$ or an impulse response with impact model as the unit velocity given.

3.9 Calculate the response of

$$3\ddot{x}(t) + 12\dot{x}(t) + 12x(t) = 3\delta(t)$$

for zero initial conditions. The units are in Newtons. Plot the response.

Solution: Dividing the equation of motion by 3 reveals;

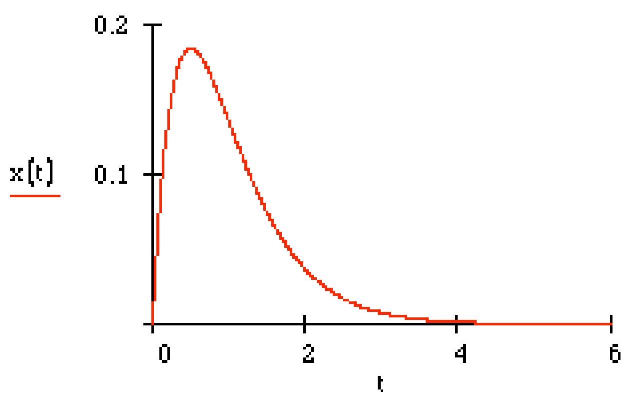
$$\omega_n = \sqrt{4} = 2 \text{ rad/s} \quad \zeta = \frac{12}{2(3)(2)} = 1 \Rightarrow \text{critically damped}$$

$$\hat{F} = 3 \qquad v_0 = \frac{F\Delta t}{m}, \qquad x_0 = 0$$

$$x = (a_1 + a_2 t)e^{-\omega_n t} \quad a_1 = 0 \quad a_2 = \frac{F\Delta t}{m}$$

$$\Rightarrow x(t) = \frac{\hat{F}}{m} t e^{-2t} = \frac{3}{3} t e^{-2t} = t e^{-2t}$$

$$x(t) := t \cdot e^{-2 \cdot t}$$



3.10 Compute the response of the system:

$$3\ddot{x}(t) + 12\dot{x}(t) + 12x(t) = 3\delta(t)$$

subject to the initial conditions $x(0) = 0.01$ m and $v(0) = 0$. The units are in Newtons. Plot the response.

Solution: From the previous problem the system is critically damped with a solution of the form

$$x(t) = (a_1 + a_2 t)e^{-2t}.$$

Applying the given initial conditions yields

$$x(0) = 0.01 = a_1 \quad \text{and} \quad \dot{x}(0) = 0 = -2(0.01 + a_2 0) + a_2$$

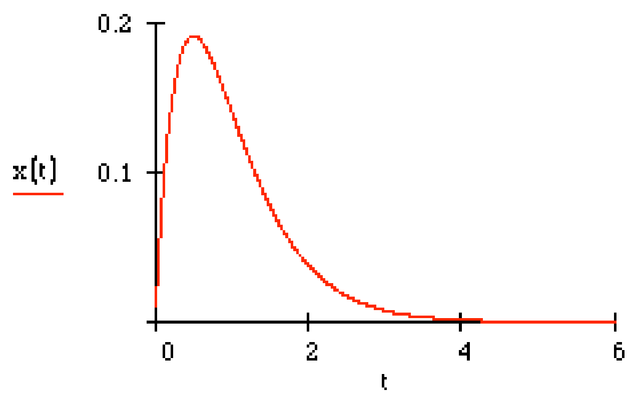
$$\Rightarrow x(t) = (0.01 + 0.02t)e^{-2t}$$

Next add to this the solution due to the unit impulse, which was calculated in Problem 3.9 to get:

$$x(t) = te^{-2t} + (0.01 + 0.02t)e^{-2t}$$

$$\Rightarrow \underline{x(t) = (0.01 + 1.02t)e^{-2t}}$$

$$x(t) := (0.01 + 1.02 \cdot t) \cdot e^{-2 \cdot t}$$



3.11 Calculate the response of the system

$$3\ddot{x}(t)+6\dot{x}(t)+12x(t)=3\delta(t)-\delta(t-1)$$

subject to the initial conditions $x(0)$ =0.01 m and $v(0)$ = 1 m/s. The units are in Newtons. Plot the response.

Solution: First compute the natural frequency and damping ratio:

$$\omega_n=\sqrt{\frac{12}{3}}=2 \text{ rad/s},\ \ \zeta=\frac{6}{2\cdot 2\cdot 3}=0.5,\ \ \omega_d=2\sqrt{1-0.5^2}=1.73 \text{ rad/s}$$

so that the system is underdamped. Next compute the responses to the two impulses:

$$x_1(t)=\frac{\hat{F}}{m\omega_d}e^{-\zeta\omega_n t}\sin\omega_d t=\frac{3}{3(1.73)}e^{-(t-1)}\sin 1.73(t-1)=0.577e^{-t}\sin 1.73t, t>0$$

$$x_2(t)=\frac{\hat{F}}{m\omega_d}e^{-\zeta\omega_n (t-1)}\sin\omega_d (t-1)=\frac{1}{3(1.73)}e^{-t}\sin 1.73t=0.193e^{-(t-1)}\sin 1.73(t-1), t>1$$

Now compute the response to the initial conditions from Equation (1.36)

$$x_h(t)=Ae^{-\zeta\omega_n t}\sin(\omega_d t+\phi)$$

$$A=\sqrt{\frac{(v_0+\zeta\omega_n x_0)^2+(x_0\omega_d)^2}{\omega_d^2}},\qquad \phi=\tan^{-1}\left[\frac{x_0\omega_d}{v_0+\zeta\omega_n x_0}\right]=0.071 \text{ rad}$$

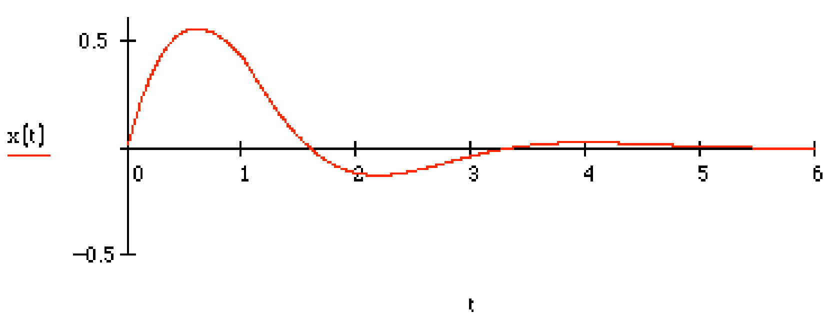
$$\Rightarrow x_h(t)=0.5775e^{-t}\sin(t+0.017)$$

Using the Heaviside function the total response is

$$x(t)=0.577e^{-t}\sin 1.73t+0.583e^{-t}\sin(t+0.017)+0.193e^{-(t-1)}\sin 1.73(t-1)\Phi(t-1)$$

This is plotted below in Mathcad:

$$x(t) := \left(\frac{e^{-\zeta\cdot \omega n\cdot t}}{\omega d}\cdot \sin(\omega d\cdot t)+A\cdot e^{-\zeta\cdot\omega n\cdot t}\cdot\sin(\omega d\cdot t+\phi)\right)+\left[\frac{e^{-\zeta\cdot\omega n\cdot(t-1)}}{-3\cdot\omega d}\cdot\sin[\omega d\cdot(t-1)]\right]\cdot\Phi(t-1)$$



Note the slight bump in the response at t = 1 when the second impact occurs.

3.12 A chassis dynamometer is used to study the unsprung mass of an automobile as illustrated in Figure P3.12 and discussed in Example 1.4.1 and again in Problem 1.64. Compute the maximum magnitude of the center of the wheel due to an impulse of 5000 N applied over 0.01 seconds. Assume the wheel mass is m = 15 kg, the spring stiffness is k = 500,000 N/m, the shock absorber provides a damping ratio of $\zeta = 0.3$, and the rotational inertia is J = 2.323 kg m^2. Compute and plot the response of the wheel system to an impulse of 5000 N over 0.01 s. Compare the undamped maximum amplitude to that of the maximum amplitude of the damped system (use r = 0.457 m).

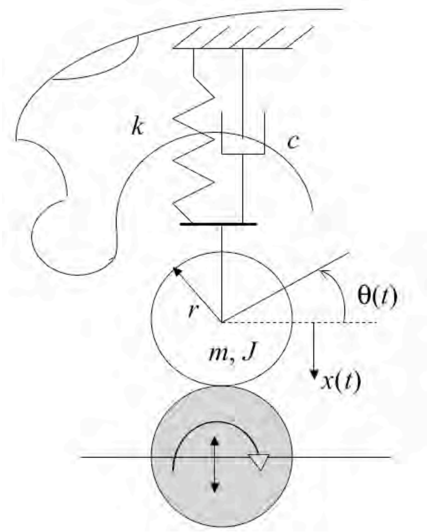


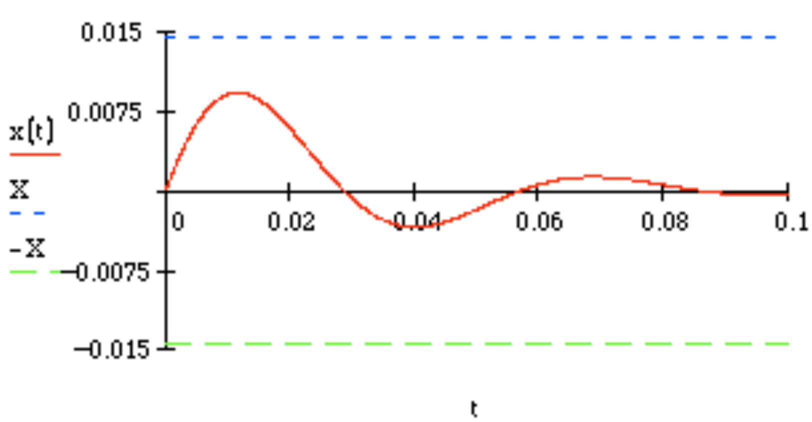
Figure P3.12 Simple model of an automobile suspension system mounted on a chassis dynamometer. The rotation of the car's wheel/tire assembly (of radius r) is given by $\theta(t)$ and is vertical deflection by $x(t)$.

Solution: With the values given the natural frequency, damped natural frequency, and impulse are calculated to be:

$$\omega_n = \sqrt{\frac{k}{m + J/r^2}} = 117.67 \text{ rad/s } = 18.73 \text{ Hz}, \;\; \omega_d = 112.25 \text{ rad/s}, \;\; X = \frac{F\Delta t}{(m + J/r^2)\omega_n} = 0.014 \text{ m}$$

The response is then plotted as

$$x(t) := \frac{F \cdot \Delta t}{\left(m + \frac{J}{r}\right) \cdot \omega n} \cdot e^{-\zeta \cdot \omega n \cdot t} \cdot \sin(\omega d \cdot t)$$



Note that the maximum amplitude of the undamped system, *X*, is not achieved.

3.13 Consider the effect of damping on the bird strike problem of Example 3.1.1. Recall from the example that the bird strike causes the camera to vibrate out of limits. Adding damping will cause the magnitude of the response to decrease but may not be able to keep the camera from vibrating past the 0.01 m limit. If the damping in the aluminum is modeled as $\zeta = 0.05$, approximately how long before the camera vibration reduces to the required limit? (Hint: plot the time response and note the value for time after which the oscillations remain below 0.01 m).

Solution: Using the values given in Example 3.1.1 and equations (3.7) and (3.8), the response has the form

$$x(t) = \frac{m_b v}{m\omega_n} e^{-\zeta\omega_n t} \sin\omega_d t = 0.026 e^{-13.07t} \sin 260.976t$$

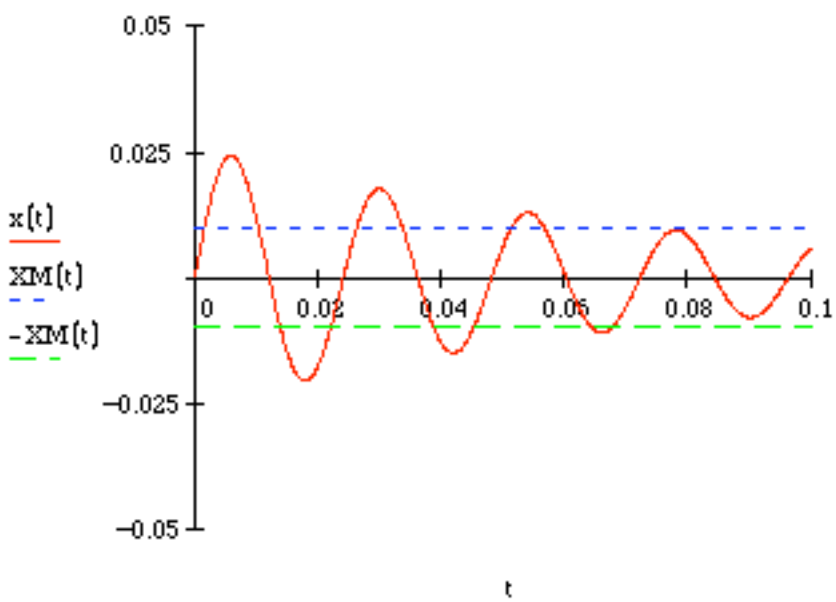
Here m_b is the mass of the bird and m is the mass of the camera. This is plotted in Mathcad below

$$X := \frac{mb \cdot v}{mc \cdot \omega n} \qquad X = 0.026 \cdot m$$

$$\zeta := 0.05 \qquad \zeta \cdot \omega n = 13.065 \cdot sec^{-1}$$

$$Y := 0.026 \qquad wn := 261.303 \qquad wd := wn \cdot \sqrt{1 - \zeta^2} \qquad wd = 260.976$$

$$x(t) := Y \cdot e^{-\zeta \cdot wn \cdot t} \cdot \sin(wd \cdot t) \qquad XM(t) := 0.01$$



From the plot, the amplitude remains below 0.01 m after about <u>0.057 s.</u>

3.14 Consider the jet engine and mount indicated in Figure P3.14 and model it as a mass on the end of a beam as done in Figure 1.24. The mass of the engine is usually fixed. Find a expression for the value of the transverse mount stiffness, *k*, as a function of the relative speed of the bird, *v*, the bird mass, the mass of the engine and the maximum displacement that the engine is allowed to vibrate.

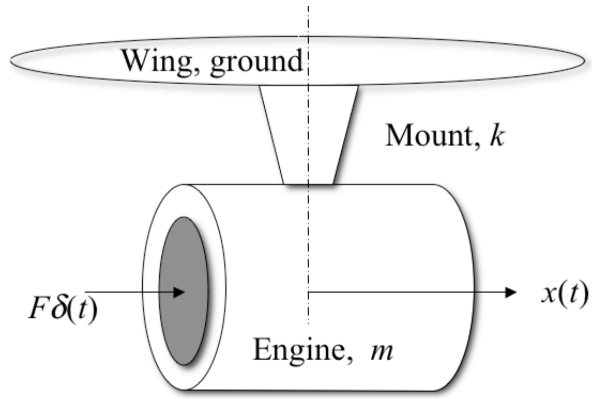


Figure P3.14 Model of a jet engine in transverse vibration due to a bird strike.

Solution: The equation of motion is

$$m\ddot{x}(t) + kx(t) = \hat{F}\delta(t)$$

From equations (3.7) and (3.8) the magnitude of the response is

$$\left|X\right| = \frac{\hat{F}}{m\omega_n}$$

for an undamped system. If the bird is moving with momentum $m_b v$ then:

$$\left|X\right| = \frac{m_b v}{m\omega_n} \Rightarrow \left|X\right| = \frac{m_b v}{\sqrt{mk}} \Rightarrow \underline{k = \frac{1}{m}\left(\frac{m_b v}{\left|X\right|}\right)^2}$$

This can be used to provide some guidance in designing the engine mount.

Problems and Solutions for Section 3.2 (3.15 through 3.25)

3.15 Calculate the response of an overdamped single-degree-of-freedom system to an arbitrary non-periodic excitation.

Solution: From Equation (3.12): $x(t) = \int_0^t F(\tau)h(t-\tau)d\tau$

For an overdamped SDOF system (see Problem 3.4)

$$h(t-\tau) = \frac{1}{2m\omega_n\sqrt{\zeta^2-1}}e^{-\zeta\omega_n(t-\tau)}\left(e^{\omega_n\sqrt{\zeta^2-1}(t-\tau)} - e^{-\omega_n\sqrt{\zeta^2-1}(t-\tau)}\right)d\tau$$

$$x(t) = \int_0^t F(\tau)\frac{1}{2m\omega_n\sqrt{\zeta^2-1}}e^{-\zeta\omega_n(t-\tau)}\left(e^{\omega_n\sqrt{\zeta^2-1}(t-\tau)} - e^{-\omega_n\sqrt{\zeta^2-1}(t-\tau)}\right)d\tau$$

$$\Rightarrow x(t) = \frac{e^{-\zeta\omega_n}}{2m\omega_n\sqrt{\zeta^2-1}}\int_0^t F(\tau)e^{\zeta\omega_n\tau}\left(e^{\omega_n\sqrt{\zeta^2-1}(t-\tau)} - e^{-\omega_n\sqrt{\zeta^2-1}(t-\tau)}\right)d\tau$$

3.16 Calculate the response of an underdamped system to the excitation given in Figure P3.16.
Plot of a pulse input of the form $f(t) = F_0 \sin t$.

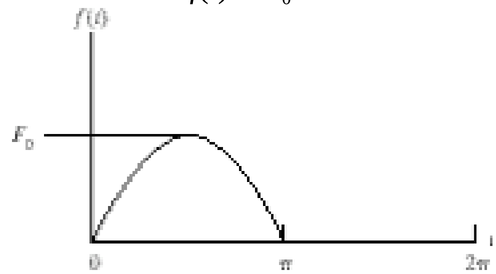


Figure P3.16

Solution:

$$x(t) = \frac{1}{m\omega_d} e^{-\zeta\omega_n t} \int_0^t \left[F(\tau) e^{\zeta\omega_n \tau} \sin\omega_d (t-\tau) \right] d\tau$$

$$F(t) = F_0 \sin(t) \qquad t < \pi \quad (\text{From Figure P3.16})$$

For $t \le \pi$, $$x(t) = \frac{F_0}{m\omega_d} e^{-\zeta\omega_n t} \int_0^t \left(\sin\tau e^{\zeta\omega_n \tau} \sin\omega_d (t-\tau) \right) d\tau$$

$$x(t) = \frac{F_0}{m\omega_d} e^{-\zeta\omega_n t} \times$$

$$\left[\frac{1}{2\left[1+2\omega_d+\omega_n^2\right]} \left\{ e^{\zeta\omega_n t} \left[(\omega_d - 1)\sin t - \zeta\omega_n \cos t \right] - (\omega_d - 1)\sin\omega_d t - \zeta\omega_n \cos\omega_d t \right\} \right.$$

$$\left. + \frac{1}{2\left[1+2\omega_d+\omega_n^2\right]} \left\{ e^{\zeta\omega_n t} \left[(\omega_d - 1)\sin t - \zeta\omega_n \cos t \right] + (\omega_d - 1)\sin\omega_d t - \zeta\omega_n \cos\omega_d t \right\} \right]$$

For $\tau > \pi,:\ \int_0^t f(\tau)h(t-\tau)d\tau = \int_0^\pi f(\tau)h(t-\tau)d\tau + \int_\pi^t (0)h(t-\tau)d\tau$

$$x(t)=\frac{F_0}{m\omega_d}e^{-\zeta\omega_n t}\int_0^{\pi}\left(\sin\tau e^{\zeta\omega_n\tau}\sin\omega_d(t-\tau)\right)d\tau$$

$$=\frac{F_0}{m\omega_d}e^{-\zeta\omega_n t}\times$$

$$\left[\frac{1}{2\left[1+2\omega_d+\omega_n{}^2\right]}\left\{\begin{matrix}e^{\zeta\omega_n t}\left[(\omega_d-1)\sin\left[\omega_d(t-\pi)\right]-\zeta\omega_n\cos\left[\omega_d(t-\pi)\right]\right]\\ -(\omega_d-1)\sin\omega_d t-\zeta\omega_n\cos\omega_d t\end{matrix}\right\}\right.$$

$$\left.+\frac{1}{2\left[1+2\omega_d+\omega_n{}^2\right]}\left\{\begin{matrix}e^{\zeta\omega_n t}\left[(\omega_d+1)\sin\left[\omega_d(t-\tau)\right]+\zeta\omega\cos\left[\omega_d(t-\pi)\right]\right]\\ +(\omega_d-1)\sin\omega_d t-\zeta\omega_n\cos\omega_d t\end{matrix}\right\}\right]$$

Alternately, one could take a Laplace Transform approach and assume the under-damped system is a mass-spring-damper system of the form

$$m\ddot{x}(t)+c\dot{x}(t)+kx(t)=F(t)$$

The forcing function given can be written as

$$F(t)=F_0\left(H(t)-H(t-\pi)\right)\sin(t)$$

Normalizing the equation of motion yields

$$\ddot{x}(t)+2\zeta\omega_n\dot{x}(t)+\omega_n^2x(t)=f_0\left(H(t)-H(t-\pi)\right)\sin(t)$$

where $f_0=\frac{F_0}{m}$ and *m*, *c* and *k* are such that $0<\zeta<1$.

Assuming initial conditions, transforming the equation of motion into the Laplace domain yields

$$X(s)=\frac{f_0\left(1+e^{-\pi s}\right)}{\left(s^2+1\right)\left(s^2+2\zeta\omega_n s+\omega_n^2\right)}$$

The above expression can be converted to partial fractions

$$X(s)=f_0\left(1+e^{-\pi s}\right)\left(\frac{As+B}{s^2+1}\right)+f_0\left(1+e^{-\pi s}\right)\left(\frac{Cs+D}{s^2+2\zeta\omega_n s+\omega_n^2}\right)$$

where *A*, *B*, *C*, and *D* are found to be

$$A = \frac{-2\zeta\omega_n}{\left(1-\omega_n^2\right)^2 + \left(2\zeta\omega_n\right)^2}$$

$$B = \frac{\omega_n^2 - 1}{\left(1-\omega_n^2\right)^2 + \left(2\zeta\omega_n\right)^2}$$

$$C = \frac{2\zeta\omega_n}{\left(1-\omega_n^2\right)^2 + \left(2\zeta\omega_n\right)^2}$$

$$D = \frac{\left(1-\omega_n^2\right) + \left(2\zeta\omega_n\right)^2}{\left(1-\omega_n^2\right)^2 + \left(2\zeta\omega_n\right)^2}$$

Notice that $X(s)$ can be written more attractively as

$$\begin{aligned} X(s) &= f_0\left(\frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+2\zeta\omega_n s+\omega_n^2}\right) + f_0 e^{-\pi s}\left(\frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+2\zeta\omega_n s+\omega_n^2}\right) \\ &= f_0\left(G(s) + e^{-\pi s}G(s)\right) \end{aligned}$$

Performing the inverse Laplace Transform yields

$$x(t) = f_0\left(g(t) + H(t-\pi)g(t-\pi)\right)$$

where $g(t)$ is given below

$$g(t) = A\cos(t) + B\sin(t) + Ce^{-\zeta\omega_n t}\cos(\omega_d t) + \left(\frac{D - C\zeta\omega_n}{\omega_d}\right)e^{-\zeta\omega_n t}\sin(\omega_d t)$$

ω_d is the damped natural frequency, $\omega_d = \omega_n\sqrt{1-\zeta^2}$.

Let m=1 kg, c=2 kg/sec, k=3 N/m, and F_0=2 N. The system is solved numerically. Both exact and numerical solutions are plotted below

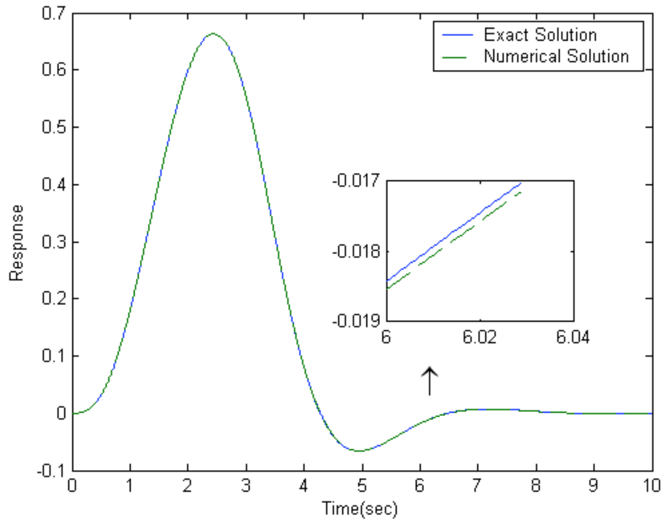


Figure 1 Analytical vs. Numerical Solutions

Below is the code used to solve this problem

```
% Establish a time vector
t=[0:0.001:10];

% Define the mass, spring stiffness and damping coefficient
m=1;
c=2;
k=3;

% Define the amplitude of the forcing function
F0=2;

% Calculate the natural frequency, damping ratio and normalized force amplitude
zeta=c/(2*sqrt(k*m));
wn=sqrt(k/m);
f0=F0/m;

% Calculate the damped natural frequency
wd=wn*sqrt(1-zeta^2);

% Below is the common denominator of A, B, C and D (partial fractions
% coefficients)
dummy=(1-wn^2)^2+(2*zeta*wn)^2;

% Hence, A, B, C, and D are given by
A=-2*zeta*wn/dummy;
B=(wn^2-1)/dummy;
C=2*zeta*wn/dummy;
```

```
D=((1-wn^2)+(2*zeta*wn)^2)/dummy;

% EXACT SOLUTION
%
***********************************************************************
*
%
***********************************************************************
*
for i=1:length(t)
   % Start by defining the function g(t)
   g(i)=A*cos(t(i))+B*sin(t(i))+C*exp(-zeta*wn*t(i))*cos(wd*t(i))+((D-
C*zeta*wn)/wd)*exp(-zeta*wn*t(i))*sin(wd*t(i));
   % Before t=pi, the response will be only g(t)
   if t(i)<pi
      xe(i)=f0*g(i);
      % d is the index of delay that will correspond to t=pi
      d=i;
   else
      % After t=pi, the response is g(t) plus a delayed g(t). The amount
      % of delay is pi seconds, and it is d increments
      xe(i)=f0*(g(i)+g(i-d));
   end;
end;

% NUMERICAL SOLUTION
%
***********************************************************************
*
%
***********************************************************************
*

% Start by defining the forcing function
for i=1:length(t)
   if t(i)<pi
      f(i)=f0*sin(t(i));
   else
      f(i)=0;
   end;
end;

% Define the transfer functions of the system
% This is given below
%        1
% --------------------------
```

```
% s^2+2*zeta*wn+wn^2

% Define the numerator and denominator
num=[1];
den=[1 2*zeta*wn wn^2];
% Establish the transfer function
sys=tf(num,den);

% Obtain the solution using lsim
xn=lsim(sys,f,t);

% Plot the results
figure;
set(gcf,'Color','White');
plot(t,xe,t,xn,'--');
xlabel('Time(sec)');
ylabel('Response');
legend('Forcing Function','Exact Solution','Numerical Solution');
text(6,0.05,'\uparrow','FontSize',18);
axes('Position',[0.55 0.3/0.8 0.25 0.25])
plot(t(6001:6030),xe(6001:6030),t(6001:6030),xn(6001:6030),'--');
```

3.17 Speed bumps are used to force drivers to slow down. Figure P3.17 is a model of a car going over a speed bump. Using the data from Example 2.4.1 and an undamped model of the suspension system (k = 4 x 10^5 N/m, m = 1007 kg), find an expression for the maximum relative deflection of the car's mass versus the velocity of the car. Model the bump as a half sine of length 40 cm and height 20 cm. Note that this is a moving base problem.

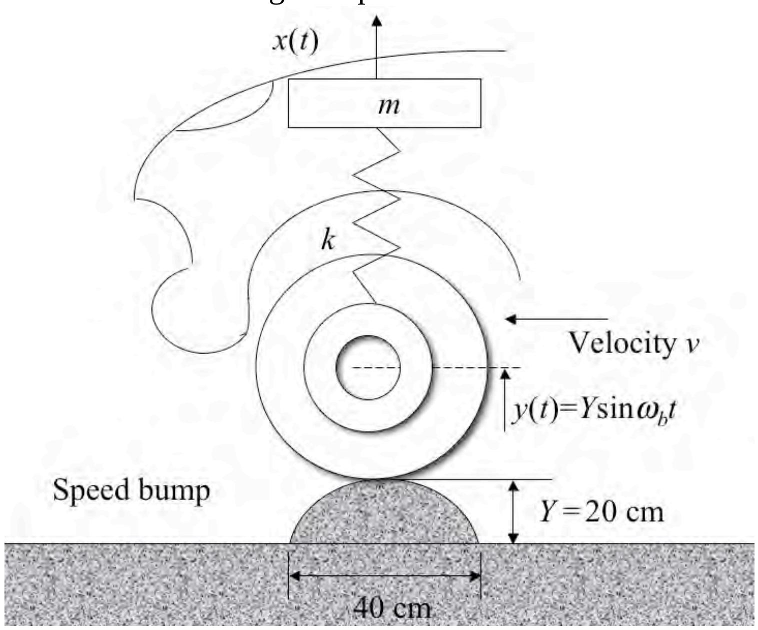


Figure P3.17 Model of a car driving over a speed bump.

Solution: This is a base motion problem, so the first step is to translate the equation of motion into a useable form. Summing forces yields in the vertical direction yields

$$m\ddot{x}(t) + k\big(x(t) - y(t)\big) = 0$$

were the displacement $y(t)$ is prescribed. Next defined the relative displacement to be $z(t)$ = x(t)-$y(t)$, the relative motion between the car's wheel and body. The equation of motion becomes:

$$m\ddot{z}(t) + m\ddot{y}(t) + kz(t) = 0 \Rightarrow m\ddot{z}(t) + kz(t) = -m\ddot{y}(t)$$

Substitution of the form of $y(t)$ into this last expression yields:

$$m\ddot{z}(t) + kz(t) = mY\omega_b^2 \sin\omega_b t\big(\Phi(t) - \Phi(t - t_1)\big)$$

where Φ is the Heavyside step function and ω_b is the frequency associated with the bump. The relationship between the bump frequency and the car's constant velocity is

$$\omega_b = \frac{2\pi}{2\ell}v = \frac{\pi}{\ell}v$$

where v is the speed of the car in m/s. For constant velocity, the time $t_1 = v\ell$, when the car finishes going over the bump.

Here, $z(t)$ is From equation (3.13) with zero damping the solution is:

$$z(t) = \frac{1}{m\omega_n}\int_0^t f(t-\tau)\sin\omega_n\tau d\tau \qquad t < t_1$$

Substitution of $f(t) = y(t)$ yields:

$$z(t) = \frac{Y\omega_b^2}{\omega_n}\int_0^t \sin(\omega_b t - \omega_b\tau)\sin\omega_n\tau d\tau =$$

$$= \frac{Y\omega_b^2}{\omega_n}\frac{1}{2}\left[\frac{\sin\big(\omega_b t - (\omega_n + \omega_b)\tau\big)}{-(\omega_n + \omega_b)} - \frac{\sin\big(\omega_b t + (\omega_n - \omega_b)\tau\big)}{\omega_n - \omega_b}\right]_0^t$$

$$= \frac{Y\omega_b^2}{\omega_n}\frac{1}{\omega_n^2 - \omega_b^2}\big(\omega_n\sin\omega_b t - \omega_b\sin\omega_n t\big) \qquad t < t_1$$

where the integral has been evaluated symbolically. Clearly a resonance situation prevails. Consider two cases, high speed $(\omega_b >> \omega_n)$ and low speed $((\omega_b << \omega_n))$ as when the two frequencies are near each other and obvious maximum occurs.

For high speed, the amplitude can be approximated as

$$\frac{Y\omega_b^2}{\omega_n}\frac{\omega_b}{\omega_n^2 - \omega_b^2}\Big((\omega_n/\omega_b)\sin\omega_b t - \sin\omega_n t\Big) \approx \frac{Y\omega_b^2}{\omega_n}\frac{\omega_b}{\omega_n^2 - \omega_b^2}\sin\omega_n t$$

For the values given, this has magnitude:

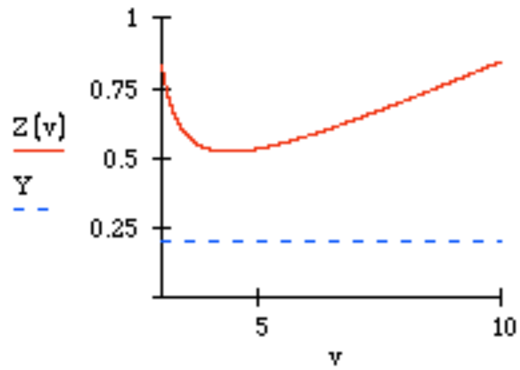
$$|Z(v)| \approx \left| \frac{Y\left(\frac{\pi}{\ell}\right)^3 v^3}{\omega_n\left(\omega_n^2 - \omega_b^2\right)} \right|$$

This increases with the cube of the velocity. Thus the faster the car is going the more sever the bump is (larger relative amplitude of vibration), hence serving to slow motorist down. A plot of magnitude versus speed shows bump size is amplified by the suspension system.

$k := 4 \cdot 10^5$ $\quad m := 1007$ $\quad \omega n := \sqrt{\frac{k}{m}}$ $\quad \omega n = 19.93$

$L := 0.4$ $\quad Y := 0.2$

$$Z(v) := \frac{Y \cdot \left(\frac{\pi}{L}\right)^2 \cdot v^2}{\omega n} \cdot \left[\frac{\left(\frac{\pi}{L}\right) \cdot v}{\left| \omega n^2 - \left(\frac{\pi}{L}\right)^2 \cdot v^2 \right|} \right]$$

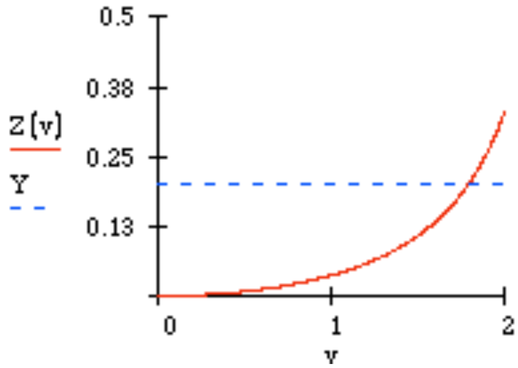


For slow speed, magnitude becomes

$$|Z(v)| \approx \left| \frac{Y\left(\frac{\pi}{\ell}\right)^2 v^2 \omega_n}{\omega_n\left(\omega_n^2 - \omega_b^2\right)} \right|$$

A plot of the approximate magnitude versus speed is given below

$$Z(v) := \frac{Y \cdot \left(\frac{\pi}{L}\right)^2 \cdot v^2}{\omega n} \cdot \frac{\omega n}{\left|\omega n^2 - \left(\frac{\pi}{L}\right)^2 \cdot v^2\right|}$$



Clearly at speeds above the designed velocity there is strong amplification of the bump's magnitude, causing discomfort to the driver and passengers, encouraging a slow speed when passing over the bumb.

3.18 Calculate and plot the response of an undamped system to a step function with a finite rise time of t_1 for the case m = 1 kg, k = 1 N/m, t_1 = 4 s and F_0 = 20 N. This function is described by

$$F(t) = \begin{cases} \dfrac{F_0 t}{t_1} & 0 \le t \le t_1 \\ F_0 & t > t_1 \end{cases}$$

Solution: Working in Mathcad to perform the integrals the solution is:

3.14

$m := 1 \quad k := 1 \quad F0 := 20 \quad \omega n := \sqrt{\frac{k}{m}}$

$$x1(t) := \frac{F0}{m\cdot\omega n\cdot 4}\cdot\left(\int_0^t \tau\cdot\sin(t-\tau)\,d\tau\right)$$

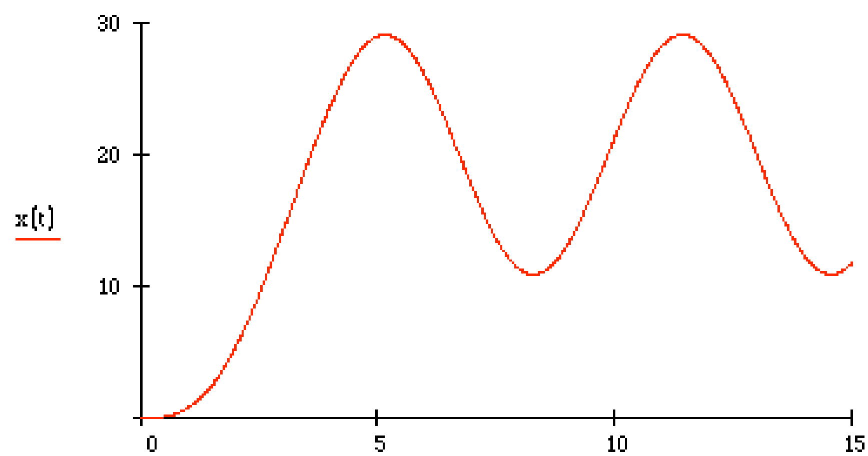
$$x1(t) := \frac{1}{4}\cdot\frac{F0}{(m\cdot\omega n)}\cdot(t-\sin(t))$$

$$x2(t) := \int_4^t \left[\frac{F0}{m\cdot\omega n}\cdot\left(1-\frac{\tau}{4}\right)\cdot\sin(t-\tau)\right]d\tau$$

$$x2(t) := \frac{-1}{4}\cdot(t-4)\cdot\frac{F0}{(m\cdot\omega n)} + \frac{1}{4}\cdot\sin(t-4)\cdot\frac{F0}{(m\cdot\omega n)}$$

$$x(t) := x1(t) + x2(t)\cdot\Phi(t-4)$$

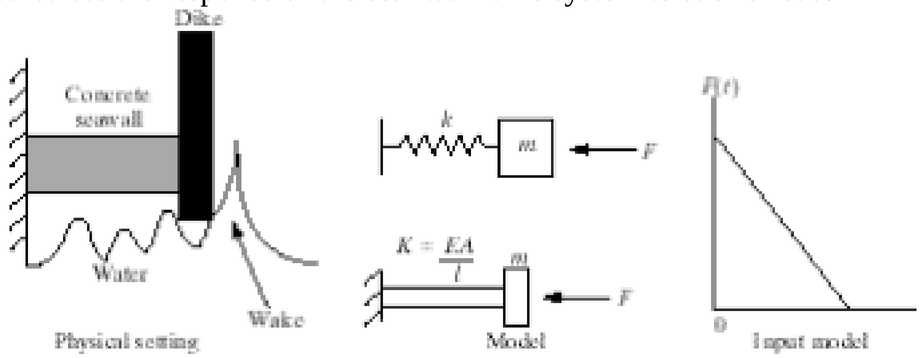
$$\frac{1}{4}\cdot\frac{F0}{(m\cdot\omega n)} = 5 \qquad \frac{F0}{(m\cdot\omega n)} = 20$$



3.19 A wave consisting of the wake from a passing boat impacts a seawall. It is desired to calculate the resulting vibration. Figure P3.19 illustrates the situation and suggests a model. This force in Figure P3.19 can be expressed as

$$F(t) = \begin{cases} F_0\left(1 - \dfrac{t}{t_0}\right) & 0 \le t \le t_0 \\ 0 & t > t_0 \end{cases}$$

Calculate the response of the seal wall-dike system to such a load.



Solution: From Equation (3.12): $x(t) = \int_0^t F(\tau)h(t-\tau)d\tau$

From Problem 3.18, $h(t-\tau) = \dfrac{1}{m\omega_n}\sin\omega_n(t-\tau)$ for an undamped system

For $t < t_0$:

$$x(t) = \frac{1}{m\omega_n}\left[\int_0^t F_0\left(1-\frac{\tau}{t_0}\right)\sin\omega_n(t-\tau)d\tau\right]$$

$$x(t) = \frac{F_0}{m\omega_n}\left[\int_0^t \sin\omega_n(t-\tau)d\tau - \frac{1}{t_0}\int_0^t \tau\sin\omega_n(t-\tau)d\tau\right]$$

After integrating and rearranging,

$$x(t) = \frac{F_0}{kt_0}\left[\frac{1}{\omega_n}\sin\omega_n t - t\right] + \frac{F_0}{k}\left[1-\cos\omega_n t\right] \quad t < t_0$$

For $t > t_0$: $\int_0^t f(\tau)h(t-\tau)d\tau = \int_0^{t_0} f(\tau)h(t-\tau)d\tau + \int_{t_0}^t (0)h(t-\tau)d\tau$

$$x(t) = \frac{1}{m\omega_n}\left[\int_0^{t_0} F_0\left(1-\frac{\tau}{t_0}\right)\sin\omega_n(t-\tau)d\tau\right]$$

$$x(t) = \frac{F_0}{m\omega_n}\left[\int_0^{t_0} \sin\omega_n(t-\tau)d\tau - \frac{1}{t_0}\int_0^{t_0} \tau\sin\omega_n(t-\tau)d\tau\right]$$

After integrating and rearranging,

$$x(t) = \frac{F_0}{kt_0\omega_n}\left[\sin\omega_n t - \sin\omega_n (t - t_0)\right] - \frac{F_0}{k}\left[\cos\omega_n t\right] \quad t > t_0$$

3.20 Determine the response of an undamped system to a ramp input of the form $F(t) = F_0 t$, where F_0 is a constant. Plot the response for three periods for the case m = 1 kg, k = 100 N/m and F_0 = 50 N.

Solution: From Eq. (3.12): $x(t) = \int_0^t F(\tau)h(t-\tau)d\tau$

From Problem 3.8, $h(t-\tau) = \frac{1}{m\omega_n}\sin\omega_n(t-\tau)$ for an undamped system.

Therefore,

$$x(t) = \frac{1}{m\omega_n}\left[\int_0^t (F_0\tau)\sin\omega_n(t-\tau)d\tau\right] = \frac{F_0}{m\omega_n}\int_0^t \tau\sin\omega_n(t-\tau)d\tau$$

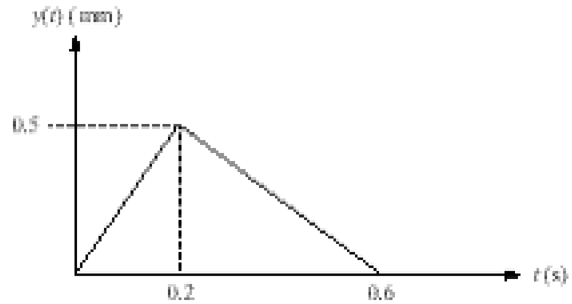
After integrating and rearranging,

$$x(t) = \frac{F_0}{m\omega_n}\left[\frac{\tau}{\omega_n} - \frac{1}{\omega_n^2}\sin\omega_n\tau\right] = \frac{F_0}{k}t - \frac{F_0}{k\omega_n}\sin\omega_n t$$

Using the values m = 1 kg, k = 100 kg, and F_0 = 50 N yields

$$x(t) = 0.5t - .05\sin(10t) \text{ m}$$

3. 21 A machine resting on an elastic support can be modeled as a single-degree-of-freedom, spring-mass system arranged in the vertical direction. The ground is subject to a motion $y(t)$ of the form illustrated in Figure P3.221. The machine has a mass of 5000 kg and the support has stiffness 1.5×10^3 N/m. Calculate the resulting vibration of the machine.



Solution: Given $m = 5000$ kg, $k = 1.5 \times 10^3$ N/m, $\omega_n = \sqrt{k/m} = 0.548$ rad/s and that the ground motion is given by:

$$y(t) = \begin{cases} 2.5t & 0 \le t \le 0.2 \\ 0.75 - 1.25t & 0.2 \le t \le 0.6 \\ 0 & t \ge 0.6 \end{cases}$$

The equation of motion is $m\ddot{x} + k(x - y) = 0$ or $m\ddot{x} + kx = ky = F(t)$ The impulse response function computed from equation (3.12) for an undamped system is

$$h(t-\tau) = \frac{1}{m\omega_n}\sin\omega_n(t-\tau)$$

This gives the solution by integrating a yh across each time step:

$$x(t) = \frac{1}{m\omega_n}\int_0^t ky(\tau)\sin\omega_n(t-\tau)d\tau = \omega_n\int_0^t y(\tau)\sin\omega_n(t-\tau)d\tau$$

For the interval $0 \le t \le 0.2$:

$$x(t) = \omega_n\int_0^t 2.5\tau\sin\omega_n(t-\tau)d\tau$$

$$\Rightarrow \underline{x(t) = 2.5t - 4.56\sin 0.548t \text{ mm } 0 \le t \le 0.2}$$

For the interval $0.2 \le t \le 0.6$:

$$x(t) = \omega_n\int_0^{0.2} 2.5\tau\sin\omega_n(t-\tau)d\tau + \omega_n\int_{0.2}^t (0.75 - 1.25\tau)\sin\omega_n(t-\tau)d\tau$$

$$= 0.75 - 0.5\cos 0.548(t-0.2) - 1.25t + 2.28\sin 0.548(t-0.2)$$

Combining this with the solution from the first interval yields:

$$\underline{x(t) = 0.75 + 1.25t - 0.5\cos 0.548(t-0.2)}$$

$$\underline{+6.48\sin 0.548(t-0.2) - 4.56\sin 0.548(t-0.2) \text{ mm } 0.2 \le t \le 0.6}$$

Finally for the interval $t \ge 0.6$:

$$x(t) = \omega_n \int_0^{0.2} 2.5t \sin\omega_n(t-\tau)d\tau + \omega_n \int_{0.2}^{0.6} (0.75-1.25t)\sin\omega_n(t-\tau)d\tau + \omega_n \int_0^{t} (0)\sin\omega_n(t-\tau)d\tau$$

$$= -0.5\cos 0.548(t-0.2) - 2.28\sin 0.548(t-0.6) + 2.28\sin 0.548(t-0.2)$$

Combining this with the total solution from the previous time interval yields:

$$x(t) = -0.5\cos 0.548(t-0.2) + 6.84\sin 0.548(t-0.2) - 2.28\sin 0.548(t-0.6)$$
$$- 4.56\sin 0.548t \quad \text{mm } t \geq 0.6$$

3.22 Consider the step response described in Figure 3.7. Calculate t_p by noting that it occurs at the first peak, or critical point, of the curve.

Solution: Assume $t_0 = 0$. The response is given by Eq. (3.17):

$$x(t) = \frac{F_0}{k} - \frac{F_0}{k\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\cos(\omega_d t - \phi)$$

To find t_p, compute the derivative and let $\dot{x}(t) = 0$

$$\dot{x}(t) = \frac{-F_0}{k\sqrt{1-\zeta^2}}\left[-\zeta\omega_n e^{-\zeta\omega_n t}\cos(\omega_d t - \phi) + e^{-\zeta\omega_n t}(-\omega_d)\sin(\omega_d t - \phi)\right] = 0$$

$$\Rightarrow -\zeta\omega_n \cos(\omega_d t - \phi) - \omega_d \sin(\omega_d t - \phi) = 0$$

$$\Rightarrow \tan(\omega_d t - \phi) = \frac{-\zeta\omega_n}{\omega_d}$$

$\omega_d t - \phi - \pi = \tan^{-1}\left(\frac{-\zeta\omega_n}{\omega_d}\right)$ (π can be added or subtracted without changing the tangent of an angle)

$$t = \frac{1}{\omega_d}\left[\pi + \phi + \tan^{-1}\left(\frac{-\zeta\omega_n}{\omega_d}\right)\right]$$

But, $\phi = \tan^{-1}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right)$

So,

$$t = \frac{1}{\omega_d}\left[\pi + \tan^{-1}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right) - \tan^{-1}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right)\right]$$

$$t_p = \frac{\pi}{\omega_d}$$

3.23 Calculate the value of the overshoot (o.s.), for the system of Figure P3.7.

Solution:

The overshoot occurs at $t_p = \dfrac{\pi}{\omega_d}$

Substitute into Eq. (3.17):

$$x\left(t_p\right) = \frac{F_0}{k} - \frac{F_0}{k\sqrt{1-\zeta^2}} e^{-\zeta\omega_n\pi/\omega_d} \cos\left[\omega_d\left(\frac{\pi}{\omega_d}\right) - \theta\right]$$

The overshoot is

$$o.s. = x\left(t_p\right) - x_{ss}\left(t\right)$$

$$o.s. = \frac{F_0}{k} - \frac{F_0}{k\sqrt{1-\zeta^2}} e^{-\zeta\omega_n\pi/\omega_d}\left(-\cos\theta\right) - \frac{F_0}{k}$$

Since $\theta = \tan^{-1}\left(\dfrac{\zeta}{\sqrt{1-\zeta^2}}\right)$, then $\cos\theta = \sqrt{1\text{-}\zeta^2}$

$$o.s. = -\frac{F_0}{k\sqrt{1-\zeta^2}}\left(e^{-\zeta\omega_n\pi/\omega_d}\right)\left(\sqrt{1-\zeta^2}\right)$$

$$o.s. = \frac{F_0}{k} e^{-\zeta\omega_n\pi/\omega_d}$$

3.24 It is desired to design a system so that its step response has a settling time of 3 s and a time to peak of 1 s. Calculate the appropriate natural frequency and damping ratio to use in the design.

Solution:

Given $t_s = 3\text{s}, t_p = 1\text{s}$

Settling time:

$$t_s = \frac{3.5}{\zeta\omega_n} = 3 \text{ s } \Rightarrow \zeta\omega_n = \frac{3.5}{3} = 1.1667 \text{ rad/s}$$

Peak time:

$$t_p = \frac{\pi}{\omega_d} = 1 \text{ s} \quad \Rightarrow \omega_d = \omega_n\sqrt{1-\zeta^2} = \pi \text{ rad/s}$$

$$\Rightarrow \omega_n\sqrt{1-\left(\frac{1.1667}{\omega_n}\right)^2} = \pi \Rightarrow \omega_n^{\,2}\left[1-\left(\frac{1.1667}{\omega_n}\right)^2\right] = \pi^2$$

$$\Rightarrow \omega_n^2\left[1-\frac{1.3611}{\omega_n^2}\right] = \pi^2 \Rightarrow \omega_n^2 - 1.311 = \pi^2 \Rightarrow \underline{\omega_n = 3.35 \text{ rad/s}}$$

Next use the settling time relationship to get the damping ratio:

$$\zeta = \frac{1.1667}{\omega_n} = \frac{1.1667}{3.35} \Rightarrow \underline{\boldsymbol{\zeta = 0.3483}}$$

3.25 Plot the response of a spring-mass-damper system for this input of Figure 3.8 for the case that the pulse width is the natural period of the system (i.e., $t_1 = \pi/\omega_n$).

Solution:

The values from Figure 3.7 will be used to plot the response.

$$F_0 = 30 \text{ N}$$
$$k = 1000 \text{ N/m}$$
$$\zeta = 0.1$$
$$\omega = 3.16 \text{ rad/s}$$

From example 3.2.2 and Figure 3.7, with $t_1 = \dfrac{\pi}{\omega}$ we have for $t = 0$ to t_1,

$$x(t) = \frac{F_0}{k} - \frac{F_0 e^{-\zeta\omega_n t}}{k\sqrt{1-\zeta^2}}\cos(\omega_d t - \phi) \qquad \text{where } \phi = \tan^{-1}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right)$$

$x(t) = .03 - .03015e^{-.316t}\cos(3.144t - .1002) \qquad 0 < t \leq t_1$

For $t > t_1$,

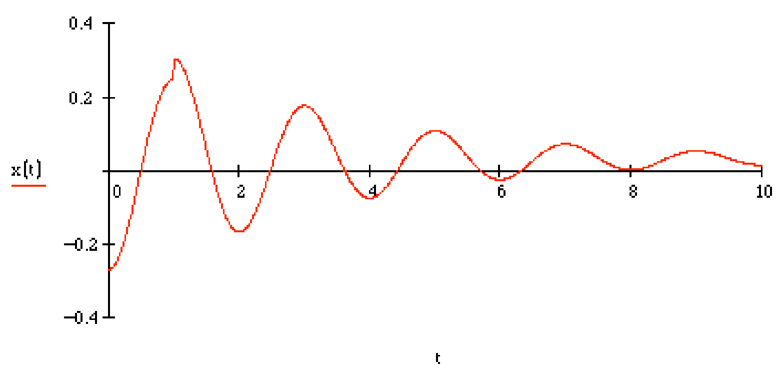
$$x(t) = \frac{F_0 e^{-\zeta\omega_n t}}{k\sqrt{1-\zeta^2}}\left\{e^{\zeta\omega_n t_1}\cos\left[\omega_d\left(t - \frac{\pi}{\omega_n}\right) - \phi\right] - \cos(\omega_d t - \phi)\right\}$$

$x(t) = 0.0315e^{-.316t}\{1.3691\cos(3.144t - 3.026) - \cos(3.144t - .1002)\}\ t > t_1$

The plot in Mathcad follows:

$\omega := 3.144$

$x(t) := 0.03 - 0.301 \cdot e^{-.316 \cdot t} \cdot \cos(\omega \cdot t - .1002) + \left[(0.0315) \cdot e^{.316 \cdot t} \cdot \left(1.3691 \cdot \cos(\omega \cdot t - 3.226) - \cos(\omega \cdot t - .1002)\right)\right] \cdot \Phi\left(t - \frac{\pi}{\omega}\right)$



Problems and Solutions Section 3.3 (problems 3.26-3.32)

3.26 Derive equations (3.24). (3.25) and (3.26) and hence verify the equations for the Fourier coefficient given by equations (3.21), (3.22) and (3.23).

Solution: For $n \neq m$, integration yields:

$$\int_0^T \sin n\omega_T t \sin m\omega_T t dt = \left[\frac{\sin(n-m)\omega_T t}{\omega_T 2(n-m)} - \frac{\sin(n+m)\omega_T t}{\omega_T 2(n+m)}\right]\Bigg|_0^T$$
$$= \frac{\sin\left[(n-m)\left(\frac{2\pi}{T}\right)T\right]}{2(n-m)\omega_T} - \frac{\sin\left[(n+m)\left(\frac{2\pi}{T}\right)T\right]}{2(n+m)\omega_T}$$
$$= \frac{\sin\left[(n-m)(2\pi)\right]}{2(n-m)\omega_T} - \frac{\sin\left[(n+m)(2\pi)\right]}{2(n+m)\omega_T} = 0$$

Since m and n are integers, the sine terms are 0, so this is equal to 0.

Equation (3.24), for $m = n$:

$$\int_0^T \sin^2 n\omega_T t dt = \left[\frac{1}{2}t - \frac{1}{4n\omega_T}\sin(2n\omega_T t)\right]\Bigg|_0^T = \frac{T}{2} - \frac{T}{8n\pi}\sin\left[2\pi\left(\frac{2\pi}{T}\right)T\right]$$
$$= \frac{T}{2} - \frac{T}{8n\pi}\sin\left[4n\pi\right] = \frac{T}{2}$$

Since n is an integer, the sine term is 0, so this is equal to $T/2$.

So, $$\int_0^T \sin n\omega_T t \sin m\omega_T t dt = \begin{cases} 0 & m \neq n \\ T/2 & m = n \end{cases}$$

Equation (3.25), for $m \neq n$

$$\int_0^T \cos n\omega_T t \cos m\omega_T t dt = \left[\frac{\sin(n-m)\omega_T t}{2(n-m)\omega_T} - \frac{\sin(n+m)\omega_T t}{2(\eta+m)\omega_T}\right]\Bigg|_0^T$$

$$= \frac{\sin\left[(n-m)\left(\frac{2\pi}{T}\right)T\right]}{2(n-m)\omega_T} - \frac{\sin\left[(n+m)\left(\frac{2\pi}{T}\right)T\right]}{2(n+m)\omega_T}$$

$$= \frac{\sin\left[(n-m)(2\pi)\right]}{2(n-m)\omega_T} - \frac{\sin\left[(n+m)(2\pi)\right]}{2(n+m)\omega_T} = 0$$

Since *m* and *n* are integers, the sine terms are 0, so this is equal to 0.

Equation (3.25), for *m* = *n* becomes:

$$\int_0^T \cos^2 n\omega_T t dt = \left[\frac{1}{2}t + \frac{1}{4n\omega_T}\sin(2n\omega_T t)\right]\Bigg|_0^T = \frac{T}{2} + \frac{T}{8n\pi}\sin\left[2n\left(\frac{2\pi}{T}\right)T\right]$$

$$= \frac{T}{2} + \frac{T}{8n\pi}\sin\left[4n\pi\right] = \frac{T}{2}$$

Since *n* is an integer, the sine term is 0, so this is equal to *T*/2.

So, $\int_0^T \cos n\omega_T t \cos m\omega_T t dt = \begin{cases} 0 & m \neq n \\ T/2 & m = n \end{cases}$

Equation (3.26), for $m \neq n$:

$$\int_0^T \cos n\omega_T t \sin m\omega_T t dt = \left[\frac{\cos(n-m)\omega_T t}{2\omega_T(n-m)} - \frac{\cos(n+m)\omega_T t}{2\omega_T(n+m)}\right]\Bigg|_0^T$$

$$= \frac{\cos\left[(n-m)\left(\frac{2\pi}{T}\right)T\right]}{2(n-m)\omega_T} - \frac{\cos\left[(n+m)\left(\frac{2\pi}{T}\right)T\right]}{2(n+m)\omega_T} - \frac{1}{2(m-n)\omega_T} + \frac{1}{2(m+n)\omega_T}$$

$$= \frac{\cos\left[(n-m)(2\pi)\right]}{2(n-m)\omega_T} - \frac{\cos\left[(n+m)(2\pi)\right]}{2(n+m)\omega_T} - \frac{1}{2(m-n)\omega_T} + \frac{1}{2(m+n)\omega_T} = 0$$

Since *n* is an integer, the cosine term is 1, so this is equal to 0.

So, $\int_0^T \cos n\omega_T t \sin m\omega_T t dt = 0$

Equation (3.26) for $n = m$ becomes:

$$\int_0^T \cos n\omega_T t \sin n\omega_T t dt = \left[\frac{1}{2n\omega_T}\sin^2 n\omega_T t\right]_0^T = \frac{T}{4n\pi}\sin^2 2\pi n = 0$$

$$\text{Thus } \int_0^T \cos n\omega_T t \sin n\omega_T t dt = 0$$

3.27 Calculate b_n from Example 3.3.1 and show that $b_n = 0$, $n = 1,2,\ldots,\infty$ for the triangular force of Figure 3.12. Also verify the expression a_n by completing the integration indicated. (*Hint*: Change the variable of integration from t to $x = 2\pi nt/T$.)

Solution: From Equation (3.23), $b_n = \dfrac{2}{T}\int_0^T F(t)\sin n\omega_T t dt$. Computing the integral yields:

$$b_n = \frac{2}{T}\left[\int_0^{T/2}\left(\frac{4}{T}t-1\right)\sin n\omega_T t dt + \int_{T/2}^{T}\left[1-\frac{4}{T}\left(t-\frac{T}{2}\right)\right]\sin n\omega_T t dt\right]$$

$$b_n = \frac{2}{T}\left[\frac{4}{T}\int_0^{T/2} t\sin n\omega_T t dt - \int_0^{T/2}\sin n\omega_T t dt + 3\int_{T/2}^{T}\sin n\omega_T t dt - \frac{4}{T}\int_{T/2}^{T} t\sin n\omega_T t dt\right]$$

Substitute $x = n\omega_T t = \dfrac{2\pi n}{T}t$

$$b_n = \frac{1}{\pi n}\left[\frac{2}{\pi n}\int_0^{\pi n} x\sin x dx - \int_0^{\pi n}\sin x dx + 3\int_{\pi n}^{2\pi n}\sin x dx - \frac{2}{\pi n}\int_{\pi n}^{2\pi n} x\sin x dx\right]$$

$$= \frac{1}{\pi n}\left[\frac{2}{\pi n}\left(\sin x - x\cos x\right)\Big|_0^{\pi n} + \cos x\Big|_0^{\pi n} - 3\cos x\Big|_{\pi n}^{2\pi n} - \frac{2}{\pi n}\left(\sin x - x\cos x\right)\Big|_{\pi n}^{2\pi n}\right]$$

$$= \frac{1}{\pi n}\left[\frac{2}{\pi n}\left(-\pi n\cos\pi n\right) + \cos\pi n - 1 - 3 + 3\cos\pi n - \frac{2}{\pi n}\left(-2\pi n + \pi n\cos\pi n\right)\right]$$

$$= \frac{1}{\pi n}\left[-2\cos\pi n + 4\cos\pi n - 4 + 4 - 2\cos\pi n\right] = \frac{1}{\pi n}\left[0\right] = 0$$

From equation (3.22), $a_n = \dfrac{2}{T}\int_0^T F(t)\cos n\omega_T t dt$

$$a_n = \frac{2}{T}\left[\int_0^{T/2}\left(\frac{4}{T}t-1\right)\cos n\omega_T t dt + \int_{T/2}^{T}\left[1-\frac{4}{T}\left(t-\frac{T}{2}\right)\right]\cos n\omega_T t dt\right]$$

$$a_n = \frac{2}{T}\left[\frac{4}{T}\int_0^{T/2} t\cos n\omega_T t dt - \int_0^{T/2}\cos n\omega_T t dt + 3\int_{T/2}^{T}\cos n\omega_T t dt - \frac{4}{T}\int_{T/2}^{T} t\cos n\omega_T t dt\right]$$

Substitute $x = n\omega_T t = \dfrac{2\pi n}{T}t$

$$a_n = \frac{1}{\pi n}\left[\frac{2}{\pi n}\int_0^{\pi n} x\cos x dx - \int_0^{\pi n} \cos x dx + 3\int_{\pi n}^{2\pi n} \cos x dx - \frac{2}{\pi n}\int_{\pi n}^{2\pi n} x\cos x dx\right]$$

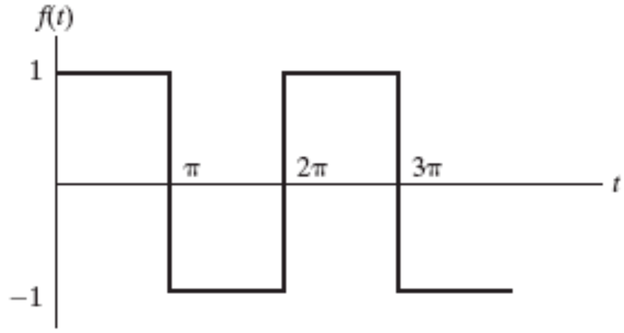
$$= \frac{1}{\pi n}\left[\frac{2}{\pi n}(\cos x + x\sin x)\Big| - \sin x\Big|_0^{\pi n} + 3\sin x\Big|_{\pi n}^{2\pi n} - \frac{2}{\pi n}(\cos x - \sin x)\Big|_{\pi n}^{2\pi n}\right]$$

$$= \frac{1}{\pi n}\left[\frac{2}{\pi n}(\cos \pi n - 1) - \frac{2}{\pi n}(1 - \cos \pi n)\right]$$

$$= \frac{2}{\pi^2 n^2}\left[\cos \pi n - 1 - 1 + \cos \pi n\right]$$

$$= \frac{4}{\pi^2 n^2}\left[\cos \pi n - 1\right] = \begin{cases} 0 & n \text{ even} \\ \dfrac{-8}{\pi^2 n^2} & n \text{ odd} \end{cases}$$

3.28 Determine the Fourier series for the rectangular wave illustrated in Figure P3.28.



Solution: The square wave of period T is described by

$$F(t) = \begin{cases} 1 & 0 \le t \le \pi \\ -1 & \pi \le t \le 2\pi \end{cases}$$

Determine the coefficients a_0, a_n, b_n from direct integration:

$$a_0 = \frac{2}{T}\int_0^T F(t)\,dt$$

$$= \frac{2}{2\pi}\left[\int_0^{\pi} (1)\,dt + \int_{\pi}^{2\pi} (-1)\,dt\right]$$

$$= \frac{1}{\pi}\left[t\Big|_0^{\pi} - t\Big|_{\pi}^{2\pi}\,dt\right]$$

$$= \frac{1}{\pi}\left[\pi - 2\pi + \pi\right] = \frac{1}{\pi}(0) \qquad \Rightarrow a_0 = 0$$

$$a_n = \frac{2}{T}\int_0^T F(t)\cos n\omega_T t\,dt, \text{ where } \omega_T = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$$

$$= \frac{2}{2\pi}\left[\int_0^{\pi} \cos nt\,dt - \int_{\pi}^{2\pi} \cos nt\,dt\right] = \frac{1}{\pi}\left[\frac{1}{n}\sin nt\ \Big|_0^{\pi} - \frac{1}{n}\sin nt\ \Big|_{\pi}^{2\pi}\right]$$

$$= \frac{1}{\pi n}\left[\sin(n\pi) - \sin(n2\pi) + \sin(n\pi)\right] = 0$$

$$b_n = \frac{2}{T}\int_0^T F(t)\sin\omega_T t\,dt = \frac{2}{2\pi}\left[\int_0^{\pi} \sin nt\,dt - \int_{\pi}^{2\pi} \sin nt\,dt\right]$$

$$= \frac{1}{\pi}\left[\frac{-1}{n}\cos nt\ \Big|_0^{\pi} - \frac{1}{n}\cos nt\ \Big|_{\pi}^{2\pi}\right] = \frac{1}{\pi n}\left[-\cos n\pi + 1 - 1 - \cos n\pi\right] = \frac{2}{\pi n}\left[1 - \cos n\pi\right]$$

If n is even, $\cos n\pi = 1$. If n is odd, $\cos n\pi = -1$

$$\text{So, } b_n = \begin{cases} 0 & n \text{ even} \\ \dfrac{4}{\pi n} & n \text{ odd} \end{cases}$$

Thus the Fourier Series collapses to a sine series of the form

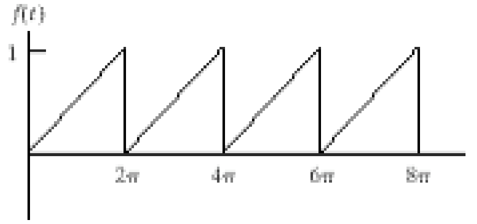
$$F(t) = \sum_{n=1}^{\infty} b_n \sin nt = \sum_{n=1,3,\cdots}^{\infty} \frac{4}{n\pi} \sin nt$$

The Vibration Toolbox can also be used:

```
t=0:pi/100:2*pi-pi/100;
f=-2*floor(t/pi)+1;
vtb3_3(f',t',100)
[a,b]=vtb3_3(f',t',100)
```

Note that vtb3_3 always gives some error on the order of delta t (.01 in this case). Using a smaller delta t reduced the error.

3.29 Determine the Fourier series representation of the sawtooth curve illustrated in Figure P3.29.



Solution: The sawtooth curve of period T is

$$F(t) = \frac{1}{2\pi}t \qquad 0 \le t \le 2\pi$$

Determine coefficients a_0, a_n, b_n :

$$a_0 = \frac{2}{T}\int_0^T F(t)dt = \frac{2}{2\pi}\int_0^{2\pi}\left(\frac{1}{2\pi}t\right)dt = \left(\frac{1}{2\pi^2}\right)\frac{1}{2}t^2\Big|_0^{2\pi}$$
$$= \frac{1}{4\pi^2}\left[4\pi^2 - 0\right] = 1$$

$$a_n = \frac{2}{T}\int_0^T F(t)\cos n\omega_T t dt, \text{ where } \omega_T = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$$
$$= \frac{2}{2\pi}\left[\int_0^{2\pi}\left(\frac{1}{2\pi}t\right)\cos nt dt\right] = \frac{1}{2\pi^2}\left[\int_0^{2\pi} t\cos nt dt\right]$$
$$= \frac{1}{2\pi^2}\left[\frac{1}{n^2}\cos nt + \frac{1}{n}t\sin nt\right]\Bigg|_0^{2\pi} = \frac{1}{2\pi^2}\left[\frac{1}{n^2}(1-1) + \frac{1}{n}(0-0)\right] = 0$$

$$b_n = \frac{2}{T}\int_0^T F(t)\sin n\omega_T t dt = \frac{2}{2\pi}\left[\int_0^{2\pi}\left(\frac{1}{2\pi}t\right)\sin nt dt\right] = \frac{1}{2\pi^2}\left[\int_0^{2\pi} t\sin nt dt\right]$$
$$= \frac{1}{2\pi^2}\left[\frac{1}{n^2}\sin nt - \frac{1}{n}t\cos nt\right]\Bigg|_0^{2\pi} = \frac{1}{2\pi^2}\left[\frac{1}{n^2}(0-0) - \frac{1}{n}(2\pi - 0)\right]$$
$$= \frac{1}{2\pi^2}\left(\frac{-2\pi}{n}\right) = \frac{-1}{\pi n}$$

Fourier Series

$$F(t) = \frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{-1}{\pi n}\right)\sin nt$$
$$F(t) = \frac{1}{2} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin nt$$

3.30 Calculate and plot the response of the base excitation problem with base motion specified by the velocity

$$\dot{y}(t) = 3e^{-t/2}\Phi(t) \text{ m/s}$$

where $\Phi(t)$ is the unit step function and m = 10 kg, ζ = 0.01, and k = 1000 N/m. Assume that the initial conditions are both zero.

Solution: Given:

$$\dot{y}(t) = 3e^{-t/2}\mu(t) \text{ m/s}$$
$$m = 10 \text{ kg}, \ \zeta = 0.01, \ k = 1000 \text{ N/m}$$
$$x(0) = \dot{x}(0) = 0$$

From Equation (2.61):

$$m\ddot{x} + c(\dot{x} - \dot{y}) + k(x - y) = 0$$
$$m\ddot{x} + c\ddot{x} + kx = c\dot{y} + ky$$

Integrate by parts to find $y(t)$:

$$y(t) = \int \ \dot{y}(t)dt = 3e^{-t/2}\mu(t)dt$$

Let

$$u = \mu(t) \qquad dv = 3e^{-t/2}dt$$
$$du = \delta(t)dt \qquad v = -6e^{-t/2}$$

When

$$t > 0, \mu(t) = 1, \ \text{so} \ y(t) = 6\left(1 - e^{-1/2}\right)$$

So, $m\ddot{x} + c\dot{x} + kx = c\left(3e^{-t/2}\right) + 6k\left(1 - e^{t/2}\right)$

Since $c = 2\zeta\sqrt{km} = 2$ kg/s,

$$10\ddot{x} + 2\dot{x} + 1000x = 6000 - 5994e^{-t/2}$$

The solution is given by equation (3.13):

$$x(t) = \frac{1}{m\omega_d} e^{-\zeta\omega_n t} \int_0^t \left[F(\tau) e^{\zeta\omega_n \tau} \sin\omega_d (t-\tau) \right] d\tau$$

$$\omega_n = \sqrt{\frac{k}{m}} = 10 \text{ rad/s}$$

$$\omega_\text{d} = \omega_n \sqrt{1-\zeta^2} = 10 \text{ rad/s}$$

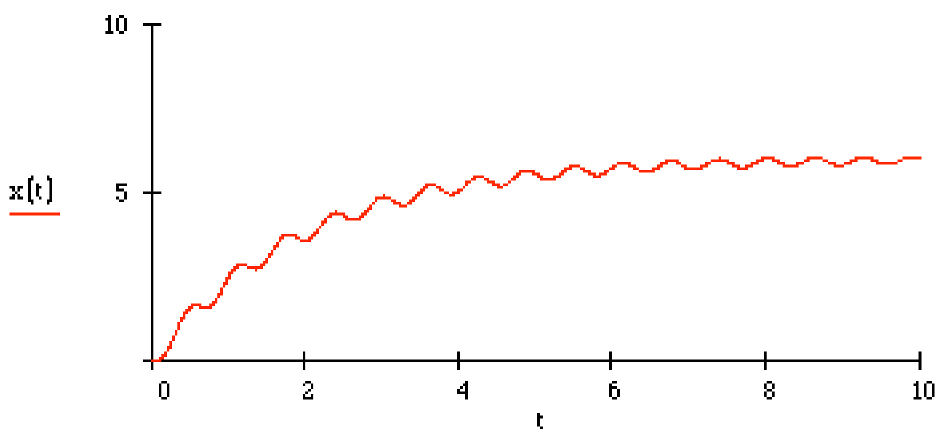
$$F(t) = 6000 - 5994 e^{-t/2}$$

$$x(t) = \frac{1}{100} e^{-0.1t} \int_0^t \left[\left(6000 - 5994 e^{-\tau/2}\right) e^{0.1\tau} \sin\left(10(t-\tau)\right) \right] d\tau$$

$$x(t) = 60 e^{-0.1t} \left\{ \int_0^t e^{0.1t} \sin\left[10(t-\tau)\right] d\tau - \int_0^t e^{-0.4t} \sin\left[10(t-\tau)\right] d\tau \right\}$$

After integrating and rearranging

$$x(t) = 6 - 5.979 e^{-t/2} - 0.0295 \cos 10t - 0.2990 \sin 10t \text{ m}$$



3.31 Calculate and plot the total response of the spring-mass-damper system of Figure 2.1 with m = 100 kg, ζ = 0.1 and k = 1000 N/m to the signal of Figure 3.12, with maximum force of 1 N. Assume that the initial conditions are zero and let $T = 2\pi$ s.

Solution: Given:

$$m = 100 \text{ kg}, \; k = 1000 \text{ N/m}, \zeta = 0.1, T = 2\pi \text{ s}, \; F_{\max} = 1N,$$

$$x(0) = \dot{x}(0) = 0, \quad \omega_n = \sqrt{\frac{k}{m}} = 3.16 \text{ rad/s}, \; \omega_d = \omega\sqrt{1-\zeta^2} = 3.15 \text{ rad/s}, \quad \omega_T = \frac{2\pi}{T} = 1 \text{ rad/s}$$

From example 3.3.1 and Figure 3.10,

$$F(t)\sum_{n=1}^{\infty} a_n \cos nt, \qquad a_n = \begin{cases} 0 & n \text{ even} \\ \dfrac{-8}{\pi^2 n^2} & n \text{ odd} \end{cases}$$

So, $m\ddot{x} + c\dot{x} + kx = \sum_{n=1}^{\infty} a_n \cos nt \qquad (n \text{ odd})$

The total solution is

$$x(t) = x_h(t) + \sum_{n=1}^{\infty} x_{cn}(t) \qquad (n \text{ odd})$$

From equation (3.33),

$$x_{cn}(t) = \frac{a_n / m}{\left[\left[\omega_n^2 - (n\omega_T)^2\right]^2 + \left[2\zeta\omega_n n\omega_T\right]^2\right]^{1/2}} \cos\left(n\omega_T t - \phi_n\right)$$

$$\phi_n = \tan^{-1}\left(\frac{2\zeta\omega_n n\omega_T}{\omega_n^2 - n^2\omega_T^2}\right) = \tan^{-1}\left(\frac{0.6325n}{10-n^2}\right)$$

$$x_{cn}(t) = \frac{-0.00811}{n^2\left[n^4 - 19.6n^2 + 100\right]^{1/2}} \cos\left[nt - \tan^{-1}\left(\frac{0.6325n}{10-n^2}\right)\right]$$

So,

$$x(t) = Ae^{\zeta\omega_n t}\sin(\omega_d t - \theta) + \sum_{n=1}^{\infty}\left[\frac{-0.00811}{n^2\left[n^4 - 19.6n^2 + 100\right]^{1/2}}\cos\left[nt - \tan^{-1}\left(\frac{0.6325n}{10 - n^2}\right)\right]\right] \quad (n \text{ odd})$$

$$\dot{x}(t) = -\zeta\omega_n Ae^{-\zeta\omega_n t}\sin(\omega_d t - \theta)$$

$$+\omega_d Ae^{-\zeta\omega_n t}\cos(\omega_d t - \theta) + \sum_{n=1}^{\infty}\left[\frac{0.00811}{n\left[n^4 - 19.6n^2 + 100\right]^{1/2}}\sin nt - \tan^{-1}\frac{0.6325n}{10 - n^2}\right] \quad (n \text{ odd})$$

$$x(0) = 0 = -A\sin\theta + \sum_{n=1}^{\infty}\left[\frac{-0.00811}{n^2\left[n^4 - 19.6n^2 + 100\right]^{1/2}}\cos\left[nt - \tan^{-1}\left(\frac{0.6325n}{10 - n^2}\right)\right]\right] \quad (n \text{ odd})$$

$$0 = -A\sin\theta - 0.00110$$

$$\dot{x}(0) = 0 = \zeta\omega_n A\sin\theta + \omega_d A\cos\theta$$

$$+\sum_{n=1}^{\infty}\left[\frac{-0.000569}{\left[n^4 - 19.6n^2 + 100\right]^{1/2}\left[0.00493n^2 + 1\right]}\right] \quad (n \text{ odd})$$

$$0 = \zeta\omega_n A\sin\theta + \omega_d A\cos\theta - 0.001186$$

So A = 0.00117 m and θ = - 1.232 rad.

The total solution is:

$$x(t) = 0.00117e^{-0.316t}\sin(3.15t + 1.23)$$

$$+\sum_{n=1}^{\infty}\left[\frac{-0.00811}{n^2\left[n^4 - 19.6n^2 + 100\right]^{1/2}}\cos\left[nt - \tan^{-1}\left(\frac{0.6325n}{10 - n^2}\right)\right]\right] \text{ m} \quad (n \text{ odd})$$

3.32 Calculate the total response of the system of Example 3.3.2 for the case of a base motion driving frequency of ω_b = 3.162 rad/s.

Solution: Let ω_b = 3.162 rad/s. From Example 3.3.2,

$$F(t) = cY\omega_b \cos\omega_b t + kY\sin\omega_b t = 1.581\cos(3.162t) + 50\sin(3.162t)$$

Also,

$$\omega_n = \sqrt{\frac{k}{m}} = 31.62 \text{ rad/s and } \zeta = \frac{c}{2\sqrt{km}} = 0.158$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 31.22 \text{ rad/s}$$

The solution is

$$x(t) = Ae^{-\zeta\omega_n t}\sin(\omega_d t + \theta) + \omega_n Y\left[\frac{\omega_n^2 + (2\zeta\omega_b)^2}{(\omega_n^2 - \omega_b^2)^2 + (2\zeta\omega_n\omega_b)^2}\right]^{1/2}\cos(\omega_b t - \phi_1 - \phi_2)$$

$$x(t) = Ae^{-5t}\sin(31.22t + \theta) + 0.0505\cos(3.162t - \phi_1 - \phi_2)$$

$$\phi_1 = \tan^{-1}\left(\frac{2\zeta\omega_n\omega_b}{\omega_n^2 - \omega_b^2}\right) = 0.0319 \text{ rad}$$

$$\phi_2 = \tan^{-1}\left(\frac{\omega_n}{2\zeta\omega_b}\right) = 1.54 \text{ rad}$$

So,

$$x(t) = Ae^{-5t}\sin(31.22t + \theta) + 0.0505\cos(3.162t - 1.57)$$

$$\dot{x}(t) = -5Ae^{-5t}\sin(31.22t + \theta) + 31.22Ae^{-5t}\cos(31.22t + \theta) - 0.16\sin(3.162t - 1.57)$$

$$\Rightarrow x(0) = 0.01 = A\sin\theta + 0.0505(0)$$

$$\Rightarrow \dot{x}(0) = 3 - 5A\sin\theta + 31.22A\cos\theta + 0.16(1)$$

So, $A = 0.0932$ m and $\theta = 0.107$ rad

The total solution is

$$x(t) = 0.0932e^{-5t}\sin(31.22t + 0.107) + 0.0505\cos(3.162t - 1.57) \text{ m}$$

Problems and Solutions for Section 3.4 (3.35 through 3.38)

3.35 Calculate the response of

$$m\ddot{x} + c\dot{x} + kx = F_0\Phi(t)$$

where $\Phi(t)$ is the unit step function for the case with $x_0 = v_0 = 0$. Use the Laplace transform method and assume that the system is underdamped.

Solution:

Given:

$$m\ddot{x} + c\dot{x} + kx = F_0\mu(t)$$

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{F_0}{m}\mu(t) \qquad (\zeta < 1)$$

Take Laplace Transform:

$$s^2 X(s) + 2\zeta\omega_n sX(s) + \omega_n^2 X(s) = \frac{F_0}{m}\left(\frac{1}{s}\right)$$

$$X(s) = \frac{F_0 / m}{\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)s} = \left(\frac{F_0}{m\omega_n^2}\right)\frac{\omega_n^2}{s\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)}$$

Using inverse Laplace tables,

$$x(t) = \frac{F_0}{k} - \frac{F_0}{k\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\left(\omega_n\sqrt{1-\zeta^2}\,t + \cos^{-1}(\zeta)\right)$$

3.36 Using the Laplace transform method, calculate the response of the system of Example 3.4.4 for the overdamped case ($\zeta > 1$). Plot the response for $m = 1$ kg, $k = 100$ N/m, and $\zeta = 1.5$.

Solution:

From example 3.4.4,

$$m\ddot{x} + c\dot{x} + kx = \delta(t)$$

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{1}{m}\delta(t) \qquad (\zeta > 1)$$

Take Laplace Transform:

$$s^2X(s) + 2\zeta\omega_n sX(s) + \omega_n^2 X(s) = \frac{1}{m}$$

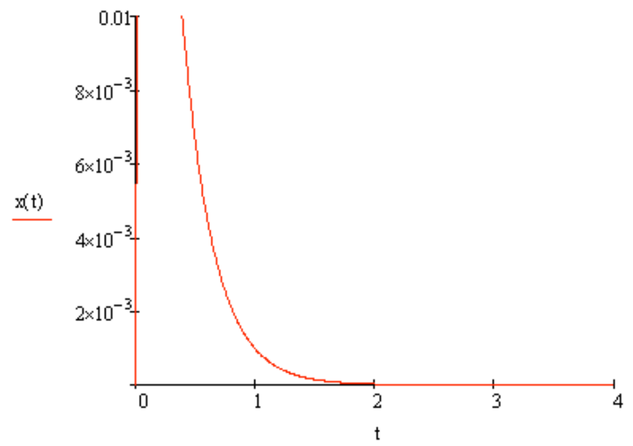
$$X(s) = \frac{1/m}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{1/m}{(s+a)(s+b)}$$

Using inverse Laplace tables, $a = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}$, $b = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}$

$$x(t) = \frac{e^{-\zeta\omega_n t}}{2m\omega_n\sqrt{\zeta^2 - 1}}\left[e^{\omega_n\sqrt{\zeta^2-1}\,t} - e^{-\omega_n\sqrt{\zeta^2-1}\,t}\right]$$

Inserting the given values yields: $x(t) = \dfrac{e^{-15t}}{22.36}\left[e^{11.18t} - e^{-11.18t}\right]$ m

$$x(t) := \frac{e^{-15 \cdot t}}{22.36} \cdot \left(e^{11.18 \cdot t} - e^{-11.18 \cdot t}\right)$$



3.37 Calculate the response of the underdamped system given by

$$m\ddot{x} + c\dot{x} + kx = F_0 e^{-at}$$

using the Laplace transform method. Assume $a > 0$ and that the initial conditions are all zero.

Solution:

Given:

$$m\ddot{x} + c\dot{x} + kx = F_0 e^{-at} \qquad a > 0, \text{ initial conditions} = 0$$

Rewrite:

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{F_0}{m}e^{-at}$$

Take Laplace Transform:

$$s^2 X(s) + 2\zeta\omega_n sX(s) + \omega_n^2 X(s) = \frac{F_0}{m}\left(\frac{1}{s+a}\right)$$

$$X(s) = \frac{F_0 / m}{\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)(s+a)}$$

For an underdamped system, the inverse Laplace Transform is

$$x(t) = \left(\frac{F_0}{m\left(2\zeta\omega_n a - \omega_n^2 - a^2\right)}\right)\left\{e^{-\zeta\omega_n t}\left[\frac{\zeta\omega_n - a}{\omega_d}\sin(\omega_d t) + \cos(\omega_d t)\right] - e^{-at}\right\}$$

3.38 Solve the following system for the response $x(t)$ using Laplace transforms:

$$100\ddot{x}(t) + 2000x(t) = 50\delta(t)$$

where the units are in Newtons and the initial conditions are both zero.

Solution:

First divide by the mass to get

$$\ddot{x} + 20x(t) = 0.5\delta(t)$$

Take the Laplace Transform to get

$$(s^2 + 20)X(s) = 0.5$$

So

$$X(s) = \frac{0.5}{s^2 + 20}$$

Taking the inverse Laplace Transform using entry 5 of Table 3.1 yields

$$X(s) = \frac{0.5}{\sqrt{20}} \cdot \frac{\omega}{s^2 + \omega^2} \quad \text{where } \omega = \sqrt{20}$$

$$\Rightarrow x(t) = \frac{1}{4\sqrt{5}} \sin\sqrt{20}t$$

Problems and Solutions Section 3.5 (3.39 through 3.42)

3.39 Calculate the mean-square response of a system to an input force of constant PSD, S_0, and frequency response function $H(\omega) = \dfrac{10}{(3 + 2j\omega)}$

Solution:

Given: $S_{ff} = S_0$ and $H(\omega) = \dfrac{10}{3 + 2j\omega}$

The mean square of the response can be found from Eqs (3.66) and (3.68):

$$\bar{x}^2 = E\left[x^2\right] = \int_{-\infty}^{\infty} \left|H(\omega)\right|^2 S_{ff}(\omega)\,d\omega$$

$$\bar{x}^2 = S_0 \int_{-\infty}^{\infty} \left|\frac{10}{3 + 2j\omega}\right|^2 d\omega$$

Using Eq. (3.67) yields

$$\bar{x}^2 = \frac{50\pi S_0}{3}$$

3.40 Consider the base excitation problem of Section 2.4 as applied to an automobile model of Example 2.4.1 and illustrated in Figure 2.16. In this problem let the road have a random stationary cross section producing a PSD of S_0. Calculate the PSD of the response and the mean-square value of the response.

Solution: Given: $S_{ff} = S_0$

From example 2.4.1: $m = 1007$ kg, $c = 2000$ kg/s, $k = 40{,}000$ N/m

$$\zeta = \frac{c}{2\sqrt{km}} = \frac{2000}{2\sqrt{40000 \bullet 1007}} = 0.157 \qquad \text{(underdamped)}$$

So,

$$H(\omega) = \frac{1}{k - m\omega^2 + jc\omega} = \frac{1}{4\times10^4 - 1007\omega^2 + 2000\,j\omega}$$

$$\left|H(\omega)\right|^2 = \frac{1}{\left(4\times10^4 - 1007\omega^2\right)^2 + \left(2000\right)^2 j\omega^2}$$

$$\left|H(\omega)\right|^2 = \frac{1}{1.01\times10^6\omega^4 - 4.06\times10^7\omega^2 + 1.6\times10^9}$$

The PSD is found from equation (3.62):

$$S_{xx}(\omega) = \left|H(\omega)\right|^2 S_{ff}(\omega)$$

$$S_{xx}(\omega) = \frac{1}{1.01\times10^6\omega^4 - 8.46\times10^7\omega^2 + 1.6\times10^9}$$

The mean square value is found from equation (3.68):

$$\bar{x}^2 = E\left[x^2\right] = \int_{-\infty}^{\infty} \left|H(\omega)\right|^2 S_{ff}(\omega)\,d\omega$$

$$\bar{x}^2 = S_0 \int_{-\infty}^{\infty} \left|\frac{1}{4\times10^4 - 1007\omega^2 + 2000\,j\omega}\right|^2 d\omega$$

Using equation (3.70) yields

$$\bar{x}^2 = \frac{\pi S_0}{8\times10^{10}}$$

3.41 To obtain a feel for the correlation functions, compute autocorrelation $R_{xx}(\tau)$ for the deterministic signal $A\sin\omega_n t$.

Solution: The autocorrelation is found from

$$R_{xx}(\tau) = \lim_{T\to\infty}\frac{1}{T}\int_{\frac{-T}{2}}^{\frac{T}{2}} A\sin(\omega_n t)A\sin(\omega_n(t+\tau))dt$$

$$= \lim_{T\to\infty}\frac{A^2}{T}\int_{\frac{-T}{2}}^{\frac{T}{2}} \sin(\omega_n t)\sin(\omega_n t)\cos(\omega_n \tau)dt$$

$$+\underbrace{\lim_{T\to\infty}\frac{A^2}{T}\int_0^T \sin(\omega_n t)\cos(\omega_n t)\sin(\omega_n \tau)dt}_{\to 0}$$

Simplifying yields:

$$R_{xx}(\tau) = \frac{A^2\cos(\omega_n\tau)}{2}$$

3.42 Verify that the average $x-\bar{x}$ is zero by using the definition given in equation (3.47).

Solution:

The definition is $\bar{f} = \lim_{T\to\infty}\frac{1}{T}\int_0^T f(t)dt$. Let

$$f(t) = x(t) - \bar{x},$$

$$\text{so that} \quad \bar{f} = \lim_{T\to\infty}\frac{1}{T}\int_0^T \left(x(t)-\bar{x}\right)dt$$

$$\bar{f} = \lim_{T\to\infty}\frac{1}{T}\int_0^T x(t)dt - \lim_{T\to\infty}\frac{1}{T}\int_0^T \bar{x}dt$$

$$= \bar{x} - \bar{x}\lim_{T\to\infty}\frac{1}{T}\int_0^T dt = \bar{x} - \bar{x} = 0$$

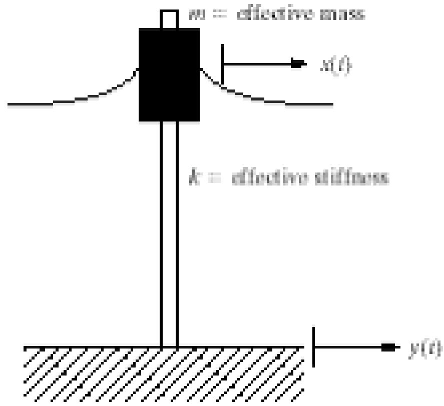
Problems and Solutions Section 3.6 (3.43 through 3.44)

3.43 A power line pole with a transformer is modeled by

$$m\ddot{x} + kx = -\ddot{y}$$

where x and y are as indicated in Figure 3.23. Calculate the response of the relative displacement $(x - y)$ if the pole is subject to an earthquake base excitation of (assume the initial conditions are zero)

$$\ddot{y}(t) = \begin{cases} A\left(1 - \dfrac{t}{t_0}\right) & 0 \le t \le 2t_0 \\ 0 & t > 2t_0 \end{cases}$$



Solution: Given: $m\ddot{x} + kx = -\ddot{y}$

$$\ddot{y} = \begin{cases} A\left(1 - \dfrac{t}{t_0}\right) & 0 \le t \le 2t_0 \\ 0 & t > 2t_0 \end{cases}$$

$$x(0) = \dot{x}(0) = 0$$

The response $x(t)$ is given by Eq. (3.12) as

$$x(t) = \int_0^t F(\tau)h(t-\tau)\,d\tau$$

where $h(t-\tau) = \dfrac{1}{m\omega_n}\sin\omega_n(t-\tau)$ for an undamped system

For $0 \le t \le 2t_0$,

$$x(t) = \int_0^t A\left(1 - \frac{\tau}{t_0}\right)\left(\frac{1}{m\omega_n}\right)\sin\omega_n(t-\tau)\,d\tau$$
$$x(t) = \frac{A}{m\omega_n^2}\left[1 - \frac{t}{t_0} + \frac{1}{t_0\omega_n}\sin\omega_n t - \cos\omega_n t\right]$$

For $t > 2t_0$,

$$x(t) = \int_0^{2t_0} A\left(1 - \frac{\tau}{t_0}\right)\left(\frac{1}{m\omega_n}\right)\sin\omega_n(t-\tau)\,d\tau$$
$$x(t) = \frac{A}{m\omega_n^2}\left[\frac{1}{t_0\omega_n}\left(\sin\omega_n t - \sin\omega_n(t-2t_0)\right) - \cos\omega_n t - \cos\omega_n(t-2t_0)\right]$$

Find $y(t)$ when $0 \le t \le 2t_0$,

$$\ddot{y}(t) = A\left(1 - \frac{t}{t_0}\right)$$
$$\dot{y}(t) = At - \frac{A}{2t_0}t^2 + C_1$$
$$y(t) = \frac{A}{2}t^2 - \frac{A}{6t_0}t^3 + C_1 t + C_2$$

Using IC's yields $C_1 = C_2 = 0$. Find $y(t)$ when $t > \mathrm{w}t_0$:

$$\ddot{y}(t) = 0$$
$$\dot{y}(t) = C_3$$
$$y(t) = C_3 t + C_4$$

Using IC's yields $C_3 = C_4 = 0$. The relative displacement $x(t) - y(t)$ is therefore:

For $0 \le t \le 2t_0$

$$x(t) - y(t) = \frac{A}{m\omega_n^2}\left[1 - \frac{t}{t_0} + \frac{1}{t_0\omega_n}\sin\omega_n t - \cos\omega_n t\right] - \frac{A}{2}t^2 + \frac{A}{6t_0}t^3$$

For $t > 2t_0$,

$$x(t) - y(t) = \frac{A}{m\omega_n^2}\left[\frac{1}{t_0\omega_n}\left(\sin\omega_n t - \sin\omega_n(t-2t_0)\right) - \cos\omega_n t - \cos\omega_n(t-2t_0)\right]$$

3.44 Calculate the response spectrum of an undamped system to the forcing function

$$F(t) = \begin{cases} F_0 \sin\dfrac{\pi t}{t_1} & 0 \le t \le t_1 \\ 0 & t > t_1 \end{cases}$$

assuming the initial conditions are zero.

Solution: Let $\omega = \pi / t_1$. The solution is the homogeneous solution $x_h(t)$ and the particular solution $x_p(t)$ or $x(t) = x_h(t) + x_p(t)$. Thus

$$x(t) = A\cos\omega_n t + B\sin\omega_n t + \left(\frac{F_0}{k - m\omega^2}\right)\sin\omega t$$

where A and B are constants and ω_n is the natural frequency of the system:
Using the initial conditions $x(0) = \dot{x}(0) = 0$ the constants A and B are

$$A = 0, \quad B = \frac{-F_0\omega}{\omega_n\left(k - m\omega^2\right)}$$

so that $x(t) = \dfrac{F_0 / k}{1 - (\omega/\omega_n)^2}\left\{\sin\omega t - \dfrac{\omega}{\omega_n}\sin\omega_n t\right\},\ 0 \le t \le t_1$

Which can be written as (where $\delta = F_0 / k$ the static deflection)

$$\frac{x(t)}{\delta} = \frac{1}{1 - \left(\dfrac{\tau}{2t_1}\right)^2}\left\{\sin\frac{\pi t}{t_1} - \frac{\tau}{2t_1}\sin\frac{2\pi t}{\tau}\right\}, 0 \le t \le t_1$$

and where $\tau = 2\pi / \omega_n$. After t_1 the solution is a free response

$$x(t) = A'\cos\omega_n t + B'\sin\omega_n t,\ t > t_1$$

where the constants A' and B' can be found by using the values of $x(t = t_1)$ and $\dot{x}(t = t_1), t > t_0$. This gives

$$x(t = t_1) = a\left[-\frac{\tau}{2t_1}\sin\frac{2\pi t_1}{\tau}\right] = A'\cos\omega_n t_1 + B'\sin\omega_n t_1$$

$$\dot{x}(t = t_1) = a\left\{-\frac{\pi}{t_1} - \frac{\pi}{t_1}\cos\frac{2\pi t_1}{\tau}\right\} = -\omega_n A'\sin\omega_n t_1 + \omega_n B'\cos\omega_n t$$

where

$$a = \frac{\delta}{1 - \left(\dfrac{\tau}{2t_1}\right)^2}$$

These are solved to yield

$$A' = \frac{a\pi}{\omega_n t_1}\sin\omega_n t_1, \quad B' = -\frac{a\pi}{\omega_n t_1}\left[1 + \cos\omega_n t_1\right]$$

So that after t_1 the solution is

$$\frac{x(t)}{\delta} = \frac{(\tau / t_1)}{2\left\{1-(\tau / 2t_1)^2\right\}}\left[\sin 2\pi\left(\frac{t_1}{\tau} - \frac{t}{\tau}\right) - \sin 2\pi\frac{t}{\tau}\right], \; t \geq t_1$$

Problems and Solutions for Section 3.7 (3.45 through 3.52)

3.45 Using complex algebra, derive equation (3.89) from (3.86) with $s = j\omega$.

Solution: From equation (3.86):

$$H(s) = \frac{1}{ms^2 + cs + k}$$

Substituting $s = j\omega$ yields

$$H(j\omega) = \frac{1}{m(j\omega)^2 + c(j\omega) + k} = \frac{1}{k - m\omega^2 - cj\omega}$$

The magnitude is given by

$$\left|H(j\omega_{dr})\right| = \left[\left(\frac{1}{m(j\omega)^2 + (cj\omega) + k}\right) = \left(\frac{1}{k - m\omega^2 - cj\omega}\right)\right]^{1/2}$$

$$\left|H(j\omega)\right| = \frac{1}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}$$ which is Eq. (3.89)

3.46 Using the plot in Figure 3.20, estimate the system's parameters *m*, *c*, and *k*, as well as the natural frequency.

Solution: From Fig. 3.20

$$\frac{1}{k} = 2 \Rightarrow k = 0.5$$

$$\omega = \omega_n = 0.25 = \sqrt{\frac{k}{m}} \Rightarrow m = 8$$

$$\frac{1}{c\omega} \approx 4.6 \Rightarrow c = 0.087$$

3.47 Using the values determined in Problem 3.46 plot the inertance transfer function's magnitude and phase for this system.

Solution: From Problem 3.46

$$\frac{1}{k} = 2 \Rightarrow k = 0.5, \omega = \omega_n = 0.25 = \sqrt{\frac{k}{m}} \Rightarrow m = 8, \frac{1}{c\omega} \approx 4.6 \Rightarrow c = 0.087$$

The inertance transfer function is given by Eq. (3.88):

$$s^2 H(s) = \frac{s^2}{ms^2 + cs + k}$$

Substitute $s = j\omega$ to get the frequency response function. The magnitude is given by:

$$\left|(j\omega)^2 H(j\omega)\right| = \frac{\omega^2}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}} = \frac{\omega^2}{\sqrt{(0.5 - 8\omega^2)^2 + (0.087\omega)^2}}$$

The phase is given by

$$\phi = \tan^{-1}\left(\frac{\text{Imaginary part of frequency response function}}{\text{Real part of frequency response function}}\right)$$

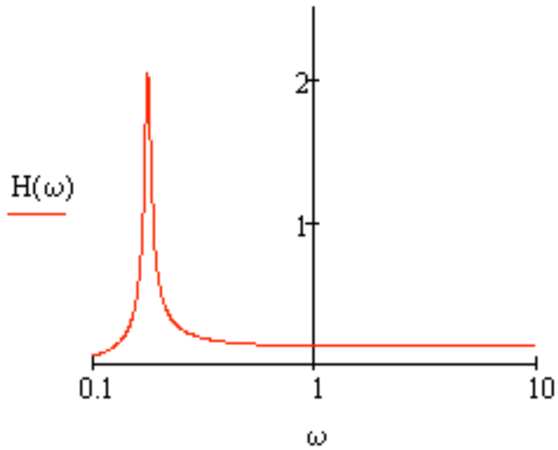
Multiply the numerator and denominator of $(j\omega)^2 H(j\omega)$ by $(k - m\omega^2) - cj\omega$ to get

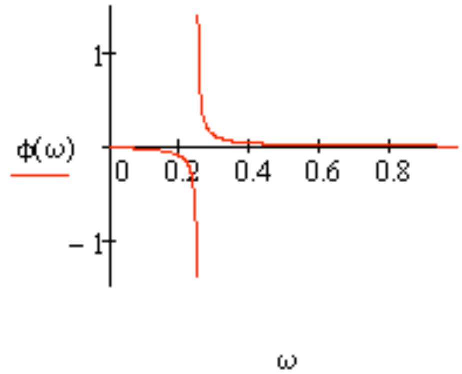
$$(j\omega)^2 H(j\omega) = \frac{-\omega^2(k - m\omega) + cj\omega^3}{(k - m\omega^2)^2 + (c\omega)^2}$$

So, $\phi = \tan^{-1}\left(\dfrac{c\omega^3}{-\omega^2(k - m\omega^2)}\right) = \tan^{-1}\left(\dfrac{0.087\omega}{8\omega^2 - 0.5}\right)$

The magnitude and phase plots are shown on a semilog scale. The plots are given in the following Mathcad session

$$\phi(\omega) := \text{atan}\left(\frac{0.087\cdot\omega}{8\cdot\omega^2 - 0.5}\right) \qquad H(\omega) := \frac{\omega^2}{\sqrt{\left(0.5^2 - 8\cdot\omega^2\right)^2 + (0.087\cdot\omega)^2}}$$





3.48 Using the values determined in Problem 3.46 plot the mobility transfer function's magnitude and phase for the system of Figure 3.20.

Solution: From Problem 3.46

$$\frac{1}{k} = 2 \Rightarrow k = 0.5, \omega = \omega_n = 0.25 = \sqrt{\frac{k}{m}} \Rightarrow m = 8, \frac{1}{c\omega} \approx 4.6 \Rightarrow c = 0.087$$

The mobility transfer function is given by equation (3.87):

$$sH(s) = \frac{s}{ms^2 + cs + k}$$

Substitute $s = j\omega$ to get the frequency response function. The magnitude is given by

$$\left|(j\omega)H(j\omega)\right| = \frac{\omega}{\sqrt{\left(k - j\omega^2\right)^2 + (c\omega)^2}} = \frac{\omega}{\sqrt{\left(0.5 - 8\omega^2\right)^2 + (0.087\omega)^2}}$$

The phase is given by

$$\phi = \tan^{-1}\left(\frac{\text{Imaginary part of frequency response function}}{\text{Real part of frequency response function}}\right)$$

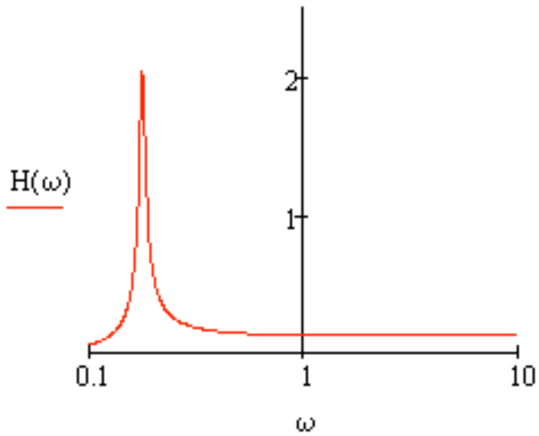
Multiply the numerator and denominator of $j\omega H(j\omega)$ by j and by $-\left(k - m\omega^2\right)j - c\omega$ to get

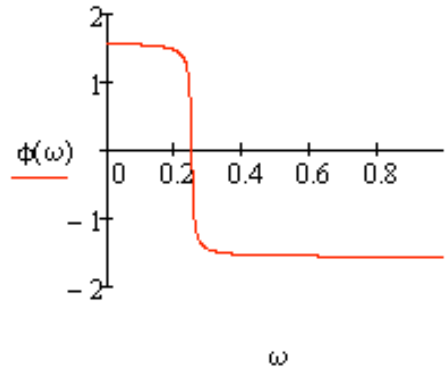
$$(j\omega)H(j\omega) = \frac{j\omega\left(k - m\omega^2\right) + c\omega^2}{\left(k - m\omega^2\right)^2 + (c\omega)^2}$$

So, $\phi = \tan^{-1}\left(\dfrac{\omega\left(k - m\omega^2\right)}{c\omega^2}\right) = \tan^{-1}\left(\dfrac{0.5 - 8\omega^2}{0.087\omega}\right)$

The magnitude and phase plots are shown on a semilog scale.

$$\phi(\omega) := \text{atan}\left(\frac{-8\cdot\omega^2 + 0.5}{0.087\cdot\omega}\right) \qquad H(\omega) := \frac{\omega^2}{\sqrt{\left(0.5^2 - 8\cdot\omega^2\right)^2 + (0.087\cdot\omega)^2}}$$





3.49 Calculate the compliance transfer function for a system described by

$$a\ddddot{x} + b\dddot{x} + c\ddot{x} + d\dot{x} + ex = f\left(t\right)$$

where $f(t)$ is the input force and $x(t)$ is a displacement.

Solution:

The compliance transfer function is $\dfrac{X\left(s\right)}{F\left(s\right)}$.

Taking the Laplace Transform yields

$$\left(as^4 + bs^3 + cs^2 + ds + e\right)X\left(s\right) = F\left(s\right)$$

So, $\dfrac{X\left(s\right)}{F\left(s\right)} = \dfrac{1}{as^4 + bs^3 + cs^2 + ds + e}$

3.50 Calculate the frequency response function for the compliance of Problem 3.49.

Solution: From problem 3.49,

$$H(s) = \frac{1}{as^4 + bs^3 + cs^2 + ds + e}$$

Substitute $s = j\omega$ to get the frequency response function:

$$H(j\omega) = \frac{1}{a(j\omega)^4 + b(j\omega)^3 + c(j\omega)^2 + d(j\omega) + e}$$

$$H(j\omega) = \frac{a\omega^4 - c\omega^2 + e - j(-b\omega^3 + d\omega)}{(a\omega^4 - c\omega^2 + e)^2 + (-b\omega^3 + d\omega)^2}$$

3.51 Plot the magnitude of the frequency response function for the system of Problem 3.49 for $a = 1, b = 4, c = 11, d = 16,$ and $e = 8$.

Solution: From Problem 3.50

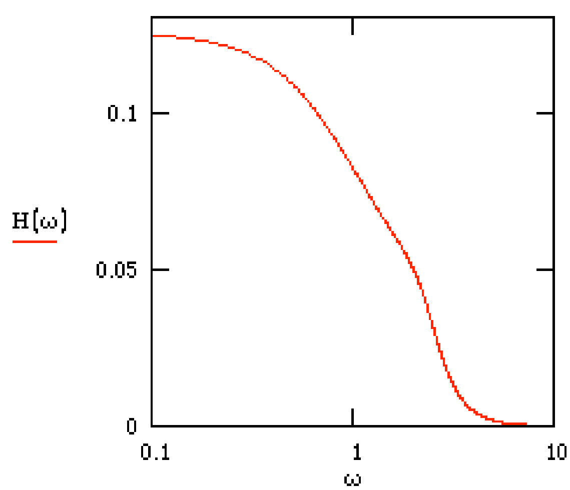
$$H(j\omega) = \frac{a\omega^4 - c\omega^2 + e - j(-b\omega^3 + d\omega)}{(a\omega^4 - c\omega^2 + e)^2 + (-b\omega^3 + d\omega)^2}$$

The magnitude is

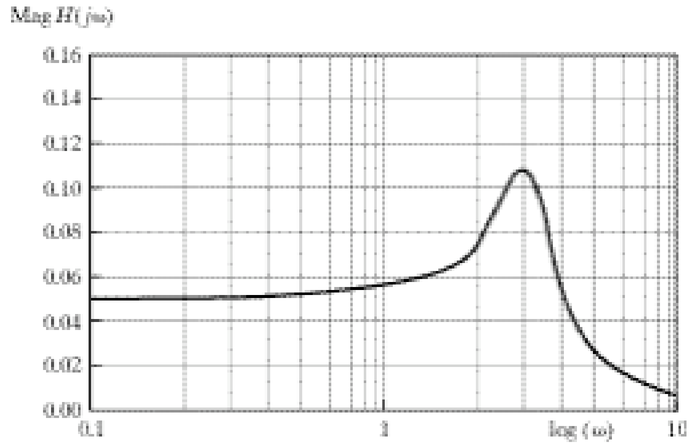
$$|H(j\omega)| = \frac{1}{\sqrt{(\omega^4 - 11\omega^2 + 8)^2 + (-4\omega^3 + 16\omega)^2}}$$

This is plotted in the following Mathcad session:

$$H(\omega) := \frac{1}{\sqrt{\left(\omega^4 - 11\cdot\omega^2 + 8\right)^2 + \left(16\cdot\omega - 4\cdot\omega^3\right)^2}}$$



3.52 An experimental (compliance) magnitude plot is illustrated in Fig. P3.52. Determine $\omega, \zeta, c, m,$ and k. Assume that the units correspond to m/N along the vertical axis.



Solution: Referring to the plot, it starts at

$$|H(\omega j)| = \frac{1}{k}$$

Thus: $$0.05 = \frac{1}{k} \Rightarrow k = 20 \text{ N/m}$$

At the peak, $\omega_n = \omega = 3$ rad/s. Thus the mass can be determined by

$$m = \frac{k}{\omega_n^2} \Rightarrow m = 2.22 \text{ kg}$$

The damping is found from

$$\frac{1}{c\omega} = 0.11 \Rightarrow c = 3.03 \text{ kg/s} \Rightarrow \zeta = \frac{c}{2\sqrt{km}} = \frac{3.03}{2\sqrt{20 \cdot 2.22}} = 0.227$$

Problems and Solutions Section 3.8 (3.53 through 3.56)

3.53 Show that a critically damped system is BIBO stable.

Solution:

For a critically damped system

$$h(t-\tau) = \frac{1}{m}(t-\tau)e^{-\omega_n(t-\tau)}$$

Let $f(t)$ be bounded by the finite constant M. Using the inequality for integrals and Equation (3.96) yields:

$$|x(t)| \le \int_0^t f(\tau)|h(t-\tau)|d\tau = \int_0^t M\frac{1}{m}\left|(t-\tau)e^{-\omega_n(t-\tau)}\right|d\tau$$

The function $h(t-\tau)$ decays exponentially and hence is bounded by some constant times $1/t$, say M_1/t. This is just a statement the exponential decays faster then "one over t" does. Thus the above expression becomes;

$$|x(t)| < M\int_0^t \frac{M_1}{t}d\tau = MM_1$$

This is bounded, so a critically damped system is BIBO stable.

3.54 Show that an overdamped system is BIBO stable.

Solution: For an overdamped system,

$$h(t-\tau)=\frac{1}{2m\omega_n\sqrt{\zeta^2-1}}e^{-\zeta\omega_n(t-\tau)}\left(e^{\left(\omega_n\sqrt{\zeta^2-1}\right)(t-\tau)}-e^{-\left(\omega_n\sqrt{\zeta^2-1}\right)(t-\tau)}\right)$$

Let $f(t)$ be bounded by M,

From equation (3.96),

$$|x(t)|\le M\int_0^t |h(t-\tau)|d\tau$$

$$|x(t)|\le M\int_0^t \frac{1}{2m\omega_n\sqrt{\zeta^2-1}}\left|e^{-\zeta\omega_n(t-\tau)}\left(e^{\left(\omega_n\sqrt{\zeta^2-1}\right)(t-\tau)}-e^{-\left(\omega_n\sqrt{\zeta^2-1}\right)(t-\tau)}\right)\right|d\tau$$

$$|x(t)|\le\frac{M}{2m\omega_n\sqrt{\zeta^2-1}}\left[\left(\frac{-1}{\omega_n\sqrt{\zeta^2-1}-\zeta\omega}\right)\left(1-e^{\left(\omega_n\sqrt{\zeta^2-1}-\zeta\omega_n\right)t}\right)\right.$$
$$\left.-\left(\frac{-1}{\omega_n\sqrt{\zeta^2-1}+\zeta\omega_n}\right)\left(1-e^{\left(\omega_n\sqrt{\zeta^2-1}-\zeta\omega_n\right)t}\right)\right]$$

Since $\omega_n\sqrt{\zeta^2-1}-\zeta\omega_n<0,$ then $1-e^{\left(\omega_n\sqrt{\zeta^2-1}-\zeta\omega_n\right)t}$ is bounded.

Also, since $-\omega_n\sqrt{\zeta^2-1}-\zeta\omega_n<0,$ then $1-e^{\left(\omega_n\sqrt{\zeta^2-1}-\zeta\omega_n\right)t}$ is bounded.

Therefore, an overdamped system is BIBO stable.

3.55 Is the solution of $2\ddot{x} + 18x = 4\cos 2t + \cos t$ Lagrange stable?

Solution: Given

$$2\ddot{x} + 18x = 4\cos 2t + \cos t$$

$$\omega_n = \sqrt{\frac{k}{m}} = 3$$

The total solution will be

$$x(t) = x_h(t) + x_{P1}(t) + x_{P2}(t)$$

From Eq. (1.3): $$x_h(t) = A\sin(\omega_n t + \phi)$$

From Eq. (2.7): $$x_{P1}(t) = \frac{f_{0_1}}{\omega_n^2 - 2^2}\cos 2t$$

and $$x_{P2}(t) = \frac{f_{0_2}}{\omega_n^2 - 1^2}\cos t$$

Adding the solutions yields

$$|x(t)| = \left| A\sin(3t + \phi) + \frac{f_{0_1}}{3^2 - 2^2}\cos 2t + \frac{f_{0_1}}{3^2 - 1^2}\cos t \right| < M$$

Since $3 \neq 2, 3 \neq 1$, and the homogeneous solution is marginally stable, this system is Lagrange stable.

3.56 Calculate the response of equation (3.99) for $x_0 = 0, v_0 = 1$ for the case that a = 4 and b = 0. Is the response bounded?

Solution: Given: $x_0 = 0, v_0 = 1, a = 4, b = 0$. From Eq. (3.99),

$$\ddot{x} + \dot{x} + 4x = ax + b\dot{x} = 4x$$

So, $\ddot{x} + \dot{x} = 0$

Let

$$x(t) = Ae^{\lambda t}$$
$$\dot{x}(t) = \lambda Ae^{\lambda t}$$
$$\ddot{x}(t) = \lambda^2 Ae^{\lambda t}$$

Substituting,

$$\lambda^2 Ae^{\lambda t} + \lambda Ae^{\lambda t} = 0$$
$$\lambda^2 + \lambda = 0$$

So, $\lambda_{1,2} = 0, -1$

The solution is

$$x(t) = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t} = A_1 + A_2 e^{-t}$$
$$\dot{x}(t) = -A_2 e^{-1}$$
$$x(0) = 0 = A_1 + A_2$$
$$\dot{x}(0) = 1 = -A_2$$

So, $A_1 = 1$ and $A_2 = -1$

Therefore,

$$x(t) = 1 - e^{-1}$$

Since $|x(t)| = |1 - e^{-t}| 1$, the response is bounded.

Problems and Solutions from Section 3.9 (3.57-3.64)

3.57*. Numerically integrate and plot the response of an underdamped system determined by m = 100 kg, k = 1000 N/m, and c = 20 kg/s, subject to the initial conditions of $x_0 = 0$ and $v_0 = 0$, and the applied force F(t) = 30Φ(t -1). Then plot the exact response as computed by equation (3.17). Compare the plot of the exact solution to the numerical simulation.

Solution: First the solution is presented in Mathcad:

Prob 3.50

$m := 100 \quad k := 1000 \quad \omega n := \sqrt{\frac{k}{m}} \quad c := 20$

$F0 := 30 \quad \omega n = 3.162 \quad \zeta := \frac{c}{2\cdot\sqrt{k\cdot m}} \quad t0 := 1 \quad \zeta = 0.032$

$\theta := \operatorname{atan}\left[\frac{\zeta}{1-\zeta^2}\right] \quad \frac{F0}{k} = 0.03 \quad \frac{F0}{k\cdot\sqrt{1-\zeta^2}} = 0.03002 \quad \zeta\cdot\omega n = 0.1$

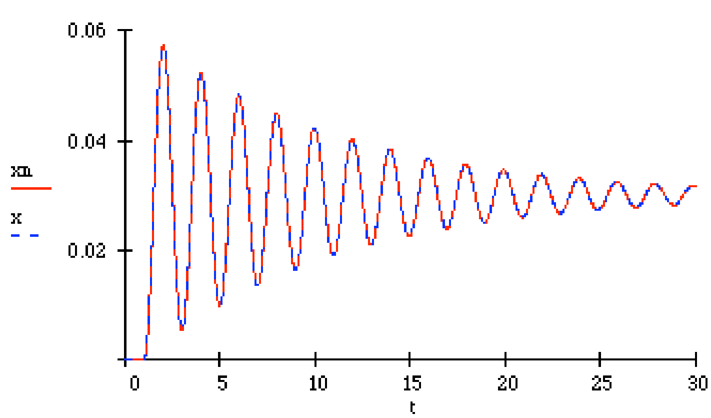
$\omega d := \omega n\cdot\sqrt{1-\zeta^2} \quad \omega d = 3.1607 \quad \theta = 0.032 \quad m := \left(\frac{k}{\omega n^2}\right)$

$$xa(t) := \left[\left[\frac{F0}{k} - \frac{F0}{k\cdot\sqrt{1-\zeta^2}}\cdot e^{-\zeta\cdot\omega n\cdot(t-t0)}\cdot\cos\left[\omega d\cdot(t-t0)-\theta\right]\right]\right]\cdot\Phi(t-t0)$$

$X := \begin{bmatrix}0\\0\end{bmatrix} \quad D(t,X) := \begin{bmatrix}X_1\\ -2\cdot\zeta\cdot\omega n\cdot X_1 - \omega n^2\cdot X_0 + \frac{F0}{m}\cdot\Phi(t-t0)\end{bmatrix}$

$Z := \mathrm{rkfixed}(X, 0, 30, 2000, D)$

$t := Z^{\langle 0\rangle} \quad x := \overrightarrow{xa(t)} \quad xn := Z^{\langle 1\rangle}$



The Matlab code to provide similar plots is given next:

```
%Numerical Solutions
%Problem #57
clc
clear
close all
%Numerical Solution
x0=[0;0];
tspan=[0 15];

[t,x]=ode45('prob57a',tspan,x0);

figure(1)
plot(t,x(:,1));
title('Problem #57');
xlabel('Time, sec.');
ylabel('Displacement, m');
hold on

%Analytical Solution
m=100;
c=20;
k=1000;
F=30;
w=sqrt(k/m);
d=c/(2*w*m);
wd=w*sqrt(1-d^2);
to=1;
phi=atan(d/sqrt(1-d^2));

%for t<to
t=linspace(0,1,3);
x=0.*t;
plot(t,x,'*');

%for t>=to
t=linspace(1,15);
x=F/k-F/(k*sqrt(1-d^2)).*exp(-d.*w.*(t-to)).*cos(wd.*(t-to)-phi);
plot(t,x,'*');
legend('Numerical', 'Analytical')

%M-file for Prob #50

function dx=prob(t,x);
[rows, cols]=size(x);dx=zeros(rows, cols);
m=100;
c=20;
k=1000;
F=30;

if t<1
   dx==0;
   else
      dx(1)=x(2);
   dx(2)=-c/m*x(2) - k/m*x(1) + F/m;
end
```

3.58*. Numerically integrate and plot the response of an underdamped system determined by m = 150 kg, and k = 4000 N/m subject to the initial conditions of x_0 = 0.01 m and v_0 = 0.1 m/s, and the applied force $F(t)$ = F(t) = 15Φ(t -1), for various values of the damping coefficient. Use this "program" to determine a value of damping that causes the transient term to die out with in 3 seconds. Try to find the smallest such value of damping remembering that added damping is usually expensive.

Solution: First the solution is given in Mathcad followed by the equivalent Matlab code.

$m := 150 \quad k := 4000 \quad \omega n := \sqrt{\frac{k}{m}} \quad c := 710 \quad x0 := 0.01 \quad v0 := 0.1$

$F0 := 15 \quad \omega n = \quad \zeta := \frac{c}{2\cdot\sqrt{k\cdot m}} \quad t0 := 1 \quad \zeta =$

$\theta := \text{atan}\left(\frac{\zeta}{1-\zeta^2}\right) \quad \theta = \quad \omega d := \omega n\cdot\sqrt{1-\zeta^2} \quad A := \sqrt{\frac{(v0+\zeta\cdot\omega n\cdot x0)^2+(x0\cdot\omega d)^2}{\omega d^2}}$

$\omega d = \quad \phi := \text{atan}\left(\frac{x0\cdot\omega d}{v0+\zeta\cdot\omega n\cdot x0}\right) \quad xh(t) := A\cdot e^{-\zeta\cdot\omega n\cdot t}\cdot\sin(\omega d\cdot t+\phi)$

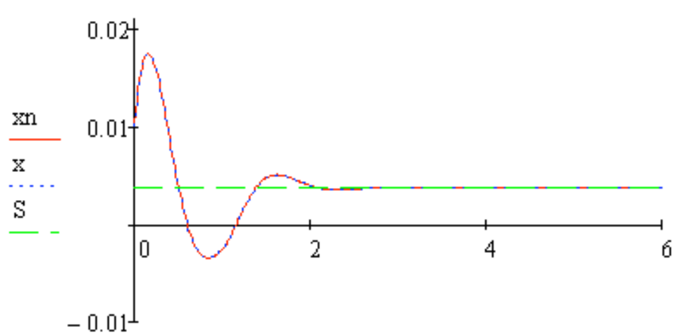
$$xa(t) := \left[\left[\frac{F0}{k}-\frac{F0}{k\cdot\sqrt{1-\zeta^2}}\cdot e^{-\zeta\cdot\omega n\cdot(t-t0)}\cdot\cos[\omega d\cdot(t-t0)-\theta]\right]\right]\cdot\Phi(t-t0)+xh(t)$$

$$xa(t) := \left[\left[\frac{F0}{k}-\frac{F0}{k\cdot\sqrt{1-\zeta^2}}\cdot e^{-\zeta\cdot\omega n\cdot(t-t0)}\cdot\cos[\omega d\cdot(t-t0)-\theta]\right]\right]\cdot\Phi(t-t0)+xh(t)$$

$$X := \begin{pmatrix} x0 \\ v0 \end{pmatrix} \quad D(t,X) := \begin{pmatrix} X_1 \\ -2\cdot\zeta\cdot\omega n\cdot X_1-\omega n^2\cdot X_0+\frac{F0}{m}\cdot\Phi(t-t0) \end{pmatrix} \quad F(t) := \frac{F0}{k}$$

$Z := \text{rkfixed}(X, 0, 30, 2000, D)$

$t := Z^{\langle 0\rangle} \quad x := \overrightarrow{xa(t)} \quad xn := Z^{\langle 1\rangle} \quad S := \overrightarrow{F(t)}$



c = 710

A value of c = 710 kg/s will do the job.

```
%Vibrations
%Numerical Solutions
%Problem #51

clc
clear
close all

%Numerical Solution

x0=[0.01;0];
tspan=[0 15];

[t,x]=ode45('prob51a',tspan,x0);

figure(1)
plot(t,x(:,1));
title('Problem #51');
xlabel('Time, sec.');
ylabel('Displacement, m');
hold on

%Analytical Solution

m=150;
c=0;
k=4000;
F=15;
w=sqrt(k/m);
d=c/(2*w*m);
wd=w*sqrt(1-d^2);
to=1;
phi=atan(d/sqrt(1-d^2));



%for t<to
t=linspace(0,1,10);
x0=0.01;
v0=0;
A=sqrt(v0^2+(x0*w)^2)/w;
theta=pi/2;
x=A.*sin(w.*t + theta);
plot(t,x,'*')

%for t>=to
t=linspace(1,15);
x2=F/k-F/(k*sqrt(1-d^2)).*exp(-d.*w.*(t-to)).*cos(wd.*(t-to)-phi);
x1=A.*sin(w.*t + theta);

x=x1+x2;
plot(t,x,'*');
legend('Numerical', 'Analytical')
%Clay
%Vibrations
%Solutions
```

```
%M-file for Prob #51

function dx=prob(t,x);
[rows, cols]=size(x);dx=zeros(rows, cols);
m=150;
c=0;
k=4000;
F=15;


if t<1
   dx(1)=x(2);
   dx(2)=-c/m*x(2)- k/m*x(1);
   else
      dx(1)=x(2);
   dx(2)=-c/m*x(2) - k/m*x(1)+ F/m;
end
```

3.59*. Solve Example 3.3.2, Figure 3.9 by numerically integrating rather then using analytical expressions, and plot the response.

Solution: Both Mathcad and Matlab solutions follow:

Prob 3.52

$x0 := 0.01 \quad v0 := 3 \quad \omega b := 3 \quad m := 1 \quad c := 10 \quad t0 := 0$

$k := 1000 \quad \omega n := \sqrt{\frac{k}{m}} \quad \zeta := \frac{c}{2\cdot\sqrt{k\cdot m}} \quad \omega n = 31.623 \quad Y := 0.05$

$\omega d := \omega n\cdot\sqrt{1-\zeta^2} \quad \omega d = 31.22499 \quad \theta := 1.53$

$A := 0.09341 \quad \phi := 0.1074$

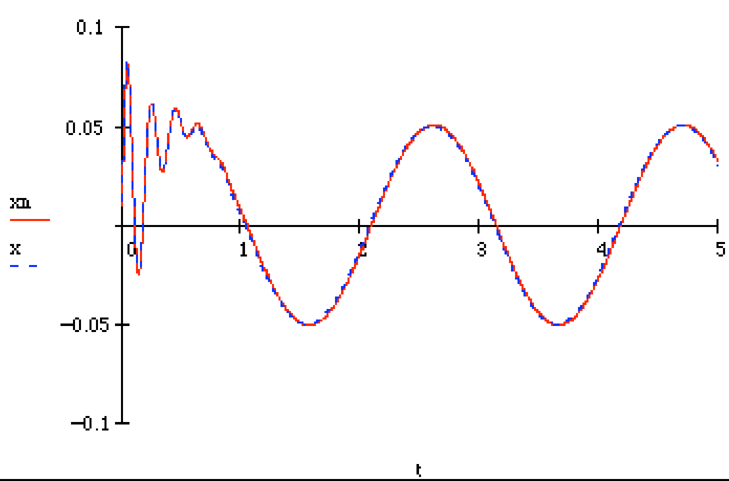
$xa(t) := A\cdot e^{-\zeta\cdot\omega n\cdot t}\sin(\omega d\cdot t+\phi) + 0.05\cdot\cos(3\cdot t-\theta)$

$X := \begin{bmatrix} x0 \\ v0 \end{bmatrix}$

$$D(t,X) := \begin{bmatrix} X_1 \\ -2\cdot\zeta\cdot\omega n\cdot X_1 - \omega n^2\cdot X_0 + \frac{c}{m}\cdot Y\cdot\omega b\cdot\cos(\omega b\cdot t) + \frac{k}{m}\cdot Y\cdot\sin(\omega b\cdot t) \end{bmatrix}$$

$Z := \mathrm{rkfixed}(X, 0, 6, 2000, D)$

$t := Z^{\langle 0\rangle} \quad x := \overrightarrow{xa(t)} \quad xn := Z^{\langle 1\rangle}$




```
%Numerical Solutions
%Problem #53
clc
clear
close all
%Numerical Solution
```

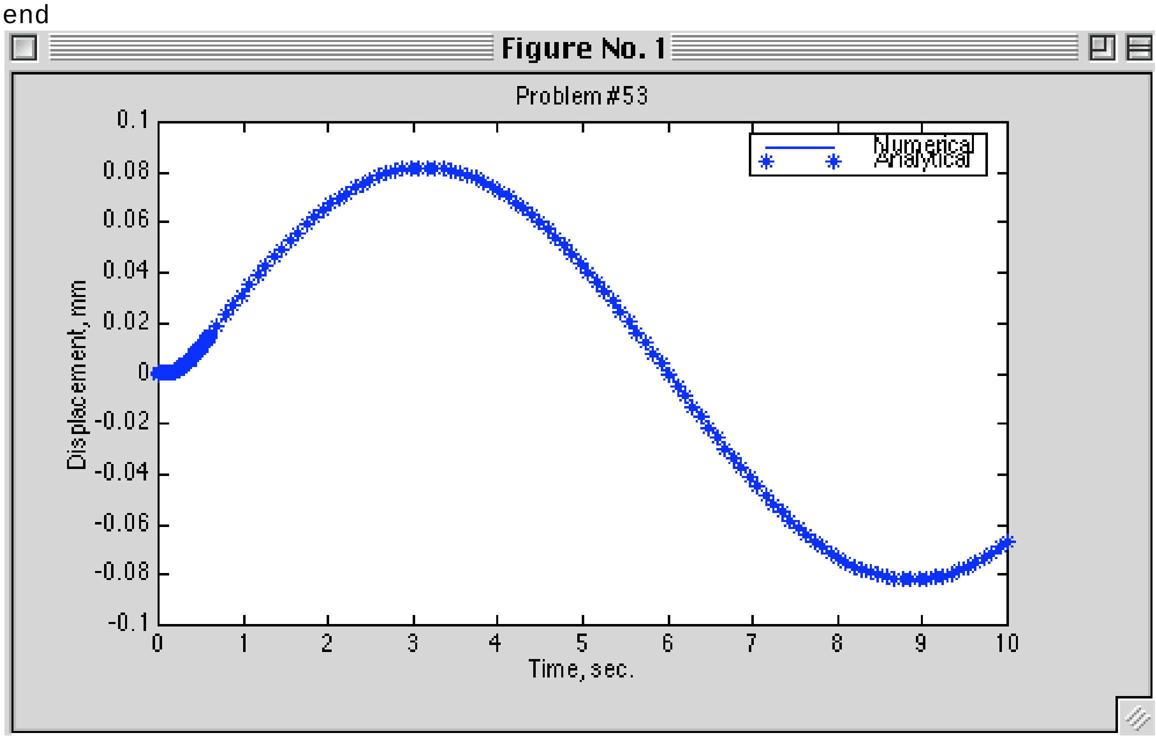
```
x0=[0;0];
tspan=[0 10];

[t,x]=ode45('prob53a',tspan,x0);

figure(1)
plot(t,x(:,1));
title('Problem #53');
xlabel('Time, sec.');
ylabel('Displacement, mm');
hold on

%Analytical Solution
t1=0.2;
t2=0.6;

%for t<to
t=linspace(0,t1);
x=2.5*t-4.56.*sin(0.548.*t);
plot(t,x,'*');

%for t1<t<t2
t=linspace(t1,t2);
x=0.75 - 1.25.*t + 6.84.*sin(0.548*(t-t1))- 4.56.*sin(0.548.*t);
plot(t,x,'*');

%for t2<t
t=linspace(t2,10);
x=6.84.*sin(0.548.*(t-t1))-2.28.*sin(0.548.*(t-t2))-
4.56.*sin(0.548.*t);
plot(t,x,'*');
legend('Numerical', 'Analytical')
%Clay
%Vibrations
%Solutions
%Clay
%Vibrations
%Solutions

%M-file for Prob #52

function dx=prob(t,x);
[rows, cols]=size(x);dx=zeros(rows, cols);
m=1;
c=10;
k=1000;
Y=0.05;
wb=3;

a=c*Y*wb;
b=k*Y;
alpha=atan(b/a);
AB=sqrt(a^2+b^2)/m;

dx(1)=x(2);
dx(2)=-c/m*x(2)- k/m*x(1)+ a/m*cos(wb*t) + b/m*sin(wb*t);
```

3.60*. Numerically simulate the response of the system of Problem 3.21 and plot the response.

Solution: The solution in Matlab is

```
%Clay
%Vibrations
%Numerical Solutions
%Problem #53

clc
clear
close all

%Numerical Solution

x0=[0;0];
tspan=[0 10];

[t,x]=ode45('prob53a',tspan,x0);

figure(1)
plot(t,x(:,1));
title('Problem #53');
xlabel('Time, sec.');
ylabel('Displacement, mm');
hold on

%Analytical Solution

t1=0.2;
t2=0.6;

%for t<to
t=linspace(0,t1);
x=2.5*t-4.56.*sin(0.548.*t);
plot(t,x,'*');

%for t1<t<t2
t=linspace(t1,t2);
x=0.75 - 1.25.*t + 6.84.*sin(0.548*(t-t1))- 4.56.*sin(0.548.*t);
plot(t,x,'*');

%for t2<t
t=linspace(t2,10);
x=6.84.*sin(0.548.*(t-t1))-2.28.*sin(0.548.*(t-t2))-
4.56.*sin(0.548.*t);
plot(t,x,'*');
legend('Numerical', 'Analytical')

%Clay
%Vibrations
%Solutions
```

```
%M-file for Prob #53

function dx=prob(t,x);
[rows, cols]=size(x);dx=zeros(rows, cols);

m=5000;
k=1.5e3;
ymax=0.5;
F=k*ymax;
t1=0.2;
t2=0.6;

if t<t1
   dx(1)=x(2);
   dx(2)= - k/m*x(1)+ F/m*(t/t1);
elseif t<t2 & t>t1
      dx(1)=x(2);
   dx(2)= - k/m*x(1)+ F/(2*t1*m)*(t2-t);
else
   dx(1)=x(2);
   dx(2)= - k/m*x(1);
end
```

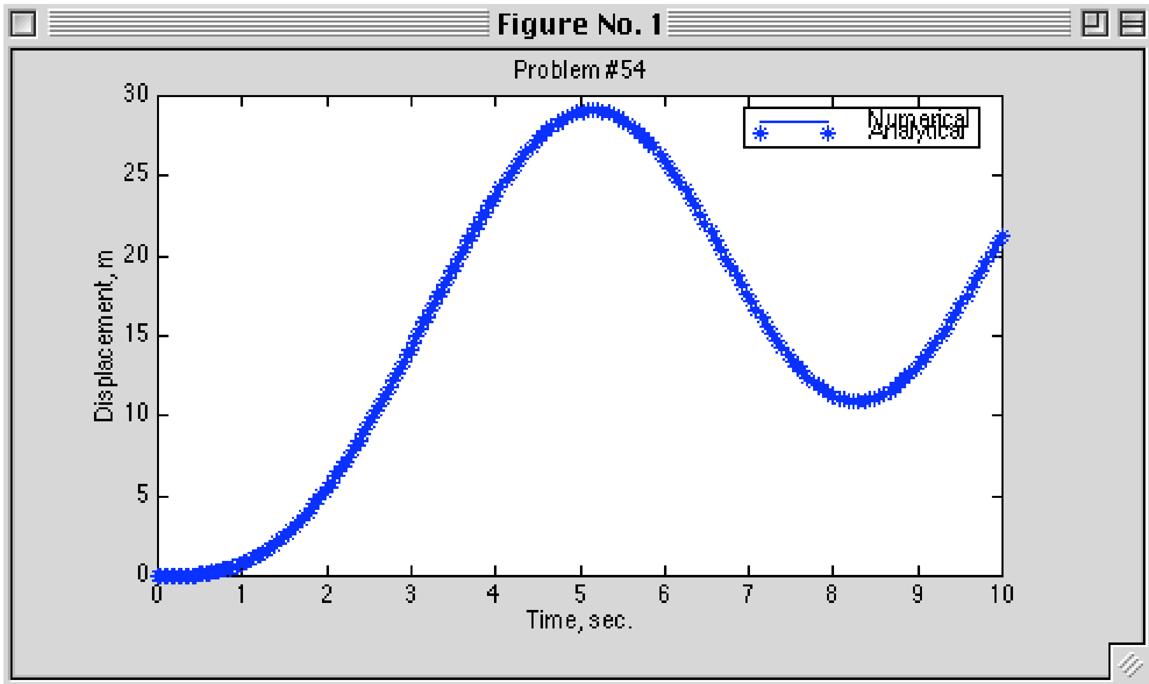


3.61*. Numerically simulate the response of the system of Problem 3.18 and plot the response.

Solution: The solution in Matlab is

```
%Clay
%Vibrations
%Numerical Solutions
%Problem #54

clc
clear
close all

%Numerical Solution

x0=[0;0];
tspan=[0 10];

[t,x]=ode45('prob54a',tspan,x0);

figure(1)
plot(t,x(:,1));
title('Problem #54');
xlabel('Time, sec.');
ylabel('Displacement, m');
hold on

%Analytical Solution

to=4;

%for t<to
t=linspace(0,to);
x=5*(t-sin(t));
plot(t,x,'*');

%for t>=to
t=linspace(to,10);
x=5*(sin(t-to)-sin(t))+20;

plot(t,x,'*');
legend('Numerical', 'Analytical')
%Clay
%Vibrations
%Solutions

%M-file for Prob #54


function dx=prob(t,x);
[rows, cols]=size(x);dx=zeros(rows, cols);

m=1;
k=1;
F=20;
```

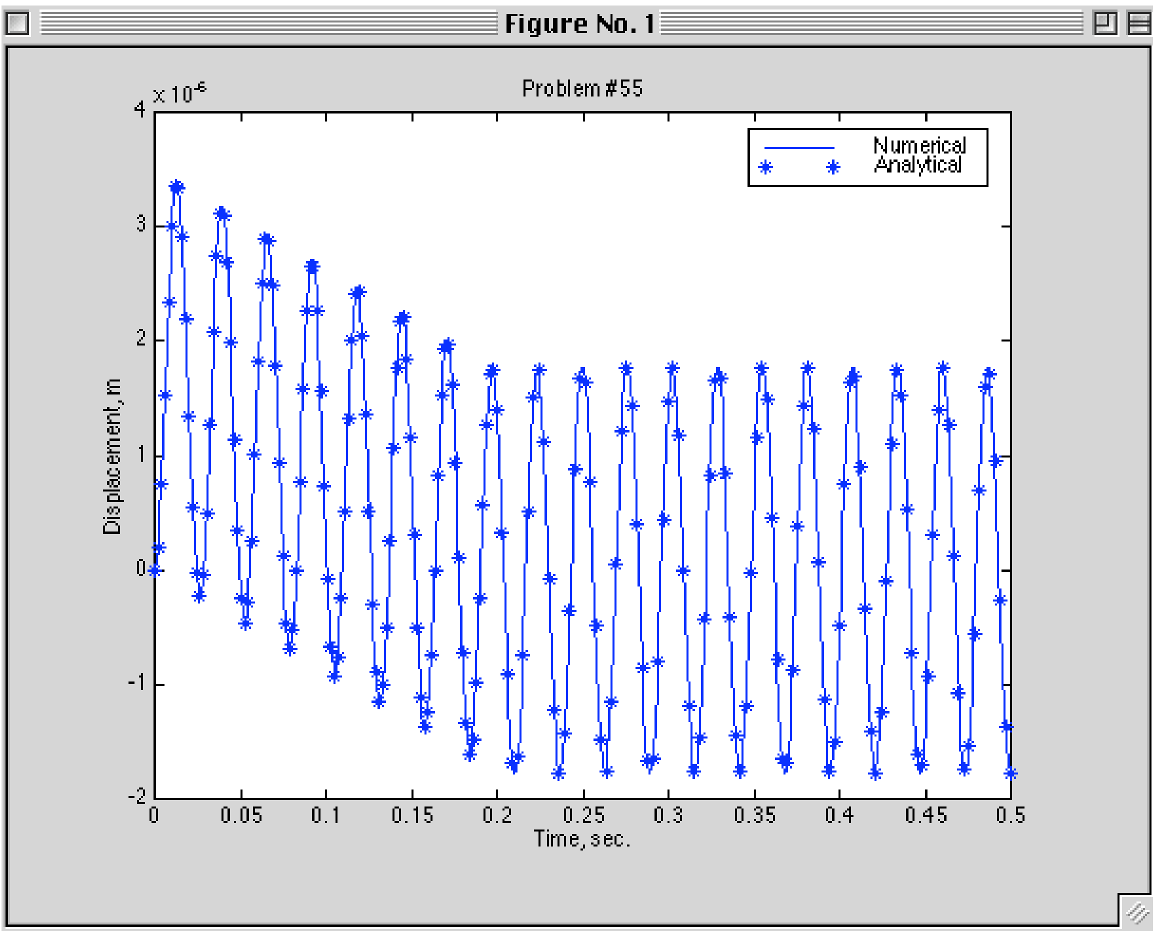
```
to=4;

if t<to
   dx(1)=x(2);
   dx(2)= - k/m*x(1)+ F/m*(t/to);
else
      dx(1)=x(2);
   dx(2)= - k/m*x(1)+ F/m;
end
```



3.62*. Numerically simulate the response of the system of Problem 3.19 for a 2 meter concrete wall with cross section 0.03 m^2 and mass modeled as lumped at the end of 1000 kg. Use F_0 = 100 N, and plot the response for the case t_0 =0.25 s.
Solution The solution in Matlab is:

```
%Numerical Solutions
%Problem #3.62

clc
clear
close all

%Numerical Solution

x0=[0;0];
tspan=[0 0.5];

[t,x]=ode45('prob55a',tspan,x0);

figure(1)
plot(t,x(:,1));
title('Problem #55');
xlabel('Time, sec.');
ylabel('Displacement, m');
hold on

%Analytical Solution

m=1000;
E=3.8e9;
A=0.03;
l=2;
k=E*A/l;
F=100;
w=sqrt(k/m);
to=0.25;


%for t<to
t=linspace(0,to);
x=F/k*(1-cos(w*t))+ F/(to*k)*(1/w*sin(w*t)-t);
plot(t,x,'*');

%for t>=to
t=linspace(to,0.5);
x=-F/k*cos(w*t)- F/(w*k*to)*(sin(w*(t-to))-sin(w*t));
plot(t,x,'*');
legend('Numerical', 'Analytical')
%Clay
%Vibrations
%Solutions

%M-file for Prob #3.62
```

```
function dx=prob(t,x);
[rows, cols]=size(x);dx=zeros(rows, cols);

m=1000;
E=3.8e9;
A=0.03;
l=2;
k=E*A/l;
F=100;
w=sqrt(k/m);
to=0.25;


if t<to
   dx(1)=x(2);
   dx(2)= - k/m*x(1) + F/m*(1-t/to);
   else
      dx(1)=x(2);
   dx(2)= - k/m*x(1);
end
```



3.63*. Numerically simulate the response of the system of Problem 3.20 and plot the response.

Solution The solution in Matlab is:

```
%Clay
%Vibrations
%Numerical Solutions
%Problem #56

clc
clear
close all

%Numerical Solution

x0=[0;0];
tspan=[0 2];

[t,x]=ode45('prob56a',tspan,x0);

figure(1)
plot(t,x(:,1));
title('Problem #56');
xlabel('Time, sec.');
ylabel('Displacement, m');
hold on

%Analytical Solution

t=linspace(0,2);
x=0.5*t-0.05*sin(10*t);
plot(t,x,'*');
legend('Numerical', 'Analytical')


%Clay
%Vibrations
%Solutions

%M-file for Prob #56


function dx=prob(t,x);
[rows, cols]=size(x);dx=zeros(rows, cols);

m=1;
k=100;
F=50;


 dx(1)=x(2);
 dx(2)= - k/m*x(1) + F/m*(t);
```

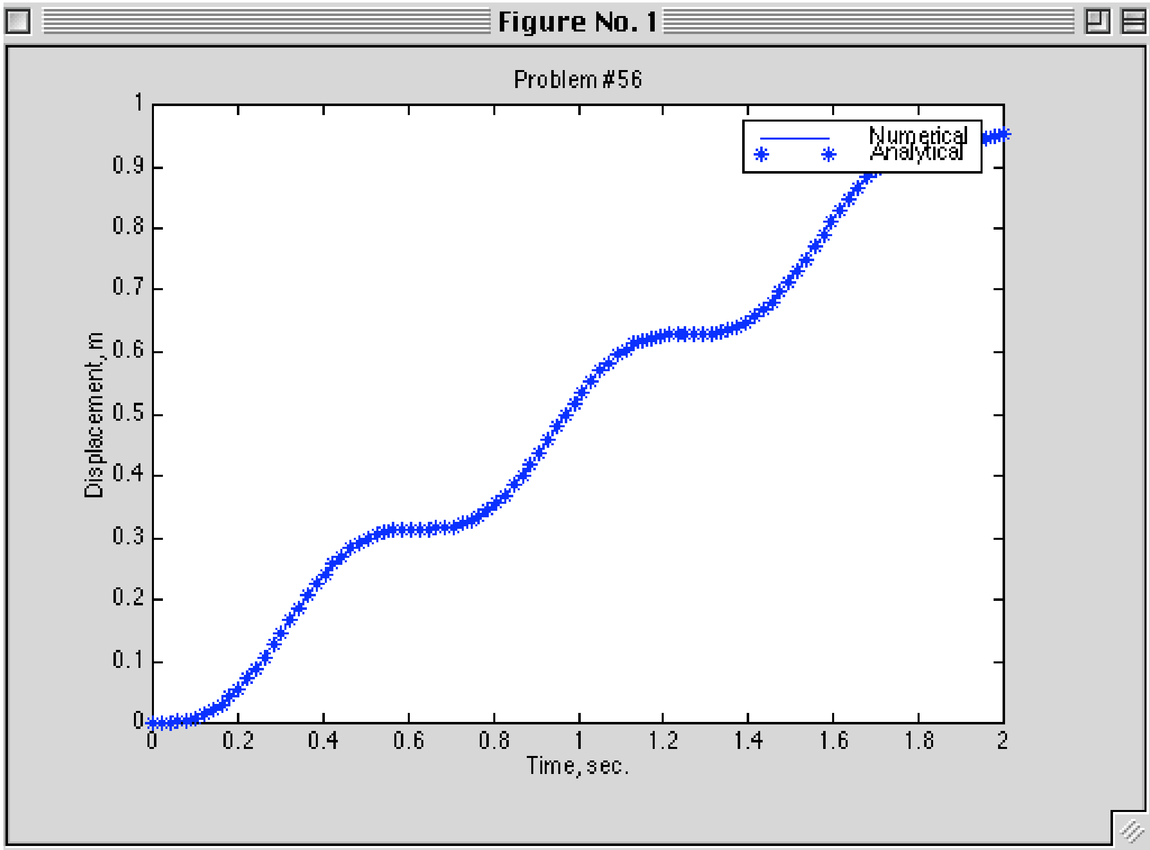


Figure No. 1
Problem #56
Numerical
Analytical
Displacement, m
Time, sec.
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2

3.64*. Compute and plot the response of the system of following system using numerical integration:

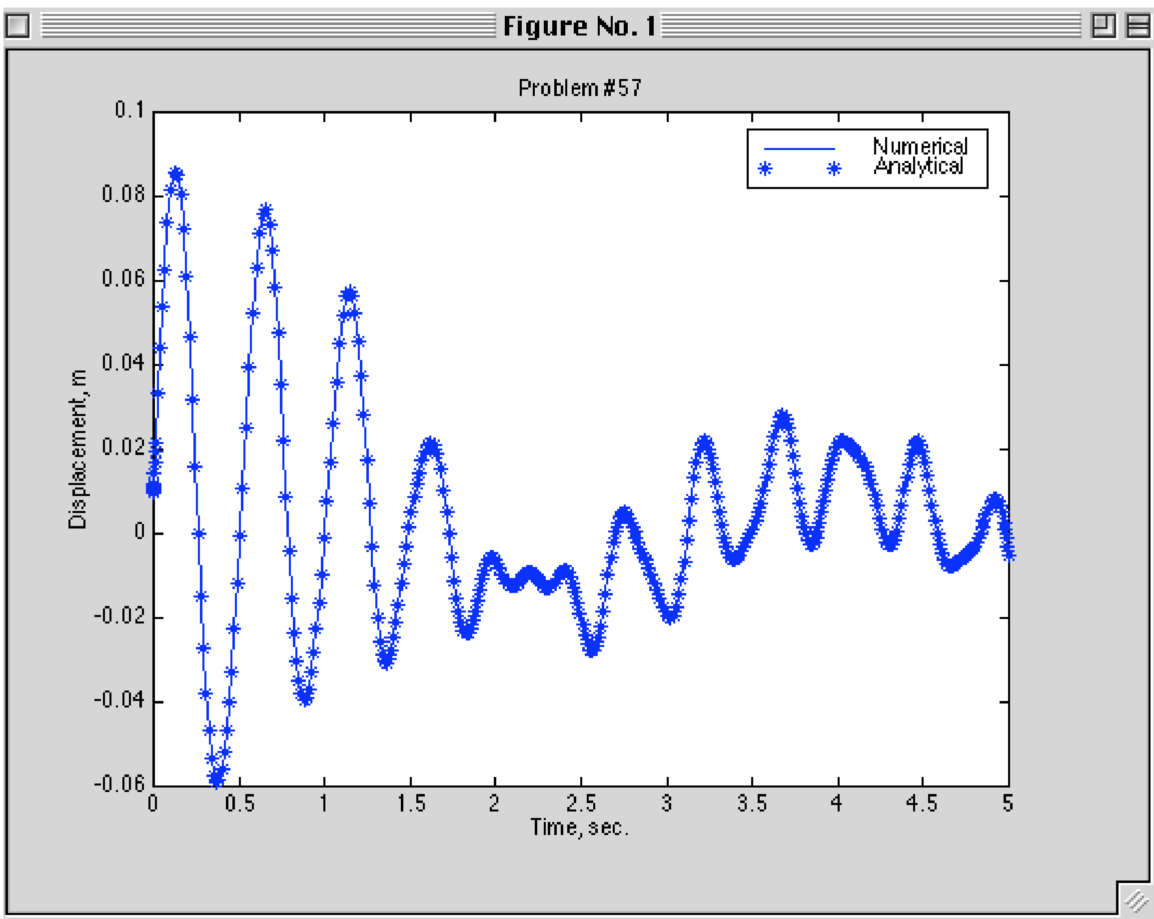
$$10\ddot{x}(t) + 20\dot{x}(t) + 1500x(t) = 20\sin 25t + 10\sin 15t + 20\sin 2t$$

with initial conditions of x_0 = 0.01 m and v_0 = 1.0 m/s.

Solution The solution in Matlab is:

```
%Clay
%Vibrations
%Numerical Solutions
%Problem #57

clc
clear
close all

%Numerical Solution

x0=[0.01;1];
tspan=[0 5];

[t,x]=ode45('prob57a',tspan,x0);

figure(1)
plot(t,x(:,1));
title('Problem #57');
xlabel('Time, sec.');
ylabel('Displacement, m');
hold on

%Analytical Solution

m=10;
c=20;
k=1500;
w=sqrt(k/m);
d=c/(2*w*m);
wd=w*sqrt(1-d^2);


Y1=0.00419;
ph1=3.04;
wb1=25;

Y2=0.01238;
ph2=2.77;
wb2=15;

Y3=0.01369;
ph3=0.0268;
wb3=2;

A=0.1047;
phi=0.1465;
```

```
x=A.*exp(-d*w.*t).*sin(wd*t+phi)+ Y1.*sin(wb1*t-ph1) + Y2*sin(wb2*t-
ph2) + Y3*sin(wb3*t-ph3);

plot(t,x,'*')
legend('Numerical', 'Analytical')
%Clay
%Vibrations
%Solutions

%M-file for Prob #57


function dx=prob(t,x);
[rows, cols]=size(x);dx=zeros(rows, cols);

m=10;
c=20;
k=1500;


dx(1)=x(2);
dx(2)= -c/m*x(2) - k/m*x(1) + 20/m*sin(25*t) + 10/m*sin(15*t) +
20/m*sin(2*t) ;
```



Problems and Solutions Section 3.10 (3.65 through 3.71)

3.65*. Compute the response of the system in Figure 3.26 for the case that the damping is linear viscous and the spring is a nonlinear soft spring of the form

$$k(x) = kx - k_1 x^3$$

and the system is subject to a excitation of the form (t_1 = 1.5 and t_2 = 1.6)

$$F(t) = 1500\left[\Phi(t - t_1) - \Phi(t - t_2)\right] \text{ N}$$

and initial conditions of x_0 = 0.01 m and v_0 = 1.0 m/s. The system has a mass of 100 kg, a damping coefficient of 30 kg/s and a linear stiffness coefficient of 2000 N/m. The value of k_1 is taken to be 300 N/m^3. Compute the solution and compare it to the linear solution (k_1 = 0). Which system has the largest magnitude? Compare your solution to that of Example 3.10.1.

Solution: The solution in Mathcad is

$x0 := 0.01$ $v0 := 1$ $m := 100$ $k := 2000$ $k1 := 300$ $c := 30$

$\omega n := \sqrt{\dfrac{k}{m}}$ $\zeta := \dfrac{c}{2\sqrt{k \cdot m}}$ $\alpha := \dfrac{k1}{m}$ $F0 := 1500$ $t1 := 1.5$ $t2 := 1.6$

$f0 := \dfrac{F0}{m}$ $\zeta = 0.034$ $\alpha = 3$

$X := \begin{pmatrix} x0 \\ v0 \end{pmatrix}$ $Y := X$

$$f(t) := f0 \cdot \Phi(t - t1) - f0 \cdot \Phi(t - t2)$$

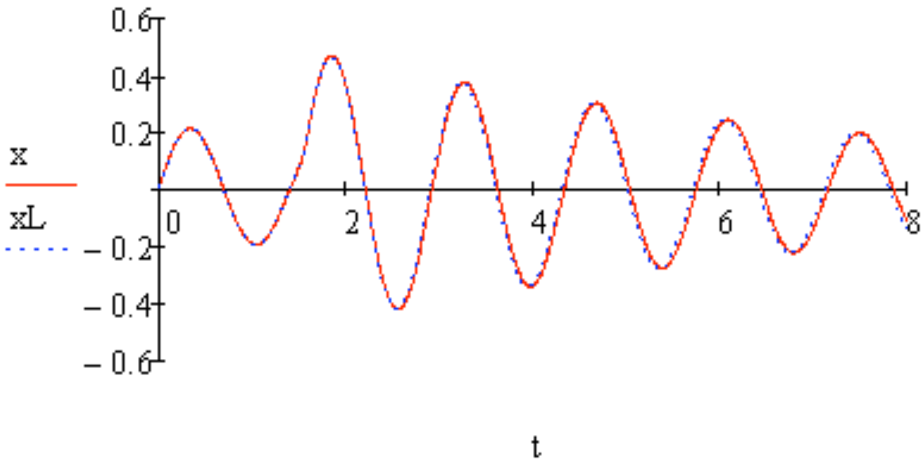
$$D(t, X) := \begin{bmatrix} X_1 \\ -2 \cdot \zeta \cdot \omega n \cdot X_1 - \omega n^2 \cdot X_0 + \left[\alpha \cdot (X_0)^3 + f(t)\right] \end{bmatrix}$$

$$L(t, Y) := \begin{bmatrix} Y_1 \\ \left(-2 \cdot \zeta \cdot \omega n \cdot Y_1 - \omega n^2 \cdot Y_0\right) + f(t) \end{bmatrix}$$

$Z := \text{rkfixed}(X, 0, 10, 2000, D)$

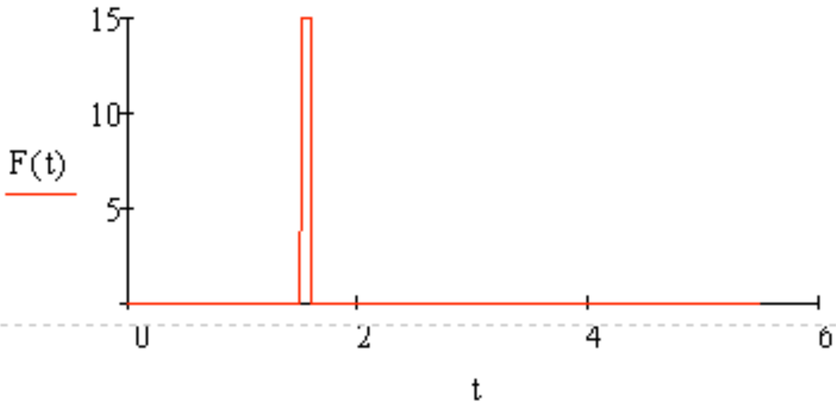
$t := Z^{\langle 0 \rangle}$ $x := Z^{\langle 1 \rangle}$ $W := \text{rkfixed}(Y, 0, 10, 2000, L)$

$xL := W^{\langle 1 \rangle}$



$t := 0, 0.001 .. 5.5$

$F(t) := f0 \cdot \Phi(t - t1) - f0 \cdot \Phi(t - t2)$



Note that for this load the load, which is more like an impulse, the linear and nonlinear responses are similar, whereas in Example 3.10.1 the applied load is a "wider" impulse and the linear and nonlinear responses differ quite a bit.

3.66*. Compute the response of the system in Figure 3.26 for the case that the damping is linear viscous and the spring is a nonlinear soft spring of the form

$$k(x) = kx - k_1 x^3$$

and the system is subject to a excitation of the form (t_1 = 1.5 and t_2 = 1.6)

$$F(t) = 1500\left[\Phi(t - t_1) - \Phi(t - t_2)\right] \text{ N}$$

and initial conditions of x_0 = 0.01 m and v_0 = 1.0 m/s. The system has a mass of 100 kg, a damping coefficient of 30 kg/s and a linear stiffness coefficient of 2000 N/m. The value of k_1 is taken to be 300 N/m^3. Compute the solution and compare it to the linear solution (k_1 = 0). How different are the linear and nonlinear responses? Repeat this for t_2 = 2. What can you say regarding the effect of the time length of the pulse?

Solution: The solution in Mathcad for the case t_2 = 1.6 is

Prob 3.59

$x0 := 0.01 \quad v0 := 1 \quad m := 100 \quad k := 2000 \quad c := 30$

$\alpha := 3 \quad F0 := 1500 \quad t1 := 1.5 \quad t2 := 1.6$

$$\omega n := \sqrt{\frac{k}{m}} \qquad \zeta := \frac{c}{2\sqrt{k \cdot m}} \qquad f0 := \frac{F0}{m} \qquad \zeta = 0.034$$

$$X := \begin{bmatrix} x0 \\ v0 \end{bmatrix} \qquad Y := X$$

$$f(t) := f0 \cdot \Phi(t - t1) - f0 \cdot \Phi(t - t2)$$

$$D(t, X) := \begin{bmatrix} X_1 \\ -2 \cdot \zeta \cdot \omega n \cdot X_1 - \omega n^2 \cdot X_0 + \left[\alpha \cdot (X_0)^3 + f(t)\right] \end{bmatrix}$$

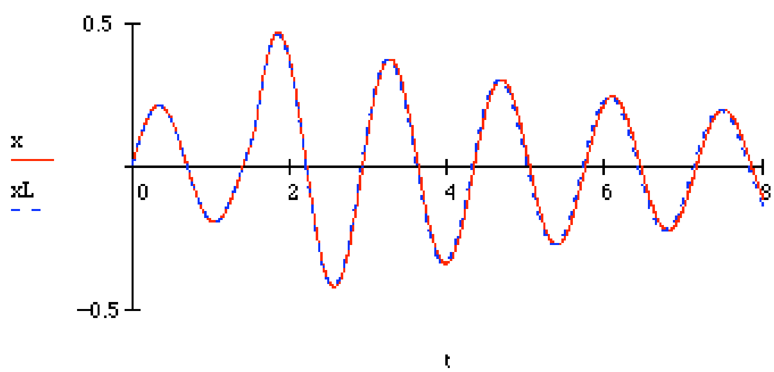
$$L(t, Y) := \begin{bmatrix} Y_1 \\ \left(-2 \cdot \zeta \cdot \omega n \cdot Y_1 - \omega n^2 \cdot Y_0\right) + f(t) \end{bmatrix}$$

$Z := \text{rkfixed}(X, 0, 10, 2000, D)$

$W := \text{rkfixed}(Y, 0, 10, 2000, L)$

$t := Z^{\langle 0 \rangle} \qquad x := Z^{\langle 1 \rangle}$

$xL := W^{\langle 1 \rangle}$



Note in this case the linear response is almost the same as the nonlinear response. Next changing the time of the pulse input to $t_2 = 2$ yields the following:

Prob 3.59

$x0 := 0.01$ $v0 := 1$ $m := 100$ $k := 2000$ $c := 30$

$\alpha := 3$ $F0 := 1500$ $t1 := 1.5$ $t2 := 2$

$\omega n := \sqrt{\dfrac{k}{m}}$ $\zeta := \dfrac{c}{2\sqrt{k \cdot m}}$ $f0 := \dfrac{F0}{m}$ $\zeta = 0.034$

$X := \begin{bmatrix} x0 \\ v0 \end{bmatrix}$ $Y := X$

$$f(t) := f0 \cdot \Phi(t - t1) - f0 \cdot \Phi(t - t2)$$

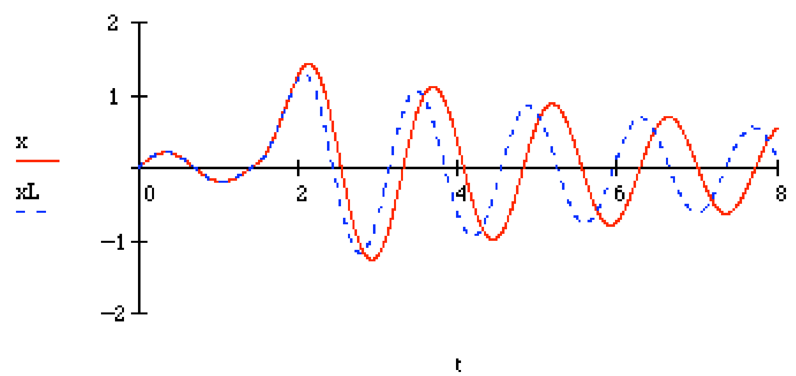
$$D(t, X) := \begin{bmatrix} X_1 \\ -2 \cdot \zeta \cdot \omega n \cdot X_1 - \omega n^2 \cdot X_0 + \left[\alpha \cdot (X_0)^3 + f(t)\right] \end{bmatrix}$$

$$L(t, Y) := \begin{bmatrix} Y_1 \\ \left(-2 \cdot \zeta \cdot \omega n \cdot Y_1 - \omega n^2 \cdot Y_0\right) + f(t) \end{bmatrix}$$

$Z := \text{rkfixed}(X, 0, 10, 2000, D)$

$t := Z^{\langle 0 \rangle}$ $x := Z^{\langle 1 \rangle}$ $W := \text{rkfixed}(Y, 0, 10, 2000, L)$

$xL := W^{\langle 1 \rangle}$



Note that as the step input last for a longer time, the response of the linear and the nonlinear becomes much different.

3.67*. Compute the response of the system in Figure 3.26 for the case that the damping is linear viscous and the spring stiffness is of the form

$$k(x) = kx - k_1x^2$$

and the system is subject to a excitation of the form (t_1 = 1.5 and t_2 = 2.5)

$$F(t) = 1500\left[\Phi(t-t_1) - \Phi(t-t_2)\right] \text{ N}$$

initial conditions of x_0 = 0.01 m and v_0 = 1 m/s. The system has a mass of 100 kg, a damping coefficient of 30 kg/s and a linear stiffness coefficient of 2000 N/m. The value of k_1 is taken to be 450 N/m^3. Which system has the largest magnitude?

Solution: The solution is computed in Mathcad as follows:

Prob 3.60

$x0 := 0.01$ $\quad v0 := 1$ $\quad m := 100$ $\quad k := 2000$ $\quad c := 30$

$\alpha := 4.5$ $\quad F0 := 1500$ $\quad t1 := 1.5$ $\quad t2 := 2.5$

$$\omega n := \sqrt{\frac{k}{m}} \qquad \zeta := \frac{c}{2\sqrt{k \cdot m}} \qquad f0 := \frac{F0}{m} \qquad \zeta = 0.034$$

$$X := \begin{bmatrix} x0 \\ v0 \end{bmatrix} \qquad Y := X$$

$$f(t) := f0 \cdot \Phi(t - t1) - f0 \cdot \Phi(t - t2)$$

$$D(t, X) := \begin{bmatrix} X_1 \\ -2 \cdot \zeta \cdot \omega n \cdot X_1 - \omega n^2 \cdot X_0 + \left[\alpha \cdot (X_0)^2 + f(t)\right] \end{bmatrix}$$

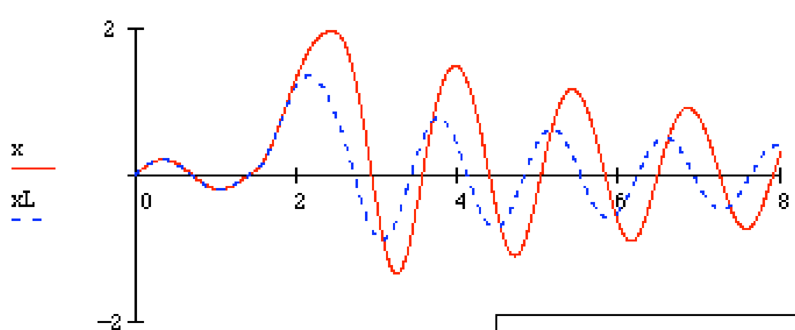
$$L(t, Y) := \begin{bmatrix} Y_1 \\ \left(-2 \cdot \zeta \cdot \omega n \cdot Y_1 - \omega n^2 \cdot Y_0\right) + f(t) \end{bmatrix}$$

$Z := \text{rkfixed}(X, 0, 10, 2000, D)$

$W := \text{rkfixed}(Y, 0, 10, 2000, L)$

$t := Z^{\langle 0 \rangle}$ $\quad x := Z^{\langle 1 \rangle}$

$xL := W^{\langle 1 \rangle}$



Note that the linear response under predicts the actual response

3.68*. Compute the response of the system in Figure 3.26 for the case that the damping is linear viscous and the spring stiffness is of the form

$$k(x) = kx + k_1x^2$$

and the system is subject to a excitation of the form (t_1 = 1.5 and t_2 = 2.5)

$$F(t) = 1500\left[\Phi(t - t_1) - \Phi(t - t_2)\right] \text{ N}$$

initial conditions of x_0 = 0.01 m and v_0 = 1 m/s. The system has a mass of 100 kg, a damping coefficient of 30 kg/s and a linear stiffness coefficient of 2000 N/m. The value of k_1 is taken to be 450 N/m^3. Which system has the largest magnitude?

Solution: The solution is calculated in Mathcad as follows:

$x0 := 0.01 \quad m := 100 \quad k := 2000$

$\omega n := \sqrt{\frac{k}{m}} \quad v0 := 1 \quad \alpha := 4.5 \quad F0 := 1500 \quad t1 := 1.5 \quad c := 30$

$X := \begin{pmatrix} x0 \\ v0 \end{pmatrix} \quad Y := X \quad \zeta := \frac{c}{2\sqrt{k \cdot m}} \quad f0 := \frac{F0}{m} \quad t2 := 2.5 \quad \zeta = 0.034$

$f(t) := f0 \cdot \Phi(t - t1) - f0 \cdot \Phi(t - t2)$

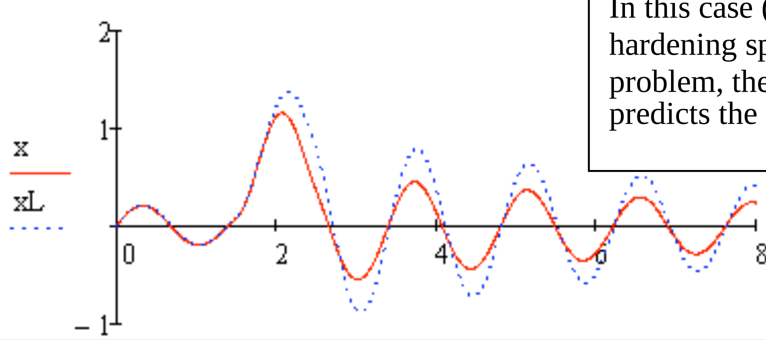
$$D(t, X) := \begin{bmatrix} X_1 \\ -2 \cdot \zeta \cdot \omega n \cdot X_1 - \omega n^2 \cdot X_0 + \left[-\alpha \cdot (X_0)^2 + f(t)\right] \end{bmatrix}$$

$Z := \text{rkfixed}(X, 0, 10, 2000, D)$

$$L(t, Y) := \begin{bmatrix} Y_1 \\ \left(-2 \cdot \zeta \cdot \omega n \cdot Y_1 - \omega n^2 \cdot Y_0\right) + f(t) \end{bmatrix}$$

$t := Z^{\langle 0 \rangle} \quad x := Z^{\langle 1 \rangle}$

$xL := W^{\langle 1 \rangle} \quad W := \text{rkfixed}(Y, 0, 10, 2000, L)$



In this case (compared to the hardening spring of the previous problem, the linear response over predicts the time history.

3.69*. Compute the response of the system in Figure 3.26 for the case that the damping is linear viscous and the spring stiffness is of the form

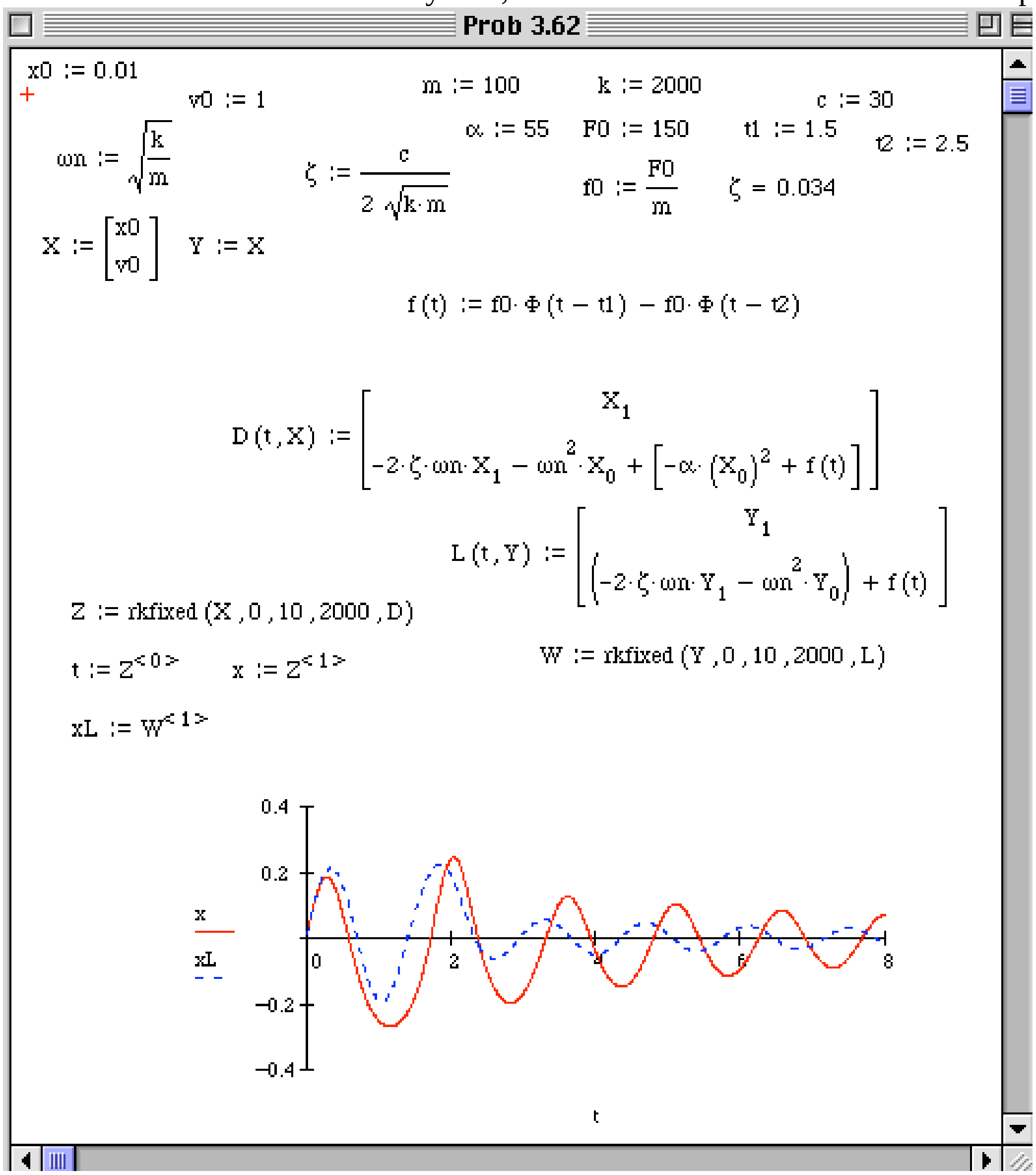
$$k(x) = kx - k_1x^2$$

and the system is subject to a excitation of the form (t_1 = 1.5 and t_2 = 2.5)

$$F(t) = 150\left[\Phi(t - t_1) - \Phi(t - t_2)\right] \text{ N}$$

initial conditions of x_0 = 0.01 m and v_0 = 1 m/s. The system has a mass of 100 kg, a damping coefficient of 30 kg/s and a linear stiffness coefficient of 2000 N/m. The value of k_1 is taken to be 5500 N/m^3. Which system has the largest magnitude transient? Which has the largest magnitude in steady state?

Solution: The solution in Mathcad is given below. Note that the linear system response is less than that of the nonlinear system, and hence underestimates the actual response.



3.70*. Compare the forced response of a system with velocity squared damping as defined in equation (2.129) using numerical simulation of the nonlinear equation to that of the response of the linear system obtained using equivalent viscous damping as defined by equation (2.131). Use as initial conditions, x_0 = 0.01 m and v_0 = 0.1 m/s with a mass of 10 kg, stiffness of 25 N/m, applied force of the form (t_1 = 1.5 and t_2 = 2.5)

$$F(t) = 15[\Phi(t - t_1) - \Phi(t - t_2)] \text{ N}$$

and drag coefficient of α = 25.

Solution: The solution calculated in Mathcad is given in the follow:

$x0 := 0.01 \quad v0 := 0.1 \quad m := 10 \quad k := 25 \quad \alpha := 25 \quad F0 := 15$

$\omega n := \sqrt{\frac{k}{m}} \quad f0 := \frac{F0}{m}$

$t1 := 1.5 \quad t2 := 2.5 \quad ceq := \sqrt{\frac{8\cdot\alpha\cdot m}{3\cdot\pi}\cdot f0}$

$\zeta := \frac{ceq}{2\sqrt{k\cdot m}} \quad f(t) := f0\cdot\Phi(t - t1) - f0\cdot\Phi(t - t2)$

$$X := \begin{pmatrix} x0 \\ v0 \end{pmatrix} \quad D(t, X) := \begin{bmatrix} X_1 \\ -\omega n^2\cdot X_0 - \frac{\alpha}{m}\cdot(X_1)^2\cdot\frac{X_1}{|X_1|} + f(t) \end{bmatrix} \quad Y := X$$

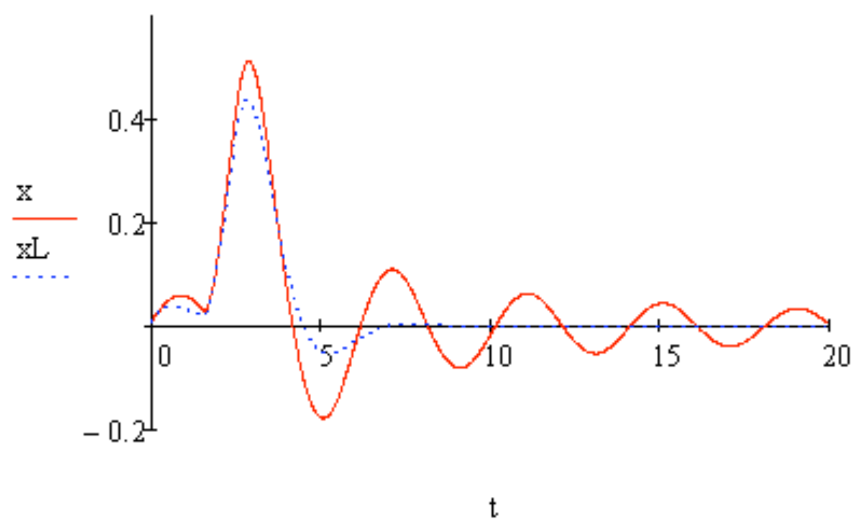
$\zeta = 0.564$

$$D1(t, Y) := \begin{bmatrix} Y_1 \\ \left(-2\cdot\zeta\cdot\omega n\cdot Y_1 - \omega n^2\cdot Y_0\right) + f(t) \end{bmatrix}$$

$Z := \text{rkfixed}(X, 0, 20, 2000, D)$

$t := Z^{\langle 0\rangle} \quad x := Z^{\langle 1\rangle} \quad W := \text{rkfixed}(Y, 0, 20, 2000, D1)$

$xL := W^{\langle 1\rangle}$



Note that the linear solution is very different from the nonlinear solution and dies out more rapidly.

3.71*. Compare the forced response of a system with structural damping (see table 2.2) using numerical simulation of the nonlinear equation to that of the response of the linear system obtained using equivalent viscous damping as defined in Table 2.2. Use the initial conditions, $x_0 = 0.01$ m and $v_0 = 0.1$ m/s with a mass of 10 kg, stiffness of 25 N/m, applied force of the form ($t_1 = 1.5$ and $t_2 = 2.5$)

$$F(t) = 15\left[\Phi(t - t_1) - \Phi(t - t_2)\right] \text{ N}$$

and solid damping coefficient of $b = 8$. Does the equivalent viscous damping linearization, over estimate the response or under estimate it?

Solution: The solution is calculated in Mathcad as follows. Note that the linear solution is an over estimate of the nonlinear response in this case.

Prob 3.64

$x0 := 0.01 \quad v0 := 0.1 \quad m := 10 \quad k := 25 \quad b := 8 \quad F0 := 15$

$\omega n := \sqrt{\frac{k}{m}} \qquad f0 := \frac{F0}{m} \qquad ceq := \frac{2 \cdot b}{\pi \cdot \omega n}$

$t1 := .5 \qquad t2 := 2.5$

$\zeta := \frac{ceq}{2\sqrt{k \cdot m}} \qquad f(t) := f0 \cdot \Phi(t - t1) - f0 \cdot \Phi(t - t2)$

$X := \begin{bmatrix} x0 \\ v0 \end{bmatrix} \qquad Y := X$

$$D(t, X) := \begin{bmatrix} X_1 \\ -\omega n^2 \cdot X_0 - \frac{b}{m} \cdot \left|X_0\right| \cdot \frac{X_1}{\left|X_1\right|} + f(t) \end{bmatrix}$$

$$L(t, Y) := \begin{bmatrix} Y_1 \\ \left(-2 \cdot \zeta \cdot \omega n \cdot Y_1 - \omega n^2 \cdot Y_0\right) + f(t) \end{bmatrix}$$

$Z := \mathrm{rkfixed}(X, 0, 20, 2000, D)$

$t := Z^{\langle 0 \rangle} \qquad x := Z^{\langle 1 \rangle} \qquad xL := W^{\langle 1 \rangle} \qquad W := \mathrm{rkfixed}(Y, 0, 20, 2000, L)$