Problems and Solutions Section 2.1 (2.1 through 2.15)

2.1 To familiarize yourself with the nature of the forced response, plot the solution of a forced response of equation (2.2) with ω = 2 rad/s, given by equation (2.11) for a variety of values of the initial conditions and ω_n as given in the following chart:

Case	x_0	v_0	f_0	ω_n
1	0.1	0.1	0.1	1
2	-0.1	0.1	0.1	1
3	0.1	0.1	1.0	1
4	0.1	0.1	0.1	2.1
5	1	0.1	0.1	1

Solution: Given: ω = 2 rad/sec.
From equation (2.11):

$$x(t) = \frac{v_0}{\omega_n}\sin\,\omega_n t + (x_0 - \frac{f_0}{\omega_n^2 - \omega^2})\cos\,\omega_n t + \frac{f_0}{\omega_n^2 - \omega^2}\cos\omega t$$

Insert the values of x_0, v_0, f_0, and ω_n for each of the five cases.

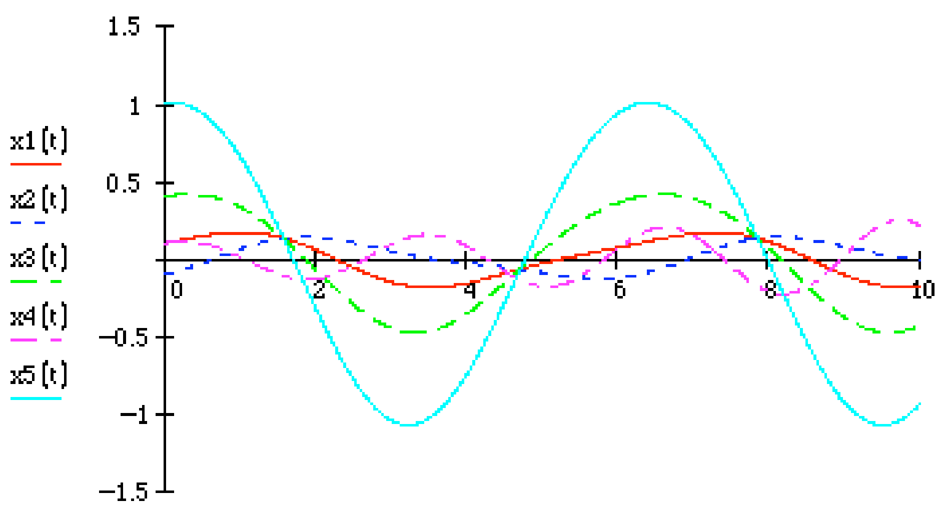
$$x1(t) := 0.1\cdot\sin(t) + 0.133\cdot\cos(t) - 0.0333\cdot\cos(2\cdot t)$$

$$x2(t) := 0.1\cdot\sin(t) - 0.0667\cdot\cos(t) - 0.0333\cdot\cos(2\cdot t)$$

$$x3(t) := 0.1\cdot\sin(t) + 0.433\cdot\cos(t) - 0.0333\cdot\cos(2\cdot t)$$

$$x4(t) := 0.0467\cdot\sin(2.1\cdot t) + 0.244\cdot\cos(2\cdot t) - 0.144\cdot\cos(2.1\cdot t)$$

$$x5(t) := 0.1\cdot\sin(t) + 1.033\cdot\cos(t) - 0.0333\cdot\cos(2\cdot t)$$



2.2 Repeat the calculation made in Example 2.1.1 for the mass of a simple spring-mass system where the mass of the spring is considered and known to be 1 kg.

Solution: Given: m_{sp} = 1 kg, Example 1.4.4 yields that the effective mass is

$$m_e = m + \frac{m_{sp}}{3} = 10 + \frac{1}{3} = 10.333 \text{ kg.}$$

Thus the natural frequency, X and the coefficients in equation (2.11) for the system now become

$$\omega_n = \sqrt{\frac{1000}{10 + 1/3}} = 9.837 \text{ rad/s}, \ \omega = 2\omega_n = 19.675 \text{ rad/s}$$

$$X = \frac{f_0}{\omega_n^2 - \omega^2} = \frac{2.338}{9.837^2 - 19.675^2} = -8.053 \times 10^{-3} \text{ m}, \ \frac{v_0}{\omega_n} = 0.02033 \text{ m}$$

Thus the response as given by equation (2.11) is

$$x(t) = 0.02033 \sin 9.837t + 8.053 \times 10^{-3} (\cos 9.837t - \cos 19.675t) \text{ m}$$

2.3 A spring-mass system is driven from rest harmonically such that the displacement response exhibits a beat of period of 0.2π s. The period of oscillation is measured to be 0.02π s. Calculate the natural frequency and the driving frequency of the system.

Solution: Given: Beat period: $T_b = 0.2\pi$ s, Oscillation period: $T_0 = 0.02\pi$ s

Equation (2.13): $$x(t) = \frac{2 f_0}{\omega_n^2 - \omega^2} \sin\left[\frac{\omega_n - \omega}{2} t\right] \sin\left[\frac{\omega_n + \omega}{2} t\right]$$

So, $$T_b = 0.2\pi = \frac{4\pi}{\omega_n - \omega}$$

$$\omega_n - \omega = \frac{4\pi}{0.2\pi} = 20 \text{ rad/s}$$

$$T_0 = 0.02\pi = \frac{4\pi}{\omega_n + \omega}$$

$$\omega_n + \omega = \frac{4\pi}{0.02\pi} = 200 \text{ rad/s}$$

Solving for ω_n and ω gives:
Natural frequency: ω_n = 110 rad/s
Driving frequency: ω = 90 rad/s

2.4 An airplane wing modeled as a spring-mass system with natural frequency 40 Hz is driven harmonically by the rotation of its engines at 39.9 Hz. Calculate the period of the resulting beat.

Solution: Given: $\omega_n = 2\pi\,(40) = 80\pi$ rad/s, $\omega = 2\pi\,(39.9) = 79.8\pi$ rad/s

Beat period: $$T_b = \frac{4\pi}{\omega_n - \omega} = \frac{4\pi}{80\pi - 79.8\pi} = 20 \text{ s.}$$

2.5 Derive Equation 2.13 from Equation 2.12 using standard trigonometric identities.

Solution: Equation (2.12): $$x(t) = \frac{f_0}{\omega_n^2 - \omega^2}\,[\cos\,\omega t - \cos\,\omega_n t]$$

Let $$A = \frac{f_0}{\omega_n^2 - \omega^2}$$

$$x(t) = A\,[\cos\omega t - \cos\,\omega_n t]$$

$$= A\,[1 + \cos\omega t - (1 + \cos\,\omega_n t)]$$

$$= A\,[2\cos^2\frac{\omega}{2}t - 2\cos^2\frac{\omega_n}{2}t\,]$$

$$= 2A\,[(\cos^2\frac{\omega}{2}t - \cos^2\frac{\omega_n}{2}\cos^2\frac{\omega}{2}t) - (\cos^2\frac{\omega_n}{2}t - \cos^2\frac{\omega_n}{2}t\,\cos^2\frac{\omega}{2}t)]$$

$$= 2A\,[(1 - \cos^2\frac{\omega_n}{2}t)\cos^2\frac{\omega}{2}t - (1 - \cos^2\frac{\omega}{2}t)\cos^2\frac{\omega_n}{2}t\,]$$

$$= 2A\,[\sin^2\frac{\omega}{2}t\,\cos^2\frac{\omega}{2}t - \cos^2\frac{\omega}{2}t\,\sin^2\frac{\omega}{2}t\,]$$

$$= 2A\,[\sin\frac{\omega_n}{2}t\,\cos\frac{\omega}{2}t - \cos\frac{\omega_n}{2}t\,\sin\frac{\omega}{2}t\,]\,[\sin\frac{\omega_n}{2}t\,\cos\frac{\omega}{2}t - \cos\frac{\omega_n}{2}t\,\sin\frac{\omega}{2}t\,]$$

$$= 2A\sin\left[\frac{\omega_n - \omega}{2}t\right]\sin\left[\frac{\omega_n + \omega}{2}t\right]$$

$$x(t) = \frac{2f_0}{\omega_n^2 - \omega^2}\sin\left[\frac{\omega_n - \omega}{2}t\right]\sin\left[\frac{\omega_n + \omega}{2}t\right]$$ which is Equation (2.13).

2.6 Compute the total response of a spring-mass system with the following values: k = 1000 N/m, m = 10 kg, subject to a harmonic force of magnitude F_0 = 100 N and frequency of 8.162 rad/s, and initial conditions given by x_0 = 0.01 m and v_0 = 0.01 m/s. Plot the response.

Solution: Given: k = 1000 N/m, m = 10 kg, F_0=100 N, ω = 8.162 rad/s
x_0=0.01m, v_0=0.01 m/s
From Eq. (2.11):

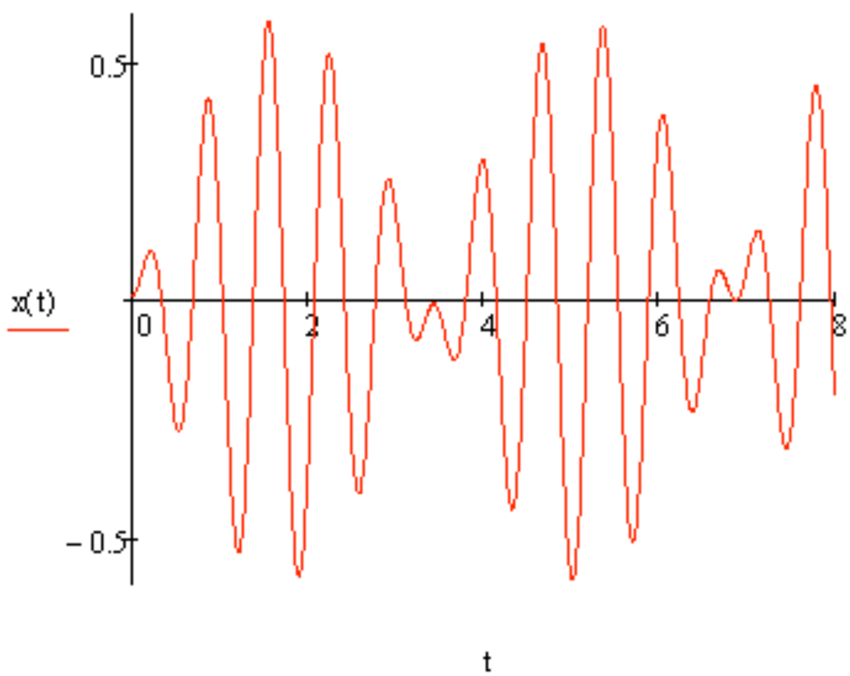
$$x(t) = \frac{v_0}{\omega_n}\sin\,\omega_n t + (x_0 - \frac{f_0}{\omega_n{}^2 - \omega^2})\cos\,\omega_n t + \frac{f_0}{\omega_n{}^2 - \omega^2}\cos\omega t$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1000}{10}} = 10\ rad\,/\,s \qquad \mathrm{f_0} = \frac{\mathrm{F_0}}{\mathrm{m}} = \frac{100}{10} = 10\ \mathrm{N\,/\,m}$$

In Mathcad the solution is

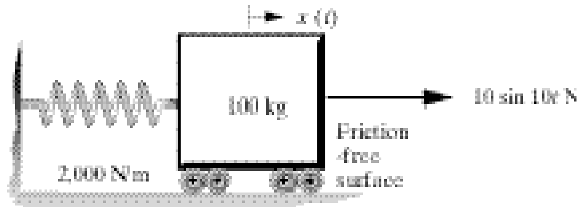
$x0 := 0.01$ $\qquad v0 := 0.01$ $\qquad \omega n := 10$ $\qquad \omega := 8.162$ $\qquad f0 := 10$

$$x(t) := \frac{v0}{\omega n}\cdot\sin(\omega n\cdot t) + \left(x0 - \frac{f0}{\omega n^2 - \omega^2}\right)\cdot\cos(\omega n\cdot t) + \frac{f0}{\left(\omega n^2 - \omega^2\right)}\cdot\cos(\omega\cdot t)$$



$$\frac{v0}{\omega n} = 1\times10^{-3} \qquad x0 - \frac{f0}{\omega n^2 - \omega^2} = -0.28956 \qquad \frac{f0}{\left(\omega n^2 - \omega^2\right)} = 0.29956$$

2.7 Consider the system in Figure P2.7, write the equation of motion and calculate the response assuming a) that the system is initially at rest, and b) that the system has an initial displacement of 0.05 m.



Solution: The equation of motion is

$$m\ddot{x} + kx = 10\sin 10t$$

Let us first determine the general solution for

$$\ddot{x} + \omega_n^2 x = f_0 \sin \omega t$$

Replacing the cosine function with a sine function in Eq. (2.4) and following the same argument, the general solution is:

$$x(t) = A_1 \sin \omega_n t + A_2 \cos \omega_n t + \frac{f_0}{\omega_n^2 - \omega^2} \sin \omega t$$

Using the initial conditions, $x(0) = x_0$ and $\dot{x}(0) = v_0$, a general expression for the response of a spring-mass system to a harmonic (sine) excitation is:

$$x(t) = \left(\frac{v_0}{\omega_n} - \frac{\omega}{\omega_n} \cdot \frac{f_0}{\omega_n^2 - \omega^2}\right) \sin \omega_n t + x_0 \cos \omega_n t + \frac{f_0}{\omega_n^2 - \omega^2} \sin \omega t$$

Given: k=2000 N/m, m=100 kg, ω=10 rad/s,

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{2000}{100}} = \sqrt{20} \text{ rad/s} = 4.472 \text{ rad/s} \qquad f_0 = \frac{F_0}{m} = \frac{10}{100} = 0.1 \text{ N/kg}$$

a) $x_0 = 0$ m, $v_0 = 0$ m/s

Using the general expression obtained above:

$$x(t) = \left(0 - \frac{10}{\sqrt{20}} \cdot \frac{0.1}{\sqrt{20}^2 - 10^2}\right) \sin \sqrt{20} t + 0 + \frac{0.1}{\sqrt{20}^2 - 10^2} \sin 10t$$

$$= 2.795 \times 10^{-3} \sin 4.472t - 1.25 \times 10^{-3} \sin 10t$$

b) $x_0 = 0.05$ m, $v_0 = 0$ m/s

$$x(t) = \left(0 - \frac{10}{\sqrt{20}} \cdot \frac{0.1}{\sqrt{20}^2 - 10^2}\right) \sin \sqrt{20} t + 0.05 \cos \sqrt{20} t + \frac{0.1}{\sqrt{20}^2 - 10^2} \sin 10t$$

$$= 0.002795 \sin 4.472t + 0.05 \cos 4.472t - 0.00125 \sin 10t$$

$$= 5.01 \times 10^{-2} \sin(4.472t + 1.515) - 1.25 \times 10^{-3} \sin 10t$$

2.8 Consider the system in Figure P2.8, write the equation of motion and calculate the response assuming that the system is initially at rest for the values $k_1 = 100$ N/m, $k_2 =$ 500 N/m and m = 89 kg.

Solution: The equation of motion is

$$m\ddot{x} + kx = 10\sin 10t \qquad \text{where} \quad k = \frac{1}{\frac{1}{k_1} + \frac{1}{k_2}}$$

The general expression obtained for the response of an underdamped spring-mass system to a harmonic (sine) input in Problem 2.7 was:

$$x(t) = \left(\frac{v_0}{\omega_n} - \frac{\omega}{\omega_n} \cdot \frac{f_0}{\omega_n^{\,2} - \omega^2}\right)\sin \omega_n t + x_0 \cos\omega_n t + \frac{f_0}{\omega_n^{\,2} - \omega^2}\sin \omega t$$

Substituting the following values

k = 1/(1/100+1/500)= 83.333 N/m, m = 89 kg ω = 10 rad/s

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{83.333}{89}} = 0.968 \text{ rad/s} \qquad f_0 = \frac{F_0}{m} = \frac{10}{89} = 0.112\ N/kg$$

and initial conditions: $x_0 = 0$, $v_0 = 0$

The response of the system is evaluated as

$$x(t) = 0.0117\sin 0.968t - 0.00113\sin 10t$$

2.9 Consider the system in Figure P2.9, write the equation of motion and calculate the response assuming that the system is initially at rest for the values θ = 30°, k = 1000 N/m and m = 50 kg.

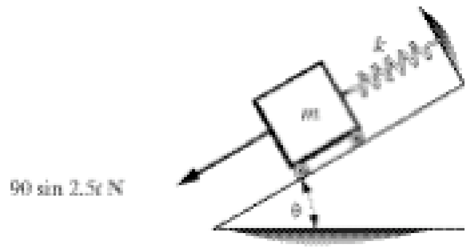
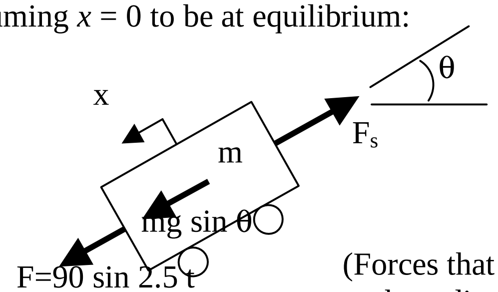


Figure P2.9

Solution: Free body diagram:
Assuming x = 0 to be at equilibrium:



(Forces that are normal to the x direction are neglected)

$$\sum F_x = m\ddot{x} = -k(x+\Delta) + mg\sin\theta + 90\sin 25t \quad (1)$$

where Δ is the static deflection of the spring. From static equilibrium in the x direction yields

$$-k\Delta + mg\sin\theta \quad (2)$$

Substitution of (2) onto (1), the equation of motion becomes

$$m\ddot{x} + kx = 90\sin 2.5t$$

The general expression for the response of a mass-spring system to a harmonic (sine) excitation (see Problem 2.7) is:

$$x(t) = \left(\frac{v_0}{\omega_n} - \frac{\omega}{\omega_n}\cdot\frac{f_0}{\omega_n^2-\omega^2}\right)\sin\omega_n t + x_0\cos\omega_n t + \frac{f_0}{\omega_n^2-\omega^2}\sin\omega t$$

Given: $v_0 = 0$, $x_0 = 0$, $\omega = 2.5$ rad/s

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1000}{50}} = \sqrt{20} = 4.472 \text{ rad/s} \quad , \quad f_0 = \frac{F_0}{m} = \frac{90}{50} = \frac{9}{5}\text{ N/kg}$$

So the response is:

$$x(t) = -0.0732\sin 4.472t + 0.1309\sin 2.5t$$

2.10 Compute the initial conditions such that the response of :

$$m\ddot{x} + kx = F_0 \cos\omega t$$

oscillates at only one frequency (ω).

Solution: From Eq. (2.11):

$$x(t) = \frac{v_0}{\omega_n}\sin \omega_n t + (x_0 - \frac{f_0}{\omega_n{}^2 - \omega^2})\cos \omega_n t + \frac{f_0}{\omega_n{}^2 - \omega^2}\cos \omega t$$

For the response of $m\ddot{x} + kx = F_0 \cos \omega t$ to have only one frequency content, namely, of the frequency of the forcing function, ω, the coefficients of the first two terms are set equal to zero. This yields that the initial conditions have to be

$$x_0 = \frac{f_0}{\omega_n{}^2 - \omega^2} \quad \text{and} \quad v_0 = 0$$

Then the solution becomes

$$x(t) = \frac{f_0}{\omega_n{}^2 - \omega^2}\cos \omega t$$

2.11 The natural frequency of a 65-kg person illustrated in Figure P.11 is measured along vertical, or longitudinal direction to be 4.5 Hz. a) What is the effective stiffness of this person in the longitudinal direction? b) If the person, 1.8 m in length and 0.58 m^2 in cross sectional area, is modeled as a thin bar, what is the modulus of elasticity for this system?

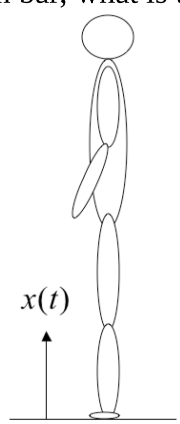


Figure P2.11 Longitudinal vibration of a person

Solution: a) First change the frequency in Hz to rad/s: $\omega_n = 4.5\frac{\text{cycles}}{\text{s}}\frac{2\pi \text{ rad}}{\text{cycles}} = 9\pi$ rad/s .

Then from the definition of natural frequency:

$$k = m\omega_n^2 = 65 \cdot (9\pi)^2 = \underline{5.196 \times 10^4 \text{ N/m}}$$

b) From section 1.4, the value of the stiffness for the longitudinal vibration of a beam is

$$k = \frac{EA}{\ell} \Rightarrow E = \frac{k\ell}{A} = \frac{\left(5.196 \times 10^4\right)(1.8)}{0.58} = \underline{1.613 \times 10^5 \text{ N/m}^2} = \underline{1.613 \times 10^5 \text{ Pa}}$$

2.12 If the person in Problem 2.11 is standing on a floor, vibrating at 4.49 Hz with an amplitude of 1 N (very small), what longitudinal displacement would the person "feel"? Assume that the initial conditions are zero.

Solution: Using equation (2.12) for a cosine excitation and zero initial conditions yields (converting the frequency from Hertz to rad/s and using the value of *k* calculated in 2.11):

$$|X| = \frac{F_0}{m}\left|\frac{1}{\omega_n^2 - \omega^2}\right| = \frac{1}{65}\left|\frac{1}{\frac{k}{m} - (4.49\cdot 2\pi)^2}\right|$$

$$= \frac{1}{65}\left|\frac{1}{\frac{5.196\times 10^4}{65} - (4.49\cdot 2\pi)^2}\right| = 0.00443347 = \underline{0.0043\text{ m}}$$

2.13 Vibration of body parts is a significant problem in designing machines and structures. A jackhammer provides a harmonic input to the operator's arm. To model this situation, treat the forearm as a compound pendulum subject to a harmonic excitation (say of mass 6 kg and length 44.2 cm) as illustrated in Figure P2.13. Consider point *O* as a fixed pivot. Compute the maximum deflection of the hand end of the arm if the jackhammer applies a force of 10 N at 2 Hz.

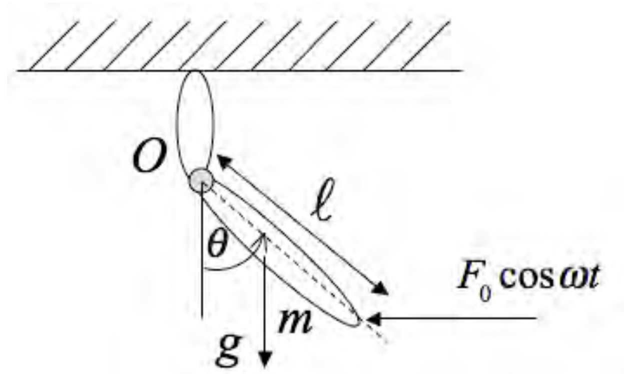


Figure P2.13 Vibration model of a forearm driven by a jackhammer

Solution: Taking moments about point *O* yields (referring to Example 1.4.6 for the inertial of a compound pendulum):

$$\frac{m\ell^2}{3}\ddot{\theta} + mg\frac{\ell}{2}\sin\theta = F_O\ell\cos\theta\cos\omega t$$

Using the linear approximation for sine and cosine and dividing through by the inertia yields:

$$\ddot{\theta} + \frac{3g}{2\ell}\theta = \frac{3F_O}{m\ell}\cos\omega t$$

Thus the natural frequency is

$$\omega_n = \sqrt{\frac{3g}{2\ell}} = \sqrt{\frac{3(9.81)}{2(0.442)}} = 5.77 \text{ rad/s} \quad (=0.92 \text{ Hz})$$

and the system is well away from resonance. Referring to equation (2.13), the amplitude for zero initial conditions is (converting the driving frequency from 2 Hertz to 2(2π) rad/s):

$$|\theta| = \left|\frac{2f_0}{\omega_n^2 - \omega^2}\right| = \left|\frac{2\left(\frac{3F_0\ell}{m\ell^2}\right)}{\frac{3g}{2\ell} - (2\cdot 2\pi)^2}\right| = \underline{0.182 \text{ rad}}$$

Note that sin(0.182) = 0.181 so the approximation made above is valid. The maximum linear displacement of the hand end of the arm is just

$$|X| = r|\theta| = 0.442\cdot 0.182 = \underline{0.08 \text{ m}}$$

2.14 Consider again the camera problem of Example 2.1.3 depicted in Figure P2.14, and determine the torsional natural frequency, the maximum torsional deflection experienced by the camera due to the wind and the linear displacement corresponding to the computed torsional deflection. Model the camera in torsional vibration as suggested in the figure where $J_P = 9.817\text{x}10^{-6}$ m^4 and $L = 0.2$ m. Use the values computed in Example 2.1.3 for the mass (m =3 kg), shaft length ($\ell = 0.55$ m), torque (M_0 = 15 x L Nm) and frequency (ω = 10 Hz). Here G is the shear modulus of aluminum and the rotational inertia of the camera is approximated by $J = mL^2$. In the example, torsion was ignored. The purpose of this problem is to determine if ignoring the torsion is a reasonable assumption or not. Please comment on this assumption based on the results of the requested calculation.

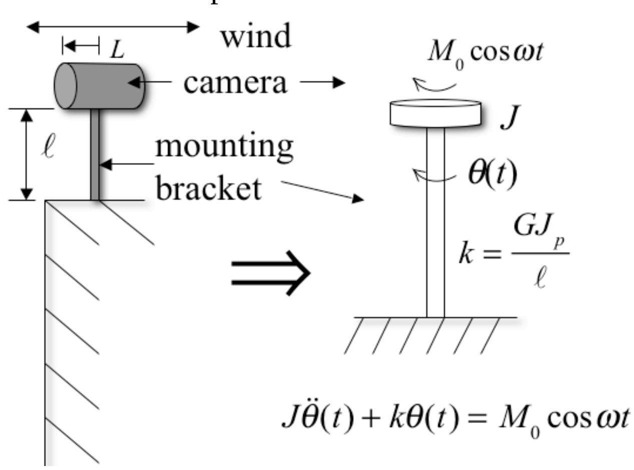


Figure P2.14 Torsional vibration of a camera

Solution: First calculate the rotational stiffness and inertia from the data given:

$$k = \frac{GJ_p}{\ell} = \frac{2.67\times 10^{10}\times 9.817\times 10^{-6}}{0.55} = 4.766\times 10^5 \text{ N}\cdot\text{m}$$

where the modulus is taken from Table 1.2 for aluminum. The inertia is approximated by

$$J = mL^2 = 3(0.2)^2 = 0.12 \text{ kg}\cdot\text{m}^2$$

The torsional natural frequency is thus

$$\omega_n = \sqrt{\frac{k}{J}} = \underline{1.993\times 10^3 \text{ rad/s}}$$

This is well away from the driving frequency. To see the effect, recall equation magnitude of the forced response given in Example 2.1.2:

$$\left|\frac{2f_0}{\omega_n^2-\omega^2}\right| = \left|\frac{2M_0/J}{\omega_n^2-\omega^2}\right| = \underline{1.26\times 10^{-5} \text{ rad}}$$

Clearly this is very small. To change this to a linear displacement of the camera tip, use

$$X = r\theta = (0.2)(1.26\times 10^{-5}) = \underline{2.52\times 10^{-6} \text{ m}}$$

well within the limit imposed on the camera's vibration requirement of 0.01 m. Thus, the assumption to ignore torsional vibration in designing the length of the mounting bracket made in example 2.1.3 is justified.

2.15 An airfoil is mounted in a wind tunnel for the purpose of studying the aerodynamic properties of the airfoil's shape. A simple model of this is illustrated in Figure P2.15 as a rigid inertial body mounted on a rotational spring, fixed to the floor with a rigid support. Find a design relationship for the spring stiffness *k* in terms of the rotational inertia, *J*, the magnitude of the applied moment, M_0, and the driving frequency, ω, that will keep the magnitude of the angular deflection less then 5°. Assume that the initial conditions are zero and that the driving frequency is such that $\omega_n^2 - \omega^2 > 0$.

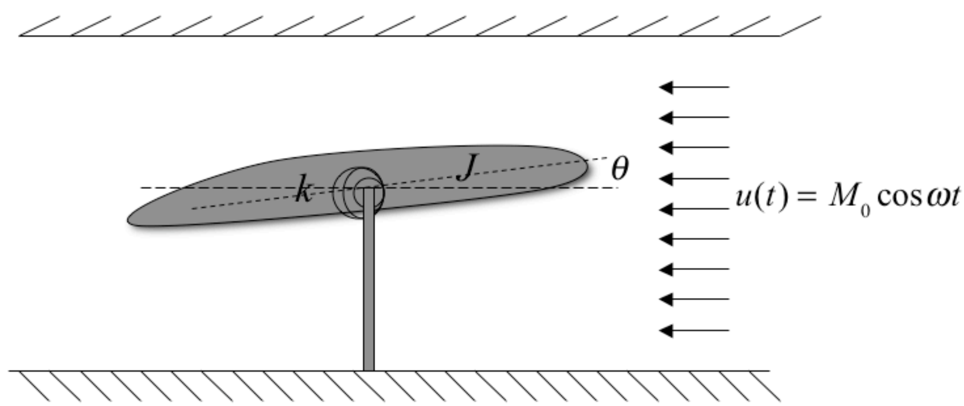


Figure P2.15 Vibration model of a wing in a wind tunnel

Solution: Assuming compatible units, the equation of motion is:

$$J\ddot{\theta}(t) + k\theta(t) = M_0\cos\omega t \Rightarrow \ddot{\theta}(t) + \frac{k}{J}\theta(t) = \frac{M_0}{J}\cos\omega t$$

From equation (2.12) the maximum deflection for zero initial conditions is

$$\theta_{\max} = \left| \frac{2M_0 / J}{\frac{k}{J} - \omega^2} \right| < 5° \frac{\pi \text{rad}}{180°} = \frac{\pi}{36} \text{rad}$$

$$\Rightarrow \frac{2M_0}{J} < \left(\frac{k}{J} - \omega^2\right) \frac{\pi}{36} \text{rad} \Rightarrow \underline{\frac{36J}{\pi}\left(\frac{2M_0}{J} + \frac{\pi\omega^2}{36}\right) < k}$$

Problems and Solutions Section 2.2 (2.16 through 2.31)

2.16 Calculate the constants A and ϕ for arbitrary initial conditions, x_0 and v_0, in the case of the forced response given by Equation (2.37). Compare this solution to the transient response obtained in the case of no forcing function (i.e. $F_0 = 0$).

Solution: From equation (2.37)

$$x(t) = Ae^{-\zeta\omega_n t}\sin(\omega_d t + \phi) + X\cos(\omega t - \theta) \Rightarrow$$
$$\dot{x}(t) = -\zeta\omega_n Ae^{-\zeta\omega_n t}\sin(\omega_d t + \phi) + A\omega_d e^{-\zeta\omega_n t}\cos(\omega_d t + \phi) - X\omega\sin(\omega t - \theta)$$

Next apply the initial conditions to these general expressions for position and velocity to get:

$$x(0) = A\sin\phi + X\cos\theta$$
$$\dot{x}(0) = -\zeta\omega_n A\sin\phi + A\omega_d\cos\phi + X\omega\sin\theta$$

Solving this system of two equations in two unknowns yields:

$$\phi = \tan^{-1}\left(\frac{(x_0 - X\cos\theta)\omega_d}{v_0 + (x_0 - X\cos\theta)\zeta\omega_n - X\omega\sin\theta}\right)$$
$$A = \frac{x_0 - X\cos\theta}{\sin\phi}$$

Recall that X has the form

$$X = \frac{F_0/m}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}} \quad \text{and} \quad \theta = \tan^{-1}\left(\frac{2\zeta\omega_n\omega}{\omega_n^2 - \omega^2}\right)$$

Now if $F_0 = 0$, then $X = 0$ and A and ϕ from above reduce to:

$$\phi = \tan^{-1}\left(\frac{x_0\omega_d}{v_0 + x_0\zeta\omega_n}\right)$$
$$A = \frac{x_0}{\sin\phi} = \sqrt{\frac{(v_0 + \zeta\omega_n x_0)^2 + (x_0\omega_d)^2}{\omega_d^2}}$$

These are identical to the values given in equation (1.38).

2.17 Show that Equations (2.28) and (2.29) are equivalent by verifying Equations (2.29) and (2.30).

Solution: From equation (2.28) and expanding the trig relation yields

$$x_p = X\cos(\omega t - \theta) = X\left[\cos\omega t\cos\theta + \sin\omega t\sin\theta\right]$$
$$= \underbrace{(X\cos\theta)}_{A_s}\cos\omega t + \underbrace{(X\sin\theta)}_{B_s}\sin\omega t$$

Now with A_s and B_s defined as indicated, the magnitude is computed:

$$X = \sqrt{A_s^2 + B_s^2}$$

and

$$\frac{B_s}{A_s} = \frac{X\sin\theta}{X\cos\theta} \Rightarrow \theta = \tan^{-1}\left(\frac{B_s}{A_s}\right)$$

2.18 Plot the solution of Equation (2.27) for the case that m = 1 kg, ζ = 0.01, ω_n = 2 rad/s. F_0 = 3 N, and ω = 10 rad/s, with initial conditions x_0 = 1 m and v_0 = 1 m/s.

Solution: The particular solution is given in equations (2.36) and (2.37). Substitution of the values given yields: $x_p = 0.03125\cos(10t + 8.333\times10^{-3})$.

Then the total solution has the form:

$$x(t) = Ae^{-0.02t}\sin(2t + \phi) + 0.03125\cos(10t + 0.008333)$$
$$= e^{-0.02t}\left(A\sin 2t + B\cos 2t\right) + 0.03125\cos(10t + 0.008333)$$

Differentiating then yields

$$\dot{x}(t) = -0.02e^{-0.02t}\left(A\sin 2t + B\cos 2t\right) + \sin(2t + \phi)$$
$$+ 2e^{-0.02t}\left(A\cos 2t - B\sin 2t\right) - 0.3125\sin(10t + 0.008333)$$

Apply the initial conditions to get:

$$x(0) = 1 = B + 0.03125\cos(0.00833) \Rightarrow \underline{B = 0.969}$$
$$\dot{x}(0) = 1 = -0.02B + 2A - 0.3125\sin(0.00833) \Rightarrow A = 0.489$$

So the solution and plot become (using Mathcad):

$F0 := 3 \qquad \omega := 10 \quad \omega n := 2 \quad \zeta := 0.02$

$$\theta := \operatorname{atan}\left(2 \cdot \frac{\zeta \cdot \omega n \cdot \omega}{\omega n^2 - \omega^2}\right) = -8.333 \times 10^{-3}$$

$$AF := \frac{F0}{\sqrt{\left(\omega n^2 - \omega^2\right)^2 + (2 \cdot \zeta \cdot \omega \cdot \omega n)^2}}$$

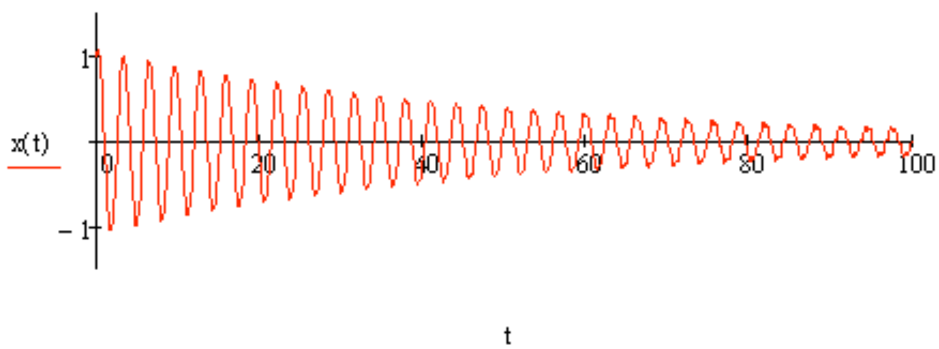
$AF = 0.03125$

$B := 1 - AF \cdot \cos(\theta) \qquad B = 0.969$

$$A1 := \frac{1 + 10 \cdot AF \cdot \sin(\theta) - 0.02 \cdot B}{2}$$

$A1 = 0.489$

$$x(t) := e^{-0.02 \cdot t} \cdot (A1 \cdot \sin(2 \cdot t) + B \cdot \cos(2 \cdot t)) + AF \cdot \cos(10 \cdot t - \theta)$$



2.19 A 100 kg mass is suspended by a spring of stiffness 30×10^3 N/m with a viscous damping constant of 1000 Ns/m. The mass is initially at rest and in equilibrium. Calculate the steady-state displacement amplitude and phase if the mass is excited by a harmonic force of 80 N at 3 Hz.

Solution: Given m = 100kg, k =30,000 N/m, c = 1000 Ns/m, F_0 = 80 N and ω = 6π rad/s:

$$f_0 = \frac{F_0}{m} = \frac{80}{100} = 0.8 \text{ m/s}^2, \quad \omega_n = \sqrt{\frac{k}{m}} = 17.32 \text{ rad/s}$$

$$\zeta = \frac{c}{2\sqrt{km}} = 0.289$$

$$X = \frac{0.8}{\sqrt{\left(17.32^2 + 36\pi^2\right)^2 + \left(2(0.289)(17.32)(6\pi)\right)^2}} = 0.0041 \text{ m}$$

Next compute the angle from

$$\theta = \tan^{-1}\left(\frac{188.702}{-55.323}\right)$$

Since the denominator is negative the angle must be found in the 4th quadrant. To find this use Window 2.3 and then in Matlab type atan2(188.702,-55.323) or use the principle value and add π to it. Either way the phase is θ =1.856 rad.

2.20 Plot the total solution of the system of Problem 2.19 including the transient.

Solution: The total response is given in the solution to Problem 2.16. For the values given in the previous problem, and with zero initial conditions the response is determined by the formulas:

$$X = 0.0041, \quad \theta = 1.856$$

$\omega n := 17.32 \quad \zeta := 0.289$

$$\theta := \operatorname{atan}\left[\frac{2\cdot 0.289\cdot 17.32\cdot 6\cdot\pi}{17.32^2 - (6\cdot\pi)^2}\right] + \pi$$

$X := 0.0041$

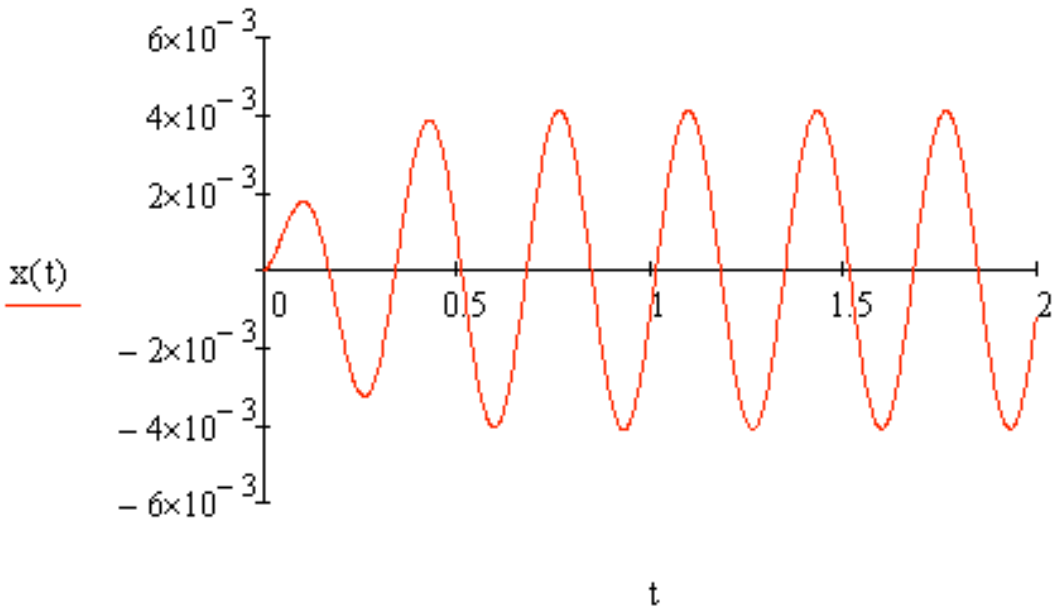
$\omega d := \omega n\cdot\sqrt{1-\zeta^2}$ $\qquad \theta = 1.856$

$$\phi := \operatorname{atan}\left(\frac{-X\cdot\cos(\theta)\cdot\omega d}{-X\cdot\cos(\theta)\cdot\zeta\cdot\omega n - X\cdot\omega n\cdot\sin(\theta)}\right) + \pi \qquad \phi = 2.844$$

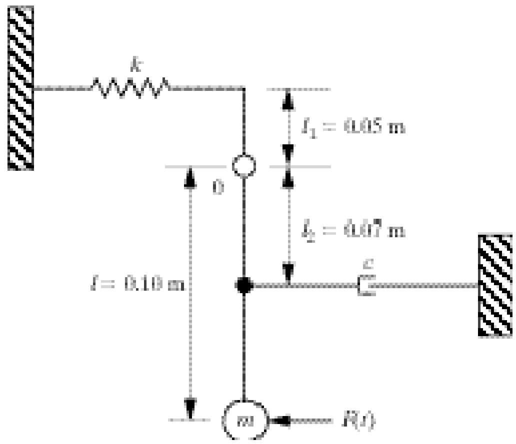
$$A := \frac{-X\cdot\cos(\theta)}{\sin(\phi)} \qquad A = 3.934\times 10^{-3}$$

$$x(t) := \left(A\cdot e^{-\zeta\cdot\omega n\cdot t}\cdot\sin(\omega d\cdot t + \phi)\right) + X\cdot\cos(6\cdot\pi\cdot t - \theta)$$

Plotting the result in Mathcad yields



2.21 Consider the pendulum mechanism of Figure P2.21 which is pivoted at point O. Calculate both the damped and undamped natural frequency of the system for small angles. Assume that the mass of the rod, spring, and damper are negligible. What driving frequency will cause resonance?



Solution: Assume the driving frequency to be harmonic of the standard form. To get the equation of motion take the moments about point O to get:

$$\sum M_0 = J\ddot{\theta}(t) = m\ell^2\ddot{\theta}(t)$$

$$= -k\ell_1 \sin\theta(\ell_1\cos\theta) - c\ell_2\dot{\theta}(\ell_2\cos\theta)$$

$$- mg(\ell\sin\theta) + F_0\cos\omega t(\ell\cos\theta)$$

Rearranging and approximating sinθ ~ θ and cosθ ~1 yields:

$$m\ell^2\ddot{\theta}(t) + c\ell_2^2\dot{\theta}(t) + (k\ell_1^2 + mg\ell)\theta(t) = F_0\ell\cos\omega t$$

Dividing through by the coefficient of the inertia term and using the standard definitions for ζ and ω yields:

$$\omega_n = \sqrt{\frac{k\ell_1^2 + mg\ell}{m\ell^2}} \text{ which is the resnonant frequency}$$

$$\zeta = \frac{c\ell_2^2}{2\sqrt{(k\ell_1^2 + mg\ell)mg\ell}}$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = \sqrt{\frac{k\ell_1^2 + mg\ell}{m\ell^2}\left(1 - \frac{c^2\ell_2^4}{4(k\ell_1^2 + mg\ell)mg\ell}\right)}$$

2.22 Consider the pivoted mechanism of Figure P2.21 with k = 4 x 10^3 N/m. l_1 = 0.05 m. l_2 = 0.07 m. and l = 0.10 m. and m = 40 kg. The mass of the beam is 40 kg; it is pivoted at point 0 and assumed to be rigid. Design the dashpot (i.e. calculate c) so that the damping ratio of the system is 0.2. Also determine the amplitude of vibration of the steady-state response if a 10-N force is applied to the mass, as indicated in the figure, at a frequency of 10 rad/s.

Solution: This is similar to the previous problem with the mass of the beam included this time around. The equation of motion becomes:

$$m_{eq}\ddot{\theta} + c_{eq}\dot{\theta} + k_{eq}\theta = F_0\ell\cos\omega t$$

Here:

$$m_{eq} = m\ell^2 + \frac{1}{3}(\ell^3 + \ell_1^3)\frac{m_b}{\ell + \ell_1} = 0.5 \text{ kg}\cdot\text{m}^2$$

$$c_{eq} = c\ell_2^2 = 0.25c$$

$$k_{eq} = k\ell_1^2 + mg\ell + \frac{1}{2}(\ell - \ell_1)m_b g = 4.326\times10^3 \text{ Nm}$$

Using the formula the damping ratio and these numbers:

$$\zeta = \frac{\ell_2^2 c}{2\sqrt{m_{eq}k_{eq}}} = 0.2 \Rightarrow c = 3.797\cdot10^3 \text{ kg/s}$$

Next compute the amplitude:

$$X = \frac{10/0.5}{\sqrt{(k_{eq}/m_{eq} - 10^2)^2 + (2\cdot0.2\cdot10\cdot\omega_n)^2}} = 2.336\times10^3 \text{ rad}$$

2.23 In the design of Problem 2.22, the damping ratio was chosen to be 0.2 because it limits the amplitude of the forced response. If the driving frequency is shifted to 11 rad/s, calculate the change in damping coefficient needed to keep the amplitude less than calculated in Problem 2.22.

Solution: In this case the frequency is far away from resonance so the change in driving frequency does not matter much. This can also be seen numerically by the following Mathcad session.

$L1 := 0.05 \qquad k := 4\cdot 10^3 \qquad L2 := 0.07 \qquad L := 0.1$

$m := 40 \qquad mb := 40$

$$meq := m\cdot L^2 + \frac{1}{3}\cdot\left(L^3 + L1^3\right)\cdot\frac{mb}{L + L1} \qquad meq = 0.5 \qquad g := 9.81$$

$c := L2^2 \qquad c = 4.9\cdot 10^{-3}$

$$keq := k\cdot L1^2 + m\cdot g + \frac{1}{2\cdot(L - L1)}\cdot(mb\cdot g) \qquad keq = 4.326\cdot 10^3$$

$$ceq := \frac{0.2\cdot 2\cdot\sqrt{meq\cdot keq}}{c} \qquad ceq = 3.797\cdot 10^3$$

$$\omega n := \sqrt{\frac{keq}{meq}} \qquad \omega n = 93.02 \qquad X := 10\cdot\frac{L}{meq}\cdot\frac{1}{\sqrt{\left(\omega n^2 - 11^2\right)^2 + (2\cdot .2\cdot \omega n\cdot 11)^2}}$$

$$X = 2.341\cdot 10^{-4}$$

The new amplitude is only slightly larger in this case. The problem would be more meaningful if the driving frequency is near resonance. Then the shift in amplitude will be more substantial and added damping may improve the response.

2.24 Compute the forced response of a spring-mass-damper system with the following values: c = 200 kg/s, k = 2000 N/m, m = 100 kg, subject to a harmonic force of magnitude F_0 = 15 N and frequency of 10 rad/s and initial conditions of x_0 = 0.01 m and v_0 = 0.1 m/s. Plot the response. How long does it take for the transient part to die off?

Solution:
Calculate the parameters

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{2000}{100}} = 4.472 \text{ rad/s} \qquad f_0 = \frac{F_0}{m} = \frac{15}{100} = 0.15 \text{ N/kg}$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 4.472\sqrt{1-0.224^2} = 4.359 \text{ rad/s}$$

$$\zeta = \frac{c}{2m\omega_n} = \frac{200}{2\cdot 100\cdot 4.472} = 0.224$$

Initial conditions: $x_0 = 0.01$ m, $v_0 = 0.1$ m/s

Using equation (2.38) and working in Mathcad yields

$$x(t) = e^{-t}(0.0104\cos 4.359t + 0.025\sin 4.359t) + 1.318\times 10^{-6}(0.335\cos 10t + 37.7\sin 10t)$$

$c := 200 \quad k := 2000 \quad m := 100 \quad F0 := 15 \quad \omega := 10$

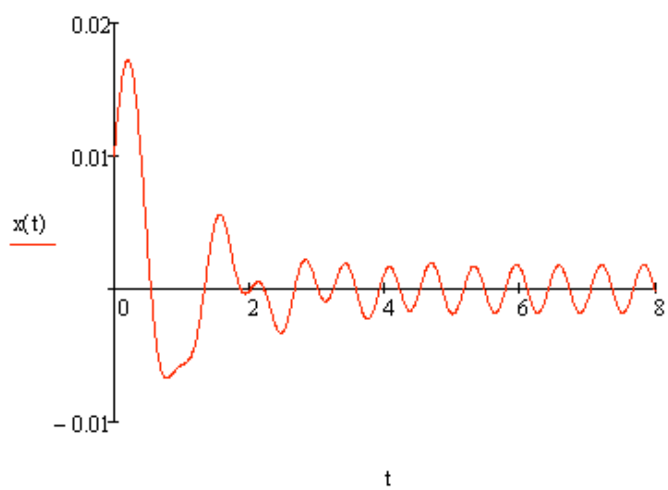
$\omega n := \sqrt{\frac{k}{m}} \quad x0 := 0.01 \quad v0 := 0.1 \quad f0 := \frac{F0}{m} \quad \zeta := \frac{c}{2\cdot m\cdot \omega n} \quad \omega d := \omega n\cdot\sqrt{1-\zeta^2}$

$$A := x0 - \frac{f0\cdot(\omega n^2 - \omega^2)}{(\omega n^2-\omega^2)^2 + (2\cdot\zeta\cdot\omega\cdot\omega n)^2}$$

$$B := \frac{\zeta\cdot\omega n}{\omega d}\cdot\left[\frac{f0\cdot(\omega n^2-\omega^2)}{(\omega n^2-\omega^2)^2+(2\cdot\zeta\cdot\omega\cdot\omega n)^2}\right] - \frac{2\cdot\zeta\cdot\omega n\cdot\omega^2}{\omega d\cdot\left[(\omega n^2-\omega^2)^2+(2\cdot\zeta\cdot\omega\cdot\omega n)^2\right]} + \frac{v0}{\omega d}$$

$$C := \frac{f0}{(\omega n^2-\omega^2)^2+(2\cdot\zeta\cdot\omega\cdot\omega n)^2}$$

$$x(t) := e^{-\zeta\cdot\omega n\cdot t}\cdot(A\cdot\cos(\omega d\cdot t) + B\cdot\sin(\omega d\cdot t)) + C\cdot\left[(\omega n^2-\omega^2)\cdot\cos(\omega\cdot t) + 2\cdot\zeta\cdot\omega n\cdot\omega\cdot\sin(\omega\cdot t)\right]$$



a plot of m vs seconds. The time for the amplitude of the transient response to be reduced, for example, to 0.1 % of the initial (t = 0) amplitude can be determined by:

$$e^{-t} = 0.001, \text{ then } t = -\ln 0.001 = 6.908 \text{ sec}$$

2.25 Show that Equation (2.38) collapses to give Equation (2.11) in the case of zero damping.

Solution:
Eq. (2.38):

$$x(t)=e^{-\zeta\omega_n t}\left\{\begin{array}{l}\left\{(x_0-\dfrac{f_0(\omega_n^2-\omega^2)}{(\omega_n^2-\omega^2)^2+(2\zeta\omega_n\omega)^2})\cos\omega_d t\right. \\ \qquad +\left(\begin{array}{l}\dfrac{\zeta\omega_n}{\omega_d}(x_0-\dfrac{f_0(\omega_n^2-\omega^2)}{(\omega_n^2-\omega^2)^2+(2\zeta\omega_n\omega)^2}) \\ -\dfrac{2\zeta\omega_n\omega^2 f_0}{\omega_d\left[(\omega_n^2-\omega^2)^2+(2\zeta\omega_n\omega)^2\right]}+\dfrac{v_0}{\omega_d}\end{array}\right)\sin\omega_d t\end{array}\right\}$$

$$+\frac{f_0}{(\omega_n^2-\omega^2)^2+(2\zeta\omega_n\omega)^2}\left[(\omega_n^2-\omega^2)\cos\omega t+2\zeta\omega_n\omega\sin\omega t\right]$$

In case of $\zeta = 0$, this equation becomes:

$$x(t)=1\cdot\left\{\begin{array}{l}\left\{(x_0-\dfrac{f_0}{(\omega_n^2-\omega^2)+0})\cos\omega_d t\right. \\ +\left(0-0+\dfrac{v_0}{\omega_d}\right)\sin\omega_d t\end{array}\right\}+\frac{f_0}{(\omega_n^2-\omega^2)}\cos\omega t$$

$$=\frac{v_0}{\omega_n}\sin\omega_n t+(x_0-\frac{f_0}{\omega_n^2-\omega^2})\cos\omega_n t+\frac{f_0}{\omega_n^2-\omega^2}\cos\omega t$$

(Note: $\omega_d = \omega_n$ for $\zeta = 0$)

2.26 Derive Equation (2.38) for the forced response of an underdamped system.

Solution:
From Sec. 1.3, the homogeneous solution is:

$$x_h(t)=e^{-\zeta\omega_n t}(A_1\sin\omega_d t+A_2\cos\omega_d t)$$

From equations (2.29) and (2.35), the particular solution is:

$$x_p(t)=\frac{(\omega_n^2-\omega^2)f_0}{(\omega_n^2-\omega^2)^2+(2\zeta\omega_n\omega)^2}\cos\omega t+\frac{2\zeta\omega_n\omega f_0}{(\omega_n^2-\omega^2)^2+(2\zeta\omega_n\omega)^2}\sin\omega t$$

Then the general solution is:

$$x(t)=x_h(t)+x_p(t)=e^{-\zeta\omega_n t}(A_1\sin\omega_d t+A_2\cos\omega_d t)$$

$$+\frac{(\omega_n^2-\omega^2)f_0}{(\omega_n^2-\omega^2)^2+(2\zeta\omega_n\omega)^2}\cos\omega t+\frac{2\zeta\omega_n\omega f_0}{(\omega_n^2-\omega^2)^2+(2\zeta\omega_n\omega)^2}\sin\omega t$$

Using the initial conditions, $x(0) = x_0$ and $\dot{x}(0) = v_0$, the constants, A_1 and A_2, are determined:

$$A_2 = x_0 - \frac{(\omega_n^2 - \omega^2) f_0}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}$$

$$A_1 = \frac{v_0}{\omega_d} + \frac{\omega}{\omega_d} \cdot \frac{2\zeta\omega_n\omega f_0}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2} + \zeta\frac{\omega_n}{\omega_d}\left(x_0 - \frac{(\omega_n^2 - \omega^2) f_0}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}\right)$$

Then, Eq. (2.30) is obtained by substituting the expressions for A_1 and A_2 into the general solution and simplifying the resulting equation.

2.27 Compute a value of the damping coefficient *c* such that the steady state response amplitude of the system in Figure P2.27 is 0.01 m.

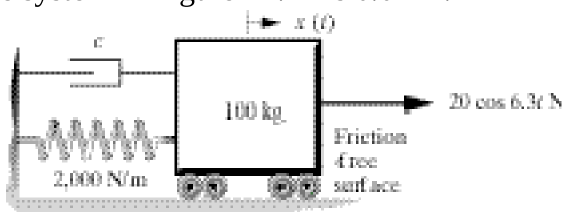


Figure P2.27

Solution:

From Eq. (2.39), the amplitude of the steady state response is given by

$$X = \frac{f_0}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}}$$

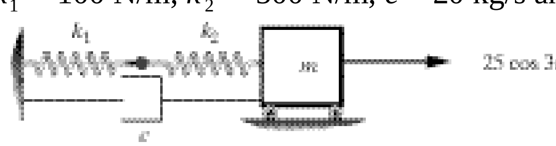
Then substitute, $2\zeta\omega_n = c/m$, $c = \sqrt{\frac{F_0^2}{\omega^2 \cdot X^2} - m^2\frac{(\omega_n^2 - \omega^2)^2}{\omega^2}}$ into this equation and solve for *c*:

Given:

$X = 0.01\text{m}$ $\qquad \omega = 6.3\,\text{rad/s}$ $\qquad F_0 = 20\,\text{N}$ $\qquad m = 100\,\text{kg}$

$$\omega_n^2 = \frac{k}{m} = \frac{2000}{100} = 20\ (\text{rad/s})^2 \Rightarrow \underline{c = 55.7\ \text{kg/s}}$$

2.28 Compute the response of the system in Figure P2.28 if the system is initially at rest for the values k_1 = 100 N/m, k_2 = 500 N/m, *c* = 20 kg/s and *m* = 89 kg.



Solution:

The equation of motion is:

$$m\ddot{x} + c\dot{x} + kx = 25\cos 3t \qquad \text{where} \quad k = \frac{1}{1/k_1 + 1/k_2}$$

Using Eq. (2.37) in an alternative form, the general solution is:

$$x(t) = e^{-\zeta\omega_n t}(A_1 \sin\omega_d t + A_2 \cos\omega_d t) + X\cos(\omega t - \theta)$$

where

$$X = \frac{f_0}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}} = \frac{25/89}{\sqrt{(0.966^2 - 3^2)^2 + (2\cdot 0.116\cdot 0.966\cdot 3)^2}} = 0.0347\,m$$

$$\theta = \tan^{-1}\cdot\frac{2\zeta\omega_n\omega}{\omega_n^2 - \omega^2} = \tan^{-1}\cdot\frac{2\cdot 0.116\cdot 0.966\cdot 3}{0.966^2 - 3^2} = 3.058\,rad \qquad \text{(see Window 2.3)}$$

Using the initial conditions, $x(0) = 0$ and $\dot{x}(0) = 0$, the constants, A_1 and A_2, are determined:

$A_2 = 0.0345$ $\qquad$ $A_1 = -0.005$

Given: $c = 20$ kg/sec, $m = 89$ kg

$$k = \frac{1}{1/k_1 + 1/k_2} = \frac{1}{1/100 + 1/500} = 83\,\text{N/m}$$

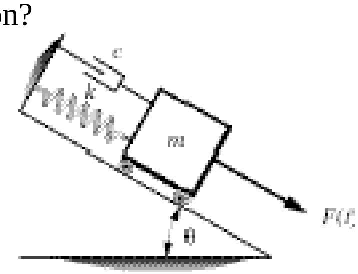
$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{83}{89}} = 0.966\,\text{rad/s} \qquad \zeta = \frac{c}{2m\omega_n} = \frac{20}{2\cdot 89\cdot 0.966} = 0.116$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 0.966\sqrt{1-0.116^2} = 0.9595\,\text{rad/s}$$

Substituting the values into the general solution:

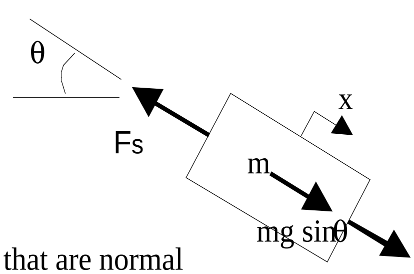
$$x(t) = e^{-0.112t}(-0.005\sin 0.9595t + 0.0345\cos 0.9595t) + 0.0347\cos(3t - 3.058)$$

2.29 Write the equation of motion for the system given in Figure P2.29 for the case that $F(t) = F\cos\omega t$ and the surface is friction free. Does the angle θ effect the magnitude of oscillation?



Solution:

Free body diagram:



(Forces that are normal to the x direction are neglected)

Assuming x = 0 to be at the equilibrium:

$$\sum F_x = F + mg\sin\theta - F_s = m\ddot{x}$$

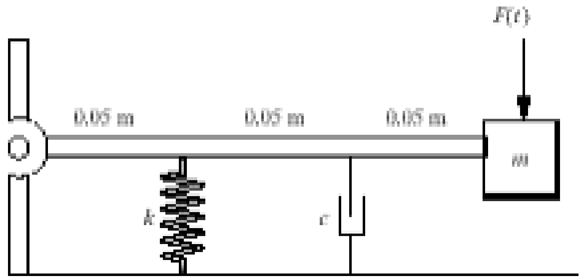
where $F_s = k(x + \frac{mg\sin\theta}{k})$ and $F(t) = F\cos\omega t$

Then the equation of motion is:

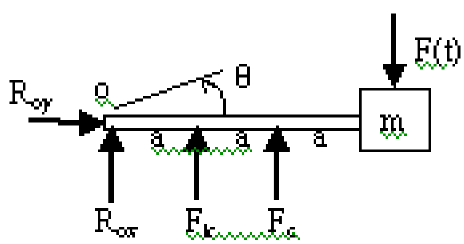
$m\ddot{x} + kx = F\cos\omega t$

Note that the equation of motion does not contain θ which means that the magnitude of the response is not affected by the angle of the incline.

2.30 A foot pedal for a musical instrument is modeled by the sketch in Figure P2.30. With k = 2000 N/m, c = 25 kg/s, m = 25 kg and $F(t)$ = 50 cos $2\pi t$N, compute the steady state response assuming the system starts from rest. Also use the small angle approximation.



Solution: Free body diagram of pedal follows:



Summing the moments with respect to the point, O:

$$\sum M_0 = F(3\cdot a) - F_c(2\cdot a) - F_s(a) = I_o\ddot{\theta}$$

$$\text{where} \quad I_o = m(3a)^2 = 9a^2m \;, \quad F_s = ka\sin\theta$$

$$F_c = c(2\cdot a\cdot\sin\theta)' = 2ca\cos\theta\dot{\theta}$$

Substituting these equations and simplifying ($\sin\theta \approx \theta$, cosθ =1,for small θ):

$$9a^2m\ddot{\theta} + 4a^2c\dot{\theta} + a^2k\theta = 3aF(t)$$

Given: k = 2000 N/m, c = 25kg/s , m = 25 kg , $F(t) = 50\cos 2\pi t$, a = 0.05 m

The equation of motion becomes: $0.5625\ddot{\theta} + 0.25\dot{\theta} + 5\theta = 7.5\cos 2\pi t$

Observing the equation of motion, equivalent mass, damping and stiffness coefficients are:

$c_{eq} = 0.25$, $m_{eq} = 0.5625$, $k_{eq} = 5$, $f_0 = \frac{F_0}{m_{eq}} = \frac{7.5}{0.5625} = 13.33$, $\omega = 2\pi$

$\omega_n = \sqrt{\frac{k_{eq}}{m_{eq}}} = \sqrt{\frac{5}{0.5625}} = 2.981$ $\qquad \zeta = \frac{c_{eq}}{2m_{eq}\omega_n} = 0.0745$

From Eq. (2.36), the steady-state response is:

$$\theta(t) = \frac{f_{0eq}}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}} \cos(\omega t - \tan^{-1}\frac{2\zeta_{eq}\omega_n\omega}{\omega_n^2 - \omega^2})$$

$$\Rightarrow \underline{\theta(t) = 0.434\cos(2\pi t - 3.051)\ \text{ rad}}$$

2.31 Consider the system of Problem 2.15, repeated here as Figure P2.31 with the effects of damping indicated. The physical constants are J =25 kg m^2, k = 2000 N/m, and the applied moment is 5 Nm at 1.432 Hz acting through the distance r = 0.5 m. Compute the magnitude of the steady state response if the measured damping ratio of the spring system is ζ = 0.01. Compare this to the response for the case where the damping is not modeled (ζ = 0).

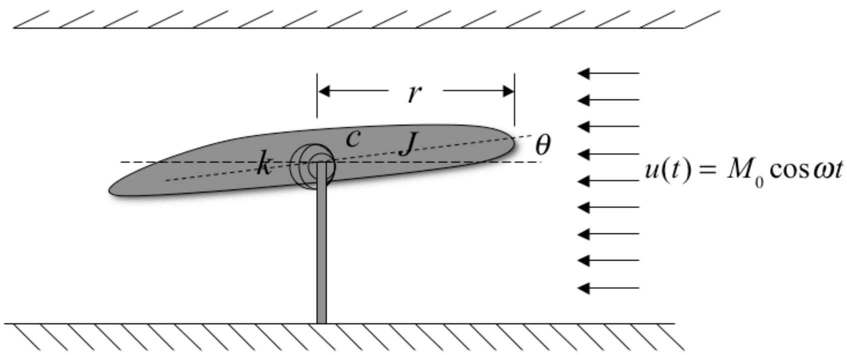


Figure P2.31 Model of an airfoil in at wind tunnel including the effects of damping.

Solution From equation (2.39) the magnitude of the steady state response for an underdamped system is

$$|\theta| = \frac{M_0 / J}{\sqrt{\left(\frac{k}{J} - \omega^2\right)^2 + \left(2\zeta\omega_n\omega\right)^2}}$$

Substitution of the given values yields (here $X = r\theta$)

$$|\theta| = 0.2 \text{ rad and } X = 0.1 \text{ m for } \zeta = 0$$

$$|\theta| = 0.106 \text{ rad and } X = 0.053 \text{ m for } \zeta = 0.01$$

where X is the vertical displacement of the wing tip. Thus a small amount of damping can greatly reduce the amplitude of vibration.

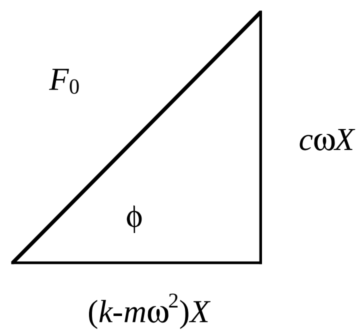
Problems and Solutions Section 2.3 (2.32 through 2.36)

2.32 Referring to Figure 2.10, draw the solution for the magnitude *X* for the case *m* = 100 kg, *c* = 4000 N s/m, and *k* = 10,000 N/m. Assume that the system is driven at resonance by a 10-N force.

Solution:
Given: m = 100 kg, c = 4000 N s/m, k = 10000 N/m, F_o = 10 N,

$$\omega = \omega_n = \sqrt{\frac{k}{m}} = 10 \text{ rad/s}$$

$$\phi = \tan^{-1}\left[\frac{cw}{k - m\omega^2}\right] = \tan^{-1}\left[\frac{(40,000)}{(10,000 - 10,000)}\right] = 90° = \frac{\pi}{2} \text{ rad}$$



From the figure:

$$X = \frac{F_o}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}} = \frac{10}{\sqrt{(10,000 - 10,000)^2 + (40,000)^2}}$$

X = 0.00025 m

2.33 Use the graphical method to compute the phase shift for the system of Problem 2.32 if $\omega = \omega_n/2$ and again for the case $\omega = 2\omega_n$.

Solution:

From Problem 2.32 $\omega_n = 10$ rad/s

(a) $\omega = \dfrac{\omega_n}{2} = 5$ rad/s

$$X = \frac{10}{\sqrt{(10{,}000 - 2500)^2 + (20{,}000)^2}} = .000468\ \text{m}$$

$kX = (10{,}000)(.000468) = 4.68$ N

$c\omega X = (4000)(5)(.000468) = 9.36$ N

$m\omega^2 X = (100)(5)^2(.000468) = 1.17$ N

From the figure given in problem 2.32:

$$\phi = \tan^{-1}\left[\frac{9.36}{4.68 - 1.17}\right] = 69.4° = 1.21\text{rad}$$

(b) $\omega = 2\omega_n = 20$ rad/s

$$X = \frac{10}{\sqrt{(10000 - 40000)^2 + (80000)^2}} = .000117\ \text{m}$$

$kX = (10000)(.000117) = 1.17$ N

$c\omega X = (4000)(20)(.000117) = 9.36$ N

$m\omega^2 X = (100)(20)^2(.000117) = 4.68$ N

From the figure:

$$\phi = \tan^{-1}\left[\frac{9.36}{1.17 - 4.68}\right] = -69.4° = -1.21\text{rad}$$

2.34 A body of mass 100 kg is suspended by a spring of stiffness of 30 kN/m and dashpot of damping constant 1000 N s/m. Vibration is excited by a harmonic force of amplitude 80 N and a frequency of 3 Hz. Calculate the amplitude of the displacement for the vibration and the phase angle between the displacement and the excitation force using the graphical method.

Solution:

Given: m = 100kg, k = 30 kN/m, F_o = 80 N, c = 1000 Ns/m,

$\omega = 3(2\pi)$ = 18.85 rad/s

kX = 30000 X

$c\omega X$ = 18850 X

$m\omega^2 X$ =35530 X

Following the figure given in problem 2.32:

$$\phi = \tan^{-1}\left[\frac{c\omega X}{\left(k - m\omega^2\right)X}\right]$$

$$\phi = \tan^{-1}\left[\frac{(18850)X}{(30000 - 35530)X}\right] = 106.4° = 1.86 \text{ rad}$$

Also from the figure, $$X = \frac{F_0}{\sqrt{\left(k - m\omega^2\right)^2 + \left(c\omega\right)^2}}$$

$$X = \frac{80}{\sqrt{(30000 - 35530)^2 + (18850)^2}} = 0.00407 \text{ m}$$

2.35 Calculate the real part of equation (2.55) to verify that it yields equation (2.36) and hence establish the equivalence of the exponential approach to solving the damped vibration problem.

Solution:

Equation (2.55) $$x_p(t) = \frac{F_o}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}} e^{j(\omega t - \theta)}$$

where $\theta = \tan^{-1}\left[\dfrac{c\omega}{k - m\omega^2}\right]$

Using Euler's Rule: $$x_p(t) = \frac{F_o}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}} [\cos(\omega t - \theta) + j\sin(\omega t - \theta)]$$

The real part is: $$x_p(t) = \frac{F_o}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}} \cos(\omega t - \theta)$$

Rearranging: $$x_p(t) = \frac{F_o / m}{\sqrt{(\omega^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}} \cos\left(\omega t - \tan^{-1}\left[\frac{2\zeta\omega_n\omega}{\omega_n^2 - \omega^2}\right]\right)$$

which is Equation (2.36).

2.36 Referring to equation (2.56) and Appendix B, calculate the solution $x(t)$ by using a table of Laplace transform pairs and show that the solution obtained this way is equivalent to (2.36).

Solution: Taking the Laplace transform of the equation of motion is given in Equation (2.56): $$X_p = (ms^2 + cs + k)X(s) = \frac{F_o s}{s^2 + \omega^2}$$

Solving this expression algebraically for X yields

$$X(s) = \frac{F_0 s}{(ms^2 + cs + k)(s^2 + \omega^2)} = \frac{f_0 s}{(s^2 + 2\zeta\omega_n s + \omega^2)(s^2 + \omega^2)}$$

Using Laplace Transform pairs from the table, this last expression is changed into the time domain to get:

$$x(t) = \frac{f_0}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}} \cos(\omega t\text{-}\theta)$$

Problems and Solutions Section 2.4 (2.37 through 2.50)

2.37 A machine weighing 2000 N rests on a support as illustrated in Figure P2.37. The support deflects about 5 cm as a result of the weight of the machine. The floor under the support is somewhat flexible and moves, because of the motion of a nearby machine, harmonically near resonance (r =1) with an amplitude of 0.2 cm. Model the floor as base motion, and assume a damping ratio of ζ = 0.01, and calculate the transmitted force and the amplitude of the transmitted displacement.

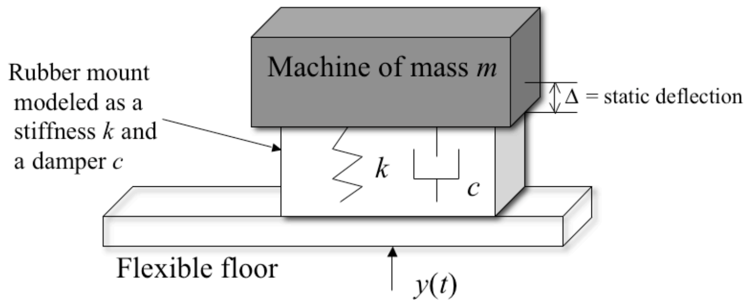


Figure P2.37

Solution:

Given: Y = 0.2 cm, ζ = 0.01, r = 1, mg = 2000N. The stiffness is computed from the static deflection and weight:

Deflection of 5 cm implies: $k = \dfrac{mg}{\Delta} = \dfrac{mg}{5\text{cm}} = \dfrac{2000}{0.05}$ = 40,000 N/m

Transmitted displacement from equation (2.70): $X = Y\left[\dfrac{1+(2\zeta r)^2}{(1-r^2)^2+(2\zeta r)^2}\right]^{1/2}$ = 10 cm

Transmitted force from equation (2.77): $F_T = kYr^2\left[\dfrac{1+(2\zeta r)^2}{(1-r^2)^2+(2\zeta r)^2}\right]^{1/2}$ = 4001N

2.38 Derive Equation (2.70) from (2.68) to see if the author has done it correctly.

Solution:

Equation (2.68) states:

$$x_p(t) = \omega_n Y\left[\frac{\omega_n^2+(2\zeta\omega_b)^2}{(\omega_n^2-\omega_b^2)^2+(2\zeta\omega_n\omega_b)^2}\right]^{1/2}\cos(\omega_b t-\theta_1-\theta_2)$$

The magnitude is: $X = \omega_n Y\left[\dfrac{\omega_n^2+(2\zeta\omega_b)^2}{(\omega_n^2-\omega_b^2)^2+(2\zeta\omega_n\omega_b)^2}\right]^{1/2}$

$$= \omega_n Y \left[\frac{(\omega_n^{-4})(\omega_n^2 + (2\zeta\omega_b)^2)}{(\omega_n^{-4})((\omega_n^2 - \omega_b^2)^2 + (2\zeta\omega_n\omega_b)^2)} \right]^{1/2}$$

$$= \omega_n Y \left[\frac{(\omega_n^{-2})(1 + (2\zeta r)^2)}{(1 - r^2)^2 + (2\zeta r)^2} \right]^{1/2} \Rightarrow$$

$$= \omega_n Y \frac{1}{\omega_n} \left[\frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2} \right]^{1/2} \Rightarrow$$

$$X = Y \left[\frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2} \right]^{1/2}$$

This is equation (2.71).

2.39 From the equation describing Figure 2.13, show that the point ($\sqrt{2}$, 1) corresponds to the value TR > 1 (i.e., for all $r < \sqrt{2}$, TR > 1).

Solution:

Equation (2.71) is $TR = \frac{X}{Y} = \left[\frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2} \right]^{1/2}$

Show $TR > 1$ for $r < \sqrt{2}$

$$TR = \frac{X}{Y} = \left[\frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2} \right]^{1/2} > 1$$

$$\frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2} > 1$$

$$1 + (2\zeta r)^2 > (1 - r^2)^2 + (2\zeta r)^2$$

$$1 > (1 - r^2)^2$$

Take the real solution:

$$1 - r^2 < +1 \text{ or } 1 - r^2 < -1 \Rightarrow$$
$$-r^2 > -2 \Rightarrow r^2 < 2 \Rightarrow r < \sqrt{2}$$

2.40 Consider the base excitation problem for the configuration shown in Figure P2.40. In this case the base motion is a displacement transmitted through a dashpot or pure damping element. Derive an expression for the force transmitted to the support in steady state.

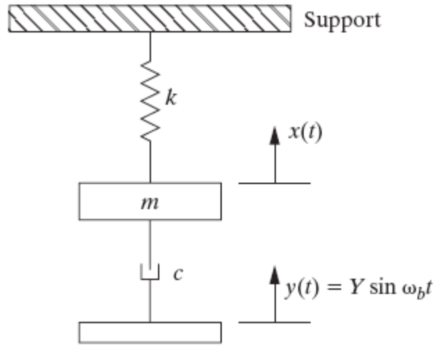


Figure P2.40

Solution: The entire force passes through the spring. Thus the support sees the force $F_T = kX$ where X is the magnitude of the displacement. From equation (2.65)

$$F_T = kX = \frac{2\zeta\omega_n\omega_b kY}{\sqrt{(\omega_n^2 - \omega_b^2)^2 + (2\zeta\omega_n\omega_b)^2}}$$

$$= \frac{2\zeta r k Y}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

2.41 A very common example of base motion is the single-degree-of-freedom model of an automobile driving over a rough road. The road is modeled as providing a base motion displacement of $y(t) = (0.01)\sin(5.818t)$ m. The suspension provides an equivalent stiffness of $k = 4 \times 10^5$ N/m, a damping coefficient of $c = 40 \times 10^3$ kg/s and a mass of 1007 kg. Determine the amplitude of the absolute displacement of the automobile mass.

Solution:

From the problem statement we have (working in Mathcad)

$\omega b := 5.818$ $\quad k := 4 \cdot 10^5$ N/m $\quad c := 40 \cdot 10^3$ kg/s

$Y := 0.01$ m $\quad m := 1007$ kg

$\omega n := \sqrt{\frac{k}{m}}$ $\quad \zeta := \frac{c}{2 \cdot \sqrt{m \cdot k}}$ $\quad \omega n = 19.93$ $\quad \zeta = 0.997$ $\quad r := \frac{\omega b}{\omega n}$ $\quad r = 0.292$

still underdamped, but very high damping. From equation (2.70)

$$X := Y \cdot \sqrt{\frac{1 + (2 \cdot \zeta \cdot r)^2}{\left(1 - r^2\right)^2 + (2 \cdot \zeta \cdot r)^2}} \qquad X = 0.011 \text{ m}$$

2.42 A vibrating mass of 300 kg, mounted on a massless support by a spring of stiffness 40,000 N/m and a damper of unknown damping coefficient, is observed to vibrate with a 10-mm amplitude while the support vibration has a maximum amplitude of only 2.5 mm (at resonance). Calculate the damping constant and the amplitude of the force on the base.

Solution:

Given: m = 300 kg, k = 40,000 N/m, $\omega_b = \omega_n (r = 1)$, X = 10 mm, Y = 2.5 mm.
Find damping constant (Equation 2.71)

$$\frac{X}{Y} = \left[\frac{1+(2\zeta r)^2}{(1-r^2)^2+(2\zeta r)^2}\right]^{1/2} \Rightarrow \frac{10}{2.5} = \left[\frac{1+4\zeta^2}{4\zeta^2}\right]^{1/2} \Rightarrow$$

$$16 = \frac{1+4\zeta^2}{4\zeta^2} \Rightarrow \zeta^2 = \frac{1}{60} = \frac{c^2}{4km} \quad \text{or}$$

$$c = \sqrt{\frac{4(40{,}000)(300)}{60}} = 894.4 \text{ kg/s}$$

Amplitude of force on base: (equation (2.76))

$$F_T = kYr^2\left[\frac{1+(2\zeta r)^2}{\left(1-r^2\right)^2+\left(2\zeta r\right)^2}\right]^{1/2} \Rightarrow$$

$$F_T = (40{,}000)(0.0025)(1)^2\left[\frac{1+4\left(\frac{1}{60}\right)}{4\left(\frac{1}{60}\right)}\right]^{1/2} \Rightarrow$$

$$F_T = 400 \text{ N}$$

2.43 Referring to Example 2.4.1, at what speed does car 1 experience resonance? At what speed does car 2 experience resonance? Calculate the maximum deflection of both cars at resonance.

Solution:

Given: m_1 = 1007 kg, m_2 =1585 kg, k = 4x10^5 N/m; c = 2,000 kg/s, Y = 0.01m

Velocity for resonance: (from Example 2.4.1)

$$\omega_b = 0.2909v \text{ (}v\text{ in km/h)}$$

Car 1: $\omega_1 = \sqrt{\dfrac{k}{m}} = \sqrt{\dfrac{4\times 10^4}{1007}} = \omega_b = 0.2909 v_1$

$$v_1 = 21.7 \text{ km/h}$$

Car 2: $\omega_2 = \sqrt{\dfrac{k}{m}} = \sqrt{\dfrac{4\times 10^4}{1585}} = \omega_b = 0.2909 v_2$

$$v_2 = 17.3 \text{ km/h}$$

Maximum deflection: (Equation 2.71 with r = 1)

$$X = Y\left[\frac{1+4\zeta^2}{4\zeta^2}\right]^{1/2} \Rightarrow$$

Car 1: $\zeta_1 = \dfrac{c}{2\sqrt{km_1}} = \dfrac{2000}{2\sqrt{(4\times 10^5)(1007)}} = 0.158$

$$X_1 = (0.01)\left[\frac{1+4(0.158)^2}{4(0.158)^2}\right]^{1/2} = 0.033 \text{ m}$$

Car 2: $\zeta_2 = \dfrac{c}{2\sqrt{km_2}} = \dfrac{2000}{2\sqrt{(4\times 10^4)(1585)}} = 0.126$

$$X_2 = (0.01)\left[\frac{1+4(0.126)^2}{4(0.126)^2}\right]^{1/2} = 0.041 \text{ m}$$

2.44 For cars of Example 2.4.1, calculate the best choice of the damping coefficient so that the transmissibility is as small as possible by comparing the magnitude of ζ = 0.01, ζ = 0.1 and ζ = 0.2 for the case r = 2. What happens if the road "frequency" changes?

Solution:

From Equation 2.62, with r = 2, the displacement transmissibility is:

$$\frac{X}{Y} = \left[\frac{1+(2\zeta r)^2}{(1-r^2)^2+(2\zeta r)^2}\right]^{1/2} = \left[\frac{1+16\zeta^2}{9+16\zeta^2}\right]^{1/2}$$

For $\zeta = 0.01$, $\frac{X}{Y} = 0.334$

For $\zeta = 0.1$, $\frac{X}{Y} = 0.356$

For $\zeta = 0.2$, $\frac{X}{Y} = 0.412$

The best choice would be $\zeta = 0.01$.

If the road frequency increases, the lower damping ratio would still be the best choice. However, if the frequency decreases, a higher damping ratio would be better because it would approach resonance.

2.45 A system modeled by Figure 2.12**,** has a mass of 225 kg with a spring stiffness of 3.5 $\times 10^4$ N/m. Calculate the damping coefficient given that the system has a deflection (X) of 0.7 cm when driven at its natural frequency while the base amplitude (Y) is measured to be 0.3 cm.

Solution:

Given: m = 225 kg, k = 3.5x10^4 N/m, X = 0.7 cm, Y = 0.3 cm, $\omega = \omega_b$.
Base excitation: (Equation (2.71) with r = 1)

$$\frac{X}{Y} = \left[\frac{1+4\zeta^2}{4\zeta^2}\right]^{1/2} \Rightarrow \frac{0.7}{0.3} = \left[\frac{1+4\zeta^2}{4\zeta^2}\right]^{1/2} \Rightarrow$$

$$\zeta = 0.237 = \frac{c}{2\sqrt{km}}$$

$$c = (0.237)(2)[(3.5\text{x}10^4)(225)]^{1/2}$$

$$\underline{c = 1331 \text{ kg/s}}$$

2.46 Consider Example 2.4.1 for car 1 illustrated in Figure P2.46, if three passengers totaling 200 kg are riding in the car. Calculate the effect of the mass of the passengers on the deflection at 20, 80, 100, and 150 km/h. What is the effect of the added passenger mass on car 2?

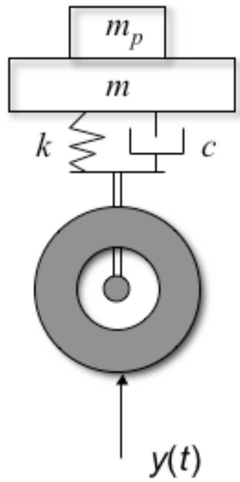


Figure P2.46 Model of a car suspension with the mass of the occupants, m_p, included.

Solution:

Add a mass of 200 kg to each car. From Example 2.4.1, the given values are:
m_1 = 1207 kg, m_2 = 1785 kg, $k = 4\text{x}10^4$ N/m; c = 2,000 kg/s, $\omega_b = 0.29v$.

Car 1: $\omega_1 = \sqrt{\frac{k}{m}} = \sqrt{\frac{4\times10^4}{1207}} = 5.76$ rad/s

$$\zeta_1 = \frac{c}{2\sqrt{km_1}} = \frac{2000}{2\sqrt{(4\times10^5)(1207)}} = 0.144$$

Car 2: $\omega_2 = \sqrt{\frac{k}{m}} = \sqrt{\frac{4\times10^4}{1785}} = 4.73$ rad/s

$$\zeta_2 = \frac{c}{2\sqrt{km_2}} = \frac{2000}{2\sqrt{(4\times10^5)(1785)}} = 0.118$$

Using Equation (2.71): $X = Y\left[\frac{1+(2\zeta r)^2}{(1-r^2)^2+(2\zeta r)^2}\right]^{1/2}$ produces the following:

Speed (km/h)	ω_b (rad/s)	r_1	r_2	x_1 (cm)	x_2 (cm)
20	5.817	1.01	1.23	3.57	1.77
80	23.271	3.871	4.71	0.107	0.070
100	29.088	5.05	6.15	0.072	0.048
150	2.40	7.58	9.23	0.042	0.028

At lower speeds there is little effect from the passengers weight, but at higher speeds the added weight reduces the amplitude, particularly in the smaller car.

2.47 Consider Example 2.4.1. Choose values of *c* and *k* for the suspension system for car 2 (the sedan) such that the amplitude transmitted to the passenger compartment is as small as possible for the 1 cm bump at 50 km/h. Also calculate the deflection at 100 km/h for your values of *c* and *k*.

Solution:

For car 2, m = 1585 kg.
Also, ω_b = 0.2909(50) = 14.545 rad/s and Y = 0.01 m.
From equation (2.70),

$$X = Y\left[\frac{1+(2\zeta r)^2}{(1-r^2)^2+(2\zeta r)^2}\right]^{1/2}$$

From Figure 2.9, we can choose a value of r away from resonance and a low damping ratio. Choose r = 2.5 and ζ =0.05.

So, $r = 2.5 = \dfrac{\omega_b}{\omega} = \dfrac{14.545}{\sqrt{k/1585}}$

k = 53,650 N/m

$\zeta = 0.05 = \dfrac{c}{2\sqrt{km}}$

c = 922.2 kg/s

So, $$X = (0.01)\left[\frac{1+[2(0.05)(2.5)]^2}{\left(1-\left(2.5\right)^2\right)^2+[2(0.05)(2.5)]^2}\right]^{1/2} = 0.00196 \text{ m}$$

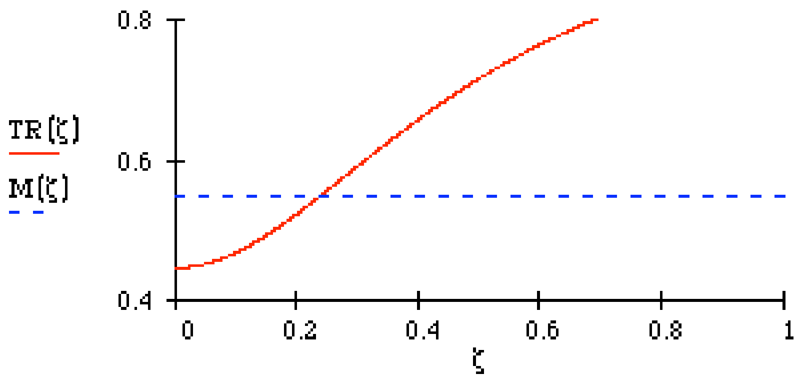
At 100 km/h, ω_b = 29.09 rad/s and $r = \dfrac{\omega_b}{\sqrt{k/m}} = 5.$

2.48 Consider the base motion problem of Figure 2.12. a) Compute the damping ratio needed to keep the displacement magnitude transmissibility less then 0.55 for a frequency ratio of $r = 1.8$. b) What is the value of the force transmissibility ratio for this system?

Solution: Working with equation (2.71), make a plot of TR versus ζ and use equation (2.77) to compute the value of the force transmissibility. The following Mathcad session illustrates the procedure.

$$r := 1.8$$

$$TR(\zeta) := \sqrt{\frac{1 + (2\cdot\zeta\cdot r)^2}{(1 - r^2)^2 + (2\cdot\zeta\cdot r)^2}} \qquad M(\zeta) := 0.55$$



$$F(\zeta) := r^2\cdot TR(\zeta)$$

$$F(.2) = 1.697$$

From the plot a value of $\zeta = 0.2$ keeps the displacement transmissibility less then 0.55 as desired. The value of the force transmissibility is then 1.697. Precise values can be found by equating the above expression to 0.55.

2.49 Consider the effect of variable mass on an aircraft landing suspension system by modeling the landing gear as a moving base problem similar to that shown in Figure P2.46 for a car suspension. The mass of a regional jet is 13, 236 kg empty and its maximum takeoff mass is 21,523 kg. Compare the maximum deflection for a wheel motion of magnitude 0.50 m and frequency of 35 rad/s, for these two different masses. Take the damping ratio to be $\zeta = 0.1$ and the stiffness to be 4.22×10^6 N/m.

Solution: Using a Mathcad worksheet the following calculations result:

$Y := 0.5 \cdot m \quad k = 4.22 \times 10^{6} kg \cdot s^{-2} \quad mf := 21523kg \quad me := 13236 \cdot kg$

From equation (2.70):

$$\omega b := 35 \cdot \frac{rad}{sec}$$

$$re := \frac{\omega b}{\sqrt{\frac{k}{me}}} \qquad re = 1.96 \qquad \zeta := 0.1$$

$$\sqrt{\frac{k}{mf}} = 14.002 s^{-1}$$

$$Xe := Y \cdot \sqrt{\frac{1 + (2 \cdot \zeta \cdot re)^2}{\left(1 - re^2\right)^2 + (2 \cdot \zeta \cdot re)^2}} \qquad Xe = 0.187 m$$

$$rf := \frac{\omega b}{\sqrt{\frac{k}{mf}}} \qquad rf = 2.5$$

$$Xf := Y \cdot \sqrt{\frac{1 + (2 \cdot \zeta \cdot rf)^2}{\left(1 - rf^2\right)^2 + (2 \cdot \zeta \cdot rf)^2}} \qquad Xf = 0.106 m$$

Note that if the suspension stiffness were defined around the full case, when empty the plane would bounce with a larger amplitude then when full. Note Mathcad does not have a symbol for a Newton so the units on stiffness above are kg/sec^2 in order to allow Mathcad to compute the units.

2.50 Consider the simple model of a building subject to ground motion suggested in Figure P2.50. The building is modeled as a single degree of freedom spring-mass system where the building mass is lumped atop of two beams used to model the walls of the building in bending. Assume the ground motion is modeled as having amplitude of 0.1 m at a frequency of 7.5 rad/s. Approximate the building mass by 10^5 kg and the stiffness of each wall by 3.519 x 10^6 N/m. Compute the magnitude of the deflection of the top of the building.

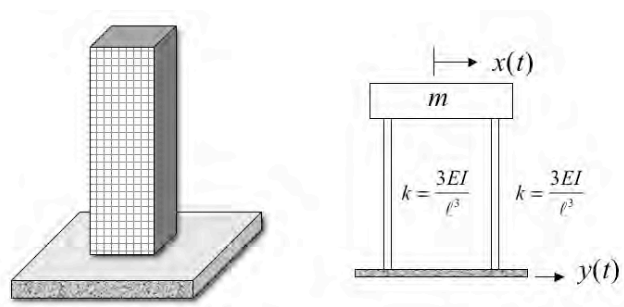


Figure P2.50 A simple model of a building subject to ground motion, such as an earthquake.

Solution: The equation of motion is

$$m\ddot{x}(t) + 2kx(t) = 0.1\cos 7.5t$$

The natural frequency and frequency ratio are

$$\omega_n = \sqrt{\frac{2k}{m}} = 8.389 \text{ rad/s} \quad \text{and} \quad r = \frac{\omega}{\omega_n} = \frac{7.5}{8.389} = 0.894$$

The amplitude of the steady state response is given by equation (2.70) with $\zeta = 0$ in this case:

$$X = Y\left|\frac{1}{1-r^2}\right| = 0.498 \text{ m}$$

Thus the earthquake will cause serious motion in the building and likely break.

Problems and Solutions Section 2.5 (2.51 through 2.58)

2.51 A lathe can be modeled as an electric motor mounted on a steel table. The table plus the motor have a mass of 50 kg. The rotating parts of the lathe have a mass of 5 kg at a distance 0.1 m from the center. The damping ratio of the system is measured to be ζ = 0.06 (viscous damping) and its natural frequency is 7.5 Hz. Calculate the amplitude of the steady-state displacement of the motor, assuming ω_r = 30 Hz.

Soltuion:

Given: m = 50 kg, $m_o = 5$, e = 0.1m, $\zeta = 0.06$, $\omega_n = 7.5\text{Hz}$

Let ω_r=30 Hz

So, $r = \dfrac{\omega_r}{\omega_n} = 4$

From Equation (2.84),

$$X = \frac{m_o e}{m}\frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} = \frac{(5)(0.1)}{50}\frac{4^2}{\sqrt{(1-4^2)^2 + [2(0.06)(4)]^{-2}}}$$

X = 0.011m

X = 1.1 cm

2.52 The system of Figure 2.18 produces a forced oscillation of varying frequency. As the frequency is changed, it is noted that at resonance, the amplitude of the displacement is 10 mm. As the frequency is increased several decades past resonance the amplitude of the displacement remains fixed at 1 mm. Estimate the damping ratio for the system.

Solution: Equation (2.84) is

$$X = \frac{m_o e}{m}\frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

At resonance, X = 10 mm = $\dfrac{m_o e}{m}\dfrac{1}{2\zeta}$

$$\frac{10m}{m_o e} = \frac{1}{2\zeta}$$

When r is very large, $\dfrac{Xm}{m_o e} = 1$ and X = 1 mm, so

$$\frac{m}{m_o e} = 1$$

Therefore, 10(1) = $\dfrac{1}{2\zeta}$

$$\zeta = 0.05$$

2.53 An electric motor (Figure P2.53) has an eccentric mass of 10 kg (10% of the total mass) and is set on two identical springs (k = 3200 /m). The motor runs at 1750 rpm, and the mass eccentricity is 100 mm from the center. The springs are mounted 250 mm apart with the motor shaft in the center. Neglect damping and determine the amplitude of the vertical vibration.

Solution:
Given m_0 = 10 kg, m= 100 kg, k = 2x3.2 N/mm, , e = 0.1 m

$$\omega_r = 1750\frac{\text{rev}}{\text{min}}\left(\frac{\text{min}}{60\,\text{sec}}\frac{2\pi\ \text{rad}}{\text{rev}}\right) = 183.26\frac{\text{rad}}{\text{s}}\ \text{rad/s}$$

Vertical vibration:

$$\omega_n = \sqrt{\frac{2(3.2)(1000)}{100}} = 8\ \text{rad/s}$$

$$r = \frac{\omega_r}{\omega_n} = \frac{183.3}{8} = 22.9$$

From equation (2.84)

$$X = e\frac{m_0}{m}\frac{r^2}{|1-r^2|} == 0.01\ \text{m}$$

2.54 Consider a system with rotating unbalance as illustrated in Figure P2.53. Suppose the deflection at 1750 rpm is measured to be 0.05 m and the damping ratio is measured to be ζ = 0.1. The out-of-balance mass is estimated to be 10%. Locate the unbalanced mass by computing e.

Solution: Given: X = 0.05 m, $\zeta = 0.1$, $m_e = 0.1m$, and from the solution to problem 2.53 the frequency ratio is calculated to be r = 22.9. Solving the rotating unbalance Equation (2.84) for e yields:

$$X = \frac{m_0 e}{m}\frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \Rightarrow e = \frac{mX}{m_0}\frac{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}{r^2} = \underline{0.499\ \text{m}}$$

This sort of calculation can be introduced to discuss the application of machinery diagnostics if time permits. Machinery diagnostics deals with determining the location and extend of damage from measurements of the response and input.

2.55 A fan of 45 kg has an unbalance that creates a harmonic force. A spring-damper system is designed to minimize the force transmitted to the base of the fan. A damper is used having a damping ratio of ζ = 0.2. Calculate the required spring stiffness so that only 10% of the force is transmitted to the ground when the fan is running at 10,000 rpm.

Solution: The equation of motion of the fan is

$$m\ddot{x} + c\dot{x} + kx = m_0 e\omega^2 \sin(\omega t + \theta)$$

The steady state solution as given by equation (2.84) is

$$x(t) = \frac{m_0 e}{m} \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \sin \omega t$$

where r is the standard frequency ratio. The force transmitted to the ground is

$$F(t) = kx + c\dot{x} = \frac{m_0 e}{m} \frac{kr^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \sin \omega t + \frac{m_0 e}{m} \frac{c\omega r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \cos \omega t$$

Taking the magnitude of this quantity, the magnitude of the force transmitted becomes

$$F_0 = \frac{m_0 e}{m} \frac{r^2 \sqrt{k^2 + c^2\omega^2}}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} = m_0 e\omega \frac{\sqrt{1+(2\zeta r)^2}}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

From equation (2.81) the magnitude of the force generated by the rotating mass F_r is

$$F_r = m_0 e\omega^2$$

The limitation stated in the problem is that $F_0 = 0.1F_r$, or

$$m_0 e\omega^2 \frac{\sqrt{1+(2\zeta r)^2}}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} = 0.1 m_0 e\omega^2$$

Setting $\zeta = 0.2$ and solving for r yields:

$$r^4 - 17.84r^2 - 99 = 0$$

which yields only one positive solution for r^2, which is

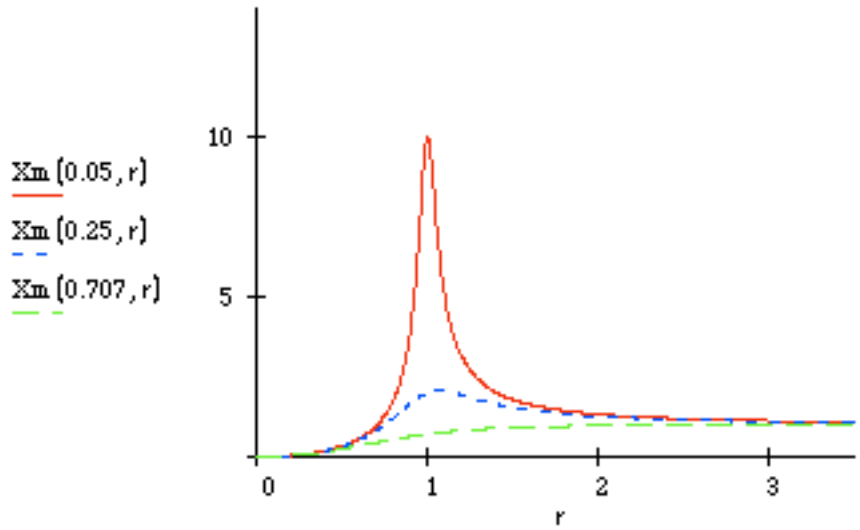
$$r^2 = 22.28 = \frac{\omega^2}{k/m} \Rightarrow \frac{k}{m} = \left(\frac{10000 \times 2\pi}{60}\right)^2 \frac{1}{22.28}$$

$$\Rightarrow k = 45\left(\frac{10000 \times 2\pi}{60}\right)^2 \frac{1}{22.28} = 2.21 \times 10^6 \text{ N/m}$$

2.56 Plot the normalized displacement magnitude versus the frequency ratio for the out of balance problem (i.e., repeat Figure 2.20) for the case of ζ = 0.05.

Solution: Working in Mathcad using equation (2.84) yields:

$$Xm(\zeta, r) := \frac{r^2}{\sqrt{\left(1 - r^2\right)^2 + (2\cdot\zeta\cdot r)^2}}$$



2.57 Consider a typical unbalanced machine problem as given in Figure P2.57 with a machine mass of 120 kg, a mount stiffness of 800 kN/m and a damping value of 500 kg/s. The out of balance force is measured to be 374 N at a running speed of 3000 rev/min. a) Determine the amplitude of motion due to the out of balance. b) If the out of balance mass is estimated to be 1% of the total mass, estimate the value of the *e*.

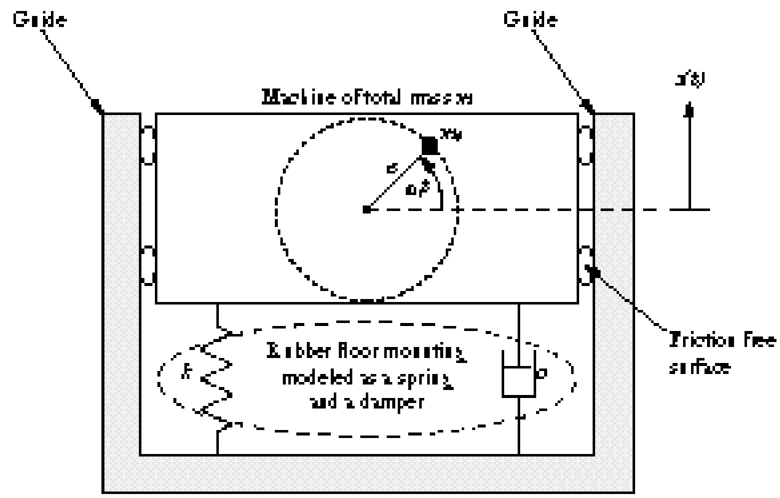


Figure P2.57 Typical unbalance machine problem.

Solution:

a) Using equation (2.84) with $m_0 e = F_0/\omega_r^2$ yields:

$k := 800 \cdot 1000 \qquad m := 120 \qquad c := 500 \qquad F0 := 374$

$\omega r := 100 \cdot \pi \qquad \omega n := \sqrt{\frac{k}{m}} \qquad \zeta := \frac{c}{2 \cdot \sqrt{k \cdot m}} \qquad k = 8 \cdot 10^5$

$r := \frac{\omega r}{\omega n} \qquad \omega n = 81.65 \qquad r = 3.848 \qquad \zeta = 0.026$

$$X := \frac{F0}{\omega r^2 \cdot m} \cdot \frac{r^2}{\sqrt{\left(1 - r^2\right)^2 + \left(2 \cdot \zeta \cdot r\right)^2}} \qquad X = 3.386 \cdot 10^{-5}$$

b) Use the fact that $F_0 = m_0 e \omega_r^2$ to get

$$e := \frac{F0}{\omega r^2 \cdot (0.01 \cdot m)} \qquad e = 3.158 \cdot 10^{-3}$$

in meters.

2.58 Plot the response of the mass in Problem 2.57 assuming zero initial conditions.

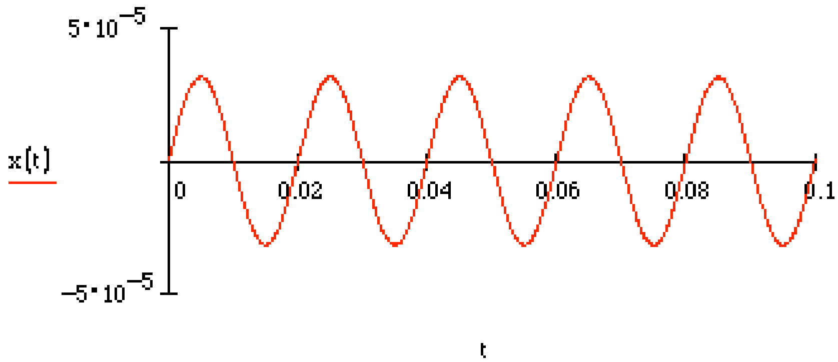
Solution: The steady state response is the particular solution given by equation (2.84) and is plotted here in Mathcad:

$m := 120 \qquad k := 120 \qquad c := 500 \qquad F0 := 374$

$$\omega n := \sqrt{\frac{k}{m}} \qquad \omega r := \frac{3000 \cdot 2 \cdot \pi}{60} \qquad r := \frac{\omega r}{\omega n} \qquad \zeta := \frac{c}{2 \cdot \sqrt{m \cdot k}}$$

$$X := \frac{F0}{\omega r^2 \cdot m} \cdot \frac{r^2}{\sqrt{\left(1 - r^2\right)^2 + (2 \cdot \zeta \cdot r)^2}} \qquad \theta := \operatorname{atan}\left(\frac{2 \cdot \zeta \cdot r}{1 - r^2}\right)$$

$$x(t) := X \cdot \sin(\omega r \cdot t - \theta)$$



Problems and Solutions Section 2.6 (2.59 through 2.62)

2.59 Calculate damping and stiffness coefficients for the accelerometer of Figure 2.23 with moving mass of 0.04 kg such that the accelerometer is able to measure vibration between 0 and 50 Hz within 5%. (*Hint*: For an accelerometer it is desirable for $Z/\omega_b^2 Y$ = constant.)

Solution: Use equation (2.90):

Given: m = 0.04 kg with error < 5%

$0.2f = 50 \text{ Hz} \rightarrow f = 250 \text{ Hz} \rightarrow \omega = 2\pi f = 1570.8$ rad/s

Thus, $k = m\omega^2 = 98{,}696$ N/m

When $r = .2$, $$0.95 < \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} < 1.05 \qquad (\pm 5\% \text{ error})$$

This becomes $0.8317 + 0.1444\zeta^2 < 1 < 1.016 + 0.1764\zeta^2$

Therefore, $$\zeta = 0.7 = \frac{c}{2\sqrt{km}}$$

$$c = 2(.7)\sqrt{(98696)(.04)}$$

c = 87.956 Ns/m

2.60 The damping constant for a particular accelerometer of the type illustrated in Figure 2.23 is 50 N s/m. It is desired to design the accelerometer (i.e., choose m and k) for a maximum error of 3% over the frequency range 0 to 75 Hz.

Solution: Given $0.2f = 75 \text{ Hz} \rightarrow f = 375 \text{ Hz} \rightarrow \omega_n = 2\pi f = 2356.2$ rad/s. Using equation (2.93) when $r = 0.2$:

$$0.97 < \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} < 1.03 \qquad (\pm 3\% \text{ error})$$

This becomes $0.8671 + 0.1505\zeta^2 < 1 < 0.9777 + 0.1697\zeta^2$

Therefore, $0.3622 < \zeta < 0.9395$

Choose $$\zeta = 0.7 = \frac{c}{2m\omega} = \frac{50}{2m(2356.2)}$$

m = 0.015 kg

$k = m\omega_n^2 = 8.326 \times 10^4$ N/m

2.61 The accelerometer of Figure 2.23 has a natural frequency of 120 kHz and a damping ratio of 0.2. Calculate the error in measurement of a sinusoidal vibration at 60 kHz.

Solution:

Given: ω = 120 kHz, $\zeta = .2$, ω_b = 60 kHz

So, $$\frac{1}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}=\frac{1}{\sqrt{(1-.5^2)^2+(2(.2)(.5))^2}}=1.288>1$$

The error is $\frac{1.288-1}{1}\times 100\%$ = 28.8%

2.62 Design an accelerometer (i.e., choose *m*, *c* and *k*) configured as in Figure 2.23 with very small mass that will be accurate to 1% over the frequency range 0 to 50 Hz.

Solution:

Given: error < 1% , 0.2f = 50 Hz $\rightarrow$ f = 250 Hz $\rightarrow$ $\omega = 2\pi f$ = 1570.8 rad/s

When r =0.2, $$0.99<\frac{1}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}<1.01 \qquad (\pm\ 1\%\ \text{error})$$

This becomes $0.9032 + 0.1568\zeta^2 < 1 < 0.9401 + 0.1632\zeta^2$

Therefore, $0.6057 < \zeta < 0.7854$

Choose m = 0.01 kg , then $k = m\omega^2$ = 24,674 N/m

Thus $\zeta = 0.7 = \frac{c}{2\sqrt{km}}$ implies that: c = 21.99 Ns/m

Problems and Solutions Section 2.7 (2.63 through 2.79)

2.63 Consider a spring-mass sliding along a surface providing Coulomb friction, with stiffness 1.2×10^4 N/m and mass 10 kg, driven harmonically by a force of 50 N at 10 Hz. Calculate the approximate amplitude of steady-state motion assuming that both the mass and the surface that it slides on, are made of lubricated steel.

Solution: Given: m = 10 kg, k = 1.2x10^4 N/m, F_o = 50 N, ω =10(2π) = 20π rad/s

$$\omega = \sqrt{\frac{k}{m}} = 34.64 \text{ rad/s}$$

for lubricated steel, μ = 0.07

From Equation (2.109) $$X = \frac{F_o}{k}\frac{\sqrt{1-\left[\frac{4\mu mg}{\pi(F_{o})}\right]^2}}{\left|(1-r^2)\right|}$$

$$X = \frac{50}{1.2\times 10^4}\frac{\sqrt{1-\left[\frac{4(.07)(10)(9.81)}{\pi(50)}\right]^2}}{\left|(1-\left(\frac{20\pi}{34.64}\right)^2)\right|}$$

$$X = 1.79\times 10^{-3}\text{ m}$$

2.64 A spring-mass system with Coulomb damping of 10 kg, stiffness of 2000 N/m, and coefficient of friction of 0.1 is driven harmonically at 10 Hz. The amplitude at steady state is 5 cm. Calculate the magnitude of the driving force.

Solution:

Given: m = 10 kg, k = 2000 N/m, μ = 0.1, ω=10(2π) = 10(2π) = 20π rad/s,

$$\omega_n = \sqrt{\frac{k}{m}} = 14.14 \text{ rad/s}, \quad X = 5 \text{ cm}$$

Equation (2.108) $$X = \frac{\frac{F_0}{k}}{\sqrt{(1-r^2)^2 + \left[\frac{4\mu mg}{\pi kX}\right]^2}} \Rightarrow F_0 = Xk\sqrt{(1-r^2)^2 + \left[\frac{4\mu mg}{\pi kX}\right]^2}$$

$$F_0 = (0.05)(2000)\sqrt{\left(1-\left[\frac{20\pi}{14.14}\right]^2\right)^2 + \left(\frac{4(0.1)(10)(9.81)}{\pi(2000)(.05)}\right)^2} = 1874 \text{ N}$$

2.65 A system of mass 10 kg and stiffness 1.5×10^4 N/m is subject to Coulomb damping. If the mass is driven harmonically by a 90-N force at 25 Hz, determine the equivalent viscous damping coefficient if the coefficient of friction is 0.1.

Solution:

Given: m = 10 kg, k = 1.5x10^4 N/m, F_0 = 90 N, ω = 25(2π) = 50π rad/s,

$$\omega_n = \sqrt{\frac{k}{m}} = 38.73 \text{ rad/s}, \ \mu = 0.1$$

Steady-state Amplitude using Equation (2.109) is

$$X = \frac{F_0}{k}\frac{\sqrt{1-\left[\dfrac{4\mu mg}{\pi(F_o)}\right]^2}}{\left|(1-r^2)\right|} = \frac{90}{1.5\times10^4}\frac{\sqrt{1-\left[\dfrac{4(0.1)(10)(9.81)}{\pi(90)}\right]^2}}{\left|1-\left(\dfrac{50\pi}{38.73}\right)^2\right|} = 3.85\times10^{-4} \text{ m}$$

From equation (2.105), the equivalent Viscous Damping Coefficient becomes:

$$c_{eq} = \frac{4\mu mg}{\pi\omega X} = \frac{4(0.1)(10)(9.81)}{\pi(50\pi)(3.85\times10^{-4})} = 206.7 \text{ Ns/m}$$

2.66 a. Plot the free response of the system of Problem 2.65 to initial conditions of $x(0) = 0$ and $\dot{x}(0) = |F_0/m| = 9$ m/s using the solution in Section 1.10**.**

b. Use the equivalent viscous damping coefficient calculated in Problem 2.65 and plot the free response of the "equivalent" viscously damped system to the same initial conditions.

Solution: See Problem 2.65

(a) $x(0) = 0$ and $\dot{x}(0) = \frac{F_o}{m} = 9$ m/s

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{1.5x10^4}{10}} = 38.73 \text{ rad/s}$$

From section 1.10:

$$m\ddot{x} + kx = \mu mg \text{ for } \dot{x} < 0$$
$$m\ddot{x} + kx = -\mu mg \text{ for } \dot{x} > 0$$

Let $F_d = \mu mg = (0.1)(10)(9.81) = 9.81$ N

To start, $\dot{x}(0) = \omega_n B_1 = 9$

Therefore, $A_1 = \frac{F_d}{k}$ and $B_1 = \frac{9}{\omega_n}$

So, $x(t) = \frac{F_d}{k}\cos\omega_n t + \frac{9}{\omega}\sin\omega_n t - \frac{F_d}{k}$

This will continue until $\dot{x} = 0$, which occurs at time t_1:

$$x(t) = A_2\cos\omega_n t + B_2\sin\omega_n t + \frac{F_d}{k}$$
$$\dot{x}(t) = -\omega_n A_2\sin\omega_n t + \omega_n B_2\cos\omega_n t$$
$$x(t_1) = A_2\cos\omega_n t_1 + B_2\sin\omega_n t_1 + \frac{F_d}{k}$$
$$\dot{x}(t_1) = 0 = -\omega_n A_2\sin\omega_n t_1 + \omega_n B_2\cos\omega_n t_1$$

Therefore, $A_2 = \left(x(t_1) - F_d/k\right)\cos\omega_n t_1$ and $B_2 = \left(x(t_1) - F_d/k\right)\sin\omega_n t_1$

So, $x(t) = \left[\left(x(t_1) - F_d/k\right)\cos\omega_n t_1\right]\cos\omega_n t + \left[\left(x(t_1) - F_d/k\right)\sin\omega_n t_1\right]\sin\omega_n t + \frac{F_d}{k}$

Again, when $\dot{x} = 0$ at time t_2, the motion will reverse:

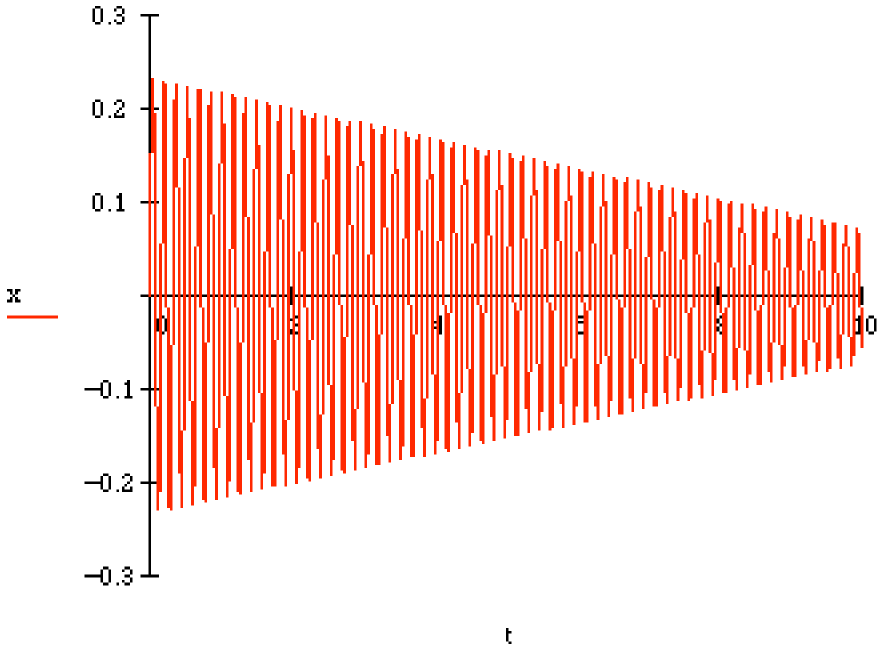
$$x(t) = A_3\cos\omega_n t + B_3\sin\omega_n t - \frac{F_d}{k}$$
$$\dot{x}(t) = -\omega_n A_3\sin\omega_n t + \omega_n B_3\cos\omega_n t$$
$$x(t_2) = A_3\cos\omega_n t_2 + B_3\sin\omega_n t_2 - \frac{F_d}{k}$$

$$\dot{x}(t_2) = 0 = -\omega_n A_3 \sin\omega_n t_2 + \omega_n B_3 \cos\omega_n t_2$$

Therefore, $A_3 = \left(x(t_2) + F_d / k\right)\cos\omega_n t_2$ and $B_3 = \left(x(t_2) - F_d / k\right)\sin\omega_n t_2$

So, $x(t) = \left[\left(x(t_2) + F_d / k\right)\cos\omega_n t_2\right]\cos\omega_n t + \left[\left(x(t_2) + F_d / k\right)\sin\omega_n t_2\right]\sin\omega_n t - \dfrac{F_d}{k}$

This continues until $\dot{x} = 0$ and $kx < \mu mg = 9.81$ N



(b) From Problem 2.65, $c_{eq} = 206.7$ kg/s

The equivalently damped system would be:

$$m\ddot{x} + c_{eq}\dot{x} + kx = 0$$

Also, $\omega_n = \sqrt{\dfrac{k}{m}} = \sqrt{\dfrac{1.5x10^4}{10}} = 38.73$ rad/s

$$\zeta = \frac{c_{eq}}{2\sqrt{km}} = \frac{206.7}{2\sqrt{(1.5x10^4)(10)}} \; 0.2668$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 37.33 \text{ rad/s}$$

The solution would be found from Equation 1.36:

$$x(t) = Ae^{-\zeta\omega_n t}\sin(\omega_d t + \phi)$$

$$\dot{x}(t) = -\zeta\omega_n Ae^{-\zeta\omega_n t}\sin(\omega_d t + \phi) + \omega_d Ae^{-\zeta\omega_n t}\cos(\omega_d t + \phi)$$

$$x(0) = A\sin\phi = 0$$

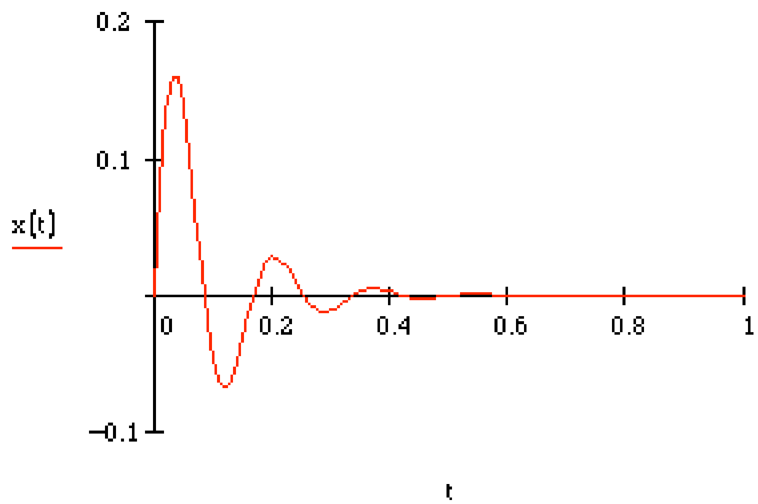
$$\dot{x}(0) = -\zeta\omega_n A\sin\phi + \omega_d A\cos\phi = 9$$

Therefore, $A = \dfrac{9}{\omega_d} = 0.2411\text{m}$ and $\phi = 0$ rad

So, $x(t) = 0.2411e^{-10.335t}\sin(37.33t)$

$t := 0, 0.01 .. 1$

$x(t) := 0.2411 \cdot e^{-10.35 \cdot t} \cdot \sin(37.33 \cdot t)$



2.67 Referring to the system of Example 2.7.1, calculate how large the magnitude of the driving force must be to sustain motion if the steel is lubricated. How large must this magnitude be if the lubrication is removed?

Solution:

From Example 2.7.1 $m = 10$ kg, $k = 1.5 \times 10^4$ N/m, $F_o = 90$ N,

$\omega = 25(2\pi) = 50\pi$ rad/s

Lubricated Steel $\mu = 0.07$

Unlubricated Steel $\mu = 0.3$

Lubricated: $F_o > \dfrac{4\mu mg}{\pi} = \dfrac{4(0.07)(10)(9.81)}{\pi}$

$F_o = 8.74$ N

Unlubricated: $F_o > \dfrac{4\mu mg}{\pi} = \dfrac{4(0.3)(10)(9.81)}{\pi}$

$F_o = 37.5$ N

2.68 Calculate the phase shift between the driving force and the response for the system of Problem 2.67 using the equivalent viscous damping approximation.

Solution:

From Problem 2.67: $m = 10$ kg, $k = 1.5 \times 10^4$ N/m, $F_o = 90$ N,

$$\omega = 25(2\pi) = 157.1 \text{ rad/s}$$

$$\omega_n = \sqrt{\frac{k}{m}} = 38.73 \text{ rad/s}$$

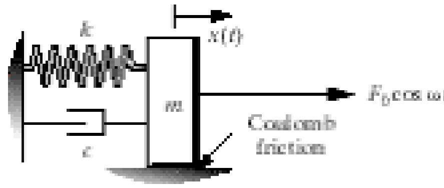
From Equation (2.111), and since r>1

$$\theta = \tan^{-1}\left[\frac{-4\mu mg}{\pi F_0\sqrt{1-\left(\frac{4\mu mg}{\pi F_o}\right)^2}}\right]$$

Since in Problem 2.67, $\pi F_o = 4\mu mg$, this reduces to

$$\theta = \tan^{-1}\left[\frac{-1}{0}\right] = \frac{-\pi}{2} \text{ rad} = -90^\circ$$

2.69 Derive the equation of vibration for the system of Figure P2.69 assuming that a viscous dashpot of damping constant *c* is connected in parallel to the spring. Calculate the energy loss and determine the magnitude and phase relationships for the forced response of the equivalent viscous system.



Solution: Sum of the forces in Figure P2.69

$$m\ddot{x} = -kx - c\dot{x} - \mu mg\,\text{sgn}(\dot{x})$$

$$m\ddot{x} + c\dot{x} + \mu mg\,\text{sgn}\;(\dot{x}) + kx = 0$$

Assume the mass is moving to the left $(\dot{x}(0) = 0, x(0) = x_0)$

$$m\ddot{x} - c\dot{x} + \mu mg + kx = 0$$

$$\ddot{x} + 2\zeta\omega_n\dot{x} - \mu g + \omega_n^2 x = 0$$

The solution of the form:

$$x(t) = ae^{rt} + \frac{\mu g}{\omega_n{}^2}$$

Substituting:

$$ar^2e^{rt} + 2\zeta\omega_n are^{rt} - \mu g + \omega_n{}^2 ae^{rt} + \mu g = 0$$

$$r^2 + 2\zeta\omega_n r + \omega_n{}^2 = 0$$

$$r = \frac{-2\zeta\omega_n \pm \sqrt{4\zeta^2\omega_n{}^2 - 4\omega_n{}^2}}{2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

So, $$x(t) = a_1e^{(-\zeta\omega_n + \omega_n\sqrt{\zeta^2-1})t} + a_1e^{(-\zeta\omega_n - \omega_n\sqrt{\zeta^2-1})t} + \frac{\mu g}{\omega_n^2}$$

$$x(t) = e^{-\zeta\omega_n t}(a_1e^{-\zeta\omega_d t} + a_2e^{-\zeta\omega_d t}) + \frac{\mu g}{\omega_n{}^2}$$

$$x(t) = Xe^{-\zeta\omega_n t}\sin(\omega_d t + \theta) + \frac{\mu g}{\omega_n{}^2}$$

Initial conditions

$$x(0) = X\sin(\theta) + \frac{\mu g}{\omega_n{}^2} = x_o$$

$$\dot{x}(0) = X(-\zeta\omega_n)(\sin\theta) + X\omega_d\cos\theta = 0$$

$$-X\zeta\omega_n\sin\theta + X\omega_d\cos\theta = 0$$

$$\tan\theta = \frac{\omega_d}{\zeta\omega_n} \Rightarrow \theta = \tan^{-1}\left[\frac{\omega_d}{\zeta\omega_n}\right]$$

$$X = \frac{\left(x_o - \frac{\mu g}{\omega_n^2}\right)\sqrt{\omega_d^2 + (\zeta\omega_n)^2}}{\omega_d}$$

$$x(t) = \frac{\left(x(0) - \frac{\mu g}{\omega_n^2}\right)\sqrt{\omega_d^2 + (\zeta\omega_n)^2}}{\omega_d} e^{-\zeta\omega_n t} \sin\left(\omega_d t + \tan^{-1}\left[\frac{\omega_d}{\zeta\omega_n}\right]\right) + \frac{\mu g}{\omega_n^2} \quad (1)$$

This will occur until $\dot{x}(t) = 0$:

$$\dot{x}(t) = X(-\zeta\omega_n)e^{-\zeta\omega_n t}\sin(\omega_d t + \theta) + A_0 e^{-\zeta\omega_n t}\omega_d\cos(\omega_d t + \theta) = 0$$

$$-\zeta\omega_n \sin(\omega_d t + \theta) + \omega_d\cos(\omega_d t + \theta) = 0$$

$$\Rightarrow \tan(\omega_d t + \theta) = \frac{\omega_d}{\zeta\omega_n}$$

$$t = \frac{\pi}{\omega_d}$$

So Equation (1) is valid from $0 \le t \le \frac{\pi}{\omega_d}$

For motion to the right

Initial conditions (From Equation (1)):

$$x\left(\frac{\pi}{\omega_d}\right) = Xe^{-\zeta\omega_n\left(\frac{\pi}{\omega_d}\right)}\cos\theta + \frac{\mu g}{\omega_n^2} = \frac{\left(x(0) - \frac{\mu g}{\omega_n^2}\right)\zeta\omega_n}{\omega_d} e^{-\zeta\omega_n\left(\frac{\pi}{\omega_d}\right)} + \frac{\mu g}{\omega_n^2}$$

$$\dot{x}\left(\frac{\pi}{\omega_d}\right) = 0$$

$$x(t) = A_1 e - ^{-\zeta\omega_n t}\sin(\omega_d t + \theta_1) - \frac{\mu g}{\omega_n^2}$$

$$x(0) = A_1\sin\theta_1 - \frac{\mu g}{\omega_n^2} = \frac{\left(x(0) - \frac{\mu g}{\omega_n^2}\right)\zeta\omega}{\omega_d} e^{-\zeta\omega_n\left(\frac{\pi}{\omega_d}\right)} + \frac{\mu g}{\omega_n^2}$$

$$\dot{x}(0) = A_1(-\zeta\omega_n)\sin\theta_1 + X\omega_d\cos\theta_1 = 0$$

Solution: $x(t) = A_1 e^{-\zeta\omega_n t}\sin(\omega_d t + \theta_1) - \frac{\mu g}{\omega_n^2}$

$$A_1 = \frac{\sqrt{\omega_d^2 + (\zeta\omega_n)^2}}{\omega_d}\left[\frac{\left(x(0) - \frac{\mu g}{\omega_n^2}\right)\zeta\omega_n}{\omega_d} e^{-\zeta\omega_n\left(\frac{\pi}{\omega_d}\right)} + \frac{\mu g}{\omega_n^2}\right]$$

$$\theta = \tan^{-1}\left[\frac{\omega_d}{\zeta\omega_n}\right]$$

Forced Case:

$$m\ddot{x} - c\dot{x} + \mu mg\,\mathrm{sgn}(\dot{x}) + kx = F_o\cos(\omega t)$$

Approximate Steady-state Response:

$$x_{ss}(t) = X\sin(\omega t - \theta)$$

Energy Dissipated per Cycle:

$$\Delta E = \int F_d dx = \int_{2\pi}^{\frac{2\pi}{\omega}}\left[c\dot{x}\frac{dx}{dt} + \mu mg\,\mathrm{sgn}\,\dot{x}\frac{dx}{dt}\right]dt$$

$$= \int_{2\pi}^{\frac{2\pi}{\omega}}(c\dot{x}^2 dt) + \mu mg\int_{2\pi}^{\frac{2\pi}{\omega}}\mathrm{sgn}(\dot{x})\dot{x}dt$$

$$\Delta E = \pi c\omega X^2 + 4\mu mgX$$

This results in an equivalent viscously damped system:

$$\ddot{x} + 2(\zeta + \zeta_{eq})\omega_n\dot{x} + \omega_n{}^2 x = F_o\cos\omega t$$

where $\zeta_{eq} = \dfrac{2\mu g}{\pi\omega_n\omega X}$

The magnitude is:

$$X = \frac{\frac{F_0}{k}}{\sqrt{(1-r)^2 + (2(\zeta + \zeta_{eq})r)^2}}$$

Solving for X:

$$X = \frac{\left(\frac{8\mu gcr^2}{\pi k\omega}\right) + \sqrt{\left(\frac{8\mu gcr^2}{\pi k\omega}\right) - 2\left[(1-r^2)^2 + \frac{c^2r^2}{km}\right]\left[\left(\frac{4\mu gr}{\pi\omega_n\omega}\right)^2 - \left(\frac{F_0}{k}\right)^2\right]}}{4\left[(1-r^2)^2 + \frac{c^2r^2}{km}\right]}$$

The phase is:

$$\theta = \tan^{-1}\left[\frac{2(\zeta + \zeta_{eq})r}{1-r^2}\right] = \tan^{-1}\left[\frac{2\zeta r + \frac{4\mu gr}{\pi\omega_n\omega X}}{1-r^2}\right]$$

2.70 A system of unknown damping mechanism is driven harmonically at 10 Hz with an adjustable magnitude. The magnitude is changed, and the energy lost per cycle and amplitudes are measured for five different magnitudes. The measured quantities are:

$\Delta E(J)$	0.25	0.45	0.8	1.16	3.0
$X\ (M)$	0.01	0.02	0.04	0.08	0.15

Is the damping viscous or Coulomb?

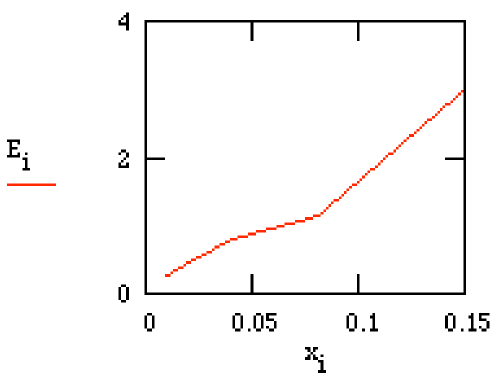
Solution:

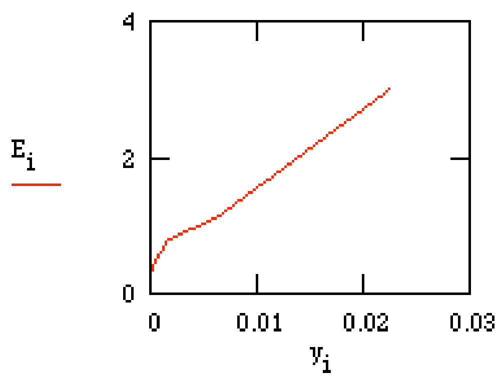
For viscous damping, $\Delta E = \pi c \omega X^2$

For Coulomb damping, $\Delta E = 4\mu m g X$

$$x := \begin{bmatrix} 0.01 \\ 0.02 \\ 0.04 \\ 0.08 \\ 0.15 \end{bmatrix} \qquad i := 0, 1 .. 4 \qquad y_i := (x_i)^2$$

$$E := \begin{bmatrix} 0.25 \\ 0.45 \\ 0.8 \\ 1.16 \\ 3 \end{bmatrix}$$





For the data given, a plot of ΔE vs X^2 yields a curve, while ΔE vs X yields a straight line. Therefore, the damping is likely Coulomb in nature

2.71 Calculate the equivalent loss factor for a system with Coulomb damping.

Solution:

Loss Factor: $\eta = \dfrac{\Delta E}{2\pi U_{\max}}$

For Coulomb damping: $\Delta E = 4\mu mgX$

$$U_{\max} = \frac{1}{2}kX^2$$

$$\eta = \frac{4\mu mgX}{2\pi\left(\frac{1}{2}kX^2\right)} = \frac{4\mu mg}{\pi kX}$$

Substituting for X (from Equation 2.109):

$$\eta = \frac{4\mu mg}{\pi F_o}\frac{\left|1-r^2\right|}{\sqrt{1-\left(\frac{4\mu mg}{\pi F_o}\right)^2}}$$

2.72 A spring-mass system (m = 10 kg, $k = 4\times10^3$ N/m) vibrates horizontally on a surface with coefficient of friction μ = 0.15. When excited harmonically at 5 Hz, the steady-state displacement of the mass is 5 cm. Calculate the amplitude of the harmonic force applied.

Solution: Given: m = 10 kg, $k = 4\times10^3$ N/m, μ = 0.15, X = 5 cm = 0.05 m,

$$\omega = 5(2\pi) = 10\pi \text{ rad/s}, \ \omega_n = \sqrt{\frac{k}{m}} = 20 \text{ rad/s}$$

Equation (2.109)

$$X = \frac{\frac{F_0}{k}}{\sqrt{(1-r^2)^2+\left(\frac{4\mu mg}{\pi kX}\right)^2}} \Rightarrow$$

$$F_o = kX\sqrt{(1-r^2)^2+\left(\frac{4\mu mg}{\pi kX}\right)^2} = (0.05)(4\times10^3)\sqrt{\left(1-\left(\frac{10\pi}{20}\right)^2\right)^2+\left(\frac{4(0.15)(10)(9.81)}{\pi(4x10^3)(0.05)}\right)^2}$$

$$F_o = 294 \text{ N}$$

2.73 Calculate the displacement for a system with air damping using the equivalent viscous damping method.

Solution:

The equivalent viscous damping for air is given by Equation (2.131):

$$c_{eq} = \frac{8}{3\pi}\alpha\omega X$$

From Equation 2.31:

$$X = \frac{F_o}{\sqrt{\left(\omega_n^2 - \omega^2\right)^2 + \left(2\zeta\omega_n\omega\ \right)^2}} = \frac{F_o}{\sqrt{\left(\omega_n^2 - \omega^2\right)^2 + \left(\frac{c_{eq}}{m}\omega_n\right)^2}}$$

$$X = \frac{F_o}{\sqrt{\left(\omega_n^2 - \omega^2\right)^2 + \left(\frac{8}{3\pi m}\alpha\omega X\right)^2}} = \frac{F_o m}{k\sqrt{(1-r^2)^2 + \left(\frac{8}{3\pi m}\alpha r^2 X\right)^2}}$$

Solving for X and taking the real solution:

$$X = \frac{\sqrt{-\frac{1}{2}(1-r^2)^2 + \frac{1}{2}\sqrt{(1-r^2)^2 + \left(\frac{16F_o\alpha r^2}{3\pi km}\right)^2}}}{\left(\frac{8\alpha r^2}{3\pi m}\right)}$$

2.74 Calculate the semimajor and semiminor axis of the ellipse of equation (2.119). Then calculate the area of the ellipse. Use c = 10 kg/s, ω = 2 rad/s and X = 0.01 m.

Solution: The equation of an ellipse usually appears when the plot of the ellipse is oriented along with the x axis along the principle axis of the ellipse. Equation (2.1109) is the equation of an ellipse rotated about the origin. If k is known, the angle of rotation can be computed from formulas given in analytical geometry. However, we know from the energy calculation that the stiffness does not effect the amount of energy dissipated. Thus only the orientation of the ellipse is effected by the stiffness, not its area or axis. Thus we can use this fact to answer the question. First re-write equation (2.119) with k = 0 to get:

$$F^2 + c^2\omega^2 x^2 = c^2\omega^2 X^2$$

$$\Rightarrow \left(\frac{F}{c\omega X}\right)^2 + \left(\frac{x}{X}\right)^2 = 1$$

This is the equation of an ellipse with major axis a and minor axis b given by

$$a = X = 0.01 \text{ m}, \quad \text{and} \quad b = c\omega X = 0.2 \text{ kg m/s}^2$$

The area, and hence energy lost per cycle through the damper then becomes

$$\pi c\omega_n X^2 = (3.14159)(10)(2)(.0001) = 0.006283 \text{ Joules.}$$

Alternately, realized that Equation 2.119 is that of ellipse rotated by an angle θ defined by tan2θ = -2k/($c^2\omega_n^2 + k^2 - 1$). Then match the ellipse to standard form, read off the major and minor axis (say a and b) and calculate the area from πab. See the following web site for an elipse http://mathworld.wolfram.com/Ellipse.html

2.75 The area of a force deflection curve of Figure P2.28 is measured to be 2.5 N- m, and the maximum deflection is measured to be 8 mm. From the "slope" of the ellipse the stiffness is estimated to be 5×10^4 N/m. Calculate the hysteretic damping coefficient. What is the equivalent viscous damping if the system is driven at 10 Hz?

Solution:

Given: Area = 2.5 N • m, k = 5x10^4 N/m, X = 8 mm, $\omega = 10(2\pi) = 20\pi$ rad/s

Hysteric Damping Coefficient:

$$\Delta E = \text{Area} = \pi k\beta X^2$$

$$2.5 = \pi(5\times 10^4)\beta(0.008)^2$$

$$\beta = 0.249$$

Equivalent Viscous Damping:

$$c_{eq} = \frac{k\beta}{\omega} = \frac{(5\times 10^4)(0.249)}{20\pi}$$

$$c_{eq} = 198 \text{ kg/s}$$

2.76 The area of the hysteresis loop of a hysterically damped system is measured to be 5 $\text{N}\bullet\text{m}$ and the maximum deflection is measured to be 1 cm. Calculate the equivalent viscous damping coefficient for a 20-Hz driving force. Plot c_{eq} versus ω for $2\pi \leq \omega \leq 100\pi$ rad/s.

Solution:

Given: Area = $5\,\text{N}\bullet\text{m}$, X = 1 cm, $\omega = 20(2\pi) = 40\pi$ rad/s

Hysteric Damping Coefficient:

$$\Delta E = \text{Area} = \pi k \beta X^2$$

$$5 = \pi k\beta (0.01)^2$$

$$k\beta = 15{,}915 \text{ N/m}$$

Equivalent Viscous Damping:

$$c_{eq} = \frac{k\beta}{\omega} = \frac{15915}{40\pi}$$

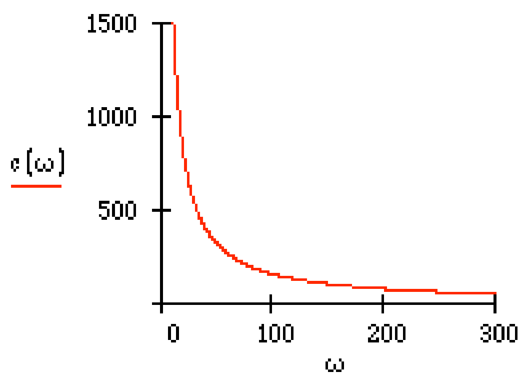
$$c_{eq} = 126.65 \text{ kg/s}$$

To plot, rearrange so that

$$\pi c_{eq} \omega X^2 = \Delta E$$

$$c_{eq} = \frac{\Delta E}{\pi \omega X^2} = \frac{5}{\pi\omega(.01)^2} = \frac{50{,}000}{\pi\omega}$$

$$c(\omega) := \frac{50000}{\pi\cdot\omega}$$



2.77 Calculate the nonconservative energy of a system subject to both viscous and hysteretic damping.

Solution:

$$\Delta E = \Delta E_{hys} + \Delta E_{visc}$$
$$\Delta E = \pi c\omega X^2 + k\pi\beta X^2$$
$$\Delta E = (c\omega + k\beta)\pi X^2$$

2.78 Derive a formula for equivalent viscous damping for the damping force of the form, F_d = $c(\dot{x})^n$ where n is an integer.

Solution:

Given: $F_d = c(\dot{x})^n$

Assume the steady-state response $x = X\sin\omega t$.

The energy lost per cycle is given by Equation (2.99) as:

$$\Delta E = \oint F_d dx = \int_0^{\frac{2\pi}{\omega}} c(\dot{x})^n \dot{x}dt = c\int_0^{\frac{2\pi}{\omega}} (\dot{x})^{n+1} dt$$

Substituting for $\dot{x}$:

$$\Delta E = \int_0^{\frac{2\pi}{\omega}} \left[\omega^{n+1} X^{n+1} \cos^{n+1}(\omega t)\right] dt$$

Let $u = \omega t$:

$$\Delta E = cX^{n+1}\omega^n \int_0^{2\pi} \left(\cos^{n+1} u\right) du$$

Equating this to Equation 2.91 yields:

$$\pi c_{eq}\omega X^2 = cX^{n+1}\omega^n \int_0^{2\pi} (\cos^{n+1} u) du$$

$$c_{eq} = \frac{cX^{n-1}\omega^{n-1}}{\pi} \int_0^{2\pi} (\cos^{n+1} u) du$$

2.79 Using the equivalent viscous damping formulation, determine an expression for the steady-state amplitude under harmonic excitation for a system with both Coulomb and viscous damping present.

Solution:

$$\Delta E = \Delta E_{visc} + \Delta E_{coul}$$
$$\Delta E = \pi c \omega X^2 + 4umgX$$

Equate to Equivalent Viscously Damped System

$$\pi c_{eq} \omega X^2 = \pi c \omega X^2 + 4\mu mg$$
$$c_{eq} = \frac{\pi c \omega X + 4\mu mg}{\pi \omega X} = c + \frac{4\mu mg}{\pi \omega X} = 2\zeta_{eq}\omega_n m$$
$$\zeta_{eq} = \zeta + \frac{2\mu g}{\pi \omega \omega_n X}$$

Amplitude:

$$X = \frac{\frac{F_o}{k}}{\sqrt{(1-r^2)^2 + (2\zeta_{eq} r)^2}} = \frac{\frac{F_o}{k}}{\sqrt{(1-r^2)^2 + \left(2\zeta r + \frac{4\mu mg}{\pi k X}\right)^2}}$$

Solving for *X*:

$$X = \frac{-\left(\frac{8\mu gcr^2}{\pi k \omega}\right) + \sqrt{\left(\frac{8\mu gcr^2}{\pi k \alpha \omega}\right)^2 - 4\left[(1-r^2)^2 + \frac{c^2\ r^2}{km}\right]\left[\left(\frac{4\mu gr}{\pi \omega_n \omega}\right)^2 - \left(\frac{F_o}{k}\right)^2\right]}}{2\left[(1-r^2)^2 + \frac{c^2 r^2}{km}\right]}$$

Problems and Solutions Section 2.8 (2.80 through 2.86)

2.80*. Numerically integrate and plot the response of an underdamped system determined by *m* = 100 kg, *k* = 20,000 N/m, and *c* = 200 kg/s, subject to the initial conditions of x_0 = 0.01 m and v_0 = 0.1 m/s, and the applied force *F*(*t*) = 150cos5*t*. Then plot the exact response as computed by equation (2.33). Compare the plot of the exact solution to the numerical simulation.

Solution: The solution is presented in Matlab:

First the m file containing the state equation to integrate is set up and saved as ftp2_72.m

```
function xdot=f(t, x)
xdot=[-(200/100)*x(1)-(20000/100)*x(2)+(150/100)*cos(5*t); x(1)];
% xdot=[x(1)'; x(2)']=[-2*zeta*wn*x(1)-wn^2*x(2)+fo*cos(w*t) ; x(1)]
% which is a state space form of
% x" + 2*zeta*wn*x' + (wn^2)*x = fo*cos(w*t)    (fo=Fo/m)

clear all;
```

Then the following m file is created and run:

```
%---- numerical simulation ---
x0=[0.1; 0.01];          %[xdot(0); x(0)]
tspan=[0 10];
[t,x]=ode45('fp2_72',tspan,x0);
plot(t, x(:,2), '.');
hold on;

%--- exact solution ----
t=0: .002: 10;
m=100; k=20000; c=200; Fo=150 ; w=5
wn=sqrt(k/m); zeta=c/(2*wn*m); fo=Fo/m; wd=wn*sqrt(1-zeta^2)
x0=0.01; v0= 0.1;
xe= exp(-zeta*wn*t) .* ( (x0-fo*(wn^2-w^2)/((wn^2-w^2)^2 ...
 +(2*zeta*wn*w)^2))*cos(wd*t) ...
 + (zeta*wn/wd*( x0-fo*(wn^2-w^2)/((wn^2-w^2)^2+(2*zeta*wn*w)^2)) ...
 - 2*zeta*wn*w^2*fo/(wd*((wd^2-w^2)^2  ...
 + (2*zeta*wn*w)^2))+v0/wd)*sin(wd*t) ) ...
 + fo/((wn^2-w^2)^2+(2*zeta*wn*w)^2)*((wn^2-w^2)*cos(w*t) ...
 + 2*zeta*wn*w*sin(w*t))

plot(t, xe, 'w');
hold off;
```

This produces the following plot:

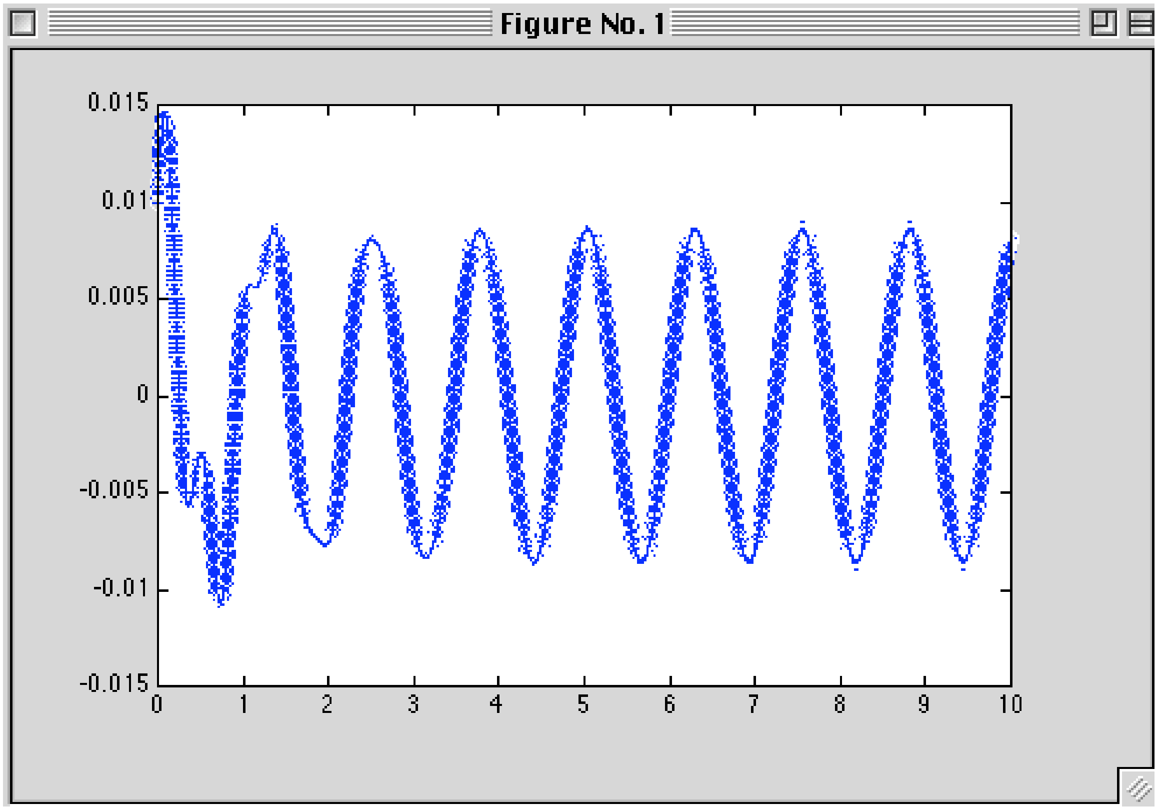


Figure No. 1
0.015
0.01
0.005
0
-0.005
-0.01
-0.015
0
1
2
3
4
5
6
7
8
9
10

2.81*. Numerically integrate and plot the response of an underdamped system determined by m = 150 kg, and k = 4000 N/m subject to the initial conditions of x_0 = 0.01 m and v_0 = 0.1 m/s, and the applied force $F(t)$ = 15cos10t , for various values of the damping coefficient. Use this "program" to determine a value of damping that causes the transient term to die out with in 3 seconds. Try to find the smallest such value of damping remembering that added damping is usually expensive.

Solution: The solution is given by the following Mathcad session. A value of c = 350 kg/s corresponding to ζ = 0.226 gives the desired result.

2.73

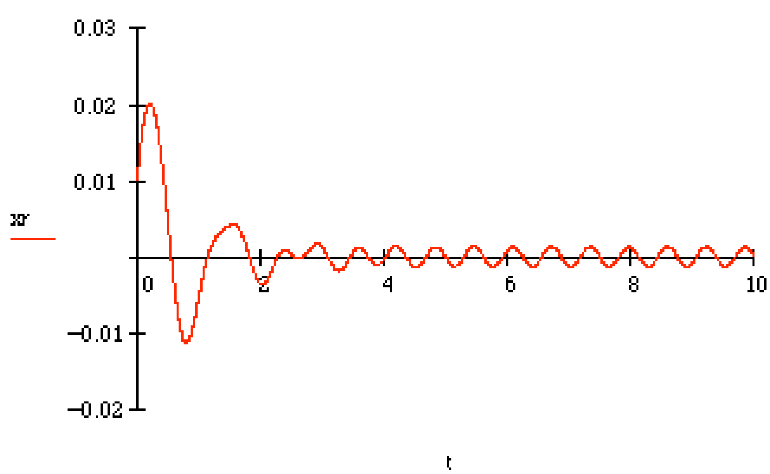
$m := 150 \qquad k := 4000 \qquad c := 350$

$\omega n := \sqrt{\frac{k}{m}} \qquad \zeta := \frac{c}{2\cdot\sqrt{k\cdot m}} \qquad F0 := 15 \qquad \omega := 10$

$$x := \begin{bmatrix} 0.01 \\ 0.1 \end{bmatrix} \qquad D(t,x) := \begin{bmatrix} x_1 \\ \left(-2\cdot\zeta\cdot\omega n\cdot x_1 - \omega n^2\cdot x_0\right) + \frac{F0}{m}\cdot\sin(\omega\cdot t) \end{bmatrix}$$

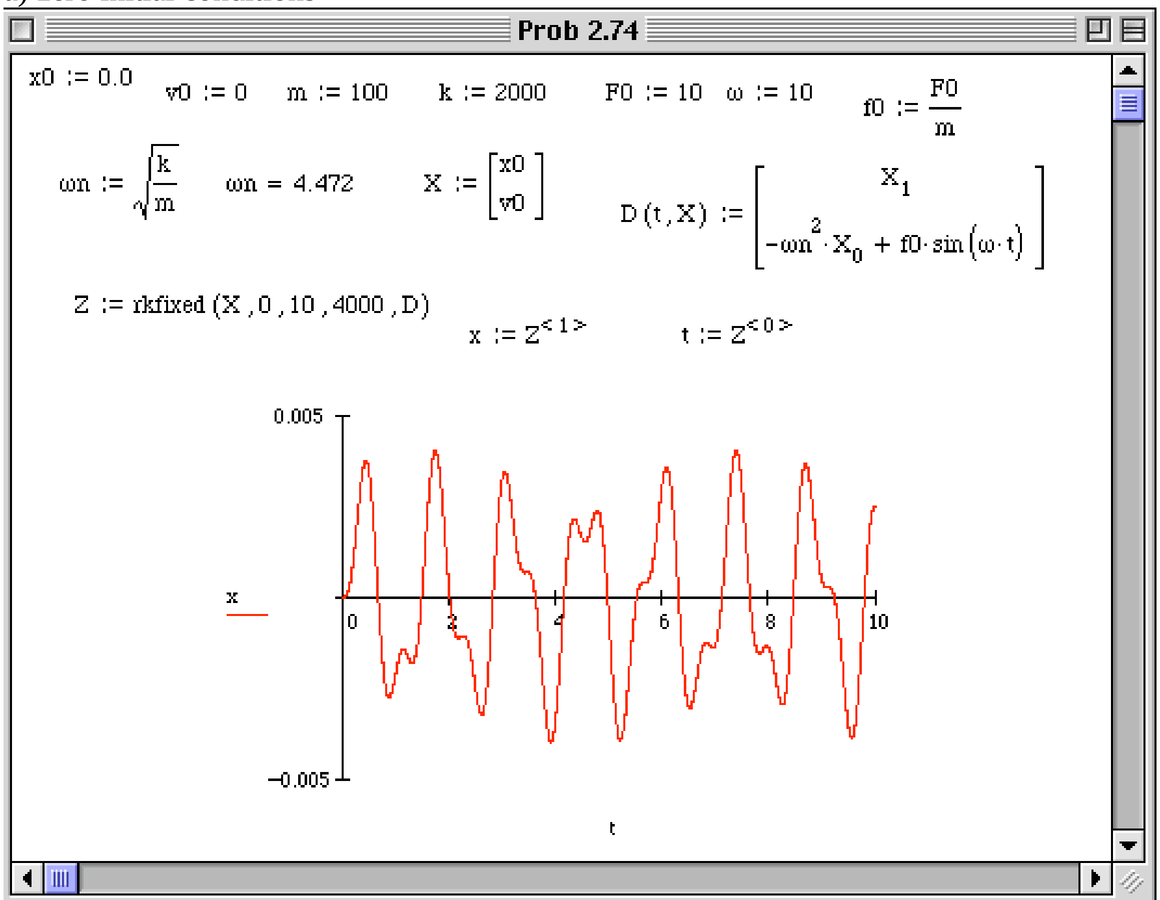
$Z := \text{rkfixed}(x, 0, 10, 4000, D)$

$t := Z^{<0>} \qquad xr := Z^{<1>} \qquad \zeta = 0.226 \qquad \omega n = 5.164$



2.82*. Solve Problem 2.7 by numerically integrating rather than using analytical expressions.

Solution: The following session in Mathcad illustrates the solution:
a) zero initial conditions



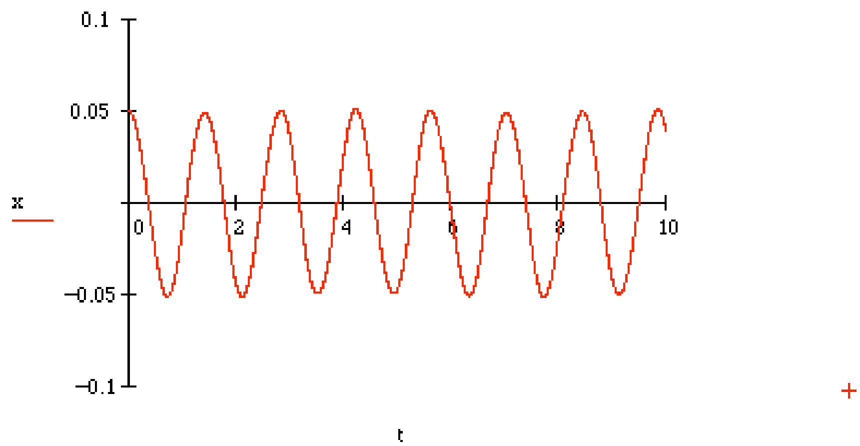
b) Using and initial condition of $x(0) = 0.05$ m. Note the difference in the response.

Prob 2.74

$x0 := 0.05 \quad v0 := 0 \quad m := 100 \quad k := 2000 \quad F0 := 10 \quad \omega := 10 \quad f0 := \frac{F0}{m}$

$\omega n := \sqrt{\frac{k}{m}} \quad \omega n = 4.472 \quad X := \begin{bmatrix} x0 \\ v0 \end{bmatrix} \quad D(t, X) := \begin{bmatrix} X_1 \\ -\omega n^2 \cdot X_0 + f0 \cdot \sin(\omega \cdot t) \end{bmatrix}$

$Z := \mathrm{rkfixed}(X, 0, 10, 4000, D) \quad x := Z^{\langle 1 \rangle} \quad t := Z^{\langle 0 \rangle}$



2.83*. Numerically simulate the response of the system of Problem 2.30.

Solution: From problem 2.30, the equation of motion is

$$9a^2 m\ddot{\theta} + 4a^2 c\cos\theta\dot{\theta} + a^2 k\sin\theta = -3aF(t)$$

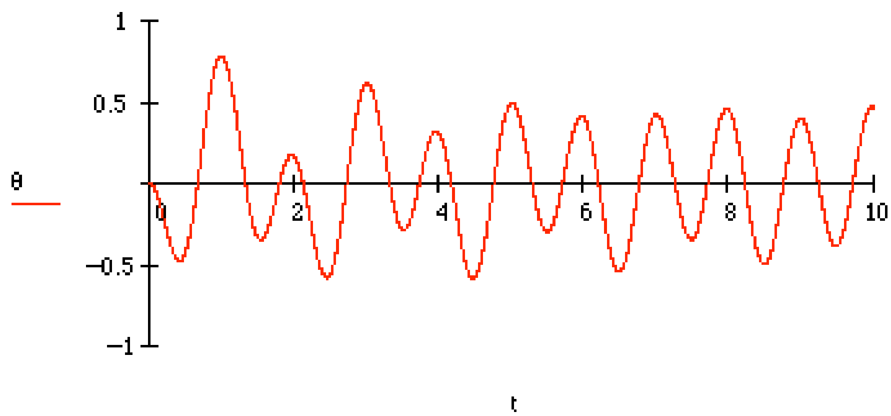
where k = 2000 kg, c = 25 kg/s , m = 25 kg , $F(t) = 50\cos 2\pi t$, a = 0.05 m

Placing the equation of motion in first order form and numerically integrating using Mathcad yields

$k := 2000$ $\quad c := 25$ $\quad m := 25$ $\quad a := 0.05$ $\quad f0 := \frac{-50}{3\cdot a\cdot m}$

$$X := \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad D(t,X) := \begin{bmatrix} X_1 \\ \frac{-4\cdot c}{9\cdot m}\cdot X_1\cdot\cos\left(X_0\right) - \frac{k}{9\cdot m}\cdot\sin\left(X_0\right) + f0\cdot\cos(2\cdot\pi\cdot t) \end{bmatrix}$$

$Z := \text{rkfixed}(X, 0, 10, 4000, D)$ $\quad t := Z^{\langle 0 \rangle}$ $\quad \theta := Z^{\langle 1 \rangle}$



2.84*. Numerically integrate the system of Example 2.8.1 for the following sets of initial conditions:

a) $x_0 = 0.0$ m and $v_0 = 0.1$ m/s

b) $x_0 = 0.01$ m and $v_0 = 0.0$ m/s

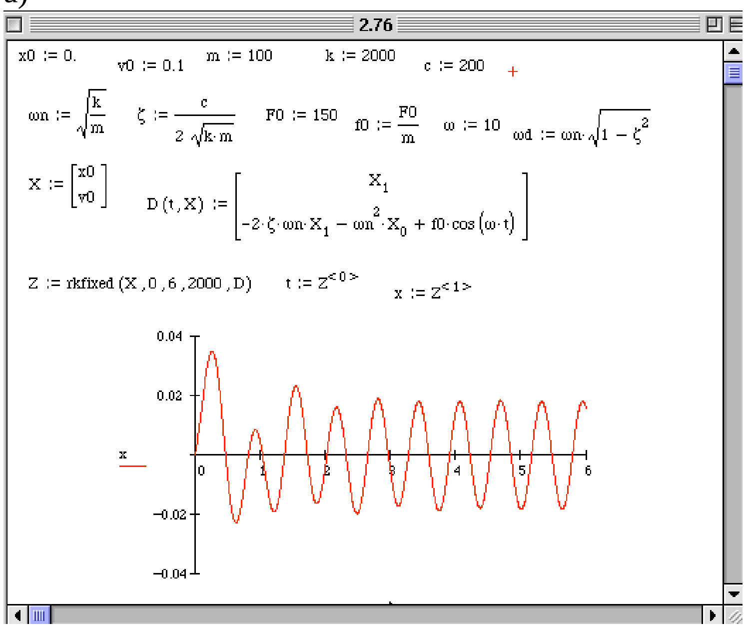
c) $x_0 = 0.05$ m and $v_0 = 0.0$ m/s

d) $x_0 = 0.0$ m and $v_0 = 0.5$ m/s

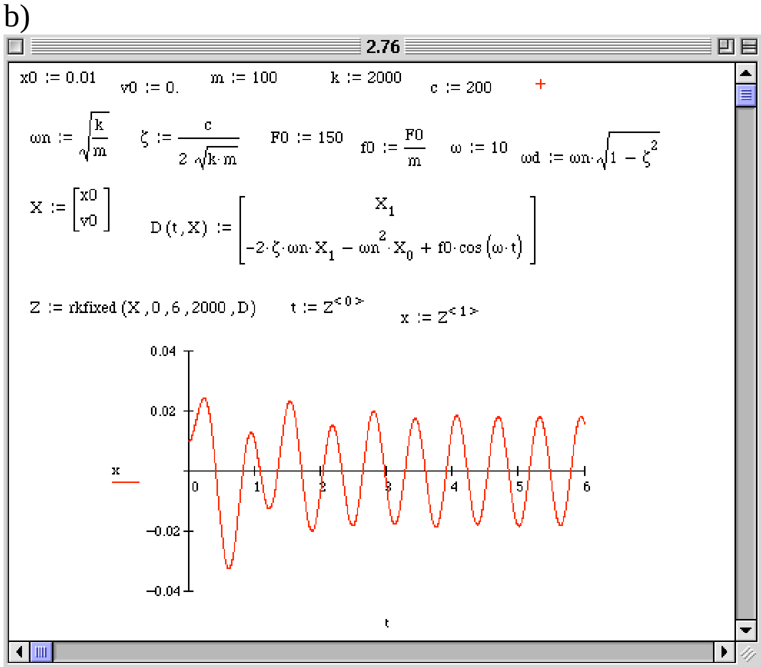
Plot these responses on the same graph and note the effects of the initial conditions on the transient part of the response.

Solution: The following are the solutions in Mathcad. Of course the other codes and Toolbox will yield the same results.

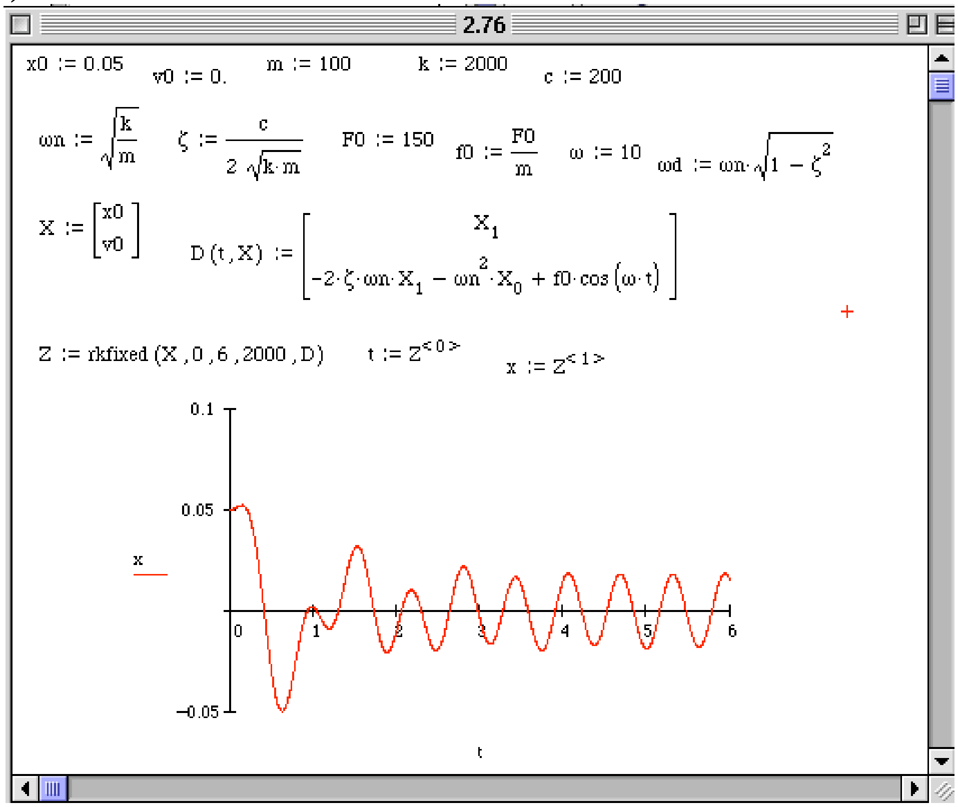
a)



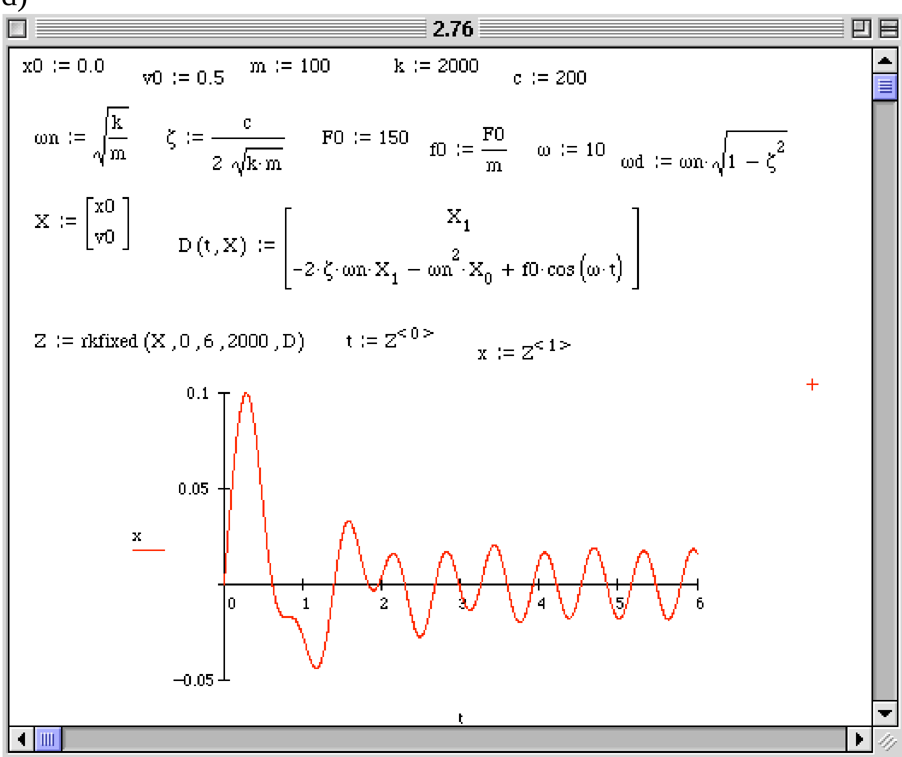
b)



c)



d)



Note the profound effect on the transient, but of course no effect on the steady state.

2.85*. A DVD drive is mounted on a chassis and is modeled as a single degree-degree-of-freedom spring, mass and damper. During normal operation, the drive (having a mass of 0.4 kg) is subject to a harmonic force of 1 N at 10 rad/s. Because of material considerations and static deflection, the stiffness is fixed at 500 N/m and the natural damping in the system is 10 kg/s. The DVD player starts and stops during its normal operation providing initial conditions to the module of $x_0 = 0.001$ m and $v_0 = 0.5$ m/s. The DVD drive must not have an amplitude of vibration larger then 0.008 m even during the transient stage. First compute the response by numerical simulation to see if the constraint is satisfied. If the constraint is not satisfied, find the smallest value of damping that will keep the deflection less then 0.008 m.

Solution: The solution is given by the following Mathcad session:

2.77

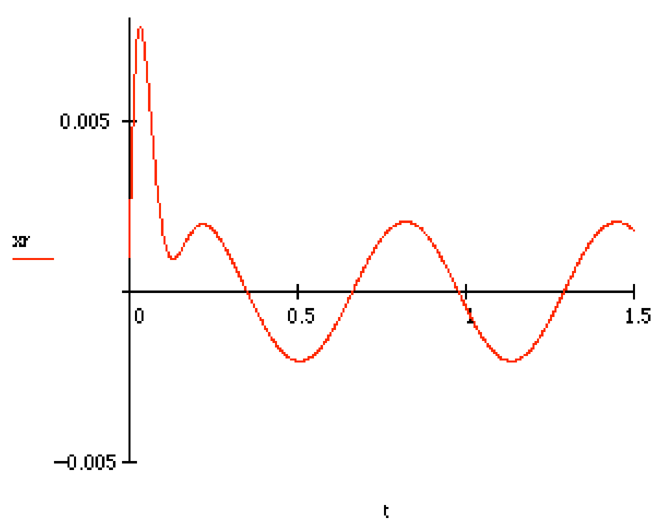
$m := 0.4 \qquad k := 500 \qquad c := 17$

$\omega n := \sqrt{\frac{k}{m}} \qquad \zeta := \frac{c}{2 \cdot \sqrt{k \cdot m}} \qquad F0 := 1 \qquad \omega := 10$

$$x := \begin{bmatrix} 0.001 \\ 0.5 \end{bmatrix} \qquad D(t, x) := \begin{bmatrix} x_1 \\ \left(-2 \cdot \zeta \cdot \omega n \cdot x_1 - \omega n^2 \cdot x_0\right) + \frac{F0}{m} \cdot \sin(\omega \cdot t) \end{bmatrix}$$

$Z := \mathrm{rkfixed}(x, 0, 1.5, 1000, D)$

$t := Z^{\langle 0 \rangle} \qquad xr := Z^{\langle 1 \rangle}$



This yields c =17 kg/s as a solution.

2.86 Use a plotting routine to examine the base motion problem of Figure 2.12 by plotting the particular solution (for an undamped system) for the three cases k = 1500 N/m, and k = 700 N/m. Also note the values of the three frequency ratios and the corresponding amplitude of vibration of each case compared to the input. Use the following values: ω_b = 4.4 rad/s, m = 100 kg, and Y = 0.05 m.

Solution; The following Mathcad worksheet shows the plotting:

$m := 100 \quad k1 := 1500 \quad \omega 1 := \sqrt{\frac{k1}{m}}$

$k2 := 700$

$k3 := 2500 \quad \omega 2 := \sqrt{\frac{k2}{m}} \quad \omega 3 := \sqrt{\frac{k3}{m}} \quad \omega 1 = 3.873 \quad \omega b := 4.4$

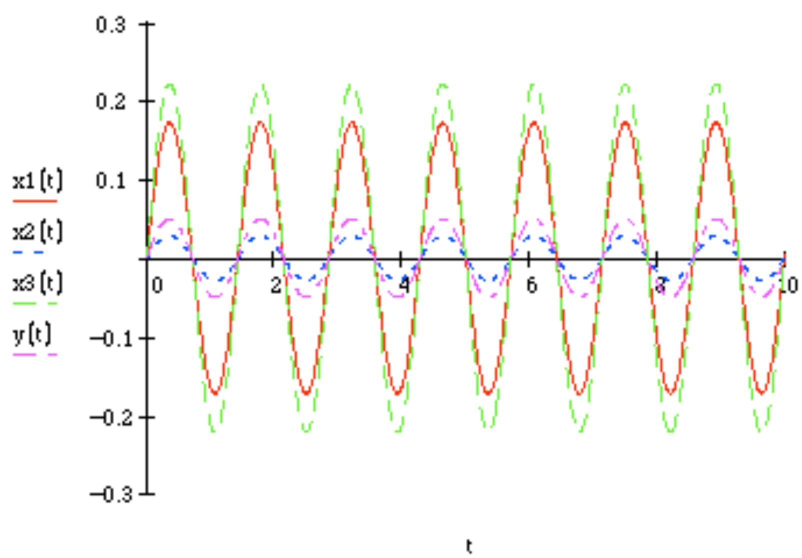
$r1 := \frac{\omega b}{\omega 1} \quad r2 := \frac{\omega b}{\omega 2} \quad r3 := \frac{\omega b}{\omega 3} \quad \omega 2 = 2.646 \quad \omega 3 = 5$

$r1 = 1.136 \quad r2 = 1.663 \quad r3 = 0.88 \quad Y := 0.05$

$y(t) := Y \cdot \sin(\omega b \cdot t)$ then from equation 2.68 and 2.70 with zero damping:

$$x1(t) := Y \cdot \frac{1}{\left|1 - r1^2\right|} \cdot \cos\left(\omega b \cdot t - \frac{\pi}{2}\right) \quad x2(t) := Y \cdot \frac{1}{\left|1 - r2^2\right|} \cdot \cos\left(\omega b \cdot t - \frac{\pi}{2}\right)$$

$$x3(t) := Y \cdot \frac{1}{\left|1 - r3^2\right|} \cdot \cos\left(\omega b \cdot t - \frac{\pi}{2}\right)$$



Note that $k2$, the softest system (smallest k) has the smallest amplitude, smaller than the amplitude of the input as predicted by the magnitude plots in section 2.3. Thus when $r > \sqrt{2}$, the amplitude is the smallest.

Problems and Solutions Section 2.9 (2.87 through 2.93)

2.87*. Compute the response of the system in Figure 2.34 for the case that the damping is linear viscous and the spring is a nonlinear soft spring of the form

$$k(x) = kx - k_1 x^3$$

and the system is subject to a harmonic excitation of 300 N at a frequency of approximately one third the natural frequency ($\omega = \omega_n/3$) and initial conditions of x_0 = 0.01 m and v_0 = 0.1 m/s. The system has a mass of 100 kg, a damping coefficient of 170 kg/s and a linear stiffness coefficient of 2000 N/m. The value of k_1 is taken to be 10000 N/m^3. Compute the solution and compare it to the linear solution (k_1 = 0). Which system has the largest magnitude?

Solution: The following is a Mathcad simulation. The green is the steady state magnitude of the linear system, which bounds the linear solution, but is exceeded by the nonlinear solution. The nonlinear solution has the largest response.

Prop 2.78

$x0 := 0.01 \quad v0 := 0.1 \quad m := 100 \quad k := 2000 \quad c := 170$

2.964
1.09

$k1 := 10000 \quad F0 := 300$

$$\omega n := \sqrt{\frac{k}{m}} \qquad \zeta := \frac{c}{2\sqrt{k \cdot m}} \qquad f0 := \frac{F0}{m} \qquad \omega := \frac{\omega n}{3}$$

$$X := \begin{bmatrix} x0 \\ v0 \end{bmatrix} \qquad Y := X$$

$$D(t, X) := \begin{bmatrix} X_1 \\ -2 \cdot \zeta \cdot \omega n \cdot X_1 - \omega n^2 \cdot X_0 + \frac{k1}{m} \cdot (X_0)^3 + f0 \cdot \cos(\omega \cdot t) \end{bmatrix}$$

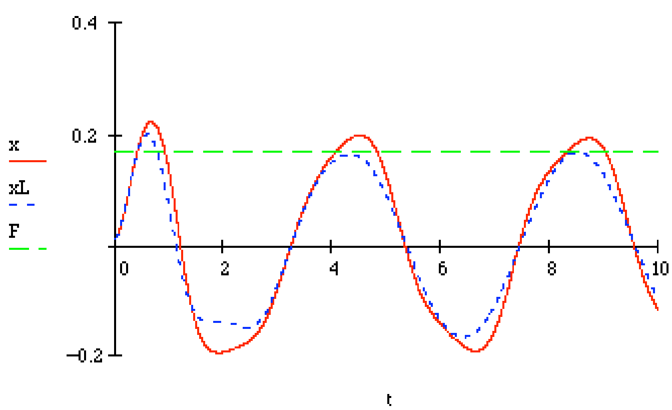
$$L(t, Y) := \begin{bmatrix} Y_1 \\ \left(-2 \cdot \zeta \cdot \omega n \cdot Y_1 - \omega n^2 \cdot Y_0\right) + f0 \cdot \cos(\omega \cdot t) \end{bmatrix}$$

$Z := \text{rkfixed}(X, 0, 10, 4000, D)$

$t := Z^{\langle 0 \rangle} \qquad x := Z^{\langle 1 \rangle} \qquad W := \text{rkfixed}(Y, 0, 10, 4000, L)$

$xL := W^{\langle 1 \rangle}$

$$d(t) := \frac{f0}{\sqrt{\left(\omega n^2 - \omega^2\right)^2 + (2 \cdot \zeta \cdot \omega n \cdot \omega)^2}} \qquad F := \overrightarrow{d(t)}$$



2.88*. Compute the response of the system in Figure 2.34 for the case that the damping is linear viscous and the spring is a nonlinear hard spring of the form

$$k(x) = kx + k_1 x^3$$

and the system is subject to a harmonic excitation of 300 N at a frequency equal to the natural frequency ($\omega = \omega_n$) and initial conditions of $x_0 = 0.01$ m and $v_0 = 0.1$ m/s. The system has a mass of 100 kg, a damping coefficient of 170 kg/s and a linear stiffness coefficient of 2000 N/m. The value of k_1 is taken to be 10000 N/m^3. Compute the solution and compare it to the linear solution ($k_1 = 0$). Which system has the largest magnitude?

Solution: The Mathcad solution appears below. Note that in this case the linear amplitude is the largest!

Prop 2.79

$x0 := 0.01 \quad v0 := 0.1 \quad m := 100 \quad k := 2000 \quad c := 170$

$\alpha := 100 \quad F0 := 300$

$\omega n := \sqrt{\frac{k}{m}} \quad \zeta := \frac{c}{2\sqrt{k \cdot m}} \quad f0 := \frac{F0}{m} \quad \omega := \frac{\omega n}{1}$

$X := \begin{bmatrix} x0 \\ v0 \end{bmatrix} \quad Y := X$

$$D(t, X) := \begin{bmatrix} X_1 \\ \left(-2 \cdot \zeta \cdot \omega n \cdot X_1 - \omega n^2 \cdot X_0\right) - \alpha \cdot (X_0)^3 + f0 \cdot \cos(\omega \cdot t) \end{bmatrix}$$

$$L(t, Y) := \begin{bmatrix} Y_1 \\ -2 \cdot \zeta \cdot \omega n \cdot Y_1 - \omega n^2 \cdot Y_0 + f0 \cdot \cos(\omega \cdot t) \end{bmatrix}$$

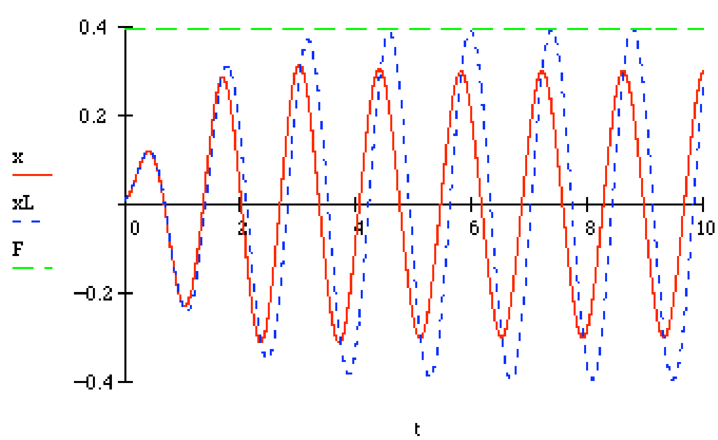
$Z := \text{rkfixed}(X, 0, 10, 4000, D)$

$t := Z^{\langle 0 \rangle} \quad x := Z^{\langle 1 \rangle} \quad W := \text{rkfixed}(Y, 0, 10, 4000, L)$

$xL := W^{\langle 1 \rangle}$

$\alpha \cdot m = 1 \cdot 10^4$

$$d(t) := \frac{f0}{\sqrt{\left(\omega n^2 - \omega^2\right)^2 + (2 \cdot \zeta \cdot \omega n \cdot \omega)^2}} \quad F := \overrightarrow{d(t)}$$



2.89*. Compute the response of the system in Figure 2.34 for the case that the damping is linear viscous and the spring is a nonlinear soft spring of the form

$$k(x) = kx - k_1x^3$$

and the system is subject to a harmonic excitation of 300 N at a frequency equal to the natural frequency ($\omega = \omega_n$) and initial conditions of $x_0 = 0.01$ m and $v_0 = 0.1$ m/s. The system has a mass of 100 kg, a damping coefficient of 15 kg/s and a linear stiffness coefficient of 2000 N/m. The value of k_1 is taken to be 100 N/m^3. Compute the solution and compare it to the hard spring solution ($k(x) = kx + k_1x^3$).

Solution: The Mathcad solution is presented, first for a hard spring, then for a soft spring

Prop 2.80

$x0 := 0.01 \quad v0 := 0.1 \quad m := 100 \quad k := 2000 \quad \alpha := 100 \quad c := 170 \quad F0 := 300$

$$\omega n := \sqrt{\frac{k}{m}} \quad \zeta := \frac{c}{2\sqrt{k \cdot m}} \quad f0 := \frac{F0}{m} \quad \omega := \frac{\omega n}{1} \quad X := \begin{bmatrix} x0 \\ v0 \end{bmatrix} \quad Y := X$$

$$D(t, X) := \begin{bmatrix} X_1 \\ \left(-2\cdot\zeta\cdot\omega n\cdot X_1 - \omega n^2\cdot X_0\right) - \alpha\cdot\left(X_0\right)^3 + f0\cdot\cos(\omega\cdot t) \end{bmatrix}$$

$$L(t, Y) := \begin{bmatrix} Y_1 \\ -2\cdot\zeta\cdot\omega n\cdot Y_1 - \omega n^2\cdot Y_0 + f0\cdot\cos(\omega\cdot t) \end{bmatrix}$$

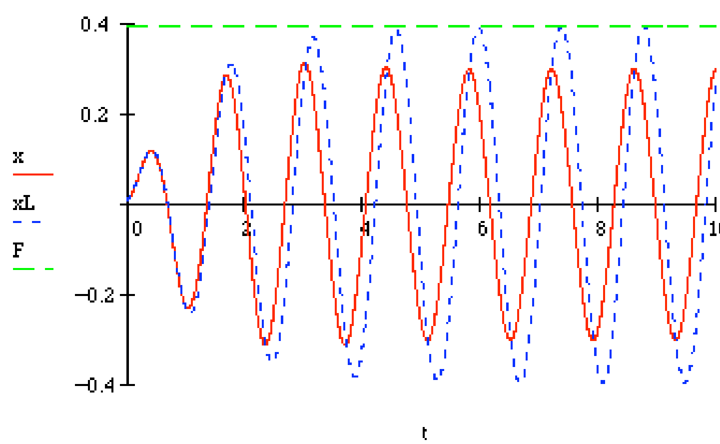
$Z := \text{rkfixed}(X, 0, 10, 4000, D)$

$t := Z^{\langle 0 \rangle} \quad x := Z^{\langle 1 \rangle} \quad W := \text{rkfixed}(Y, 0, 10, 4000, L)$

$xL := W^{\langle 1 \rangle}$

$\alpha \cdot m = 1 \cdot 10^4$

$$d(t) := \frac{f0}{\sqrt{\left(\omega n^2 - \omega^2\right)^2 + \left(2\cdot\zeta\cdot\omega n\cdot\omega\right)^2}} \quad F := \overrightarrow{d(t)}$$



Next consider the result for the soft spring and note that the nonlinear response is higher in the transient then the linear case (opposite of the hardening spring), but nearly the same in steady state as the hardening spring.

Prop 2.80

$x0 := 0.01$ $v0 := 0.1$ $m := 100$ $k := 2000$ $\alpha := -100$ $c := 170$ $F0 := 300$

$$\omega n := \sqrt{\frac{k}{m}} \quad \zeta := \frac{c}{2\sqrt{k \cdot m}} \quad f0 := \frac{F0}{m} \quad \omega := \frac{\omega n}{1} \quad X := \begin{bmatrix} x0 \\ v0 \end{bmatrix} \quad Y := X$$

$$D(t, X) := \begin{bmatrix} X_1 \\ \left(-2 \cdot \zeta \cdot \omega n \cdot X_1 - \omega n^2 \cdot X_0\right) - \alpha \cdot \left(X_0\right)^3 + f0 \cdot \cos(\omega \cdot t) \end{bmatrix}$$

$$L(t, Y) := \begin{bmatrix} Y_1 \\ -2 \cdot \zeta \cdot \omega n \cdot Y_1 - \omega n^2 \cdot Y_0 + f0 \cdot \cos(\omega \cdot t) \end{bmatrix}$$

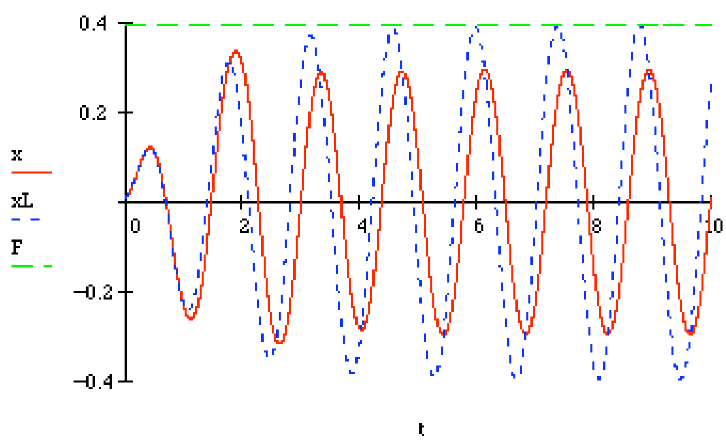
$Z := \text{rkfixed}(X, 0, 10, 4000, D)$

$t := Z^{\langle 0 \rangle}$ $x := Z^{\langle 1 \rangle}$ $W := \text{rkfixed}(Y, 0, 10, 4000, L)$

$xL := W^{\langle 1 \rangle}$

$\alpha \cdot m = -1 \cdot 10^4$

$$d(t) := \frac{f0}{\sqrt{\left(\omega n^2 - \omega^2\right)^2 + \left(2 \cdot \zeta \cdot \omega n \cdot \omega\right)^2}} \quad F := \overrightarrow{d(t)}$$

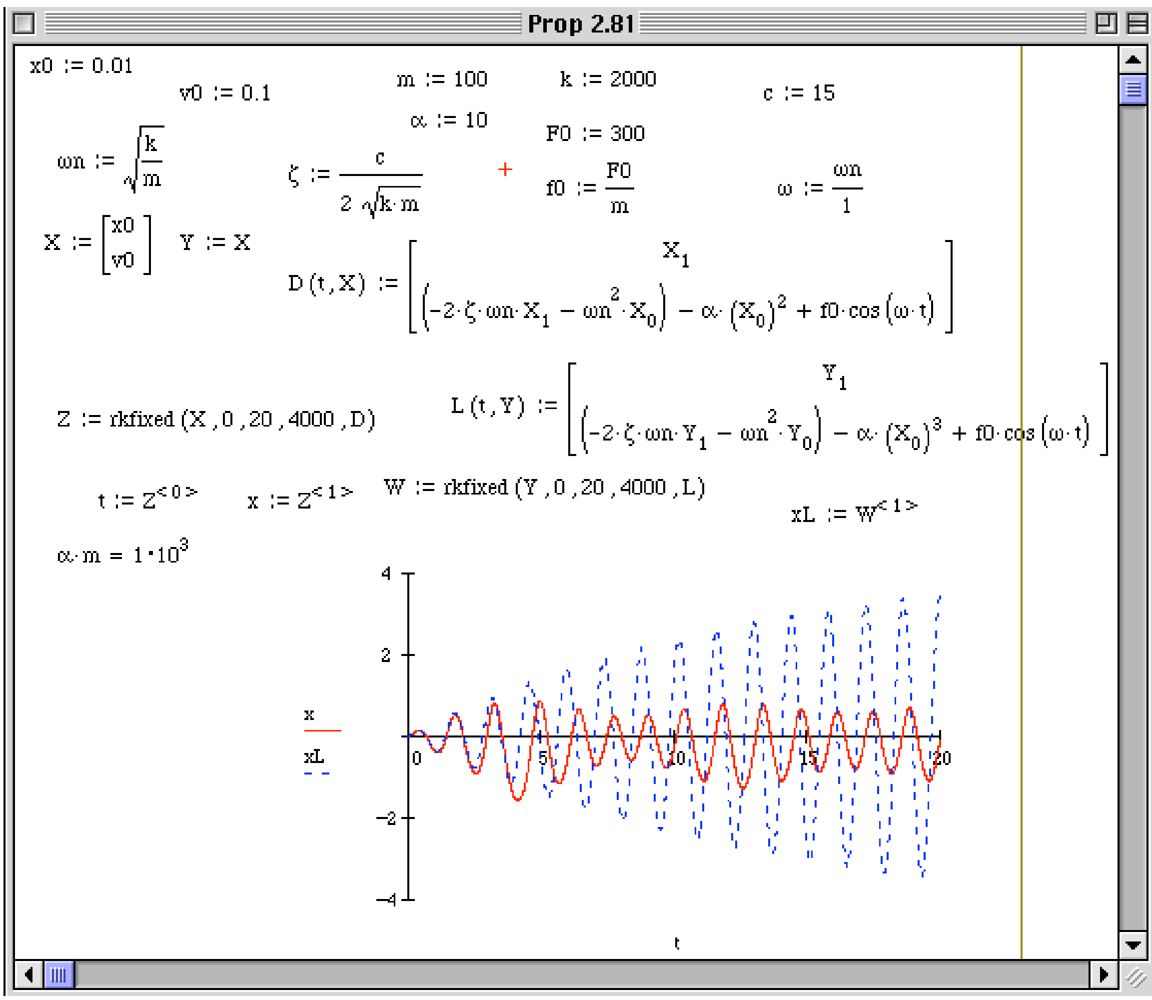


2.90*. Compute the response of the system in Figure 2.34 for the case that the damping is linear viscous and the spring is a nonlinear soft spring of the form

$$k(x) = kx - k_1x^3$$

and the system is subject to a harmonic excitation of 300 N at a frequency equal to the natural frequency ($\omega = \omega_n$) and initial conditions of $x_0 = 0.01$ m and $v_0 = 0.1$ m/s. The system has a mass of 100 kg, a damping coefficient of 15 kg/s and a linear stiffness coefficient of 2000 N/m. The value of k_1 is taken to be 1000 N/m^3. Compute the solution and compare it to the quadratic soft spring ($k(x) = kx + k_1x^2$).

Solution: The response to both the hardening and softening spring are given in the following Mathcad sessions. In each case the linear response is also shown for comparison. With the soft spring, the response is more variable, whereas the hardening spring seems to reach steady state.



2.91*. Compare the forced response of a system with velocity squared damping as defined in equation (2.129) using numerical simulation of the nonlinear equation to that of the response of the linear system obtained using equivalent viscous damping as defined by equation (2.131). Use as initial conditions, $x_0 = 0.01$ m and $v_0 = 0.1$ m/s with a mass of 10 kg, stiffness of 25 N/m, applied force of 150 cos ($\omega_n t$) and drag coefficient of $\alpha = 250$.

Solution:

Prop 2.82

$x0 := 1 \quad v0 := 0.1 \quad m := 10 \quad k := 25 \quad \alpha := 250 \quad F0 := 150$

$\omega n := \sqrt{\frac{k}{m}} \quad f0 := \frac{F0}{m} \quad \omega := 1 \cdot \omega n$

$ceq := \sqrt{\frac{8 \cdot \alpha \cdot m}{3 \cdot \pi} \cdot f0} \quad \zeta := \frac{ceq}{2\sqrt{k \cdot m}}$

$X := \begin{bmatrix} x0 \\ v0 \end{bmatrix} \quad Y := X$

$$D(t, X) := \begin{bmatrix} X_1 \\ -\omega n^2 \cdot X_0 - \frac{\alpha}{m} \cdot (X_1)^2 \cdot \frac{X_1}{|X_1|} + f0 \cdot \cos(\omega \cdot t) \end{bmatrix}$$

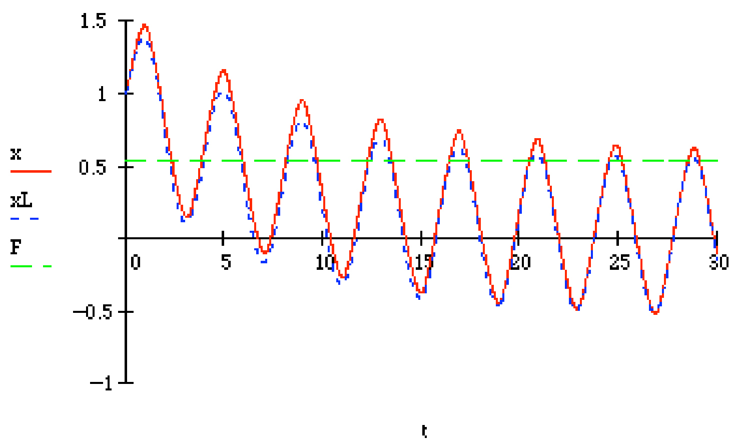
$$L(t, Y) := \begin{bmatrix} Y_1 \\ \left(-2 \cdot \zeta \cdot \omega n \cdot Y_1 - \omega n^2 \cdot Y_0\right) + f0 \cdot \cos(\omega \cdot t) \end{bmatrix}$$

$Z := \text{rkfixed}(X, 0, 30, 2000, D)$

$t := Z^{<0>} \quad x := Z^{<1>} \quad W := \text{rkfixed}(Y, 0, 30, 2000, L)$

$xL := W^{<1>}$

$$d(t) := \frac{f0}{\sqrt{\left(\omega n^2 - \omega^2\right)^2 + (2 \cdot \zeta \cdot \omega n \cdot \omega)^2}} \qquad F := \overrightarrow{d(t)}$$



2.92*. Compare the forced response of a system with structural damping (see table 2.2) using numerical simulation of the nonlinear equation to that of the response of the linear system obtained using equivalent viscous damping as defined in Table 2.2. Use as initial conditions, x_0 = 0.01 m and v_0 = 0.1 m/s with a mass of 10 kg, stiffness of 25 N/m, applied force of 150 cos ($\omega_n t$) and solid damping coefficient of b = 25.
Solution: The solution is presented here in Mathcad

Prop 2.83

$$x0 := 0.01 \quad v0 := 0.1 \quad m := 10 \quad k := 25 \quad b := 25 \quad F0 := 150$$

$$\omega n := \sqrt{\frac{k}{m}} \quad f0 := \frac{F0}{m} \quad \omega := 1\cdot\omega n \quad ceq := 2\cdot\frac{b}{\pi\cdot\omega} \quad \zeta := \frac{ceq}{2\sqrt{k\cdot m}}$$

$$X := \begin{bmatrix} x0 \\ v0 \end{bmatrix} \quad Y := X$$

$$D(t,X) := \begin{bmatrix} X_1 \\ -\omega n^2\cdot X_0 - \frac{b}{m}\cdot\left|X_0\right|\cdot\frac{X_1}{\left|X_1\right|} + f0\cdot\cos(\omega\cdot t) \end{bmatrix}$$

$$L(t,Y) := \begin{bmatrix} Y_1 \\ \left(-2\cdot\zeta\cdot\omega n\cdot Y_1 - \omega n^2\cdot Y_0\right) + f0\cdot\cos(\omega\cdot t) \end{bmatrix}$$

$$Z := \text{rkfixed}(X, 0, 25, 2500, D)$$

$$t := Z^{\langle 0\rangle} \quad x := Z^{\langle 1\rangle}$$

$$W := \text{rkfixed}(Y, 0, 25, 2500, L)$$

$$xL := W^{\langle 1\rangle}$$