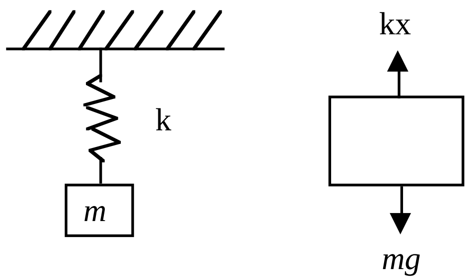
Problems and Solutions Section 1.1 (1.1 through 1.19)

1.1 The spring of Figure 1.2 is successively loaded with mass and the corresponding (static) displacement is recorded below. Plot the data and calculate the spring's stiffness. Note that the data contain some error. Also calculate the standard deviation.

m(kg)	10	11	12	13	14	15	16
x(m)	1.14	1.25	1.37	1.48	1.59	1.71	1.82

Solution:

Free-body diagram:

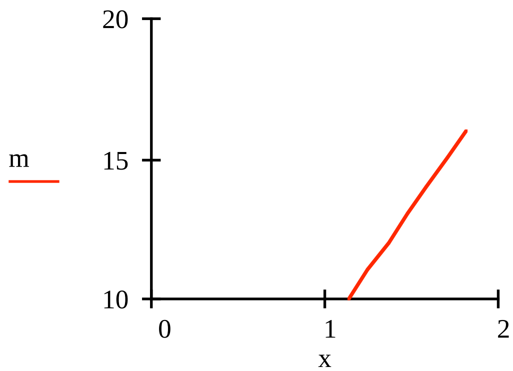


From the free-body diagram and static equilibrium:

$$kx = mg \quad (g = 9.81 m/s^2)$$

$$k = mg/x$$

$$\mu = \frac{\Sigma k_i}{n} = 86.164$$



Plot of mass in kg versus displacement in m

The sample standard deviation in computed stiffness is:

$$\sigma = \sqrt{\frac{\sum_{i=1}^{n}(k_i - \mu)^2}{n-1}} = \mathbf{0.164}$$

Computation of slope from mg/x

m(kg)	x(m)	k(N/m)
10	1.14	86.05
11	1.25	86.33
12	1.37	85.93
13	1.48	86.17
14	1.59	86.38
15	1.71	86.05
16	1.82	86.24

1.2 Derive the solution of $m\ddot{x} + kx = 0$ and plot the result for at least two periods for the case with ω_n = 2 rad/s, x_0 = 1 mm, and v_0 = $\sqrt{5}$ mm/s.

Solution:

Given:

$$m\ddot{x} + kx = 0 \qquad (1)$$

Assume: $x(t) = ae^{rt}$. Then: $\dot{x} = are^{rt}$ and $\ddot{x} = ar^2e^{rt}$. Substitute into equation (1) to get:

$$mar^2e^{rt} + kae^{rt} = 0$$

$$mr^2 + k = 0$$

$$r = \pm\sqrt{\frac{k}{m}}\; i$$

Thus there are two solutions:

$$x_1 = c_1e^{\left(\sqrt{\frac{k}{m}}i\right)t}, \text{ and } x_2 = c_2e^{\left(-\sqrt{\frac{k}{m}}i\right)t}$$

$$\text{where } \omega_n = \sqrt{\frac{k}{m}} = 2 \text{ rad/s}$$

The sum of x_1 and x_2 is also a solution so that the total solution is:

$$x = x_1 + x_2 = c_1e^{2it} + c_2e^{-2it}$$

Substitute initial conditions: x_0 = 1 mm, v_0 = $\sqrt{5}$ mm/s

$x(0) = c_1 + c_2 = x_0 = 1 \Rightarrow \underline{c_2 = 1 - c_1}$, and $v(0) = \dot{x}(0) = 2ic_1 - 2ic_2 = v_0 = \sqrt{5}$ mm/s

$\Rightarrow \underline{-2c_1 + 2c_2 = \sqrt{5}\,i}$. Combining the two underlined expressions (2 eqs in 2 unkowns):

$-2c_1 + 2 - 2c_1 = \sqrt{5}\,i \Rightarrow \underline{\underline{c_1 = \frac{1}{2} - \frac{\sqrt{5}}{4}i}}$, and $\underline{\underline{c_2 = \frac{1}{2} + \frac{\sqrt{5}}{4}i}}$

Therefore the solution is:

$$x = \left(\frac{1}{2} - \frac{\sqrt{5}}{4}i\right)e^{2it} + \left(\frac{1}{2} + \frac{\sqrt{5}}{4}i\right)e^{-2it}$$

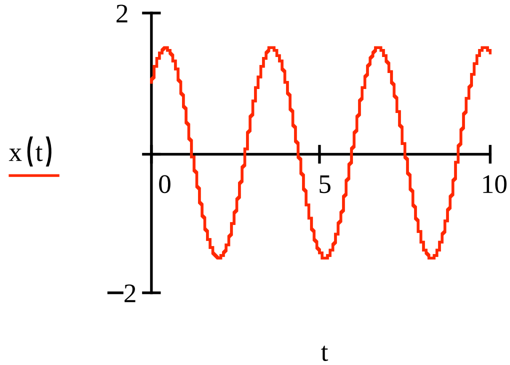
Using the Euler formula to evaluate the exponential terms yields:

$$x = \left(\frac{1}{2} - \frac{\sqrt{5}}{4}i\right)(\cos 2t + i\sin 2t) + \left(\frac{1}{2} + \frac{\sqrt{5}}{4}i\right)(\cos 2t - i\sin 2t)$$

$$\Rightarrow \underline{x(t) = \cos 2t + \frac{\sqrt{5}}{2}\sin 2t = \sqrt{\frac{3}{2}}\sin(2t + 0.7297)}$$

Using Mathcad the plot is:

$$x(t) := \cos(2\cdot t) + \frac{\sqrt{5}}{2}\cdot \sin(2\cdot t)$$



1.3 Solve $m\ddot{x} + kx = 0$ for k = 4 N/m, m = 1 kg, x_0 = 1 mm, and v_0 = 0. Plot the solution.

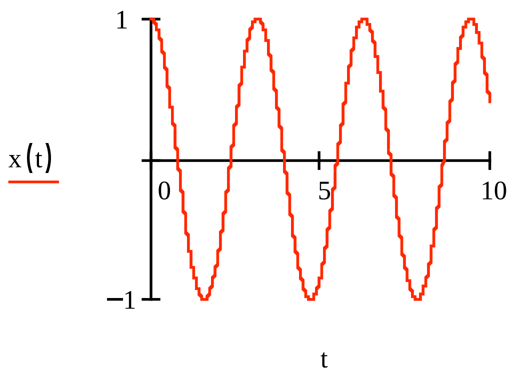
Solution:

This is identical to problem 2, except $v_0 = 0$. $\left(\omega_n = \sqrt{\dfrac{k}{m}} = 2 \text{ rad/s}\right)$. Calculating the initial conditions:

$$x(0) = c_1 + c_2 = x_0 = 1 \Rightarrow c_2 = 1 - c_1$$
$$v(0) = \dot{x}(0) = 2ic_1 - 2ic_2 = v_0 = 0 \Rightarrow c_2 = c_1$$
$$c_2 = c_1 = 0.5$$
$$x(t) = \frac{1}{2}e^{2it} + \frac{1}{2}e^{-2it} = \frac{1}{2}(\cos 2t + i\sin 2t) + \frac{1}{2}(\cos 2t - i\sin 2t)$$
$$\underline{x(t) = \cos(2t)}$$

The following plot is from Mathcad:

$x(t) := \cos(2 \cdot t)$



Alternately students may use equation (1.10) directly to get

$$x(t) = \frac{\sqrt{2^2(1)^2 + 0^2}}{2}\sin(2t + \tan^{-1}[\frac{2 \cdot 1}{0}])$$
$$= 1\sin(2t + \frac{\pi}{2}) = \cos 2t$$

1.4 The amplitude of vibration of an undamped system is measured to be 1 mm. The phase shift from $t = 0$ is measured to be 2 rad and the frequency is found to be 5 rad/s. Calculate the initial conditions that caused this vibration to occur. Assume the response is of the form $x(t) = A\sin(\omega_n t + \phi)$.

Solution:

Given: $A = 1\,\text{mm}$, $\phi = 2\,\text{rad}$, $\omega = 5\,\text{rad/s}$. For an *undamped* system:

$$x(t) = A\sin(\omega_n t + \phi) = 1\sin(5t + 2) \quad \text{and}$$
$$v(t) = \dot{x}(t) = A\omega_n \cos(\omega_n t + \phi) = 5\cos(5t + 2)$$

Setting $t = 0$ in these expressions yields:

$$x(0) = 1\sin(2) = \underline{0.9093\text{ mm}}$$
$$v(0) = 5\cos(2) = \underline{-2.081\text{ mm/s}}$$

1.5 Find the equation of motion for the hanging spring-mass system of Figure P1.5, and compute the natural frequency. In particular, using static equilibrium along with Newton's law, determine what effect gravity has on the equation of motion and the system's natural frequency.

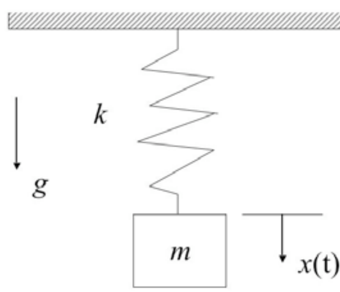
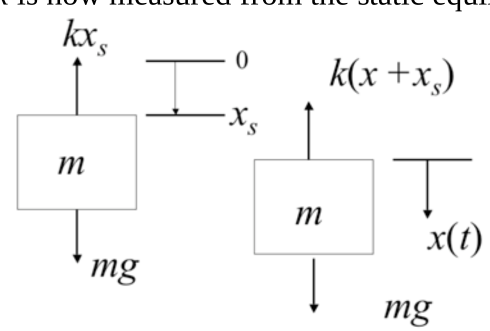


Figure P1.5

Solution:
The free-body diagram of problem system in (a) for the static case and in (b) for the dynamic case, where x is now measured from the static equilibrium position.



(a) (b)

From a force balance in the static case (a): $mg = kx_s$, where x_s is the static deflection of the spring. Next let the spring experience a dynamic deflection $x(t)$ governed by summing the forces in (b) to get

$$m\ddot{x}(t) = mg - k(x(t) + x_s) \Rightarrow m\ddot{x}(t) + kx(t) = mg - kx_s$$

$$\Rightarrow \underline{m\ddot{x}(t) + kx(t) = 0} \Rightarrow \underline{\omega_n = \sqrt{\frac{k}{m}}}$$

since $mg = kx_s$ from static equilibrium.

1.6 Find the equation of motion for the system of Figure P1.6, and find the natural frequency. In particular, using static equilibrium along with Newton's law, determine what effect gravity has on the equation of motion and the system's natural frequency. Assume the block slides without friction.

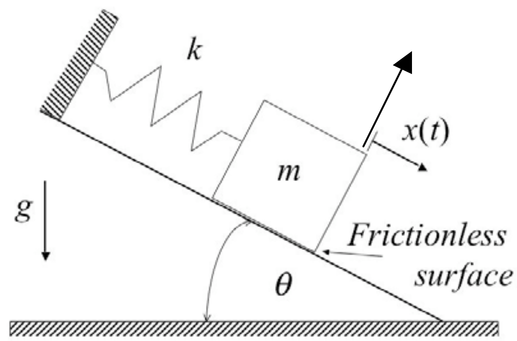
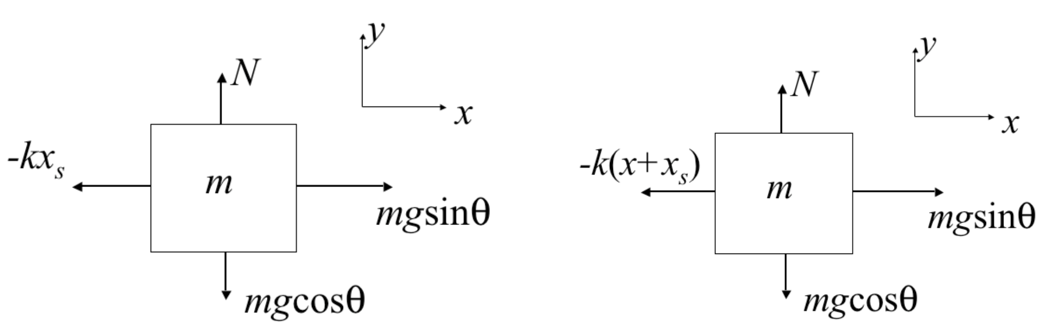


Figure P1.6

Solution:
Choosing a coordinate system along the plane with positive down the plane, the free-body diagram of the system for the static case is given and (a) and for the dynamic case in (b):



In the figures, N is the normal force and the components of gravity are determined by the angle θ as indicated. From the static equilibrium: $-kx_s + mg\sin\theta = 0$. Summing forces in (b) yields:

$$\sum F_i = m\ddot{x}(t) \Rightarrow m\ddot{x}(t) = -k(x + x_s) + mg\sin\theta$$
$$\Rightarrow m\ddot{x}(t) + kx = -kx_s + mg\sin\theta = 0$$
$$\Rightarrow \underline{m\ddot{x}(t) + kx = 0}$$
$$\Rightarrow \underline{\omega_n = \sqrt{\frac{k}{m}}} \text{ rad/s}$$

1.7 An undamped system vibrates with a frequency of 10 Hz and amplitude 1 mm. Calculate the maximum amplitude of the system's velocity and acceleration.

Solution:

Given: First convert Hertz to rad/s: $\omega_n = 2\pi f_n = 2\pi(10) = 20\pi$ rad/s. We also have that A= 1 mm.

For an undamped system:

$$x(t) = A\sin(\omega_n t + \phi)$$

and differentiating yields the velocity: $v(t) = A\omega_n \cos(\omega_n t + \phi)$. Realizing that both the sin and cos functions have maximum values of 1 yields:

$$v_{\max} = A\omega_n = 1(20\pi) = \mathbf{62.8\ mm/s}$$

Likewise for the acceleration: $a(t) = -A\omega_n^2 \sin(\omega_n t + \phi)$

$$a_{\max} = A\omega_n^2 = 1(20\pi)^2 = \mathbf{3948\ mm/s^2}$$

1.8 Show by calculation that $A\sin(\omega_n t+\phi)$ can be represented as $B\sin\omega_n t + C\cos\omega_n t$ and calculate C and B in terms of A and ϕ.

Solution:

This trig identity is useful: $\sin(a+b)=\sin a\cos b+\cos a\sin b$

Given: $A\sin(\omega_n t+\phi)=A\sin(\omega_n t)\cos(\phi)+A\cos(\omega_n t)\sin(\phi)$

$$= B\sin\omega_n t + C\cos\omega_n t$$
$$\text{where } B = A\cos\phi \quad \textit{and} \quad C = A\sin\phi$$

1.9 Using the solution of equation (1.2) in the form $x(t)=B\sin\omega_n t+C\cos\omega_n t$ calculate the values of B and C in terms of the initial conditions x_0 and v_0.

Solution:
Using the solution of equation (1.2) in the form

$$x(t) = B\sin\omega_n t + C\cos\omega_n t$$

and differentiate to get:

$$\dot{x}(t) = \omega_n B\cos(\omega_n t) - \omega_n C\sin(\omega_n t)$$

Now substitute the initial conditions into these expressions for the position and velocity to get:

$$x_0 = x(0) = B\sin(0) + C\cos(0) = C$$
$$v_0 = \dot{x}(0) = \omega_n B\cos(0) - \omega_n C\sin(0)$$
$$= \omega_n B(1) - \omega_n C(0) = \omega_n B$$

Solving for B and C yields:

$$B = \frac{v_0}{\omega_n},\ \text{and } C = x_0$$

Thus $$x(t) = \frac{v_0}{\omega_n}\sin\omega_n t + x_0\cos\omega_n t$$

1.10 Using Figure 1.6, verify that equation (1.10) satisfies the initial velocity condition.

Solution: Following the lead given in Example 1.1.2, write down the general expression of the velocity by differentiating equation (1.10):

$$x(t) = A\sin(\omega_n t + \phi) \Rightarrow \dot{x}(t) = A\omega_n \cos(\omega_n t + \phi)$$
$$\Rightarrow v(0) = A\omega_n \cos(\omega_n 0 + \phi) = A\omega_n \cos(\phi)$$

From the figure:

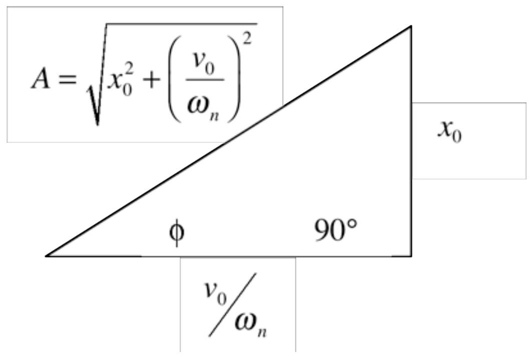


Figure 1.6

$$A = \sqrt{x_0^2 + \left(\frac{v_0}{\omega_n}\right)^2}, \quad \cos\phi = \frac{v_0/\omega_n}{\sqrt{x_0^2 + \left(\frac{v_0}{\omega_n}\right)^2}}$$

Substitution of these values into the expression for $v(0)$ yields

$$v(0) = A\omega_n \cos\phi = \sqrt{x_0^2 + \left(\frac{v_0}{\omega_n}\right)^2}\,(\omega_n)\frac{\frac{v_0}{\omega_n}}{\sqrt{x_0^2 + \left(\frac{v_0}{\omega_n}\right)^2}} = v_0$$

verifying the agreement between the figure and the initial velocity condition.

1.11 (a)A 0.5 kg mass is attached to a linear spring of stiffness 0.1 N/m. Determine the natural frequency of the system in hertz. b) Repeat this calculation for a mass of 50 kg and a stiffness of 10 N/m. Compare your result to that of part a.

Solution: From the definition of frequency and equation (1.12)

$$(a) \qquad \omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{.5}{.1}} = 0.447 \text{ rad/s}$$

$$f_n = \frac{\omega_n}{2\pi} = \frac{2.236}{2\pi} = \underline{0.071 \text{ Hz}}$$

$$(b) \qquad \omega_n = \sqrt{\frac{50}{10}} = 0.447\text{rad/s}, \ f_n = \frac{\omega_n}{2\pi} = \underline{0.071 \text{ Hz}}$$

Part (b) is the same as part (a) thus very different systems can have same natural frequencies.

1.12 Derive the solution of the single degree of freedom system of Figure 1.4 by writing Newton's law, $ma = -kx$, in differential form using $adx = vdv$ and integrating twice.

Solution: Substitute $a = vdv/dx$ into the equation of motion $ma = -kx$, to get $mvdv = -kxdx$. Integrating yields:

$$\frac{v^2}{2} = -\omega_n^2 \frac{x^2}{2} + c^2, \quad \text{where } c \text{ is a constant}$$

$$\text{or} \quad v^2 = -\omega_n^2 x^2 + c^2 \Rightarrow$$

$$v = \frac{dx}{dt} = \sqrt{-\omega_n^2 x^2 + c^2} \Rightarrow$$

$$dt = \frac{dx}{\sqrt{-\omega_n^2 x^2 + c^2}}, \quad \text{write } u = \omega_n x \text{ to get:}$$

$$t - 0 = \frac{1}{\omega_n}\int \frac{du}{\sqrt{c^2 - u^2}} = \frac{1}{\omega_n}\sin^{-1}\left(\frac{u}{c}\right) + c_2$$

Here c_2 is a second constant of integration that is convenient to write as $c_2 = -\phi/\omega_n$. Rearranging yields

$$\omega_n t + \phi = \sin^{-1}\left(\frac{\omega_n x}{c}\right) \Rightarrow$$

$$\frac{\omega_n x}{c} = \sin(\omega_n t + \phi) \Rightarrow$$

$$\underline{x(t) = A\sin(\omega_n t + \phi)}, \quad A = \frac{c}{\omega_n}$$

in agreement with equation (1.19).

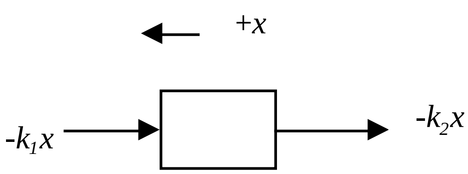
1.13 Determine the natural frequency of the two systems illustrated.

(a) (b)

Figure P1.13

Solution:

(a) Summing forces from the free-body diagram in the x direction yields:



Free-body diagram for part a

$$m\ddot{x} = -k_1 x - k_2 x \Rightarrow$$

$$m\ddot{x} + k_1 x + k_2 x = 0$$

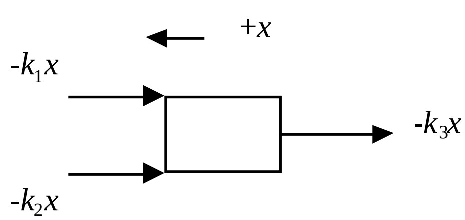
$$m\ddot{x} + x\left(k_1 + k_2\right) = 0, \text{ dividing by } m \text{ yields :}$$

$$\ddot{x} + \left(\frac{k_1 + k_2}{m}\right)x = 0$$

Examining the coefficient of x yields:

$$\omega_n = \sqrt{\frac{k_1 + k_2}{m}}$$

(b) Summing forces from the free-body diagram in the x direction yields:



Free-body diagram for part b

$$m\ddot{x} = -k_1 x - k_2 x - k_3 x, \Rightarrow$$

$$m\ddot{x} + k_1 x + k_2 x + k_3 x = 0 \Rightarrow$$

$$m\ddot{x} + (k_1 + k_2 + k_3)x = 0 \Rightarrow \ddot{x} + \frac{(k_1 + k_2 + k_3)}{m}x = 0$$

$$\Rightarrow \underline{\omega_n = \sqrt{\frac{k_1 + k_2 + k_3}{m}}}$$

1.14* Plot the solution given by equation (1.10) for the case k = 1000 N/m and m = 10 kg for two complete periods for each of the following sets of initial conditions: a) x_0 = 0 m, v_0 = 1 m/s, b) x_0 = 0.01 m, v_0 = 0 m/s, and c) x_0 = 0.01 m, v_0 = 1 m/s.

Solution: Here we use Mathcad:
a) all units in m, kg, s

$m := 10 \qquad k := 1000 \qquad \boxed{\omega_n := \sqrt{\dfrac{k}{m}}}$

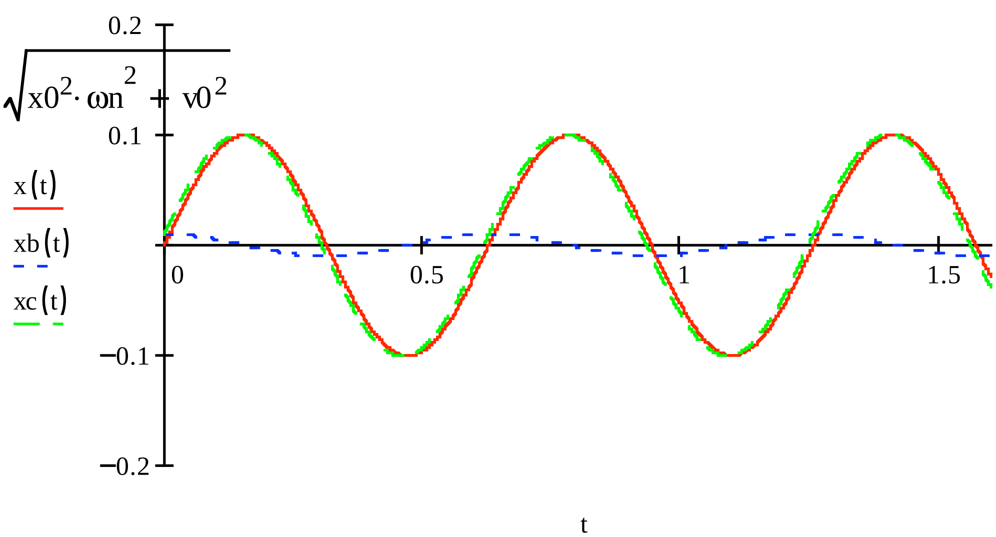
$x0 := 0.0 \qquad v0 := 1$

$$fn := \frac{\omega n}{2\cdot\pi} \qquad T := \frac{2\cdot\pi}{\omega n}$$

$$\phi := \operatorname{atan}\left(\frac{\omega n\cdot x0}{v0}\right) \qquad x(t) := A\cdot\sin(\omega n\cdot t + \phi)$$

parts b and c are plotted in the above by simply changing the initial conditions as appropriate

$$A := \frac{1}{\omega n}\cdot\sqrt{x0^2\cdot\omega n^2 + v0^2}$$



1.15* Make a three dimensional surface plot of the amplitude A of an undamped oscillator given by equation (1.9) versus x_0 and v_0 for the range of initial conditions given by $-0.1 \leq x_0 \leq 0.1$ m and $-1 \leq v_0 \leq 1$ m/s, for a system with natural frequency of 10 rad/s.

Solution: Working in Mathcad the solution is generated as follows:

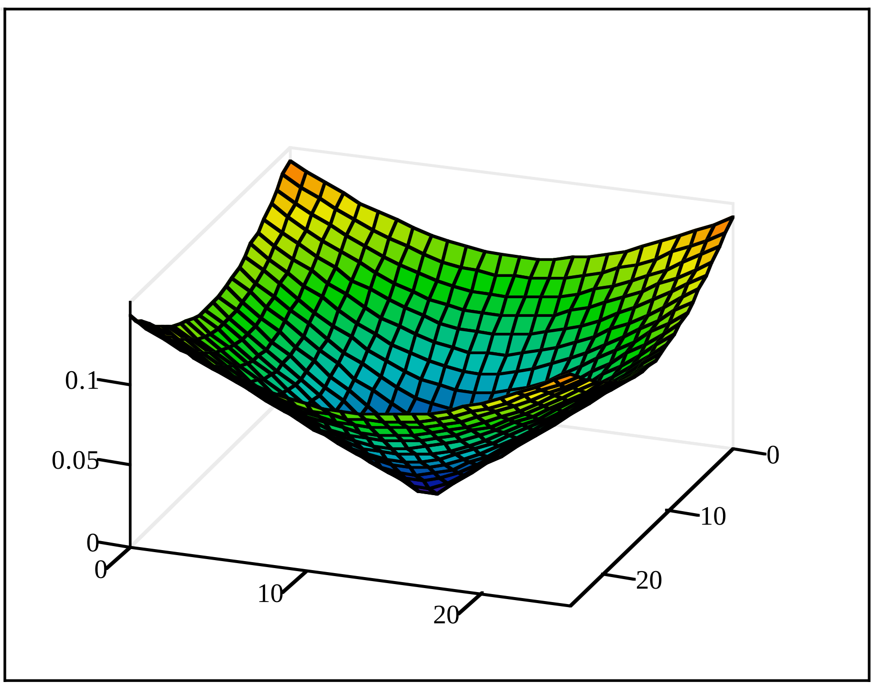
$\omega n := 10$

$N := 25$ $\quad i := 0 .. N$ $\quad j := 0 .. N$

$v0_j := -1 + \left(\frac{2}{N} \cdot j\right)$ $\quad x0_i := -0.1 + \left(\frac{0.2}{N}\right) \cdot i$

$$A(x0, v0) := \frac{1}{\omega n} \cdot \sqrt{\omega n^2 \cdot (x0)^2 + (v0)^2}$$

$M_{i,j} := A\left(x0_i, v0_j\right)$



M

Amplitude vs initial conditions

1.16 A machine part is modeled as a pendulum connected to a spring as illustrated in Figure P1.16. Ignore the mass of pendulum's rod and derive the equation of motion. Then following the procedure used in Example 1.1.1, linearize the equation of motion and compute the formula for the natural frequency. Assume that the rotation is small enough so that the spring only deflects horizontally.

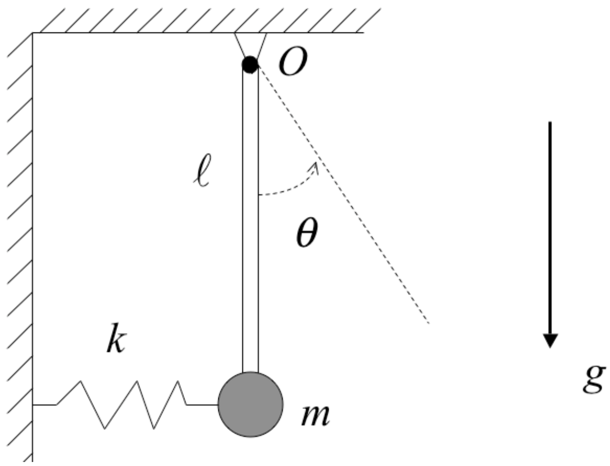
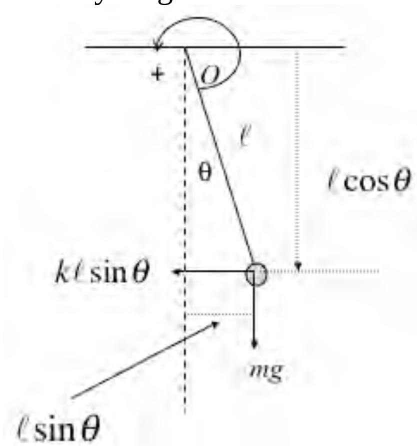


Figure P1.16

Solution: Consider the free body diagram of the mass displaced from equilibrium:



There are two forces acting on the system to consider, if we take moments about point *O* (then we can ignore any forces at *O*). This yields

$$\sum M_O = J_O\alpha \Rightarrow m\ell^2\ddot{\theta} = -mg\ell\sin\theta - k\ell\sin\theta \bullet \ell\cos\theta$$

$$\Rightarrow \underline{m\ell^2\ddot{\theta} + mg\ell\sin\theta + k\ell^2\sin\theta\cos\theta = 0}$$

Next consider the small θ approximations to that $\sin\theta \sim \theta$ and $\cos\theta = 1$. Then the linearized equation of motion becomes:

$$\ddot{\theta}(t) + \left(\frac{mg + k\ell}{m\ell}\right)\theta(t) = 0$$

Thus the natural frequency is

$$\underline{\omega_n = \sqrt{\frac{mg + k\ell}{m\ell}} \text{ rad/s}}$$

1.17 A pendulum has length of 250 mm. What is the system's natural frequency in Hertz?

Solution:

Given: l =250 mm

Assumptions: small angle approximation of sin

From Window 1.1, the equation of motion for the pendulum is as follows:

$$I_O\ddot{\theta} + mg\theta = 0, \quad \text{where } I_O = ml^2 \Rightarrow \ddot{\theta} + \frac{g}{l}\theta = 0$$

The coefficient of θ yields the natural frequency as:

$$\omega_n = \sqrt{\frac{g}{l}} = \sqrt{\frac{9.8 \text{ m/s}^2}{0.25 \text{ m}}} = 6.26 \text{ rad/s}$$

$$f_n = \frac{\omega_n}{2\pi} = 0.996 \text{ Hz}$$

1.18 The pendulum in Example 1.1.1 is required to oscillate once every second. What length should it be?

Solution:

Given: $f = 1\ Hz$ (one cycle per second)

$$\omega_n = 2\pi f = \sqrt{\frac{g}{l}}$$

$$\therefore\ l = \frac{g}{(2\pi f)^2} = \frac{9.81}{4\pi^2} = 0.248\ m$$

1.19 The approximation of sin θ = θ, is reasonable for θ less than 10°. If a pendulum of length 0.5 m, has an initial position of θ(0) = 0, what is the maximum value of the initial angular velocity that can be given to the pendulum with out violating this small angle approximation? (be sure to work in radians)

Solution: From Window 1.1, the linear equation of the pendulum is

$$\ddot{\theta}(t) + \frac{g}{\ell}\theta(t) = 0$$

For zero initial position, the solution is given in equation (1.10) by

$$\theta(t) = \frac{v_0\sqrt{\ell}}{\sqrt{g}}\sin(\sqrt{\frac{g}{\ell}}t) \Rightarrow |\theta| \leq \frac{v_0\sqrt{\ell}}{\sqrt{g}}$$

since sin is always less then one. Thus if we need θ < 10°= 0.175 rad, then we need to solve:

$$\frac{v_0\sqrt{0.5}}{\sqrt{9.81}} = 0.175$$

for v_0 which yields:

$v_0 \leq$ **0.773 rad/s.**

Problems and Solutions for Section 1.2 and Section 1.3 (1.20 to 1.51)

Problems and Solutions Section 1.2 (Numbers 1.20 through 1.30)

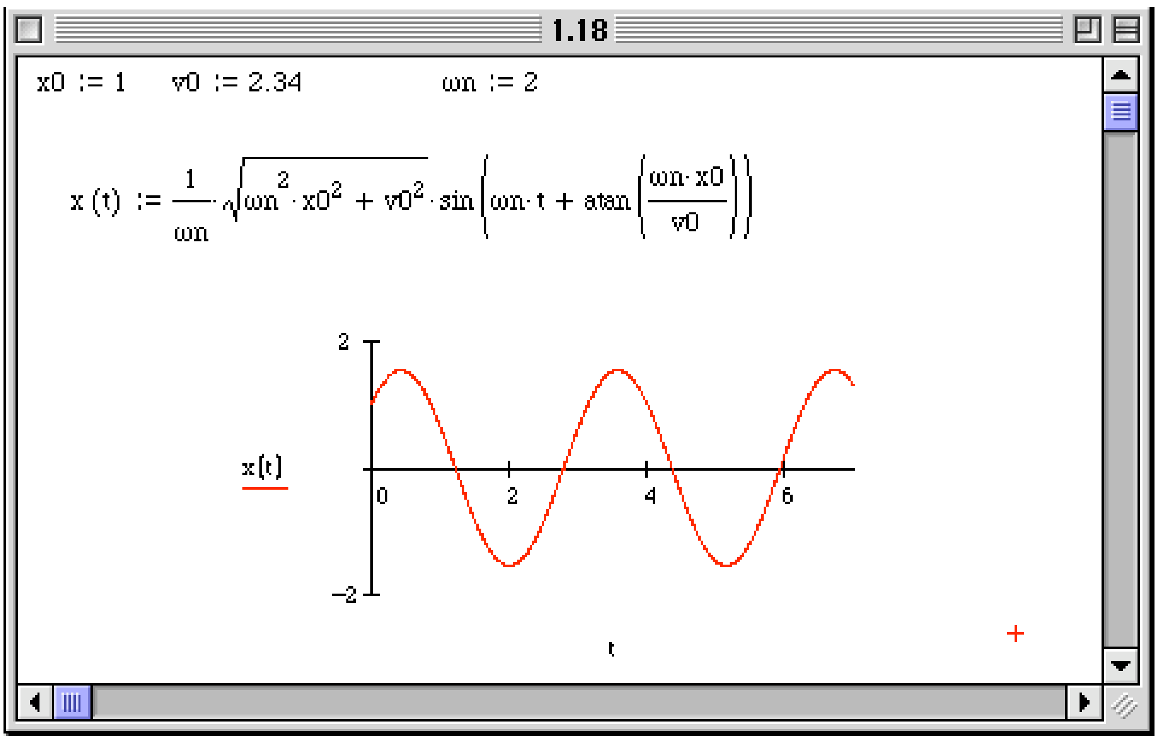
1.20* Plot the solution of a linear, spring and mass system with frequency $\omega_n = 2$ rad/s, $x_0 = 1$ mm and $v_0 = 2.34$ mm/s, for at least two periods.

Solution: From Window 1.18, the plot can be formed by computing:

$$A = \frac{1}{\omega_n}\sqrt{\omega_n^2 x_0^2 + v_0^2} = 1.54 \text{ mm}, \ \ \phi = \tan^{-1}\left(\frac{\omega_n x_0}{v_0}\right) = 40.52^\circ$$

$$x(t) = A\sin(\omega_n t + \phi)$$

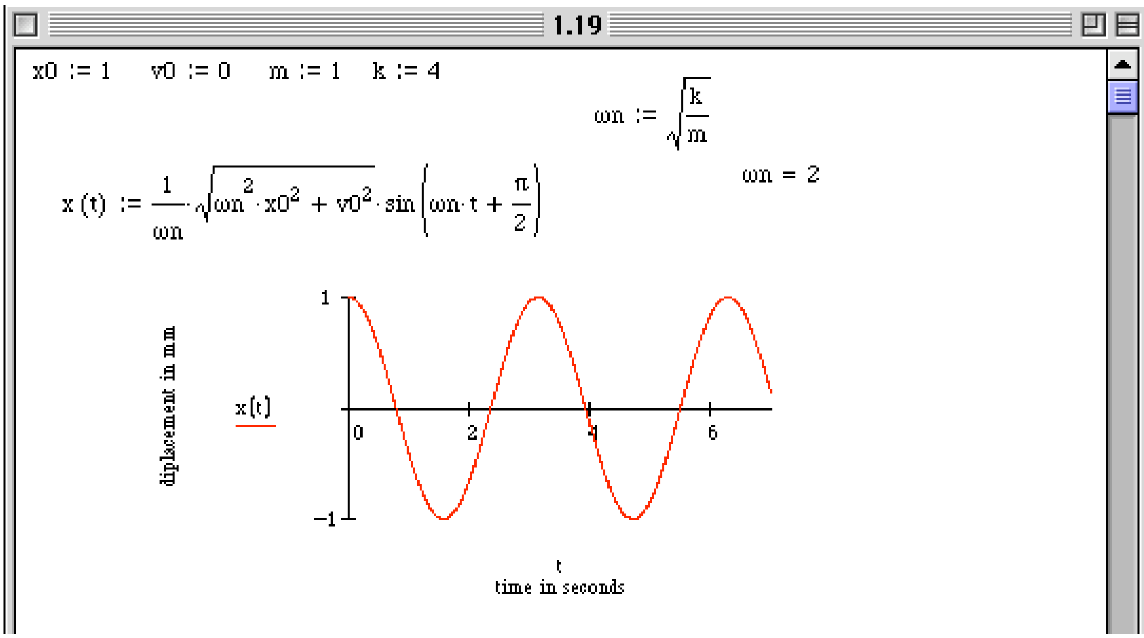
This can be plotted in any of the codes mentioned in the text. In Mathcad the program looks like.



In this plot the units are in mm rather than meters.

1.21* Compute the natural frequency and plot the solution of a spring-mass system with mass of 1 kg and stiffness of 4 N/m, and initial conditions of x_0 = 1 mm and v_0 = 0 mm/s, for at least two periods.

Solution: Working entirely in Mathcad, and using the units of mm yields:



Any of the other codes can be used as well.

1.22 To design a linear, spring-mass system it is often a matter of choosing a spring constant such that the resulting natural frequency has a specified value. Suppose that the mass of a system is 4 kg and the stiffness is 100 N/m. How much must the spring stiffness be changed in order to increase the natural frequency by 10%?

Solution: Given m =4 kg and k = 100 N/m the natural frequency is

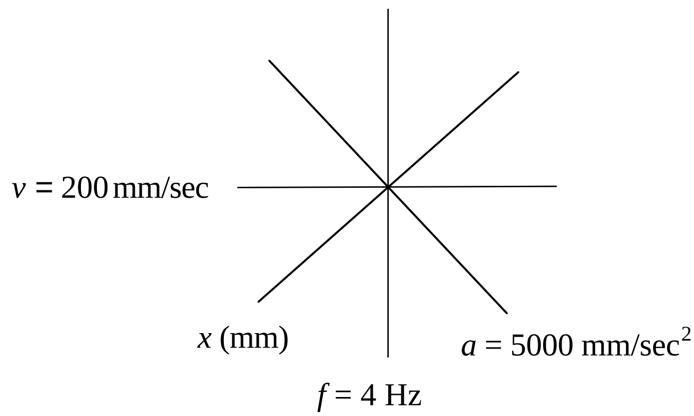
$$\omega_n = \sqrt{\frac{100}{4}} = 5 \text{ rad/s}$$

Increasing this value by 10% requires the new frequency to be 5 x 1.1 = 5.5 rad/s. Solving for k given m and ω_n yields:

$$5.5 = \sqrt{\frac{k}{4}} \Rightarrow k = (5.5)^2(4) = 121 \text{ N/m}$$

Thus the stiffness k must be increased by about 20%.

1.23 Referring to Figure 1.8, if the maximum peak velocity of a vibrating system is 200 mm/s at 4 Hz and the maximum allowable peak acceleration is 5000 mm/s^2, what will the peak displacement be?



Solution:

Given: v_{max} = 200 mm/s @ 4 Hz

a_{max} = 5000 mm/s @ 4 Hz

$x_{max} = A$

$v_{max} = A\omega_n$

$a_{max} = A\omega_n{}^2$

$$\therefore x_{\max} = \frac{v_{\max}}{\omega_n} = \frac{v_{\max}}{2\pi f} = \frac{200}{8\pi} = 7.95\ mm$$

At the center point, the peak displacement will be **x = 7.95 mm**

1.24 Show that lines of constant displacement and acceleration in Figure 1.8 have slopes of +1 and –1, respectively. If rms values instead of peak values are used, how does this affect the slope?

Solution: Let

$$x = x_{\max} \sin \omega_n t$$
$$\dot{x} = x_{\max}\omega_n \cos \omega_n t$$
$$\ddot{x} = -x_{\max}\omega_n{}^2 \sin \omega_n t$$

Peak values:

$$\dot{x}_{\max} = x_{\max}\omega_n = 2\pi f x_{\max}$$
$$\ddot{x}_{\max} = x_{\max}\omega_n^2 = (2\pi f)^2 x_{\max}$$

Location:

$$\ln \dot{x}_{\max} = \ln x_{\max} + \ln 2\pi f$$
$$\ln \dot{x}_{\max} = \ln \ddot{x}_{\max} - \ln 2\pi f$$

Since x_{max} is constant, the plot of ln $\dot{x}_{\max}$ versus ln 2πf is a straight line of slope +1. If ln $\ddot{x}_{\max}$ is constant, the plot of ln $\dot{x}_{\max}$ versus ln 2πf is a straight line of slope –1. Calculate RMS values

Let

$$x(t) = A \sin \omega_n t$$
$$\dot{x}(t) = A\omega_n \cos \omega_n t$$
$$\ddot{x}(t) = -A\omega_n{}^2 \sin \omega_n t$$

Mean Square Value: $\overline{x}^{\,2} = \lim\limits_{T\to\infty} \dfrac{1}{T}\displaystyle\int_0^T x^2(t)\,dt$

$$\overline{x}^{\,2} = \lim_{T\to\infty} \frac{1}{T}\int_0^T A^2 \sin^2 \omega_n t\; dt \;= \lim_{T\to\infty} \frac{A^2}{T}\int_0^T (1- \cos 2\omega_n t)\; dt \;= \frac{A^2}{2}$$

$$\overline{\dot{x}}^{\,2} = \lim_{T\to\infty} \frac{1}{T}\int_0^T A^2\,\omega_n^2 \cos^2 \omega_n\, t\; dt \;= \lim_{T\to\infty} \frac{A^2\omega_n^{\;2}}{T}\int_0^T \frac{1}{2}(1 + \cos 2\omega_n t)\; dt \;= \frac{A^2\omega_n^{\;2}}{2}$$

$$\overline{\ddot{x}}^{\,2} = \lim_{T\to\infty} \frac{1}{T}\int_0^T A^2\,\omega_n^4 \sin^2 \omega_n t\; dt \;= \lim_{T\to\infty} \frac{A^2\omega_n^{\;4}}{T}\int_0^T \frac{1}{2}(1 + \cos 2\omega_n t)\; dt \;= \frac{A^2\omega_n^{\;4}}{2}$$

Therefore,

$$x_{rms} = \sqrt{\overline{x}^{\,2}} = \frac{\sqrt{2}}{2}\,A$$

$$\dot{x}_{rms} = \sqrt{\overline{\dot{x}}^{\,2}} = \frac{\sqrt{2}}{2}\,A\omega_n$$

$$\ddot{x}_{rms} = \sqrt{\overline{\ddot{x}}^{\,2}} = \frac{\sqrt{2}}{2}\,A\omega_n^2$$

The last two equations can be rewritten as:

$$\dot{x}_{rms} = x_{rms}\,\omega = 2\pi f\; x_{rms}$$

$$\ddot{x}_{rms} = x_{rms}\,\omega^2 = 2\pi f\; \dot{x}_{rms}$$

The logarithms are:

$$\ln\, \dot{x}_{\max} = \ln\, x_{\max} + \ln\, 2\pi f$$

$$\ln\, \ddot{x}_{\max} = \ln\, \dot{x}_{\max} + \ln\, 2\pi f$$

The plots of $\ln\, \dot{x}_{rms}$ versus $\ln\, 2\pi f$ is a straight line of slope +1 when x_{rms} is constant, and –1 when $\ddot{x}_{rms}$ is constant. Therefore **the slopes are unchanged.**

1.25 A foot pedal mechanism for a machine is crudely modeled as a pendulum connected to a spring as illustrated in Figure P1.25. The purpose of the spring is to keep the pedal roughly vertical. Compute the spring stiffness needed to keep the pendulum at 1° from the horizontal and then compute the corresponding natural frequency. Assume that the angular deflections are small, such that the spring deflection can be approximated by the arc length, that the pedal may be treated as a point mass and that pendulum rod has negligible mass. The values in the figure are $m = 0.5$ kg, $g = 9.8$ m/s^2, $\ell_1 = 0.2$ m and $\ell_2 = 0.3$ m.

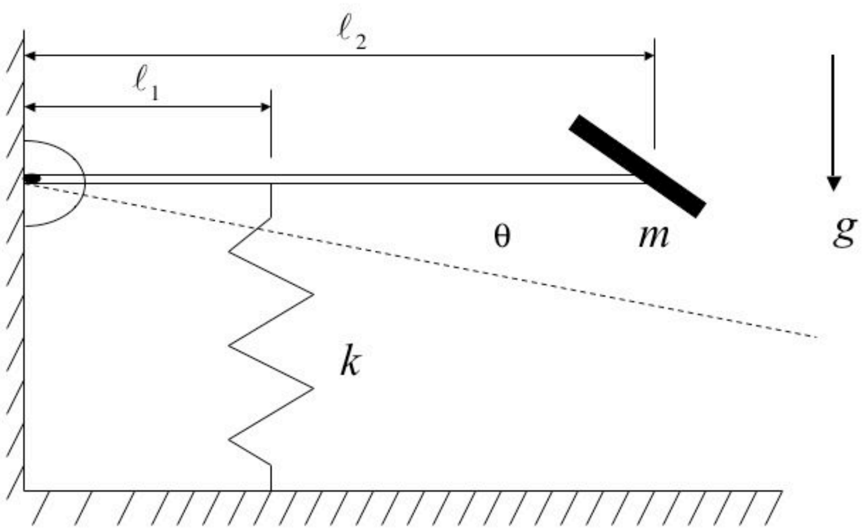
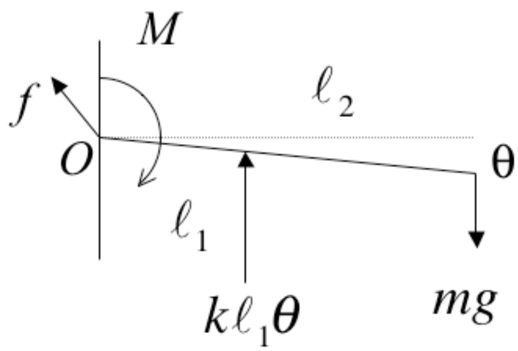


Figure P1.25

Solution: You may want to note to your students, that many systems with springs are often designed based on static deflections, to hold parts in specific positions as in this case, and yet allow some motion. The free-body diagram for the system is given in the figure.



For static equilibrium the sum of moments about point O yields (θ_1 is the static deflection):

$$\sum M_0 = -\ell_1\theta_1(\ell_1)k + mg\ell_2 = 0$$

$$\Rightarrow \ell_1^2\theta_1 k = mg\ell_2 \qquad (1)$$

$$\Rightarrow k = \frac{mg\ell_2}{\ell_1^2\theta_1} = \frac{0.5\cdot 0.3}{(0.2)^2\,\frac{\pi}{180}} = \underline{2106\text{ N/m}}$$

Again take moments about point O to get the dynamic equation of motion:

$$\sum M_O = J\ddot{\theta} = m\ell_2^2\ddot{\theta} = -\ell_1^2k(\theta+\theta_1) + mg\ell_2 = -\ell_1^2k\theta + \ell_1^2k\theta_1 - mg\ell_2\theta$$

Next using equation (1) above for the static deflection yields:

$$m\ell_2^2\ddot{\theta} + \ell_1^2k\theta = 0$$

$$\Rightarrow \ddot{\theta} + \left(\frac{\ell_1^2k}{m\ell_2^2}\right)\theta = 0$$

$$\Rightarrow \omega_n = \frac{\ell_1}{\ell_2}\sqrt{\frac{k}{m}} = \frac{0.2}{0.3}\sqrt{\frac{2106}{0.5}} = \underline{43.27\ \text{rad/s}}$$

1.26 An automobile is modeled as a 1000-kg mass supported by a spring of stiffness k = 400,000 N/m. When it oscillates it does so with a maximum deflection of 10 cm. When loaded with passengers, the mass increases to as much as 1300 kg. Calculate the change in frequency, velocity amplitude, and acceleration amplitude if the maximum deflection remains 10 cm.

Solution:

Given: $m_1 = 1000\ kg$

$m_2 = 1300\ kg$

$k = 400{,}000\ N/m$

$$x_{max} = A = 10 \text{ cm}$$

$$\omega_{n1} = \sqrt{\frac{k}{m_1}} = \sqrt{\frac{400{,}000}{1000}} = 20 \; rad/s$$

$$\omega_{n2} = \sqrt{\frac{k}{m_2}} = \sqrt{\frac{400{,}000}{1300}} = 17.54 \; rad/s$$

$$\Delta\omega = 17.54 - 20 = -2.46 \; rad/s$$

$$\Delta f = \frac{\Delta\omega}{2\pi} = \left|\frac{-2.46}{2\pi}\right| = 0.392 \; Hz$$

$v_1 = A\omega_{n1} = 10 \;cm \;\text{x}\; 20 \;rad/s = 200 \;cm/s$

$v_2 = A\omega_{n2} = 10 \;cm \;\text{x}\; 17.54 \;\text{rad/s} = 175.4 \;cm/s$

$\boldsymbol{\Delta v} = 175.4 - 200 = \mathbf{-24.6 \;cm/s}$

$a_1 = A\omega_{n1}^2 = 10 \;cm \;\text{x}\; (20 \;rad/s)^2 = 4000 \;cm/s^2$

$a_2 = A\omega_{n2}^2 = 10 \;cm \;\text{x}\; (17.54 \;\text{rad/s})^2 = 3077 \;cm/s^2$

$\boldsymbol{\Delta a} = 3077 - 4000 = \mathbf{-923 \;cm/s^2}$

1.27 The front suspension of some cars contains a torsion rod as illustrated in Figure P1.27 to improve the car's handling. (a) Compute the frequency of vibration of the wheel assembly given that the torsional stiffness is 2000 N m/rad and the wheel assembly has a mass of 38 kg. Take the distance x = 0.26 m. (b) Sometimes owners put different wheels and tires on a car to enhance the appearance or performance. Suppose a thinner tire is put on with a larger wheel raising the mass to 45 kg. What effect does this have on the frequency?

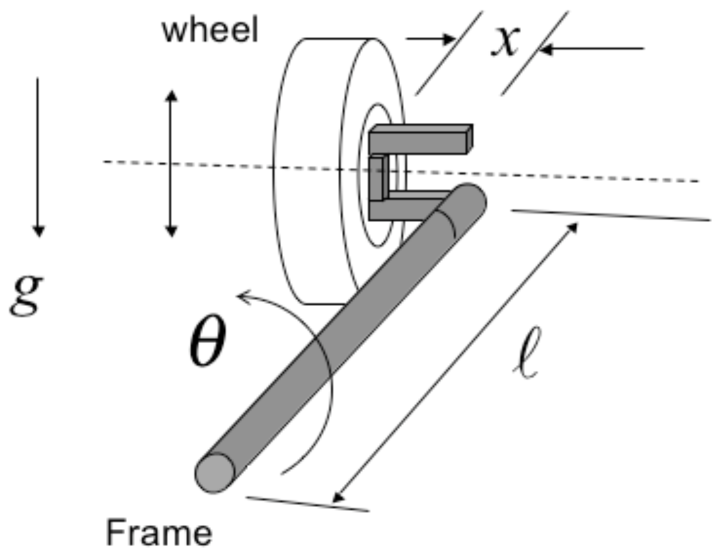


Figure P1.27

Solution: (a) Ignoring the moment of inertial of the rod, and computing the moment of inertia of the wheel as mx^2, the frequency of the shaft mass system is

$$\omega_n = \sqrt{\frac{k}{mx^2}} = \sqrt{\frac{2000 \text{ N}\cdot\text{m}}{38\cdot\text{kg } (0.26 \text{ m})^2}} = \underline{27.9 \text{ rad/s}}$$

(b) The same calculation with 45 kg will *reduce* the frequency to

$$\omega_n = \sqrt{\frac{k}{mx^2}} = \sqrt{\frac{2000 \text{ N}\cdot\text{m}}{45\cdot\text{kg } (0.26 \text{ m})^2}} = \underline{25.6 \text{ rad/s}}$$

This corresponds to about an 8% change in unsprung frequency and could influence wheel hop etc. You could also ask students to examine the effect of increasing x, as commonly done on some trucks to extend the wheels out for appearance sake.

1.28 A machine oscillates in simple harmonic motion and appears to be well modeled by an undamped single-degree-of-freedom oscillation. Its acceleration is measured to have an amplitude of 10,000 mm/s^2 at 8 Hz. What is the machine's maximum displacement?

Solution:

Given: a_{max} = 10,000 mm/s^2 @ 8 Hz

The equations of motion for position and acceleration are:

$$x = A\sin(\omega_n t + \phi) \qquad (1.3)$$

$$\ddot{x} = -A\omega_n^2 \sin(\omega_n t + \phi) \qquad (1.5)$$

The amplitude of acceleration is $A\omega_n^2 = 10{,}000$ mm/s^2 and ω_n = 2πf = 2π(8) = 16π rad/s, from equation (1.12).

The machine's displacement is $A = \dfrac{10{,}000}{\omega_n^2} = \dfrac{10{,}000}{(16\pi)^2}$

A = 3.96 mm

1.29 A simple undamped spring-mass system is set into motion from rest by giving it an initial velocity of 100 mm/s. It oscillates with a maximum amplitude of 10 mm. What is its natural frequency?

Solution:

Given: x_0 = 0, v_0 = 100 mm/s, A = 10 mm

From equation (1.9), $A = \dfrac{v_0}{\omega_n}$ or $\omega_n = \dfrac{v_0}{A} = \dfrac{100}{10}$, so that: **$\omega_n$= 10 rad/s**

1.30 An automobile exhibits a vertical oscillating displacement of maximum amplitude 5 cm and a measured maximum acceleration of 2000 cm/s^2. Assuming that the automobile can be modeled as a single-degree-of-freedom system in the vertical direction, calculate the natural frequency of the automobile.

Solution:

Given: A = 5 cm. From equation (1.15)

$$|\ddot{x}| = A\omega_n^2 = 2000 \text{ cm/s}$$

Solving for ω_n yields:

$$\omega_n = \sqrt{\frac{2000}{A}} = \sqrt{\frac{2000}{5}}$$

$$\omega_n = \mathbf{20\,rad/s}$$

Problems Section 1.3 (Numbers 1.31 through 1.46)

1.31 Solve $\ddot{x} + 4\dot{x} + x = 0$ for $x_0 = 1$ mm, $v_0 = 0$ mm/s. Sketch your results and determine which root dominates.

Solution:

Given $\ddot{x} + 4\dot{x} + x = 0$ where $x_0 = 1\,\text{mm}, v_0 = 0$

Let
$$x = ae^{rt} \Rightarrow \dot{x} = are^{rt} \Rightarrow \ddot{x} = ar^2e^{rt}$$
Substitute these into the equation of motion to get:
$$ar^2e^{rt} + 4are^{rt} + ae^{rt} = 0$$
$$\Rightarrow r^2 + 4r + 1 = 0 \Rightarrow r_{1,2} = -2 \pm \sqrt{3}$$
So
$$x = a_1e^{(-2+\sqrt{3})t} + a_2e^{(-2-\sqrt{3})t}$$
$$\dot{x} = \left(-2+\sqrt{3}\right)a_1e^{(-2+\sqrt{3})t} + \left(-2-\sqrt{3}\right)a_2e^{(-2-\sqrt{3})t}$$
Applying initial conditions yields,
$$x_0 = a_1 + a_2 \quad \Rightarrow \quad x_0 - a_2 = a_1 \qquad (1)$$
$$v_0 = \left(-2+\sqrt{3}\right)a_1 + \left(-2-\sqrt{3}\right)a_2 \qquad (2)$$
Substitute equation (1) into (2)
$$v_0 = \left(-2+\sqrt{3}\right)(x_0 - a_2) + \left(-2-\sqrt{3}\right)a_2$$
$$v_0 = \left(-2+\sqrt{3}\right)x_0 - 2\sqrt{3}\,a_2$$
Solve for a_2
$$a_2 = \frac{-v_0 + \left(-2+\sqrt{3}\right)x_0}{2\sqrt{3}}$$
Substituting the value of a_2 into equation (1), and solving for a_1 yields,
$$a_1 = \frac{v_0 + \left(2+\sqrt{3}\right)x_0}{2\sqrt{3}}$$
$$\therefore\ x(t) = \frac{v_0 + \left(2+\sqrt{3}\right)x_0}{2\sqrt{3}}e^{(-2+\sqrt{3})t} + \frac{-v_0 + \left(-2+\sqrt{3}\right)x_0}{2\sqrt{3}}e^{(-2-\sqrt{3})t}$$
The response is dominated by the root: $\underline{-2+\sqrt{3}}$ as the other root dies off very fast.

1.32 Solve $\ddot{x} + 2\dot{x} + 2x = 0$ for $x_0 = 0$ mm, $v_0 = 1$ mm/s and sketch the response. You may wish to sketch $x(t) = e^{-t}$ and $x(t) = -e^{-t}$ first.

Solution:

Given $\ddot{x} + 2\dot{x} + x = 0$ where $x_0 = 0$, $v_0 = 1$ mm/s

Let: $x = ae^{rt} \Rightarrow \dot{x} = are^{rt} \Rightarrow \ddot{x} = ar^2e^{rt}$

Substitute into the equation of motion to get

$$ar^2e^{rt} + 2are^{rt} + ae^{rt} = 0 \Rightarrow r^2 + 2r + 1 = 0 \Rightarrow r_{1,2} = -1 \pm i$$

So

$$x = c_1e^{(-1+i)\,t} + \; c_2e^{(-1-i)\,t} \Rightarrow \dot{x} = (-1+i)c_1e^{(-1+i)\,t} + (-1-i)c_2e^{(-1-i)\,t}$$

Initial conditions:

$$x_0 = x(0) = c_1 + c_2 = 0 \quad \Rightarrow \quad c_2 = -c_1 \qquad (1)$$
$$v_0 = \dot{x}(0) = (-1+i)c_1 + (-1-i)c_2 = 1 \qquad (2)$$

Substituting equation (1) into (2)

$$v_0 = (-1+i)c_1 - (-1-i)c_1 = 1$$
$$c_1 = -\frac{1}{2}i, \;\; c_2 = \frac{1}{2}i$$
$$x(t) = -\frac{1}{2}ie^{(-1+i)\,t} + \frac{1}{2}ie^{(-1-i)\,t} = -\frac{1}{2}ie^{-t}\left(e^{it} - e^{-it}\right)$$

Applying Euler's formula

$$x(t) = -\frac{1}{2}ie^{-t}\left(\cos t + i\sin t - (\cos t - i\sin t)\right)$$
$$\underline{x(t) = e^{-t}\sin t}$$

Alternately use equations (1.36) and (1.38). The plot is similar to figure 1.11.

1.33 Derive the form of λ_1 and λ_2 given by equation (1.31) from equation (1.28) and the definition of the damping ratio.

Solution:

Equation (1.28): $\lambda_{1,2} = -\dfrac{c}{2m} \pm \dfrac{1}{2m}\sqrt{c^2 - 4km}$

Rewrite, $\lambda_{1,2} = -\left(\dfrac{c}{2\sqrt{m}\sqrt{m}}\right)\left(\dfrac{\sqrt{k}}{\sqrt{k}}\right) \pm \dfrac{1}{2\sqrt{m}\sqrt{m}}\left(\dfrac{\sqrt{k}}{\sqrt{k}}\right)\left(\dfrac{c}{c}\right)\sqrt{c^2 - \left(2\sqrt{km}\right)^2\left(\dfrac{c}{c}\right)^2}$

Rearrange, $\lambda_{1,2} = -\left(\dfrac{c}{2\sqrt{km}}\right)\left(\dfrac{\sqrt{k}}{\sqrt{m}}\right) \pm \dfrac{c}{2\sqrt{km}}\left(\dfrac{\sqrt{k}}{\sqrt{m}}\right)\left(\dfrac{1}{c}\right)\sqrt{c^2\left[1 - \left(\dfrac{2\sqrt{km}}{c}\right)^2\right]}$

Substitute:

$$\omega_n = \sqrt{\frac{k}{m}} \text{ and } \zeta = \frac{c}{2\sqrt{km}} \Rightarrow \lambda_{1,2} = -\zeta\omega_n \pm \zeta\omega_n\left(\frac{1}{c}\right)c\sqrt{1 - \left(\frac{1}{\zeta^2}\right)}$$

$$\Rightarrow \lambda_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2\left[1 - \left(\frac{1}{\zeta^2}\right)\right]}$$

$$\Rightarrow \lambda_{1,2} = \underline{-\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}}$$

1.34 Use the Euler formulas to derive equation (1.36) from equation (1.35) and to determine the relationships listed in Window 1.4.

Solution:

Equation (1.35): $x(t) = e^{-\zeta\omega_n t}\left(a_1 e\right)^{j\omega_n\sqrt{1-\zeta^2}t} - a_2 e^{-j\omega_n\sqrt{1-\zeta^2}t}$

From Euler,

$$x(t) = e^{-\zeta\omega_n t}(a_1\cos\left(\omega_n\sqrt{1-\zeta^2}t\right) + a_1 j\sin\left(\omega_n\sqrt{1-\zeta^2}t\right)$$

$$+ a_2\cos\left(\omega_n\sqrt{1-\zeta^2}t\right) - a_2 j\sin\left(\omega_n\sqrt{1-\zeta^2}t\right))$$

$$= e^{-\zeta\omega_n t}\left(a_1 + a_2\right)\cos\omega_d t + j\left(a_1 - a_2\right)\sin\omega_d t$$

Let: $A_1 = \left(a_1 + a_2\right)$, $A_2 = \left(a_1 - a_2\right)$, then this last expression becomes

$$x(t) = e^{-\zeta\omega_n t}A_1\cos\omega_d t + A_2\sin\omega_d t$$

Next use the trig identity:

$$A = \sqrt{A_1 + A_2},\quad \phi = \tan^{-1}\frac{A_1}{A_2}$$

to get: $\underline{x(t) = e^{-\zeta\omega_n t}A\sin(\omega_d t + \phi)}$

1.35 Using equation (1.35) as the form of the solution of the underdamped system, calculate the values for the constants a_1 and a_2 in terms of the initial conditions x_0 and v_0.

Solution:

Equation (1.35):

$$x(t) = e^{-\zeta\omega_n t}\left(a_1 e^{j\omega_n\sqrt{1-\zeta^2}\,t} + a_2 e^{-j\omega_n\sqrt{1-\zeta^2}\,t}\right)$$

$$\dot{x}(t) = (-\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2})a_1 e^{\left(-\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}\right)t} + (-\zeta\omega_n - j\omega_n\sqrt{1-\zeta^2})a_2\, e^{\left(-\zeta\omega_n - j\omega_n\sqrt{1-\zeta^2}\right)t}$$

Initial conditions

$$x_0 = x(0) = a_1 + a_2 \Rightarrow a_1 = x_0 - a_2 \qquad (1)$$

$$v_0 = \dot{x}(0) = (-\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2})a_1 + (-\zeta\omega_n - j\omega_n\sqrt{1-\zeta^2})a_2 \qquad (2)$$

Substitute equation (1) into equation (2) and solve for a_2

$$v_0 = \left(-\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}\right)(x_0 - a_2) + \left(-\zeta\omega_n - j\omega_n\sqrt{1-\zeta^2}\right)a_2$$

$$v_0 = \left(-\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}\right)x_0 - 2j\omega_n\sqrt{1-\zeta^2}\,a_2$$

Solve for a_2

$$a_2 = \frac{-v_0 - \zeta\omega_n x_0 + j\omega_n\sqrt{1-\zeta^2}\,x_0}{2j\omega_n\sqrt{1-\zeta^2}}$$

Substitute the value for a_2 into equation (1), and solve for a_1

$$a_1 = \frac{v_0 + \zeta\omega_n x_0 + j\omega_n\sqrt{1-\zeta^2}\,x_0}{2j\omega_n\sqrt{1-\zeta^2}}$$

1.36 Calculate the constants A and ϕ in terms of the initial conditions and thus verify equation (1.38) for the underdamped case.

Solution:

From Equation (1.36),

$$x(t) = Ae^{-\zeta\omega_n t}\sin\left(\omega_d t + \phi\right)$$

Applying initial conditions ($t = 0$) yields,

$$x_0 = A\sin\phi \qquad (1)$$

$$v_0 = \dot{x}_0 = -\zeta\omega_n A\sin\phi + \omega_d A\cos\phi \qquad (2)$$

Next solve these two simultaneous equations for the two unknowns A and ϕ.

From (1),

$$A = \frac{x_0}{\sin\phi} \qquad (3)$$

Substituting (3) into (1) yields

$$v_0 = -\zeta\omega_n x_0 + \frac{\omega_d x_0}{\tan\phi} \quad \Rightarrow \quad \tan\phi = \frac{x_0\omega_d}{v_0 + \zeta\omega_n x_0} \ .$$

Hence,

$$\phi = \tan^{-1}\left[\frac{x_0\omega_d}{v_0 + \zeta\omega_n x_0}\right] \qquad (4)$$

From (3), $$\sin\phi = \frac{x_0}{A} \qquad (5)$$

and From (4), $$\cos\phi = \frac{v_0 + \zeta\omega_n x_0}{\left(x_0\omega_d\right)^2 + \left(v_0 + \zeta\omega_n x_0\right)^2} \qquad (6)$$

Substituting (5) and (6) into (2) yields,

$$A = \sqrt{\frac{(v_0 + \zeta\omega_n x_0)^2 + (x_0\omega_d)^2}{\omega_d^2}}$$

which are the same as equation (1.38)

1.37 Calculate the constants a_1 and a_2 in terms of the initial conditions and thus verify equations (1.42) and (1.43) for the overdamped case.

Solution: From Equation (1.41)

$$x(t) = e^{-\zeta\omega_n t}\left(a_1 e^{\omega_n\sqrt{\zeta^2-1}\,t} + a_2 e^{-\omega_n\sqrt{\zeta^2-1}\,t}\right)$$

taking the time derivative yields:

$$\dot{x}(t) = (-\zeta\omega_n + \omega_n\sqrt{\zeta^2-1})a_1 e^{\left(-\zeta\omega_n+\omega_n\sqrt{\zeta^2-1}\right)t} + (-\zeta\omega_n - \omega_n\sqrt{\zeta^2-1})a_2\, e^{\left(-\zeta\omega_n-\omega_n\sqrt{\zeta^2-1}\right)t}$$

Applying initial conditions yields,

$$x_0 = x(0) = a_1 + a_2 \quad \Rightarrow \quad x_0 - a_2 = a_1 \qquad (1)$$

$$v_0 = \dot{x}(0) = \left(-\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}\right)a_1 + \left(-\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}\right)a_2 \qquad (2)$$

Substitute equation (1) into equation (2) and solve for a_2

$$v_0 = \left(-\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}\right)(x_0 - a_2) + \left(-\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}\right)a_2$$

$$v_0 = \left(-\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}\right)x_0 - 2\omega_n\sqrt{\zeta^2-1}\,a_2$$

Solve for a_2

$$a_2 = \frac{-v_0 - \zeta\omega_n x_0 + \omega_n\sqrt{\zeta^2-1}\;x_0}{2\omega_n\sqrt{\zeta^2-1}}$$

Substitute the value for a_2 into equation (1), and solve for a_1

$$a_1 = \frac{v_0 + \zeta\omega_n x_0 + \omega_n\sqrt{\zeta^2-1}\;x_0}{2\omega_n\sqrt{\zeta^2-1}}$$

1.38 Calculate the constants a_1 and a_2 in terms of the initial conditions and thus verify equation (1.46) for the critically damped case.

Solution:

From Equation (1.45),

$$x(t) = (a_1 + a_2 t)e^{-\omega_n t}$$

$$\Rightarrow \dot{x}_0 = -\omega_n a_1 e^{-\omega_n t} - \omega_n a_2 t e^{-\omega_n t} + a_2 e^{-\omega_n t}$$

Applying the initial conditions yields:

$$x_0 = a_1 \qquad (1)$$

and

$$v_0 = \dot{x}(0) = a_2 - \omega_n a_1 \qquad (2)$$

solving these two simultaneous equations for the two unknowns a_1 and a_2.

Substituting (1) into (2) yields,

$$a_1 = x_0$$

$$a_2 = v_0 + \omega_n x_0$$

which are the same as equation (1.46).

1.39 Using the definition of the damping ratio and the undamped natural frequency, derive equitation (1.48) from (1.47).

Solution:

$\omega_n = \sqrt{\dfrac{k}{m}}$ thus, $\dfrac{k}{m} = \omega_n^{\;2}$

$\zeta = \dfrac{c}{2\sqrt{km}}$ thus, $\dfrac{c}{m} = \dfrac{2\zeta\sqrt{km}}{m} = 2\zeta\omega_n$

Therefore, $\ddot{x} + \dfrac{c}{m}\dot{x} + \dfrac{k}{m}x = 0$

becomes,

$\ddot{x}(t) + 2\zeta\omega_n\dot{x}(t) + \omega_n^{\;2}x(t) = 0$

1.40 For a damped system, *m*, *c*, and *k* are known to be m = 1 kg, c = 2 kg/s, k = 10 N/m. Calculate the value of ζ and ω_n. Is the system overdamped, underdamped, or critically damped?

Solution:

Given: m = 1 kg, c = 2 kg/s, k = 10 N/m

Natural frequency: $\omega_n = \sqrt{\dfrac{k}{m}} = \sqrt{\dfrac{10}{1}} = 3.16\ rad/s$

Damping ratio: $\zeta = \dfrac{c}{2\omega_n m} = \dfrac{2}{2(3.16)(1)} = 0.316$

Damped natural frequency: $\omega_d = \sqrt{10}\sqrt{1-\left(\dfrac{1}{\sqrt{10}}\right)^2} = 3.0$ rad/s

Since 0 < ζ < 1, the system is **underdamped**.

1.41 Plot $x(t)$ for a damped system of natural frequency ω_n = 2 rad/s and initial conditions x_0 = 1 mm, v_0 = 1 mm, for the following values of the damping ratio: ζ = 0.01, ζ = 0.2, ζ = 0.1, ζ = 0.4, and ζ = 0.8.

Solution:

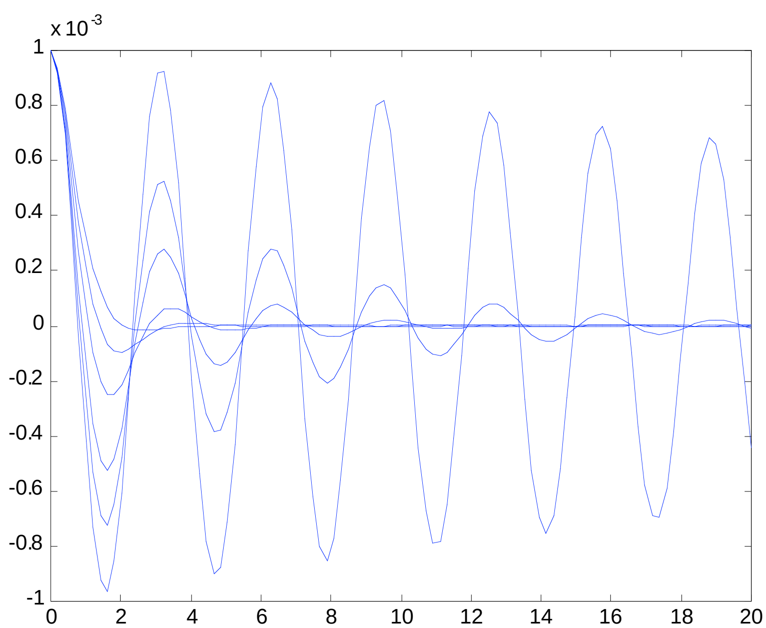
Given: ω_n = 2 rad/s, x_0 = 1 mm, v_0 = 1 mm, ζ_i = [0.01; 0.2; 0.1; 0.4; 0.8]
Underdamped cases:

$$\therefore \omega_{di} = \omega_n \sqrt{1 - \zeta_i^2}$$

From equation 1.38,

$$A_i = \sqrt{\frac{(v_0 + \zeta_i \omega_n x_0)^2 + (x_0 \omega_{di})^2}{\omega_{di}^2}} \qquad \phi_i = \tan^{-1} \frac{x_0 \omega_{di}}{v_0 + \zeta_i \omega_n x_0}$$

The response is plotted for each value of the damping ratio in the following using Matlab:



1.42 Plot the response $x(t)$ of an underdamped system with ω_n = 2 rad/s, ζ = 0.1, and v_0 = 0 for the following initial displacements: x_0 = 10 mm and x_0 = 100 mm.

Solution:

Given: ω_n = 2 rad/s, ζ = 0.1, v_0 = 0, x_0 = 10 mm and x_0 = 100 mm.

Underdamped case:

$$\therefore \omega_d = \omega_n\sqrt{1-\zeta_i^{\,2}} = 2\sqrt{1-0.1^2} = 1.99 \text{ rad/s}$$

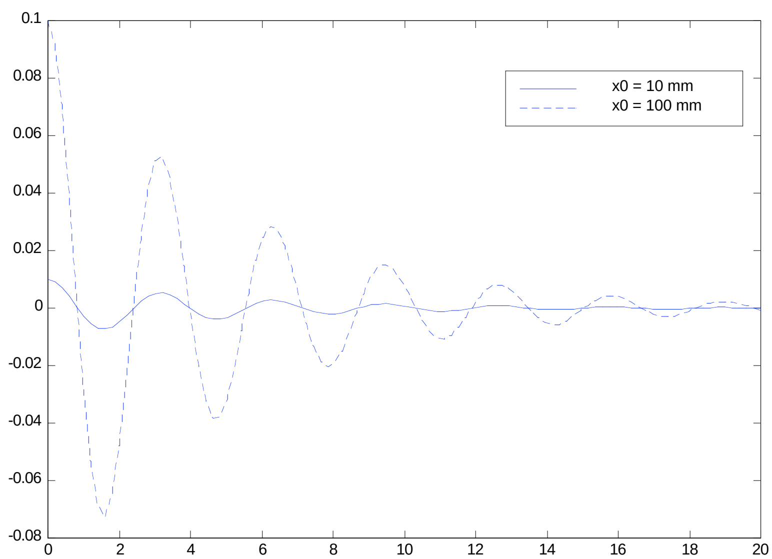
$$A = \sqrt{\frac{\left(v_0 + \zeta\omega_n x_0\right)^2 + \left(x_0\omega_d\right)^2}{\omega_d^{\,2}}} = 1.01\,x_0$$

$$\phi = \tan^{-1}\frac{x_0\omega_d}{v_0 + \zeta\omega_n x_0} = 1.47 \text{ rad}$$

where

$$x(t) = Ae^{-\zeta\omega_n t}\sin\left(\omega_d t + \phi\right)$$

The following is a plot from Matlab.



1.43 Solve $\ddot{x} - \dot{x} + x = 0$ with $x_0 = 1$ and $v_0 = 0$ for *x(t)* and sketch the response.

Solution: This is a problem with negative damping which can be used to tie into Section 1.8 on stability, or can be used to practice the method for deriving the solution using the method suggested following equation (1.13) and eluded to at the start of the section on damping. To this end let $x(t) = Ae^{\lambda t}$ the equation of motion to get:

$$(\lambda^2 - \lambda + 1)e^{\lambda t} = 0$$

This yields the characteristic equation:

$$\lambda^2 - \lambda + 1 = 0 \Rightarrow \lambda = \frac{1}{2} \pm \frac{\sqrt{3}}{2} j, \text{ where } j = \sqrt{-1}$$

There are thus two solutions as expected and these combine to form

$$x(t) = e^{0.5t}(Ae^{\frac{\sqrt{3}}{2}jt} + Be^{-\frac{\sqrt{3}}{2}jt})$$

Using the Euler relationship for the term in parenthesis as given in Window 1.4, this can be written as

$$x(t) = e^{0.5t}(A_1 \cos\frac{\sqrt{3}}{2}t + A_2 \sin\frac{\sqrt{3}}{2}t)$$

Next apply the initial conditions to determine the two constants of integration:

$$x(0) = 1 = A_1(1) + A_2(0) \Rightarrow A_1 = 1$$

Differentiate the solution to get the velocity and then apply the initial velocity condition to get

$$\dot{x}(t) =$$

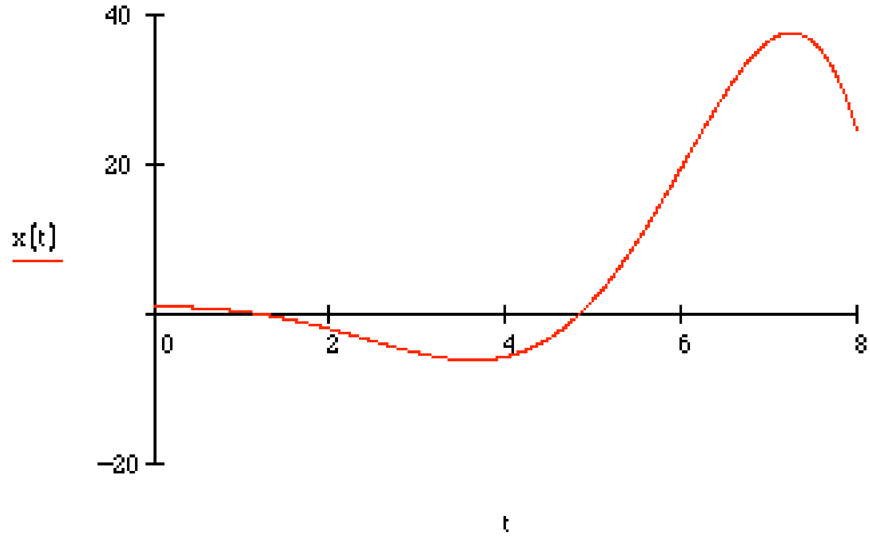
$$\frac{1}{2}e^0(A_1 \cos\frac{\sqrt{3}}{2}0 + A_2 \sin\frac{\sqrt{3}}{2}0) + e^0\frac{\sqrt{3}}{2}(-A_1 \sin\frac{\sqrt{3}}{2}0 + A_2 \cos\frac{\sqrt{3}}{2}0) = 0$$

$$\Rightarrow A_1 + \sqrt{3}(A_2) = 0 \Rightarrow A_2 = -\frac{1}{\sqrt{3}},$$

$$\Rightarrow \underline{x(t) = e^{0.5t}(\cos\frac{\sqrt{3}}{2}t - \frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t)}$$

This function oscillates with increasing amplitude as shown in the following plot which shows the increasing amplitude. This type of response is referred to as a flutter instability. This plot is from Mathcad.

$$x(t) := e^{0.5 \cdot t} \left[\cos\left(\frac{\sqrt{3}}{2} \cdot t\right) - \frac{1}{\sqrt{3}} \cdot \sin\left(\frac{\sqrt{3}}{2} \cdot t\right) \right]$$



1.44 A spring-mass-damper system has mass of 100 kg, stiffness of 3000 N/m and damping coefficient of 300 kg/s. Calculate the undamped natural frequency, the damping ratio and the damped natural frequency. Does the solution oscillate?

Solution: Working straight from the definitions:

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{3000 \text{ N/m}}{100 \text{ kg}}} = 5.477 \text{ rad/s}$$

$$\zeta = \frac{\mathrm{c}}{\mathrm{c}_{\mathrm{cr}}} = \frac{300}{2\sqrt{km}} = \frac{300}{2\sqrt{(3000)(100)}} = 0.274$$

Since ζ is less then 1, the solution is underdamped and will oscillate. The damped natural frequency is $\omega_d = \omega_n \sqrt{1-\zeta^2} = 5.27$ rad/s.

1.45 A sketch of a valve and rocker arm system for an internal combustion engine is give in Figure P1.45. Model the system as a pendulum attached to a spring and a mass and assume the oil provides viscous damping in the range of $\zeta = 0.01$. Determine the equations of motion and calculate an expression for the natural frequency and the damped natural frequency. Here J is the rotational inertia of the rocker arm about its pivot point, k is the stiffness of the valve spring and m is the mass of the valve and stem. Ignore the mass of the spring.

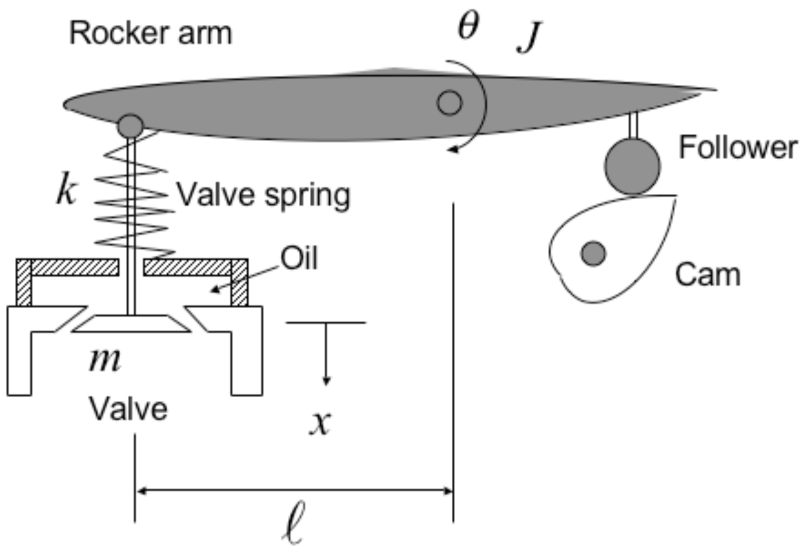
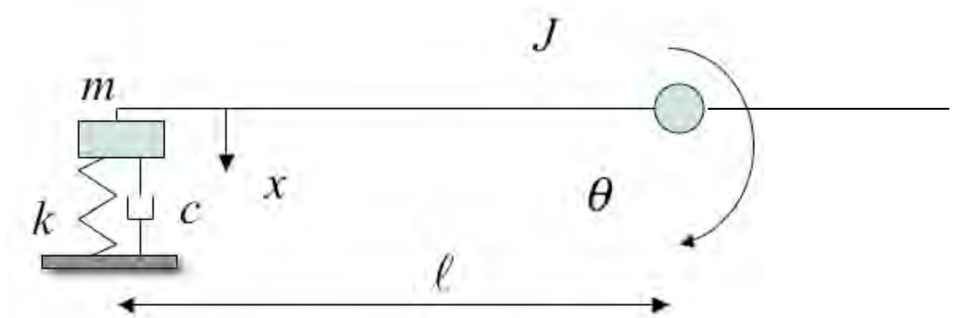


Figure P1.45

Solution: The model is of the form given in the figure. You may wish to give this figure as a hint as it may not be obvious to all students.



Taking moments about the pivot point yields:

$$(J + m\ell^2)\ddot{\theta}(t) = -kx\ell - c\dot{x}\ell = -k\ell^2\theta - c\ell^2\dot{\theta}$$

$$\Rightarrow \underline{(J + m\ell^2)\ddot{\theta}(t) + c\ell^2\dot{\theta} + k\ell^2\theta = 0}$$

Next divide by the leading coefficient to get;

$$\ddot{\theta}(t) + \left(\frac{c\ell^2}{J + m\ell^2}\right)\dot{\theta}(t) + \frac{k\ell^2}{J + m\ell^2}\theta(t) = 0$$

From the coefficient of q, the undamped natural frequency is

$$\underline{\omega_n = \sqrt{\frac{k\ell^2}{J + m\ell^2}} \text{ rad/s}}$$

From equation (1.37), the damped natural frequency becomes

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = \underline{0.99995\sqrt{\frac{k\ell^2}{J + m\ell^2}}} \sim \sqrt{\frac{k\ell^2}{J + m\ell^2}}$$

This is effectively the same as the undamped frequency for any reasonable accuracy. However, it is important to point out that the resulting response will still decay, even though the frequency of oscillation is unchanged. So even though the numerical value seems to have a negligible effect on the frequency of oscillation, the small value of damping still makes a substantial difference in the response.

1.46 A spring-mass-damper system has mass of 150 kg, stiffness of 1500 N/m and damping coefficient of 200 kg/s. Calculate the undamped natural frequency, the damping ratio and the damped natural frequency. Is the system overdamped, underdamped or critically damped? Does the solution oscillate?

Solution: Working straight from the definitions:

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1500 \text{ N/m}}{150 \text{ kg}}} = 3.162 \text{ rad/s}$$

$$\zeta = \frac{c}{c_{cr}} = \frac{200}{2\sqrt{km}} = \frac{200}{2\sqrt{(1500)(150)}} = 0.211$$

This last expression follows from the equation following equation (1.29). Since ζ is less then 1, the solution is underdamped and will oscillate. The damped natural frequency is $\omega_d = \omega_n\sqrt{1-\zeta^2} = 3.091 \text{ rad/s}$, which follows from equation (1.37).

1.47* The system of Problem 1.44 is given a zero initial velocity and an initial displacement of 0.1 m. Calculate the form of the response and plot it for as long as it takes to die out.

Solution: Working from equation (1.38) and using Mathcad the solution is:

1.42

$m := 100 \quad k := 3000 \quad c := 300$

$$\omega n := \sqrt{\frac{k}{m}} \qquad \zeta := \frac{c}{2\cdot\sqrt{m\cdot k}} \qquad \omega d := \omega n\cdot\sqrt{1-\zeta^2}$$

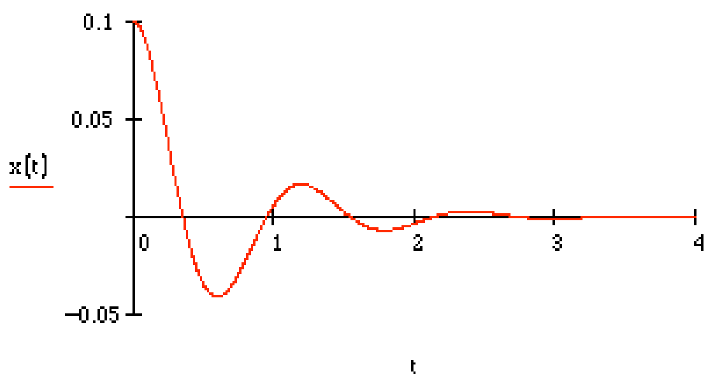
$v0 := 0 \quad x0 := 0.1$

$$A := \frac{1}{\omega d}\cdot\sqrt{(v0+\zeta\cdot x0\cdot\omega n)^2+(x0\cdot\omega d)^2} \qquad \phi := \mathrm{atan}\left(\frac{\omega d\cdot x0}{v0+\zeta\cdot\omega n\cdot x0}\right)$$

$A = 0.104 \qquad \phi = 1.293 \qquad \omega n = 5.477 \qquad \omega d = 5.268$

$\zeta = 0.274$

$$x(t) := A\cdot\sin(\omega d\cdot t+\phi)\cdot e^{-\zeta\cdot\omega n\cdot t}$$



1.48* The system of Problem 1.46 is given an initial velocity of 10 mm/s and an initial displacement of -5 mm. Calculate the form of the response and plot it for as long as it takes to die out. How long does it take to die out?

Solution: Working from equation (1.38), the form of the response is programmed in Mathcad and is given by:

1.43

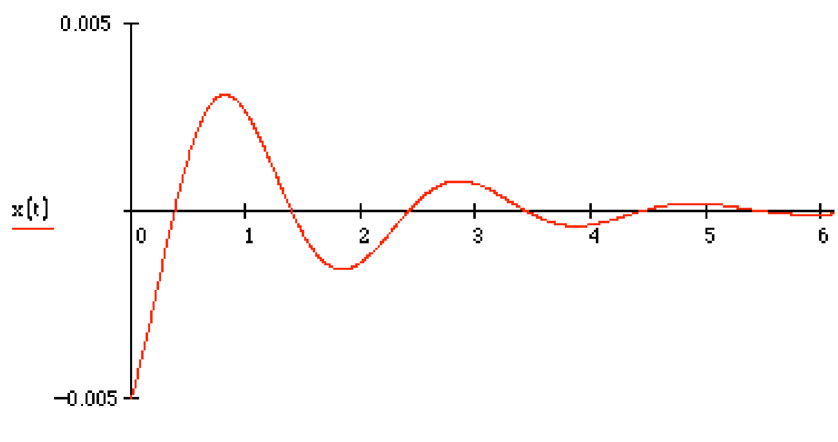
$k := 1500 \quad m := 150 \quad x0 := -0.005 \quad v0 := 0.010 \quad c := 200$

all units in m, kg, s

$$\omega n := \sqrt{\frac{k}{m}} \qquad \zeta := \frac{c}{2\cdot\sqrt{m\cdot k}} \qquad \zeta = 0.211 \qquad \omega d := \omega n\cdot\sqrt{1-\zeta^2} \qquad \omega d = 3.091$$

$$A := \frac{1}{\omega d}\cdot\sqrt{(v0+\zeta\cdot x0\cdot\omega n)^2 + (x0\cdot\omega d)^2} \qquad A = 5.445\cdot 10^{-3} \qquad T := \frac{2\cdot\pi}{\omega d}$$

$$x(t) := A\cdot\sin(\omega d\cdot t+\phi)\cdot e^{-\zeta\cdot\omega n\cdot t}$$ in meters $$\phi := \operatorname{atan}\left(\frac{\omega d\cdot x0}{v0+\zeta\cdot\omega n\cdot x0}\right)$$



It appears to take a little over 6 to 8 seconds to die out. This can also be plotted in Matlab, Mathematica or by using the toolbox.

1.49* Choose the damping coefficient of a spring-mass-damper system with mass of 150 kg and stiffness of 2000 N/m such that it's response will die out after about 2 s, given a zero initial position and an initial velocity of 10 mm/s.

Solution: Working in Mathcad, the response is plotted and the value of c is changed until the desired decay rate is meet:

$c := 800 \qquad k := 2000 \qquad v0 := 0.010 \qquad x0 := 0$

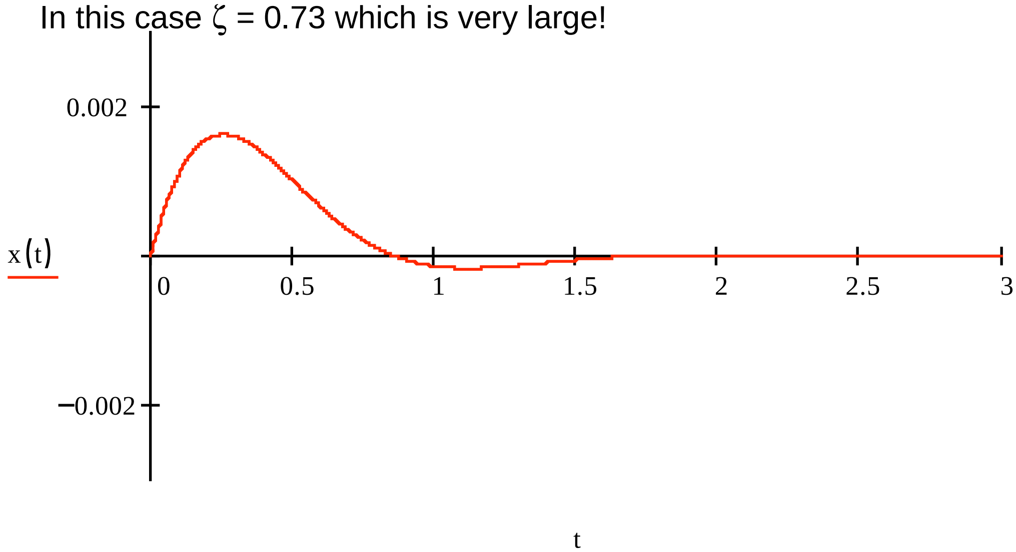
$m := 150$

$$\zeta := \frac{c}{2\cdot\sqrt{m\cdot k}} \qquad \omega n := \sqrt{\frac{k}{m}} \qquad \omega d := \omega n\cdot\sqrt{1-\zeta^2}$$

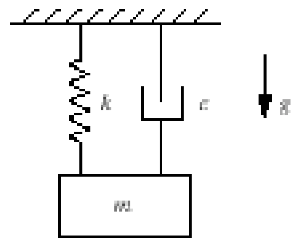
$$x(t) := A\cdot \sin(\omega n\cdot t + \phi)\cdot e^{-\zeta\cdot\omega n\cdot t}$$

$$\phi := \operatorname{atan}\left(\frac{\omega d\cdot x0}{v0 + \zeta\cdot\omega n\cdot x0}\right)$$

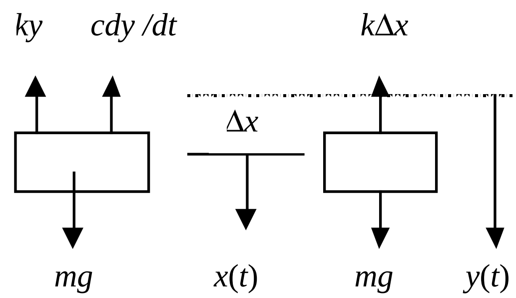
In this case ζ = 0.73 which is very large!



1.50 Derive the equation of motion of the system in Figure P1.50 and discuss the effect of gravity on the natural frequency and the damping ratio.



Solution: This requires two free body diagrams. One for the dynamic case and one to show static equilibrium.



(a) (b)

From the free-body diagram of static equilibrium (b) we have that $mg = k\Delta x$, where Δx represents the static deflection. From the free-body diagram of the dynamic case given in (a) the equation of motion is:

$$m\ddot{y}(t) + c\dot{y}(t) + ky(t) - mg = 0$$

From the diagram, $y(t) = x(t) + \Delta x$. Since Δx is a constant, differentiating and substitution into the equation of motion yields:

$$\dot{y}(t) = \dot{x}(t) \text{ and } \ddot{y}(t) = \ddot{x}(t) \Rightarrow$$

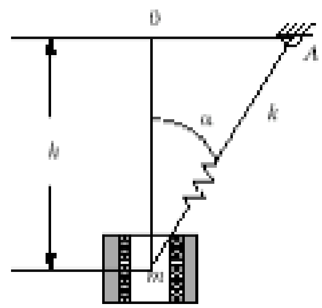
$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) + \underbrace{(k\Delta x - mg)}_{=0} = 0$$

where the last term is zero from the relation resulting from static equilibrium. Dividing by the mass yields the standard form

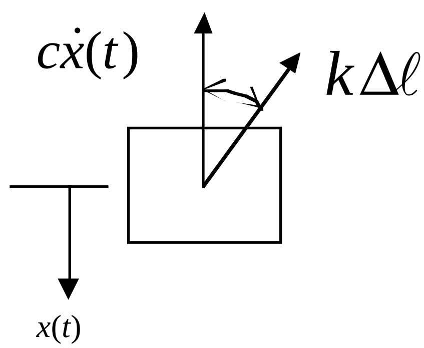
$$\ddot{x}(t) + 2\zeta\omega_n\dot{x}(t) + \omega_n^2 x(t) = 0$$

It is clear that gravity has no effect on the damping ratio ζ or the natural frequency ω_n. Not that the damping force is not present in the static case because the velocity is zero.

1.51 Derive the equation of motion of the system in Figure P1.46 and discuss the effect of gravity on the natural frequency and the damping ratio. You may have to make some approximations of the cosine. Assume the bearings provide a viscous damping force only in the vertical direction. (From the A. Diaz-Jimenez, *South African Mechanical Engineer*, Vol. 26, pp. 65-69, 1976)



Solution: First consider a free-body diagram of the system:



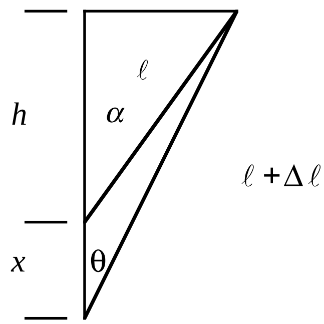
Let α be the angel between the damping and stiffness force. The equation of motion becomes

$$m\ddot{x}(t) = -c\dot{x}(t) - k(\Delta\ell + \delta_s)\cos\alpha$$

From static equilibrium, the free-body diagram (above with $c = 0$ and stiffness force $k\delta_s$) yields: $\sum F_x = 0 = mg - k\delta_s\cos\alpha$. Thus the equation of motion becomes

$$m\ddot{x} + c\dot{x} + k\Delta\ell\cos\alpha = 0 \qquad (1)$$

Next consider the geometry of the dynamic state:



From the definition of cosine applied to the two different triangles:

$$\cos\alpha = \frac{h}{\ell} \quad \text{and} \quad \cos\theta = \frac{h+x}{\ell+\Delta\ell}$$

Next assume small deflections so that the angles are nearly the same cos α = cos θ, so that

$$\frac{h}{\ell} \approx \frac{h+x}{\ell+\Delta\ell} \Rightarrow \Delta\ell \approx x\frac{\ell}{h} \Rightarrow \Delta\ell \approx \frac{x}{\cos\alpha}$$

For small motion, then this last expression can be substituted into the equation of motion (1) above to yield:

$$m\ddot{x} + c\dot{x} + kx = 0, \ \alpha \text{ and } x \text{ small}$$

Thus the frequency and damping ratio have the standard values and are not effected by gravity. If the small angle assumption is not made, the frequency can be approximated as

$$\omega_n = \sqrt{\frac{k}{m}\cos^2\alpha + \frac{g}{h}\sin^2\alpha}, \quad \zeta = \frac{c}{2m\omega_n}$$

as detailed in the reference above. For a small angle these reduce to the normal values of

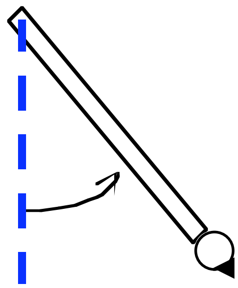
$$\omega_n = \sqrt{\frac{k}{m}}, \quad \text{and} \quad \zeta = \frac{c}{2m\omega_n}$$

as derived here.

Problems and Solutions Section 1.4 (problems 1.52 through 1.65)

1.52 Calculate the frequency of the compound pendulum of Figure 1.20(b) if a mass m_T is added to the tip, by using the energy method.

Solution Using the notation and coordinates of Figure 1.20 and adding a tip mass the diagram becomes:



If the mass of the pendulum bar is m, and it is lumped at the center of mass the energies become:

Potential Energy:

$$U = \frac{1}{2}(\ell - \ell\cos\theta)mg + (\ell - \ell\cos\theta)m_t g$$
$$= \frac{\ell}{2}(1 - \cos\theta)(mg + 2m_t g)$$

Kinetic Energy:

$$T = \frac{1}{2}J\dot{\theta}^2 + \frac{1}{2}J_t\dot{\theta}^2 = \frac{1}{2}\frac{m\ell^2}{3}\dot{\theta}^2 + \frac{1}{2}m_t\ell^2\dot{\theta}^2$$
$$= (\frac{1}{6}m + \frac{1}{2}m_t)\ell^2\dot{\theta}^2$$

Conservation of energy (Equation 1.52) requires $T + U$ = constant:

$$\frac{\ell}{2}(1 - \cos\theta)(mg + 2m_t g) + (\frac{1}{6}m + \frac{1}{2}m_t)\ell^2\dot{\theta}^2 = C$$

Differentiating with respect to time yields:

$$\frac{\ell}{2}(\sin\theta)(mg + 2m_t g)\dot{\theta} + (\frac{1}{3}m + m_t)\ell^2\dot{\theta}\ddot{\theta} = 0$$
$$\Rightarrow (\frac{1}{3}m + m_t)\ell\ddot{\theta} + \frac{1}{2}(mg + 2m_t g)\sin\theta = 0$$

Rearranging and approximating using the small angle formula $\sin\theta \sim \theta$, yields:

$$\ddot{\theta}(t) + \left(\frac{\frac{m}{2} + m_t}{\frac{1}{3}m + m_t} \frac{g}{\ell} \right) \theta(t) = 0 \Rightarrow \underline{\omega_n = \sqrt{\frac{3m + 6m_t}{2m + 6m_t}} \sqrt{\frac{g}{\ell}} \text{ rad/s}}$$

Note that this solution makes sense because if $m_t = 0$ it reduces to the frequency of the pendulum equation for a bar, and if $m = 0$ it reduces to the frequency of a massless pendulum with only a tip mass.

1.53 Calculate the total energy in a damped system with frequency 2 rad/s and damping ratio $\zeta = 0.01$ with mass 10 kg for the case $x_0 = 0.1$ and $v_0 = 0$. Plot the total energy versus time.

Solution: Given: $\omega_n = 2$ rad/s, $\zeta = 0.01$, $m = 10$ kg, $x_0 = 0.1$ mm, $v_0 = 0$.

Calculate the stiffness and damped natural frequency:

$$k = m\omega_n^2 = 10(2)^2 = 40 \text{ N/m}$$
$$\omega_d = \omega_n \sqrt{1-\zeta^2} = 2\sqrt{1-0.01^2} = 2 \text{ rad/s}$$

The total energy of the damped system is

$$E(t) = \frac{1}{2}m\dot{x}^2(t) + \frac{1}{2}kx(t)$$

where
$$x(t) = Ae^{-0.02t}\sin(2t+\phi)$$
$$\dot{x}(t) = -0.02Ae^{-0.02t}\sin(2t+\phi) + 2Ae^{-0.02t}\cos(2t+\phi)$$

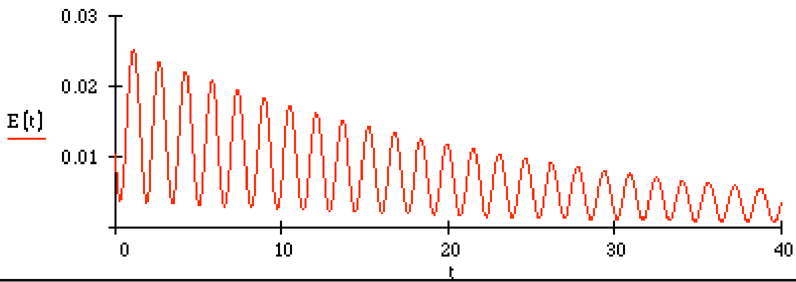
Applying the initial conditions to evaluate the constants of integration yields:

$$x(0) = 0.1 = A\sin\phi$$
$$\dot{x}(0) = 0 = -0.02A\sin\phi + 2A\cos\phi$$
$$\Rightarrow \phi = 1.56 \text{ rad/s}, \quad A = 0.1 \text{ m}$$

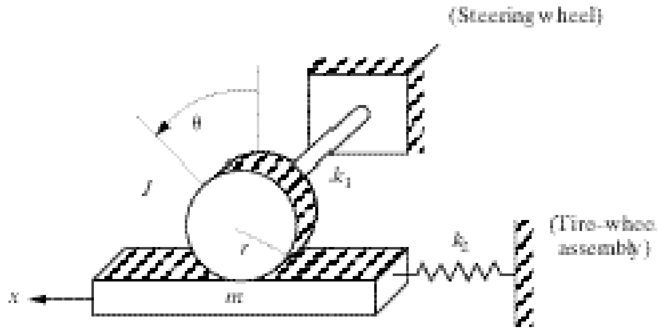
Substitution of these values into $E(t)$ yields:

1.48

$$E(t) := \left[\frac{e^{-0.04\cdot t}}{2}\cdot\left[\left(\left(0.1\cdot\sin(2\cdot t+1.56)\right)\right) + .2\cdot\cos(2\cdot t+1.56)\right]^2\right] + 0.5\cdot e^{-0.04\cdot t}\cdot\left(0.1\cdot\sin(2\cdot t+1.56)\right)^2$$



1.54 Use the energy method to calculate the equation of motion and natural frequency of an airplane's steering mechanism for the nose wheel of its landing gear. The mechanism is modeled as the single-degree-of-freedom system illustrated in Figure P1.54.



The steering wheel and tire assembly are modeled as being fixed at ground for this calculation. The steering rod gear system is modeled as a linear spring and mass system (m, k_2) oscillating in the x direction. The shaft-gear mechanism is modeled as the disk of inertia J and torsional stiffness k_2. The gear J turns through the angle θ such that the disk does not slip on the mass. Obtain an equation in the linear motion x.

Solution: From kinematics: $x = r\theta, \Rightarrow \dot{x} = r\dot{\theta}$

Kinetic energy: $$T = \frac{1}{2}J\dot{\theta}^2 + \frac{1}{2}m\dot{x}^2$$

Potential energy: $$U = \frac{1}{2}k_2 x^2 + \frac{1}{2}k_1\theta^2$$

Substitute $\theta = \frac{x}{r}: T + U = \frac{1}{2}\frac{J}{r^2}\dot{x}^2 + \frac{1}{2}m\dot{x}^2 + \frac{1}{2}k_2 x^2 + \frac{1}{2}\frac{k_1}{r^2}x^2$

Derivative: $$\frac{d(T+U)}{dt} = 0$$

$$\frac{J}{r^2}\ddot{x}\dot{x} + m\ddot{x}\dot{x} + k_2 x\dot{x} + \frac{k_1}{r^2}x\dot{x} = 0$$

$$\left[\left(\frac{J}{r^2} + m\right)\ddot{x} + \left(k_2 + \frac{k_1}{r^2}\right)x\right]\dot{x} = 0$$

Equation of motion: $$\left(\frac{J}{r^2} + m\right)\ddot{x} + \left(k_2 + \frac{k_1}{r^2}\right)x = 0$$

Natural frequency: $$\omega_n = \sqrt{\frac{k_2 + \frac{k_1}{r^2}}{\frac{J}{r^2} + m}} = \sqrt{\frac{k_1 + r^2 k_2}{J + mr^2}}$$

1.55 A control pedal of an aircraft can be modeled as the single-degree-of-freedom system of Figure P1.55. Consider the lever as a massless shaft and the pedal as a lumped mass at the end of the shaft. Use the energy method to determine the equation of motion in θ and calculate the natural frequency of the system. Assume the spring to be unstretched at θ = 0.

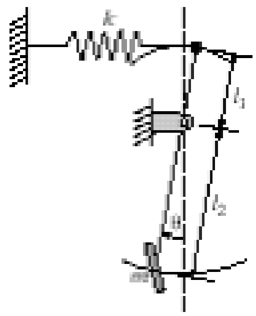


Figure P1.55

Solution: In the figure let the mass at θ = 0 be the lowest point for potential energy. Then**,** the height of the mass *m* is $(1-\cos\theta)\ell_2$.

Kinematic relation: $x = \ell_1\theta$

Kinetic Energy: $T = \frac{1}{2}m\dot{x}^2 = \frac{1}{2}m\ell_2^2\dot{\theta}^2$

Potential Energy: $U = \frac{1}{2}k(\ell_1\theta)^2 + mg\ell_2(1-\cos\theta)$

Taking the derivative of the total energy yields:

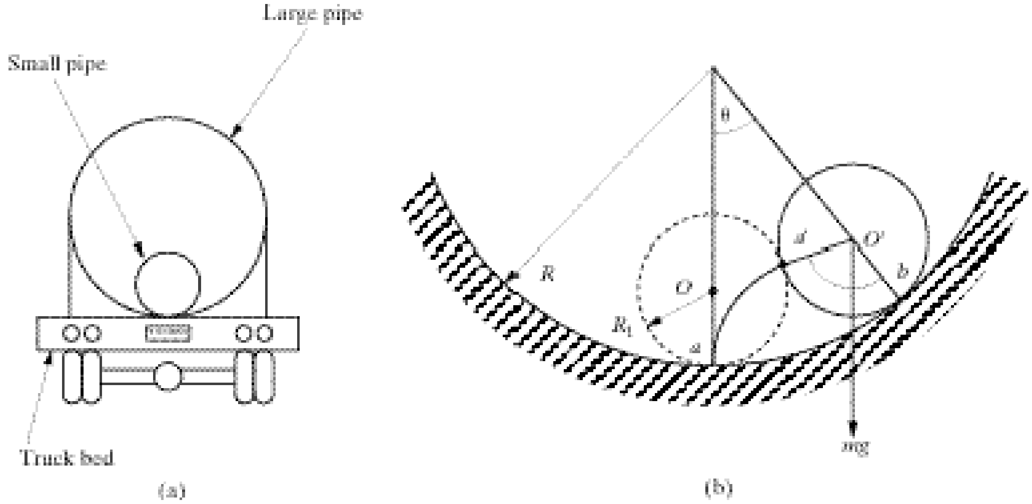
$$\frac{d}{dt}(T+U) = m\ell_2^2\dot{\theta}\ddot{\theta} + k(\ell_1^2\theta)\dot{\theta} + mg\ell_2(\sin\theta)\dot{\theta} = 0$$

Rearranging, dividing by $d\theta/dt$ and approximating sinθ with θ yields:

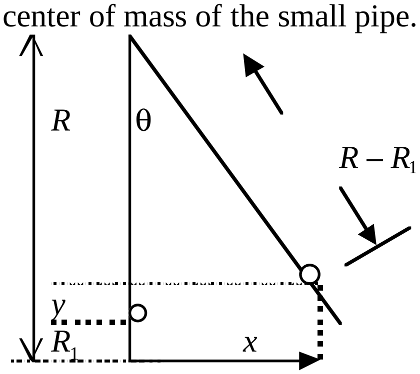
$$m\ell_2^2\ddot{\theta} + (k\ell_1^2 + mg\ell_2)\theta = 0$$

$$\Rightarrow \omega_n = \sqrt{\frac{k\ell_1^2 + mg\ell_2}{m\ell_2^2}}$$

1.56 To save space, two large pipes are shipped one stacked inside the other as indicated in Figure P1.56. Calculate the natural frequency of vibration of the smaller pipe (of radius R_1) rolling back and forth inside the larger pipe (of radius R). Use the energy method and assume that the inside pipe rolls without slipping and has a mass m.



Solution: Let θ be the angle that the line between the centers of the large pipe and the small pipe make with the vertical and let α be the angle that the small pipe rotates through. Let R be the radius of the large pipe and R_1 the radius of the smaller pipe. Then the kinetic energy of the system is the translational plus rotational of the small pipe. The potential energy is that of the rise in height of the center of mass of the small pipe.



From the drawing:

$$y + (R - R_1)\cos\theta + R_1 = R$$

$$\Rightarrow y = (R - R_1)(1 - \cos\theta)$$

$$\Rightarrow \dot{y} = (R - R_1)\sin(\theta)\dot{\theta}$$

Likewise examination of the value of x yields:

$$x = (R - R_1)\sin\theta$$

$$\Rightarrow \dot{x} = (R - R_1)\cos\theta\dot{\theta}$$

Let β denote the angle of rotation that the small pipe experiences as viewed in the inertial frame of reference (taken to be the truck bed in this case). Then the total kinetic energy can be written as:

$$T = T_{trans} + T_{rot} = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 + \frac{1}{2}I_0\dot{\beta}^2$$

$$= \frac{1}{2}m(R - R_1)^2(\sin^2\theta + \cos^2\theta)\dot{\theta}^2 + \frac{1}{2}I_0\dot{\beta}^2$$

$$\Rightarrow T = \frac{1}{2}m(R - R_1)^2\dot{\theta}^2 + \frac{1}{2}I_0\dot{\beta}^2$$

The total potential energy becomes just:

$V = mgy = mg(R - R_1)(1 - \cos\theta)$

Now it remains to evaluate the angel β. Let α be the angle that the small pipe rotates in the frame of the big pipe as it rolls (say) up the inside of the larger pipe. Then

$\beta = \theta - \alpha$

were α is the angle "rolled" out as the small pipe rolls from *a* to *b* in figure P1.56. The rolling with out slipping condition implies that arc length *a'b* must equal arc length *ab*. Using the arc length relation this yields that $R\theta = R_1\alpha$. Substitution of the expression $\beta = \theta - \alpha$ yields:

$$R\theta = R_1(\theta - \beta) = R_1\theta - R_1\beta \Rightarrow (R - R_1)\theta = -R_1\beta$$

$$\Rightarrow \beta = \frac{1}{R_1}(R_1 - R)\theta \text{ and } \dot{\beta} = \frac{1}{R_1}(R_1 - R)\dot{\theta}$$

which is the relationship between angular motion of the small pipe relative to the ground (β) and the position of the pipe (θ). Substitution of this last expression into the kinetic energy term yields:

$$T = \frac{1}{2}m(R - R_1)^2\dot{\theta}^2 + \frac{1}{2}I_0\left(\frac{1}{R_1}(R_1 - R)\dot{\theta}\right)^2$$

$$\Rightarrow T = m(R - R_1)^2\dot{\theta}^2$$

Taking the derivative of $T + V$ yields

$$\frac{d}{d\theta}(T + V) = 2m(R - R_1)^2\dot{\theta}\ddot{\theta} + mg(R - R_1)\sin\theta\dot{\theta} = 0$$

$$\Rightarrow 2m(R - R_1)^2\ddot{\theta} + mg(R - R_1)\sin\theta = 0$$

Using the small angle approximation for sine this becomes

$$2m(R - R_1)^2\ddot{\theta} + mg(R - R_1)\theta = 0$$

$$\Rightarrow \ddot{\theta} + \frac{g}{2(R - R_1)}\theta = 0$$

$$\Rightarrow \omega_n = \sqrt{\frac{g}{2(R - R_1)}}$$

1.57 Consider the example of a simple pendulum given in Example 1.4.2. The pendulum motion is observed to decay with a damping ratio of ζ = 0.001. Determine a damping coefficient and add a viscous damping term to the pendulum equation.

Solution: From example 1.4.2, the equation of motion for a simple pendulum is

$$\ddot{\theta} + \frac{g}{\ell}\theta = 0$$

So $\omega_n = \sqrt{\frac{g}{\ell}}$. With viscous damping the equation of motion in normalized form becomes:

$$\ddot{\theta} + 2\zeta\omega_n\dot{\theta} + \omega_n^2\theta = 0 \text{ or with } \zeta \text{ as given :}$$

$$\Rightarrow \ddot{\theta} + 2(.001)\omega_n\dot{\theta} + \omega_n^2\theta = 0$$

The coefficient of the velocity term is

$$\frac{c}{J} = \frac{c}{m\ell^2} = (.002)\sqrt{\frac{g}{\ell}}$$

$$c = (0.002)m\sqrt{g\ell^3}$$

1.58 Determine a damping coefficient for the disk-rod system of Example 1.4.3. Assuming that the damping is due to the material properties of the rod, determine c for the rod if it is observed to have a damping ratio of $\zeta = 0.01$.

Solution: The equation of motion for a disc/rod in torsional vibration is

$$J\ddot{\theta} + k\theta = 0$$

or $$\ddot{\theta} + \omega_n^2\theta = 0 \quad \text{where } \omega_n = \sqrt{\frac{k}{J}}$$

Add viscous damping:

$$\ddot{\theta} + 2\zeta\omega_n\dot{\theta} + \omega_n^2\theta = 0$$

$$\ddot{\theta} + 2(.01)\sqrt{\frac{k}{J}}\dot{\theta} + \omega_n^2\theta = 0$$

From the velocity term, the damping coefficient must be

$$\frac{c}{J} = (0.02)\sqrt{\frac{k}{J}}$$

$$\Rightarrow c = 0.02\sqrt{kJ}$$

1.59 The rod and disk of Window 1.1 are in torsional vibration. Calculate the damped natural frequency if $J = 1000$ m^2 · kg, $c = 20$ N· m· s/rad, and $k = 400$ N·m/rad.

Solution: From Problem 1.57, the equation of motion is

$$J\ddot{\theta} + c\dot{\theta} + k\theta = 0$$

The damped natural frequency is

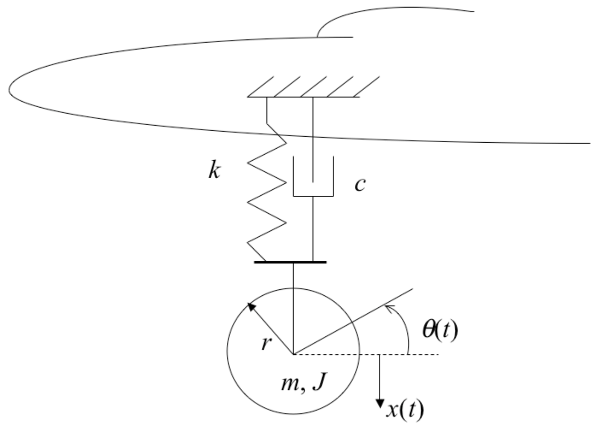
$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

where $$\omega_n = \sqrt{\frac{k}{J}} = \sqrt{\frac{400}{1000}} = 0.632 \text{ rad/s}$$

and $$\zeta = \frac{c}{2\sqrt{kJ}} = \frac{20}{2\sqrt{400 \times 1000}} = 0.0158$$

Thus the damped natural frequency is $\omega_d = 0.632$ rad/s

1.60 Consider the system of P1.60, which represents a simple model of an aircraft landing system. Assume, $x = r\theta$. What is the damped natural frequency?



Solution: From Example 1.4.1, the undamped equation of motion is

$$\left(m + \frac{J}{r^2}\right)\ddot{x} + kx = 0$$

From examining the equation of motion the natural frequency is:

$$\omega_n = \sqrt{\frac{k}{m_{eq}}} = \sqrt{\frac{k}{m + \frac{J}{r^2}}}$$

An add hoc way do to this is to add the damping force to get the damped equation of motion:

$$\left(m + \frac{J}{r^2}\right)\ddot{x} + c\dot{x} + kx = 0$$

The value of ζ is determined by examining the velocity term:

$$\frac{c}{m + \frac{J}{r^2}} = 2\zeta\omega_n \Rightarrow \zeta = \frac{c}{m + \frac{J}{r^2}} \frac{1}{2\sqrt{\frac{k}{m + \frac{J}{r^2}}}}$$

$$\Rightarrow \zeta = \frac{c}{2\sqrt{k\left(m + \frac{J}{r^2}\right)}}$$

Thus the damped natural frequency is

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = \sqrt{\frac{k}{m+\frac{J}{r^2}}}\sqrt{1-\left(\frac{c}{2\sqrt{k\left(m+\frac{J}{r^2}\right)}}\right)^2}$$

$$\Rightarrow \underline{\omega_d = \sqrt{\frac{k}{m+\frac{J}{r^2}} - \frac{c^2}{4\left(m+\frac{J}{r^2}\right)^2}}} = \frac{r}{2(mr^2+J)}\sqrt{4(kmr^2+kJ)-c^2r^2}$$

1.61 Consider Problem 1.60 with k = 400,000 N·m, m = 1500 kg, J = 100 m^2·kg, r = 25 cm, and c = 8000 N·m·s. Calculate the damping ratio and the damped natural frequency. How much effect does the rotational inertia have on the undamped natural frequency?

Solution: From problem 1.60:

$$\zeta = \frac{c}{2\sqrt{k\left(m+\frac{J}{r^2}\right)}} \text{ and } \omega_d = \sqrt{\frac{k}{m+\frac{J}{r^2}} - \frac{c^2}{4\left(m+\frac{J}{r^2}\right)^2}}$$

Given:

$$k = 4\times 10^5 \text{ Nm/rad}$$
$$m = 1.5\times 10^3 \text{ kg}$$
$$J = 100 \text{ m}^2\text{kg}$$
$$r = 0.25 \text{ m and}$$
$$c = 8\times 10^3 \text{ N}\cdot\text{m}\cdot\text{s/rad}$$

Inserting the given values yields

$$\underline{\zeta = 0.114} \text{ and } \underline{\omega_d = 11.16 \text{ rad/s}}$$

For the undamped natural frequency, $\omega_n = \sqrt{\dfrac{k}{m+J/r^2}}$

With the rotational inertia, ω_n = 36.886 rad/s

Without rotational inertia, ω_n = 51.64 rad/s

The effect of the rotational inertia is that it lowers the natural frequency by almost 33%.

1.62 Use Lagrange's formulation to calculate the equation of motion and the natural frequency of the system of Figure P1.62. Model each of the brackets as a spring of stiffness *k*, and assume the inertia of the pulleys is negligible.

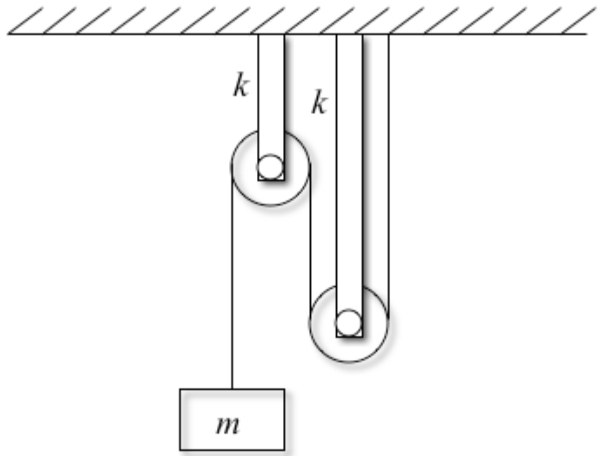


Figure P1.62

Solution: Let *x* denote the distance mass *m* moves, then each spring will deflects a distance *x*/4. Thus the potential energy of the springs is

$$U = 2\times\frac{1}{2}k\left(\frac{x}{4}\right)^2 = \frac{k}{16}x^2$$

The kinetic energy of the mass is

$$T = \frac{1}{2}m\dot{x}^2$$

Using the Lagrange formulation in the form of Equation (1.64):

$$\frac{d}{dt}\left(\frac{\partial}{\partial\dot{x}}\left(\frac{1}{2}m\dot{x}^2\right)\right) + \frac{\partial}{\partial x}\left(\frac{kx^2}{16}\right) = 0 \Rightarrow \frac{d}{dt}\left(m\dot{x}\right) + \frac{k}{8}x = 0$$

$$\Rightarrow \underline{m\ddot{x} + \frac{k}{8}x = 0} \Rightarrow \underline{\omega_n = \frac{1}{2}\sqrt{\frac{k}{2m}}} \text{ rad/s}$$

1.63 Use Lagrange's formulation to calculate the equation of motion and the natural frequency of the system of Figure P1.63. This figure represents a simplified model of a jet engine mounted to a wing through a mechanism which acts as a spring of stiffness *k* and mass m_s. Assume the engine has inertial *J* and mass *m* and that the rotation of the engine is related to the vertical displacement of the engine, *x*(*t*) by the "radius" r_0 (i.e. $x = r_0\theta$).

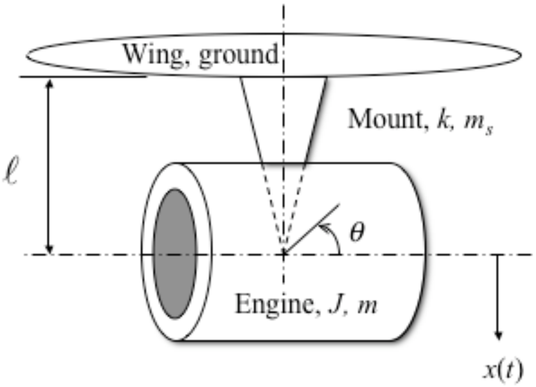


Figure P1.63

Solution: This combines Examples 1.4.1 and 1.4.4. The kinetic energy is

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}J\dot{\theta}^2 + T_{\text{spring}} = \frac{1}{2}\left(m + \frac{J}{r_0^2}\right)\dot{x}^2 + T_{\text{spring}}$$

The kinetic energy in the spring (see example 1.4.4) is

$$T_{\text{spring}} = \frac{1}{2}\frac{m_s}{3}\dot{x}^2$$

Thus the total kinetic energy is

$$T = \frac{1}{2}\left(m + \frac{J}{r_0^2} + \frac{m_s}{3}\right)\dot{x}^2$$

The potential energy is just

$$U = \frac{1}{2}kx^2$$

Using the Lagrange formulation of Equation (1.64) the equation of motion results from:

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}}\left(\frac{1}{2}\left(m + \frac{J}{r_0^2} + \frac{m_s}{3}\right)\dot{x}^2\right)\right) + \frac{\partial}{\partial x}\left(\frac{1}{2}kx^2\right) = 0$$

$$\Rightarrow \left(m + \frac{J}{r_0^2} + \frac{m_s}{3}\right)\ddot{x} + kx = 0$$

$$\Rightarrow \omega_n = \sqrt{\frac{k}{\left(m + \frac{J}{r_0^2} + \frac{m_s}{3}\right)}} \text{ rad/s}$$

1.64 Lagrange's formulation can also be used for non-conservative systems by adding the applied non-conservative term to the right side of equation (1.64) to get

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i} + \frac{\partial R_i}{\partial \dot{q}_i} = 0$$

Here R_i is the *Rayleigh dissipation function* defined in the case of a viscous damper attached to ground by

$$R_i = \frac{1}{2} c\dot{q}_i^2$$

Use this extended Lagrange formulation to derive the equation of motion of the damped automobile suspension of Figure P1.64

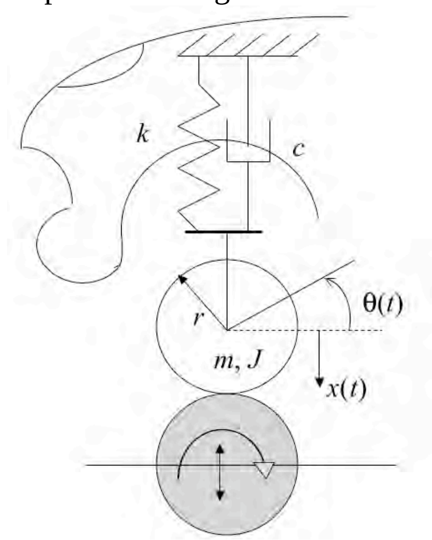


Figure P1.64

Solution: The kinetic energy is (see Example 1.4.1):

$$T = \frac{1}{2}(m + \frac{J}{r^2})\dot{x}^2$$

The potential energy is:

$$U = \frac{1}{2} kx^2$$

The Rayleigh dissipation function is

$$R = \frac{1}{2} c\dot{x}^2$$

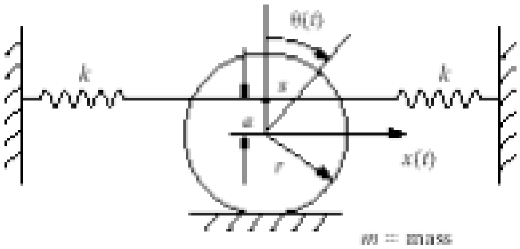
The Lagrange formulation with damping becomes

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i} + \frac{\partial R_i}{\partial \dot{q}_i} = 0$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}}\left(\frac{1}{2}(m + \frac{J}{r^2})\dot{x}^2\right)\right) + \frac{\partial}{\partial x}\left(\frac{1}{2}kx^2\right) + \frac{\partial}{\partial \dot{x}}\left(\frac{1}{2}c\dot{x}^2\right) = 0$$

$$\Rightarrow \underline{(m + \frac{J}{r^2})\ddot{x} + c\dot{x} + kx = 0}$$

1.65 Consider the disk of Figure P1.65 connected to two springs. Use the energy method to calculate the system's natural frequency of oscillation for small angles θ(t).



Solution:

Known: $x = r\theta, \dot{x} = r\dot{\theta}$ and $J_o = \frac{1}{2}mr^2$

Kinetic energy:

$$T_{rot} = \frac{1}{2}J_o \quad \dot{\theta}^2 = \frac{1}{2}\left(\frac{mr^2}{2}\right) \quad \dot{\theta}^2 = \frac{1}{4}mr^2\dot{\theta}^2$$

$$T_{trans} = \frac{1}{2}m\dot{x}^2 = \frac{1}{2}mr^2\dot{\theta}^2$$

$$T = T_{rot} + T_{trans} = \frac{1}{4}mr^2\dot{\theta}^2 + \frac{1}{2}mr^2\dot{\theta}^2 = \frac{3}{4}mr^2\dot{\theta}^2$$

Potential energy: $U = 2\left(\frac{1}{2}k\left[(a+r)\theta\right]^2\right) = k(a+r)^2\theta^2$

Conservation of energy:

$$T + U = \text{ Constant}$$

$$\frac{d}{dt}(T + U) = 0$$

$$\frac{d}{dt}\left(\frac{3}{4}mr^2\dot{\theta}^2 + k(a+r)^2\theta^2\right) = 0$$

$$\frac{3}{4}mr^2\left(2\dot{\theta}\ddot{\theta}\right) + k(a+r)^2\left(2\dot{\theta}\theta\right) = 0$$

$$\frac{3}{2}mr^2\ddot{\theta} + 2k(a+r)^2\theta = 0$$

Natural frequency:

$$\omega_n = \sqrt{\frac{k_{eff}}{m_{eff}}} = \sqrt{\frac{2k(a+r)^2}{\frac{3}{2}mr^2}}$$

$$\omega_n = 2\frac{a+r}{r}\sqrt{\frac{k}{3m}} \text{ rad/s}$$

Problems and Solutions Section 1.5 (1.66 through 1.74)

1.66 A helicopter landing gear consists of a metal framework rather than the coil spring based suspension system used in a fixed-wing aircraft. The vibration of the frame in the vertical direction can be modeled by a spring made of a slender bar as illustrated in Figure 1.21, where the helicopter is modeled as ground. Here l = 0.4 m, $E = 20 \times 10^{10}$ N/m^2, and m = 100 kg. Calculate the cross-sectional area that should be used if the natural frequency is to be f_n = 500 Hz.

Solution: From Figure 1.21

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{EA}{lm}} \tag{1}$$

and

$$\omega_n = 500 \text{ Hz}\left(\frac{2\pi \text{ rad}}{1 \text{ cycle}}\right) = 3142 \text{ rad/s}$$

Solving (1) for A yields:

$$A = \frac{\omega_n^{\,2} lm}{E} = \frac{(3142)^2(.4)(100)}{20 \times 10^{10}}$$

$$A = 0.0019 \text{ m}^2 = 19\text{cm}^2$$

1.67 The frequency of oscillation of a person on a diving board can be modeled as the transverse vibration of a beam as indicated in Figure 1.24. Let *m* be the mass of the diver (m = 100 kg) and l = 1 m. If the diver wishes to oscillate at 3 Hz, what value of *EI* should the diving board material have?

Solution: From Figure 1.24,

$$\omega_n^2 = \frac{3EI}{ml^3}$$

and

$$\omega_n = 3Hz\left(\frac{2\pi \text{ rad}}{1 \text{ cycle}}\right) = 6\pi \text{ rad/s}$$

Solving for *EI*

$$EI = \frac{\omega_n^2 ml^3}{3} = \frac{(6\pi)^2(100)(1)^3}{3} = \underline{11843.5 \text{ Nm}^2}$$

1.68 Consider the spring system of Figure 1.29. Let k_1 = k_5 = k_2 =100 N/m, k_3 = 50 N/m, and k_4 = 1 N/m. What is the equivalent stiffness?

Solution: Given: $k_1 = k_2 = k_5 = 100$ N/m, k_3 = 50 N/m, $k_4 = 1$ N/m

From Example 1.5.4

$$k_{eq} = k_1 + k_2 + k_5 + \frac{k_3 k_4}{k_3 + k_4}$$

$$\Rightarrow \underline{k_{eq} = 300.98 \text{ N/m}}$$

1.69 Springs are available in stiffness values of 10, 100, and 1000 N/m. Design a spring system using these values only, so that a 100-kg mass is connected to ground with frequency of about 1.5 rad/s.

Solution: Using the definition of natural frequency:

$$\omega_n = \sqrt{\frac{k_{eq}}{m}}$$

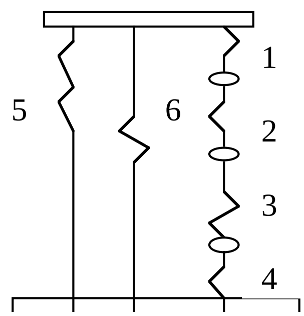
With m = 100 kg and ω_n = 1.5 rad/s the equivalent stiffness must be:

$$k_{eq} = m\omega_n^2 = (100)(1.5)^2 = 225 \text{ N/m}$$

There are many configurations of the springs given and no clear way to determine one configuration over another. Here is one possible solution. Choose two 100 N/m springs in parallel to get 200 N/m, then use four 100 N/m springs in series to get an equivalent spring of 25 N/m to put in parallel with the other 3 springs since

$$k_{eq} = \frac{1}{\frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} + \frac{1}{k_4}} = \frac{1}{4/100} = 25$$

Thus using six 100 N/m springs in the following arrangement will produce an equivalent stiffness of 225 N/m



1.70 Calculate the natural frequency of the system in Figure 1.29(a) if $k_1 = k_2 = 0$. Choose m and nonzero values of k_3, k_4, and k_5 so that the natural frequency is 100 Hz.

Solution: Given: $k_1 = k_2 = 0$ and $\omega_n = 2\pi(100) = 628.3$ rad/s

From Figure 1.29, the natural frequency is

$$\omega_n = \sqrt{\frac{k_5k_3 + k_5k_4 + k_3k_4}{m(k_3 + k_4)}} \quad \text{and} \quad k_{eq} = \left(k_5 + \frac{k_3k_4}{k_3 + k_4}\right)$$

Equating the given value of frequency to the analytical value yields:

$$\omega_n^2 = (628.3)^2 = \frac{k_5k_3 + k_5k_4 + k_3k_4}{m(k_3 + k_4)}$$

Any values of k_3, k_4, k_5, and m that satisfy the above equation will do. Again, the answer is *not unique.* One solution is

$$k_3 = 1 \text{ N/m}, k_4 = 1 \text{ N/m}, k_5 = 50{,}000 \text{ N/m, and } m = 0.127 \text{ kg}$$

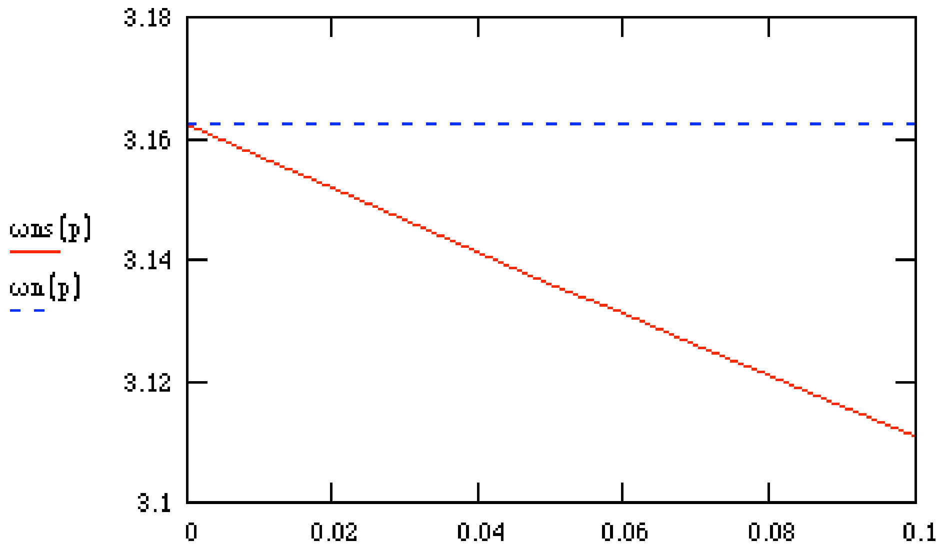
1.71* Example 1.4.4 examines the effect of the mass of a spring on the natural frequency of a simple spring-mass system. Use the relationship derived there and plot the natural frequency versus the percent that the spring mass is of the oscillating mass. Use your plot to comment on circumstances when it is no longer reasonable to neglect the mass of the spring.

Solution: The solution here depends on the value of the stiffness and mass ratio and hence the frequency. Almost any logical discussion is acceptable as long as the solution indicates that for smaller values of m_s, the approximation produces a reasonable frequency. Here is one possible answer. For

$k := 1000 \qquad m := 100$

$p := 0, 0.01 .. 0.1$

$$\omega ns(p) := \sqrt{\frac{k}{m + \frac{p \cdot m}{3}}} \qquad \omega n(p) := \sqrt{\frac{k}{m}}$$



From this plot, for these values of *m* and *k* it ***looks like a 10 % spring mass causes less then a 1 % error in the frequency.***

1.72 Calculate the natural frequency and damping ratio for the system in Figure P1.72 given the values m = 10 kg, c = 100 kg/s, k_1 = 4000 N/m, k_2 = 200 N/m and k_3 = 1000 N/m. Assume that no friction acts on the rollers. Is the system overdamped, critically damped or underdamped?

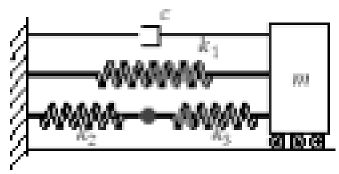


Figure P1.72

Solution: Following the procedure of Example 1.5.4, the equivalent spring constant is:

$$k_{eq} = k_1 + \frac{k_2 k_3}{k_2 + k_3} = 4000 + \frac{(200)(1000)}{1200} = 4167 \text{ N/m}$$

Then using the standard formulas for frequency and damping ratio:

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{4167}{10}} = 20.412 \text{ rad/s}$$

$$\zeta = \frac{c}{2m\omega_n} = \frac{100}{2(10)(20.412)} = 0.245$$

Thus the system is underdamped.

1.73 Repeat Problem 1.72 for the system of Figure P1.73.

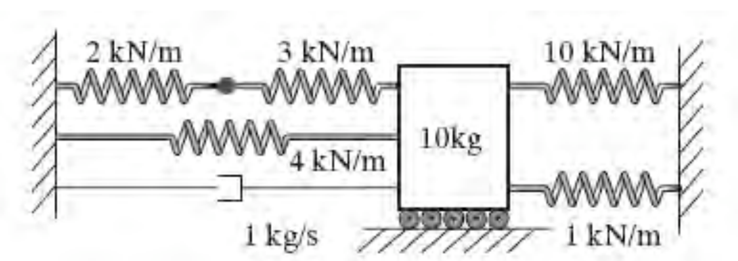


Figure P1.73

Solution: Again using the procedure of Example 1.5.4, the equivalent spring constant is:

$$k_{eq} = k_1 + k_2 + k_3 + \frac{k_4 k_5}{k_4 + k_5} = (10 + 1 + 4 + \frac{2 \times 3}{2 + 3})\text{kN/m} = 16.2 \text{ kN/m}$$

Then using the standard formulas for frequency and damping ratio:

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{16.2 \times 10^3}{10}} = 40.25 \text{ rad/s}$$
$$\zeta = \frac{c}{2m\omega_n} = \frac{1}{2(10)(40.25)} = 0.00158$$

Thus the system is underdamped.

1.74 A manufacturer makes a cantilevered leaf spring from steel (E = 2 x 10^{11} N/m^2) and sizes the spring so that the device has a specific frequency. Later, to save weight, the spring is made of aluminum (E = 7.1 x 10^{10} N/m^2). Assuming that the mass of the spring is much smaller than that of the device the spring is attached to, determine if the frequency increases or decreases and by how much.

Solution: Use equation (1.68) to write the expression for the frequency twice:

$$\omega_{\text{al}} = \sqrt{\frac{3E_{\text{al}}}{m\ell^3}} \text{ and } \omega_{\text{steel}} = \sqrt{\frac{3E_{\text{steel}}}{m\ell^3}} \text{ rad/s}$$

Dividing yields:

$$\frac{\omega_{\text{al}}}{\omega_{\text{steel}}} = \frac{\sqrt{\dfrac{3E_{\text{al}}}{m\ell^3}}}{\sqrt{\dfrac{3E_{\text{steel}}}{m\ell^3}}} = \sqrt{\frac{7.1 \times 10^{10}}{2 \times 10^{11}}} = 0.596$$

Thus the ***frequency is decreased by about 40% by using aluminum***.

Problems and Solutions Section 1.6 (1.75 through 1.81)

1.75 Show that the logarithmic decrement is equal to

$$\delta = \frac{1}{n}\ln\frac{x_0}{x_n}$$

where x_n is the amplitude of vibration after n cycles have elapsed.

Solution:

$$\ln\left[\frac{x(t)}{x(t+nT)}\right] = \ln\left[\frac{Ae^{-\zeta\omega_n t}\sin(\omega_d t+\phi)}{Ae^{-\zeta\omega_n (t+nt)}\sin(\omega_d t+\omega_d nT+\phi)}\right] \qquad (1)$$

Since $n\omega_d T = n(2\pi)$, $\sin(\omega_d t + n\omega_d T + \phi) = \sin(\omega_d t + \phi)$

Hence, Eq. (1) becomes

$$\ln\left[\frac{Ae^{-\zeta\omega_n t}\sin(\omega_d t+\phi)}{Ae^{-\zeta\omega_n (t+nT)}e^{-\zeta\omega_n nt}\sin(\omega_d t+\omega_d nt+\phi)}\right] = \ln\left(e^{\zeta\omega_n nT}\right) = n\zeta\omega_n T$$

Since $$\ln\left[\frac{x(t)}{x(t+T)}\right] = \zeta\omega_n T = \delta,$$

Then $$\ln\left[\frac{x(t)}{x(t+nT)}\right] = n\delta$$

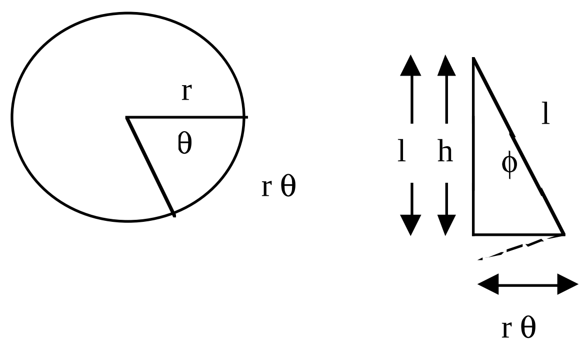
Therefore,

$$\delta = \frac{1}{n}\ln\frac{x_o}{x_n} \qquad \begin{array}{l}\leftarrow \text{original amplitude}\\ \leftarrow \text{amplitude } n \text{ cycles later}\end{array}$$

Here $x_0 = x(0)$.

1.76 Derive the equation (1.70) for the trifalar suspension system.

Solution: Using the notation given for Figure 1.29, and the following geometry:



Write the kinetic and potential energy to obtain the frequency:

Kinetic energy: $T_{\max} = \frac{1}{2} I_o \dot{\theta}^2 + \frac{1}{2} I \dot{\theta}^2$

From geometry, $x = r\theta$ and $\dot{x} = r\dot{\theta}$

$$T_{\max} = \frac{1}{2}(I_o + I)\frac{\dot{x}^2}{r^2}$$

Potential Energy:

$$U_{\max} = (m_o + m)g(l - l\cos\phi)$$

Two term Taylor Series Expansion of cos $\phi \approx 1 - \frac{\phi^2}{2}$:

$$U_{\max} = (m_o + m)gl\left(\frac{\phi^2}{2}\right)$$

For geometry, sin $\phi = \frac{r\theta}{l}$, and for small ϕ, sin $\phi = \phi$ so that $\phi = \frac{r\theta}{l}$

$$U_{\max} = (m_o + m)gl\left(\frac{r^2\theta^2}{2l^2}\right)$$

$$U_{\max} = (m_o + m)g\left(\frac{r^2\theta^2}{2l}\right) \text{ where } r\theta = x$$

$$U_{\max} = \frac{(m_o + m)g}{2l}x^2$$

Conservation of energy requires that:

$$T_{\max} = U_{\max} \quad \Rightarrow$$

$$\frac{1}{2}\frac{\left(I_o + I\right)}{r^2}\dot{x}^2 = \frac{\left(m_o + m\right)g}{2l}x^2$$

At maximum energy, x = A and $\dot{x} = \omega_n A$

$$\frac{1}{2}\frac{\left(I_o + I\right)}{r^2}\omega_n^2 A^2 = \frac{\left(m_o + m\right)g}{2l}A^2$$

$$\Rightarrow \left(I_o + I\right) = \frac{gr^2\left(m_o + m\right)}{\omega_n^2 l}$$

Substitute $\omega_n = 2\pi f_n = \dfrac{2\pi}{T}$

$$\left(I_o + I\right) = \frac{gr^2\left(m_o + m\right)}{\left(2\pi / T\right)^2 l}$$

$$I = \frac{gT^2 r^2\left(m_o + m\right)}{4\pi^2 l} - I_o$$

were T is the period of oscillation of the suspension.

1.77 A prototype composite material is formed and hence has unknown modulus. An experiment is performed consisting of forming it into a cantilevered beam of length 1 m and I = 10^{-9} m^4 with a 6-kg mass attached at its end. The system is given an initial displacement and found to oscillate with a period of 0.5 s. Calculate the modulus E.

Solution: Using equation (1.66) for a cantilevered beam,

$$T = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{ml^3}{3EI}}$$

Solving for E and substituting the given values yields

$$E = \frac{4\pi^2 ml^3}{3T^2 I} = \frac{4\pi^2(6)(1)^3}{3(.5)^2(10^{-9})}$$

$$\Rightarrow \underline{E = 3.16 \times 10^{11} \text{ N/m}^2}$$

1.78 The free response of a 1000-kg automobile with stiffness of k = 400,000 N/m is observed to be of the form given in Figure 1.32. Modeling the automobile as a single-degree-of-freedom oscillation in the vertical direction, determine the damping coefficient if the displacement at t_1 is measured to be 2 cm and 0.22 cm at t_2.

Solution: Given: x_1 = 2 cm and x_2 = 0.22 cm where $t_2 = T + t_1$

Logarithmic Decrement: $\delta = \ln \frac{x_1}{x_2} = \ln \frac{2}{0.22} = 2.207$

Damping Ratio: $\zeta = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}} = \frac{2.207}{\sqrt{4\pi^2 + (2.207)^2}} = 0.331$

Damping Coefficient: $c = 2\zeta\sqrt{km} = 2(0.331)\sqrt{(400{,}000)(1000)} = 13{,}256 \text{ kg/s}$

1.79 A pendulum decays from 10 cm to 1 cm over one period. Determine its damping ratio.

Solution: Using Figure 1.31: $x_1 = 10$ cm and $x_2 = 1$ cm

Logarithmic Decrement: $\delta = \ln \frac{x_1}{x_2} = \ln \frac{10}{1} = 2.303$

Damping Ratio: $\zeta = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}} = \frac{2.303}{\sqrt{4\pi^2 + (2.303)^2}} = 0.344$

1.80 The relationship between the log decrement δ and the damping ratio ζ is often approximated as $\delta = 2\pi\zeta$. For what values of ζ would you consider this a good approximation to equation (1.74)?

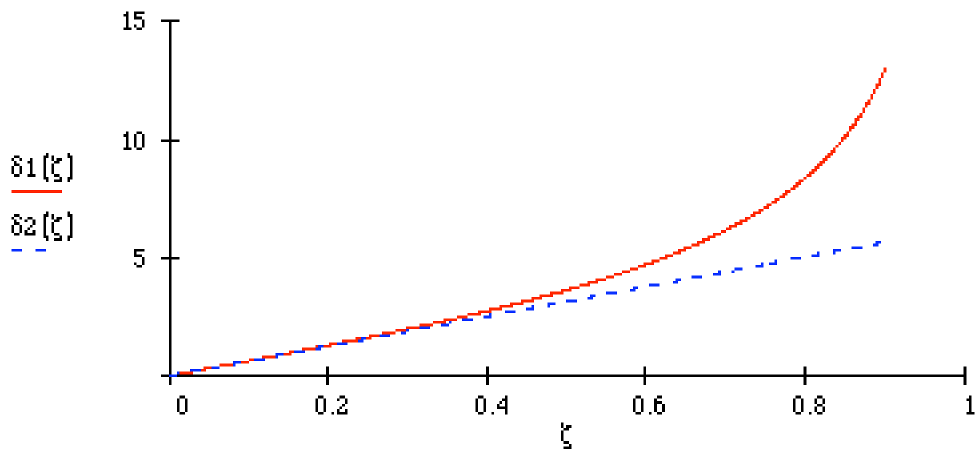
Solution: From equation (1.74), $\delta = \dfrac{2\pi\zeta}{\sqrt{1-\zeta^2}}$

For small ζ, $\delta = 2\pi\zeta$

A plot of these two equations is shown:

$\zeta := 0, 0.001 .. 0.9$

$$\delta 1(\zeta) := \frac{2\cdot\pi\cdot\zeta}{\sqrt{1-\zeta^2}} \qquad \delta 2(\zeta) := 2\cdot\pi\cdot\zeta$$



The lower curve represents the approximation for small ζ, while the upper curve is equation (1.74). The approximation appears to be valid to about ζ = 0.3.

1.81 A damped system is modeled as illustrated in Figure 1.10. The mass of the system is measured to be 5 kg and its spring constant is measured to be 5000 N/m. It is observed that during free vibration the amplitude decays to 0.25 of its initial value after five cycles. Calculate the viscous damping coefficient, *c*.

Solution:

Note that for any two consecutive peak amplitudes,

$$\frac{x_o}{x_1} = \frac{x_1}{x_2} = \frac{x_2}{x_3} = \frac{x_3}{x_4} = \frac{x_4}{x_5} = e^{\delta} \text{ by definition}$$

$$\therefore \frac{x_o}{x_5} = \frac{1}{0.25} = \frac{x_0}{x_1} \cdot \frac{x_1}{x_2} \cdot \frac{x_2}{x_3} \cdot \frac{x_3}{x_4} \cdot \frac{x_4}{x_5} = e^{5\delta}$$

So,

$$\delta = \frac{1}{5}\ln(4) = 0.277$$

and

$$\zeta = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}} = 0.044$$

Solving for c,

$$c = 2\zeta\sqrt{km} = 2(0.044)\sqrt{5000(5)}$$
$$c = 13.94 \text{ N-s/m}$$

Problems and Solutions Section 1.7 (1.82 through 1.89)

1.82 Choose a dashpot's viscous damping value such that when placed in parallel with the spring of Example 1.7.2 reduces the frequency of oscillation to 9 rad/s.

Solution:

The frequency of oscillation is $\omega_d = \omega_n\sqrt{1-\zeta^2}$

From example 1.7.2: $\omega_n = 10$ rad/s, $m = 10$ kg, and $k = 10^3$ N/m

So,
$$9 = 10\sqrt{1-\zeta^2}$$
$$\Rightarrow 0.9 = \sqrt{1-\zeta^2} \Rightarrow (0.9)^2 = 1-\zeta^2$$
$$\zeta = \sqrt{1-(0.9)^2} = 0.436$$

Then
$$c = 2m\omega_n\zeta = 2(10)(10)(0.436) = \underline{87.2\text{ kg/s}}$$

1.83 For an underdamped system, $x_0 = 0$ and $v_0 = 10$ mm/s. Determine m, c, and k such that the amplitude is less than 1 mm.

Solution: Note there are multiple correct solutions. The expression for the amplitude is:

$$A^2 = x_0^2 + \frac{(v_o + \zeta\omega_n x_o)^2}{\omega_d^2}$$

$$\text{for } x_o = 0 \Rightarrow A = \frac{v_o}{\omega_d} < 0.001\text{ m} \Rightarrow \omega_d > \frac{v_o}{0.001} = \frac{0.01}{0.001} = 10$$

So
$$\omega_d = \sqrt{\frac{k}{m}\left(1-\zeta^2\right)} > 10$$
$$\Rightarrow \frac{k}{m}\left(1-\zeta^2\right) > 100, \Rightarrow k = m\frac{100}{1-\zeta^2}$$

(1) Choose $\zeta = 0.01 \Rightarrow \dfrac{k}{m} > 100.01$

(2) Choose $\underline{m = 1\text{ kg}} \Rightarrow k > 100.01$

(3) Choose $k = 144$ N/m > 100.01

$$\Rightarrow \omega_n = \sqrt{144}\frac{\text{rad}}{\text{s}} = 12\frac{\text{rad}}{\text{s}}$$
$$\Rightarrow \omega_d = 11.99\frac{\text{rad}}{\text{s}}$$
$$\Rightarrow c = 2m\zeta\omega_n = \underline{\underline{0.24\ \frac{\text{kg}}{\text{s}}}}$$

1.84 Repeat problem 1.83 if the mass is restricted to lie between 10 kg < m < 15 kg.

Solution: Referring to the above problem, the relationship between m and k is

$$k > 1.01\text{x}10^{-4}\, m$$

after converting to meters from mm. Choose m =10 kg and repeat the calculation at the end of Problem 1.82 to get ω_n (again taking ζ = 0.01). Then k = 1000 N/m and:

$$\Rightarrow \omega_n = \sqrt{\frac{1.0\times 10^3}{10}}\ \frac{\text{rad}}{\text{s}} = 10\ \frac{\text{rad}}{\text{s}}$$

$$\Rightarrow \omega_d = 9.998\ \frac{\text{rad}}{\text{s}}$$

$$\Rightarrow c = 2m\zeta\omega_n = \underline{\underline{2.000\ \frac{\text{kg}}{\text{s}}}}$$

1.85 Use the formula for the torsional stiffness of a shaft from Table 1.1 to design a 1-m shaft with torsional stiffness of 10^5 N·m/rad.

Solution: Referring to equation (1.64) the torsional stiffness is

$$k_t = \frac{GJ_p}{\ell}$$

Assuming a solid shaft, the value of the shaft polar moment is given by

$$J_p = \frac{\pi d^4}{32}$$

Substituting this last expression into the stiffness yields:

$$k_t = \frac{G\pi d^4}{32\ell}$$

Solving for the diameter d yields

$$d = \left(\frac{k_t(32)\ell}{G\pi}\right)^{1/4}$$

Thus we are left with the design variable of the material modulus (G). Choose steel, then solve for d. For steel $G = 8 \times 10^{10}$ N/m^2. From the last expression the numerical answer is

$$d = \left[\frac{10^5 \frac{\text{Nm}}{\text{rad}}(32)(1\text{m})}{\left(8\times 10^{10}\frac{\text{N}}{\text{m}^2}\right)(\pi)}\right]^{1/4} = 0.0597 \text{ m}$$

1.86 Repeat Example 1.7.2 using aluminum. What difference do you note?

Solution:

For aluminum $G = 25 \times 10^9$ N/m^2

From example 1.7.2, the stiffness is $k = 10^3 = \dfrac{Gd^4}{64nR^3}$ and $d = .01$ m

So, $10^3 = \dfrac{(25\times 10^9)(.01)^4}{64nR^3}$

Solving for nR^3 yields: $nR^3 = 3.906 \times 10^{-3}\text{m}^3$

Choose $R = 10$ cm $= 0.1$ m, so that

$$n = \frac{3.906 \times 10^{-3}}{(0.1)^3} = 4 \text{ turns}$$

Thus, aluminum requires 1/3 fewer turns than steel.

1.87 Try to design a bar (see Figure 1.21) that has the same stiffness as the spring of Example 1.7.2. Note that the bar must remain at least 10 times as long as it is wide in order to be modeled by the formula of Figure 1.21.

Solution:

From Figure 1.21, $k = \dfrac{EA}{l}$

For steel, $E = 210 \times 10^9 \text{ N/m}^2$

From Example 1.7.2, $k = 10^3$ N/m

So, $10^3 = \dfrac{(210 \times 10^9)A}{l}$

$$l = (2.1 \times 10^8)A$$

If $A = 0.0001 \text{ m}^2$ (1 cm^2), then

$$l = (2.1 \times 10^8)(10^{-4}) = 21{,}000 \text{ m } (21\text{km or } 13 \text{ miles})$$

Not very practical at all.

1.88 Repeat Problem 1.87 using plastic ($E = 1.40 \times 10^9$ N/m^2) and rubber ($E = 7 \times 10^6$ N/m^2). Are any of these feasible?

Solution:

From problem 1.53, $k = 10^3 \text{ N/m} = \dfrac{EA}{l}$

For plastic, $E = 1.40 \times 10^9 \text{ N/m}^2$

So, $l = 140 \text{ m}$

For rubber, $E = 7 \times 10^6 \text{ N/m}^2$

So, $l = 0.7 \text{ m}$

Rubber may be feasible, plastic would not.

1.89 Consider the diving board of Figure P1.89. For divers, a certain level of static deflection is desirable, denoted by Δ. Compute a design formula for the dimensions of the board (b, h and ℓ) in terms of the static deflection, the average diver's mass, m, and the modulus of the board.

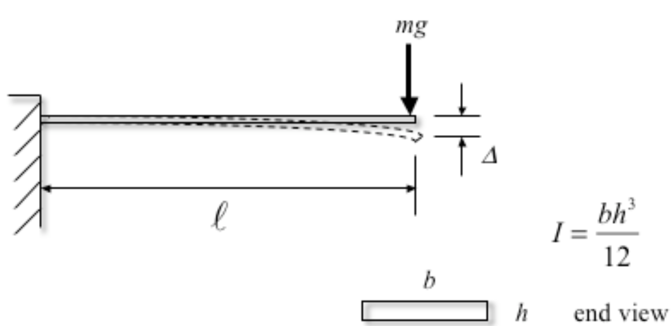


Figure P1.89

Solution: From Figure 1.15 (b), $\Delta k = mg$ holds for the static deflection. The period is:

$$T = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{m}{mg/\Delta}} = 2\pi\sqrt{\frac{\Delta}{g}} \qquad (1)$$

From Figure 1.24, we also have that

$$T = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{m\ell^3}{3EI}} \qquad (2)$$

Equating (1) and (2) and replacing I with the value from the figure yields:

$$2\pi\sqrt{\frac{m\ell^3}{3EI}} = 2\pi\sqrt{\frac{12m\ell^3}{3Ebh^3}} = 2\pi\sqrt{\frac{\Delta}{g}} \Rightarrow \underline{\frac{\ell^3}{bh^3} = \frac{\Delta E}{4mg}}$$

Alternately just use the static deflection expression and the expression for the stiffness of the beam from Figure 1.24 to get

$$\Delta k = mg \Rightarrow \Delta\frac{3EI}{\ell^3} = mg \Rightarrow \underline{\frac{\ell^3}{bh^3} = \frac{\Delta E}{4mg}}$$

Problems and Solutions Section 1.8 (1.90 through 1.93)

1.90 Consider the system of Figure 1.90 and (a) write the equations of motion in terms of the angle, θ, the bar makes with the vertical. Assume linear deflections of the springs and linearize the equations of motion. Then (b) discuss the stability of the linear system's solutions in terms of the physical constants, *m*, *k*, and ℓ. Assume the mass of the rod acts at the center as indicated in the figure.

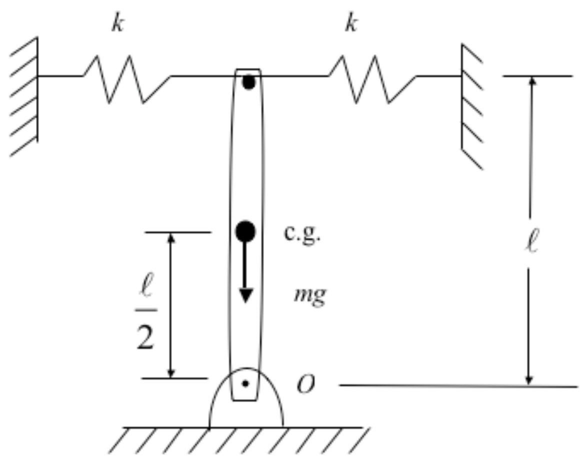


Figure P1.90

Solution: Note that from the geometry, the springs deflect a distance $kx = k(\ell \sin\theta)$ and the cg moves a distance $\ell/2 \cos\theta$. Thus the total potential energy is

$$U = 2\times\frac{1}{2}k(\ell\sin\theta)^2 - \frac{mg\ell}{2}\cos\theta$$

and the total kinetic energy is

$$T = \frac{1}{2}J_O\dot{\theta}^2 = \frac{1}{2}\frac{m\ell^2}{3}\dot{\theta}^2$$

The Lagrange equation (1.64) becomes

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\theta}}\right) + \frac{\partial U}{\partial\theta} = \frac{d}{dt}\left(\frac{m\ell^2}{3}\dot{\theta}\right) + 2k\ell\sin\theta\cos\theta - \frac{1}{2}mg\ell\sin\theta = 0$$

Using the linear, small angle approximations $\sin\theta \approx \theta$ and $\cos\theta \approx 1$ yields

$$\text{a)}\quad \underline{\frac{m\ell^2}{3}\ddot{\theta} + \left(2k\ell^2 - \frac{mg\ell}{2}\right)\theta = 0}$$

Since the leading coefficient is positive the sign of the coefficient of θ determines the stability.

b)

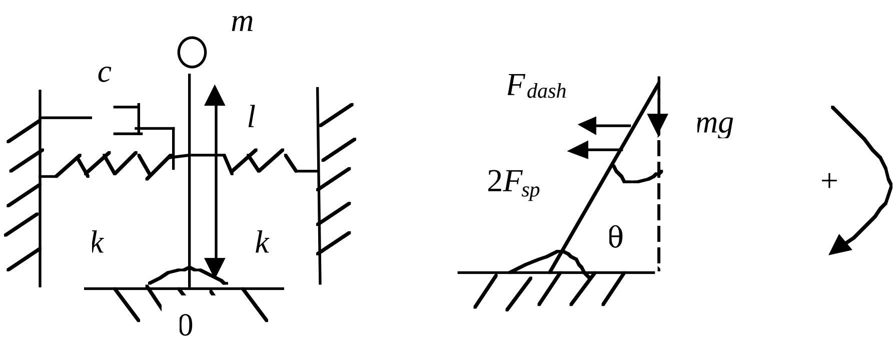
$$\text{if}\quad 2k\ell - \frac{mg}{2} > 0 \Rightarrow 4k > \frac{mg}{\ell} \Rightarrow \text{ the system is stable}$$

$$\text{if}\quad 4k = mg \Rightarrow \theta(t) = at + b \Rightarrow \text{ the system is unstable}$$

$$\text{if}\quad 2k\ell - \frac{mg}{2} < 0 \Rightarrow 4k < \frac{mg}{\ell} \Rightarrow \text{ the system is unstable}$$

Note that physically this results states that the system's response is stable as long as the spring stiffness is large enough to over come the force of gravity.

1.91 Consider the inverted pendulum of Figure 1.37 as discussed in Example 1.8.1. Assume that a dashpot (of damping rate *c*) also acts on the pendulum parallel to the two springs. How does this affect the stability properties of the pendulum?
Solution: The equation of motion is found from the following FBD:



Moment about O: $\Sigma M_o = I\ddot{\theta}$

$$ml^2\ddot{\theta} = mgl\sin\theta - 2\frac{kl}{2}\sin\theta\left(\frac{l}{2}\cos\theta\right) - c\left(\frac{l}{2}\dot{\theta}\right)\left(\frac{l}{2}\cos\theta\right)$$

When θ is small, sinθ ≈ θ and cosθ ≈ 1

$$ml^2\ddot{\theta} + \frac{cl^2}{4}\dot{\theta} + \left(\frac{kl^2}{2} - mgl\right)\theta = 0$$

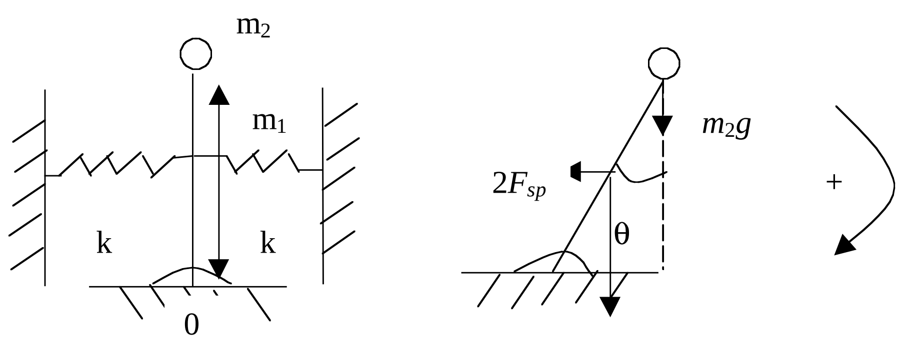
$$ml\ddot{\theta} + \frac{cl}{4}\dot{\theta} + \left(\frac{kl}{2} - mg\right)\theta = 0$$

For stability, $\frac{kl}{2} > mg$ and $c \geq 0$.

The result of ***adding a dashpot is to make the system asymptotically stable***.

1.92 Replace the massless rod of the inverted pendulum of Figure 1.37 with a solid object compound pendulum of Figure 1.20(b). Calculate the equations of vibration and discuss values of the parameter relations for which the system is stable.

Solution:



Moment about O: $\Sigma M_o = I\ddot{\theta}$

$$m_1 g\frac{l}{2}\sin\theta + m_2 gl\sin\theta - 2\frac{kl}{2}\sin\theta\left(\frac{l}{2}\cos\theta\right) = \left(\frac{1}{3}m_1 l^2 + m_2 l^2\right)\ddot{\theta}$$

When θ is small, sinθ ≈ θ and cosθ ≈ 1.

$$\left(\frac{m_1}{3} + m_2\right)l^2\ddot{\theta} + \left(\frac{kl^2}{2} - \frac{m_1}{2}gl - m_2 gl\right)\theta = 0$$

$$\left(\frac{m_1}{3} + m_2\right)l\ddot{\theta} + \left[\frac{kl}{2} - \left(\frac{m_1}{2} + m_2\right)g\right]\theta = 0$$

For stability, $\frac{kl}{2} > \left(\frac{m_1}{2} + m_2\right)g$.

1.93 A simple model of a control tab for an airplane is sketched in Figure P1.93. The equation of motion for the tab about the hinge point is written in terms of the angle θ from the centerline to be

$$J\ddot{\theta} + (c - f_d)\dot{\theta} + k\theta = 0.$$

Here J is the moment of inertia of the tab, k is the rotational stiffness of the hinge, c is the rotational damping in the hinge and $f_d\dot{\theta}$ is the negative damping provided by the aerodynamic forces (indicated by arrows in the figure). Discuss the stability of the solution in terms of the parameters c and f_d .

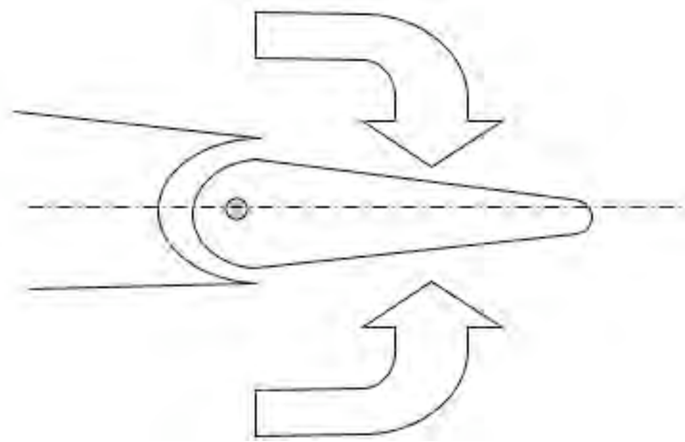


Figure P1.93 A simple model of an airplane control tab

Solution: The stability of the system is determined by the coefficient of $\dot{\theta}$ since the inertia and stiffness terms are both positive. There are three cases

Case 1 $c - f_d > 0$ and the system's solution is of the form $\theta(t) = e^{-at}\sin(\omega_n t + \phi)$ and the solution is asymptotically stable.

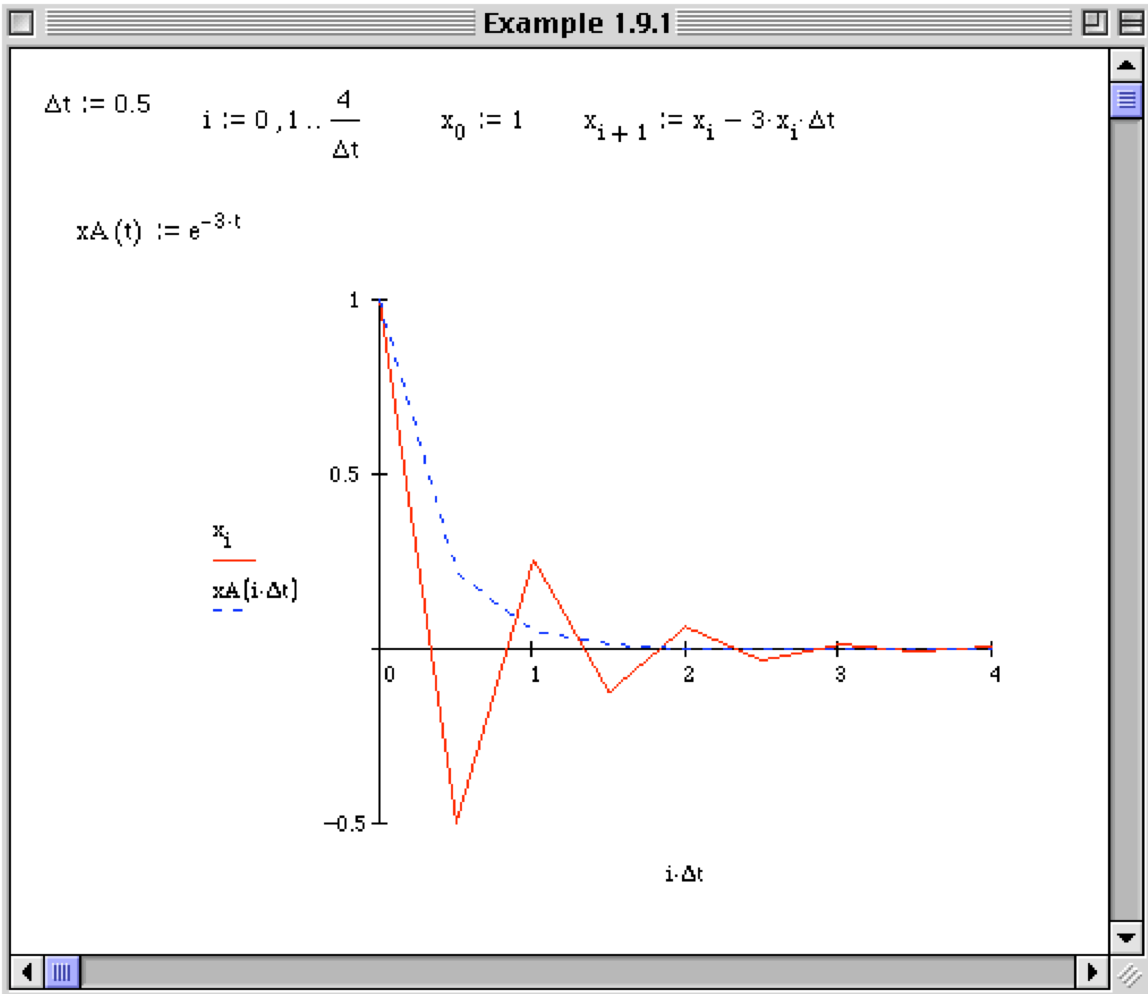
Case 2 $c - f_d < 0$ and the system's solution is of the form $\theta(t) = e^{at}\sin(\omega_n t + \phi)$ and the solution is oscillates and grows without bound, and exhibits flutter instability as illustrated in Figure 1.36.

Case 3 $c = f_d$ and the system's solution is of the form $\theta(t) = A\sin(\omega_n t + \phi)$ and the solution is stable as illustrated in Figure 1.34.

Problems and Solutions Section 1.9 (1.94 through 1.101)

1.94* Reproduce Figure 1.38 for the various time steps indicated.

Solution: The code is given here in Mathcad, which can be run repeatedly with different Δt to see the importance of step size. Matlab and Mathematica can also be used to show this.



1.95* Use numerical integration to solve the system of Example 1.7.3 with m = 1361 kg, k = 2.688 x 10^5 N/m, c = 3.81 x 10^3 kg/s subject to the initial conditions $x(0)$ = 0 and $v(0)$ = 0.01 mm/s. Compare your result using numerical integration to just plotting the analytical solution (using the appropriate formula from Section 1.3) by plotting both on the same graph.

Solution: The solution is shown here in Mathcad using an Euler integration. This can also been done in the other codes or the Toolbox:

1.84

$k := 2.688\cdot 10^5 \quad m := 1361 \quad x_0 := 0.0 \quad v_0 := 0.01 \quad c := 3.81\cdot 10^3$ all units in m, kg, s

$$\omega n := \sqrt{\frac{k}{m}} \qquad fn := \frac{\omega n}{2\cdot\pi} \qquad \zeta := \frac{c}{2\cdot\sqrt{m\cdot k}} \qquad \omega d := \omega n\cdot\sqrt{1-\zeta^2}$$

$$A := \frac{1}{\omega d}\cdot\sqrt{\left(v_0 + \zeta\cdot x_0\cdot\omega n\right)^2 + \left(x_0\cdot\omega d\right)^2} \qquad T := \frac{2\cdot\pi}{\omega d}$$

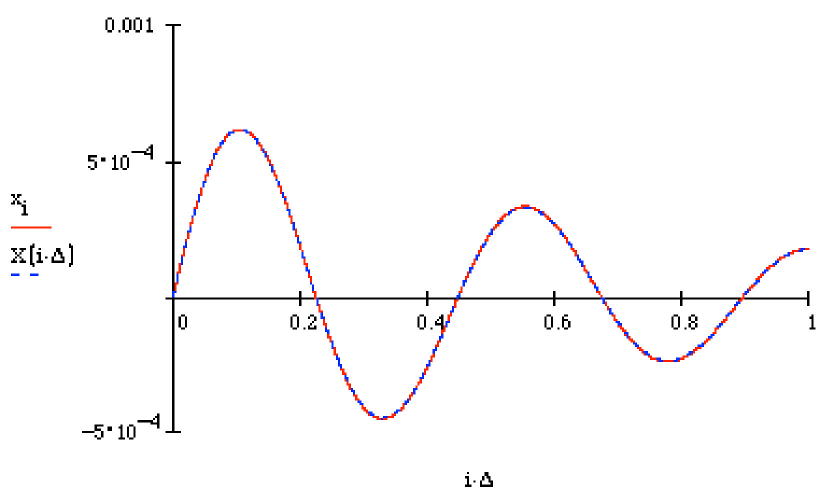
$$\phi := \operatorname{atan}\left[\frac{\omega d\cdot x_0}{v_0 + \zeta\cdot\omega n\cdot x_0}\right] \qquad \zeta = 0.1$$

$$X(t) := A\cdot\sin(\omega d\cdot t + \phi)\cdot e^{-\zeta\cdot\omega n\cdot t}$$

$$N := 10000 \qquad \Delta := \frac{6\cdot\pi}{\omega n\cdot N} \qquad y := \begin{bmatrix} 0 \\ 0.01 \end{bmatrix}$$

$$i := 0..N$$

$$\begin{bmatrix} x_{i+1} \\ v_{i+1} \end{bmatrix} := \begin{bmatrix} v_i\cdot\Delta + x_i \\ -\omega n^2\cdot(\Delta)\cdot x_i - 2\cdot\zeta\cdot\omega n\cdot v_i\cdot\Delta + v_i \end{bmatrix}$$



1.96* Consider again the damped system of Problem 1.95 and design a damper such that the oscillation dies out after 2 seconds. There are at least two ways to do this. Here it is intended to solve for the response numerically, following Examples 1.9.2, 1.9.3 or 1.9.4, using different values of the damping parameter *c* until the desired response is achieved.

Solution: Working directly in Mathcad (or use one of the other codes). Changing *c* until the response dies out within about 2 sec yields *c* =6500 kg/s or ζ = 0.17.

1.85

$k := 2.688 \cdot 10^5$ $\quad m := 1361$ $\quad x_0 := 0.0$ $\quad v_0 := 0.01$ $\quad c := 6.5 \cdot 10^3$ $\quad$ all units in m, kg, s

$$\omega n := \sqrt{\frac{k}{m}} \qquad fn := \frac{\omega n}{2 \cdot \pi} \qquad \zeta := \frac{c}{2 \cdot \sqrt{m \cdot k}} \qquad \omega d := \omega n \cdot \sqrt{1 - \zeta^2} \qquad T := \frac{2 \cdot \pi}{\omega d}$$

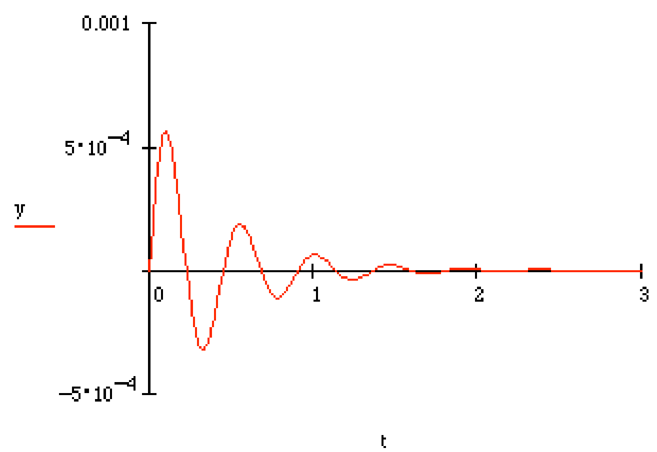
$$A := \frac{1}{\omega d} \cdot \sqrt{\left(v_0 + \zeta \cdot x_0 \cdot \omega n\right)^2 + \left(x_0 \cdot \omega d\right)^2} \qquad \phi := \operatorname{atan}\left[\frac{\omega d \cdot x_0}{v_0 + \zeta \cdot \omega n \cdot x_0}\right] \qquad y := \begin{bmatrix} x_0 \\ v_0 \end{bmatrix}$$

$$D(t, y) := \begin{bmatrix} y_1 \\ -\left(2 \cdot \zeta \cdot \omega n \cdot y_1\right) - \left(\omega n^2\right) \cdot y_0 \end{bmatrix} \qquad Z := \mathrm{rkfixed}(y, 0, 20, 1000, D)$$

$A = 7.221 \cdot 10^{-4}$ in meters $\qquad \frac{\phi}{deg} = 0$ in degrees $\qquad t := Z^{<0>}$

$\omega n = 14.054$ in rad/s $\qquad fn = 2.237$ in Hz $\qquad T = 0.454$ in sec $\qquad \zeta = 0.17$ $\qquad y := Z^{<1>}$

$\omega d = 13.849$



1.97* Consider again the damped system of Example 1.9.2 and design a damper such that the oscillation dies out after 25 seconds. There are at least two ways to do this. Here it is intended to solve for the response numerically, following Examples 1.9.2, 1.9.3 or 1.9.4, using different values of the damping parameter *c* until the desired response is achieved. Is your result overdamped, underdamped or critically damped?

Solution: The following Mathcad program is used to change *c* until the desired response results. This yields a value of *c* = 1.1 kg/s or ζ = 0.225, an underdamped solution.

1.86_1.9.2

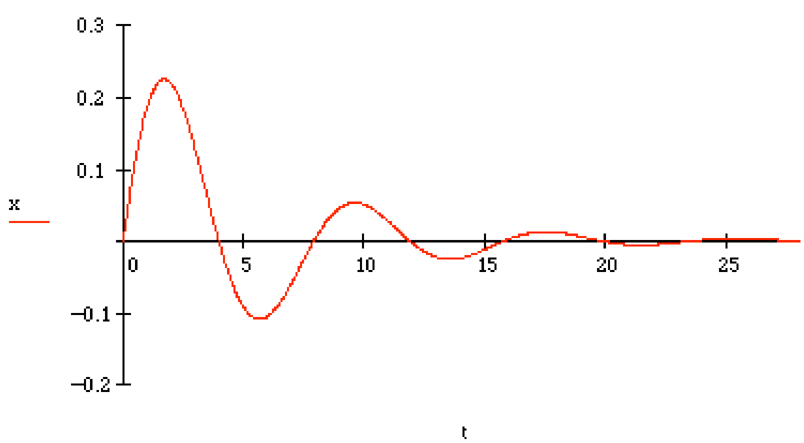
$k := 2 \quad m := 3 \quad x_0 := 0.0 \quad v_0 := 0.25 \quad c := 1.1$

all units in m, kg, s

$$\omega n := \sqrt{\frac{k}{m}} \qquad fn := \frac{\omega n}{2\cdot\pi} \qquad \zeta := \frac{c}{2\cdot\sqrt{m\cdot k}} \qquad \omega d := \omega n\cdot\sqrt{1-\zeta^2} \qquad T := \frac{2\cdot\pi}{\omega d}$$

$$x := \begin{bmatrix} x_0 \\ v_0 \end{bmatrix} \qquad D(t,x) := \begin{bmatrix} x_1 \\ -\left(2\cdot\zeta\cdot\omega n\cdot x_1\right) - \left(\omega n^2\right)\cdot x_0 \end{bmatrix} \qquad Z := \text{rkfixed}(x, 0, 28, 1000, D)$$

$$t := Z^{\langle 0 \rangle} \qquad x := Z^{\langle 1 \rangle}$$



$\omega d = 0.796 \quad \omega n = 0.816 \quad fn = 0.13 \quad T = 7.897 \quad \zeta = 0.225$

in rad/s _ _ _ _ _ in Hz . . in sec .

1.98* Repeat Problem 1.96 for the initial conditions $x(0) = 0.1$ m and $v(0) = 0.01$ mm/s.

Solution: Using the code in 1.96 and changing the initial conditions does not change the settling time, which is just a function of ζ and ω_n. Hence the value of $c = 6.5\text{x}10^3$ kg/s ($\zeta = 0.17$) as determined in problem 1.96 will still reduce the response within 2 seconds.

1.99* A spring and damper are attached to a mass of 100 kg in the arrangement given in Figure 1.9. The system is given the initial conditions $x(0) = 0.1$ m and $v(0) = 1$ mm/s. Design the spring and damper (i.e. choose k and c) such that the system will come to rest in 2 s and not oscillate more than two complete cycles. Try to keep c as small as possible. Also compute ζ.

Solution: In performing this numerical search on two parameters, several underdamped solutions are possible. Students will note that increasing k will decrease ζ. But increasing k also increases the number of cycles which is limited to two. A solution with $c = 350$ kg/s and k =2000 N/m is illustrated.

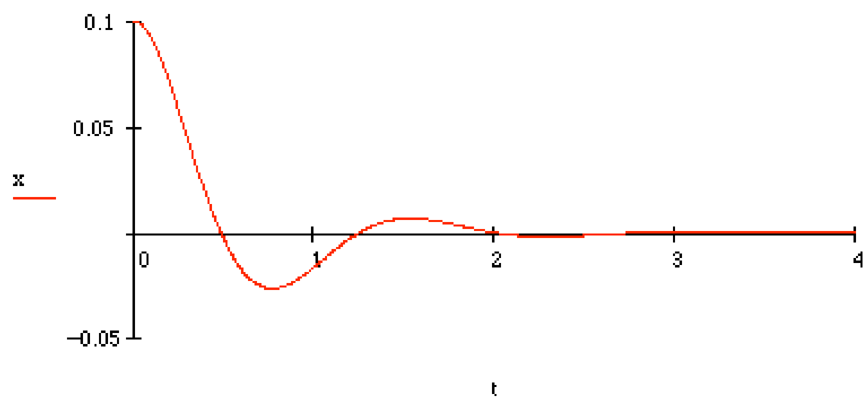
1.88

$k := 2000$ $m := 100$ $x_0 := 0.1$ $v_0 := 0.01$ $c := 350$ all units in m, kg, s

$$\omega n := \sqrt{\frac{k}{m}} \quad fn := \frac{\omega n}{2\cdot\pi} \quad \zeta := \frac{c}{2\cdot\sqrt{m\cdot k}} \quad \omega d := \omega n\cdot\sqrt{1-\zeta^2} \quad T := \frac{2\cdot\pi}{\omega d}$$

$$x := \begin{bmatrix} x_0 \\ v_0 \end{bmatrix} \quad D(t,x) := \begin{bmatrix} x_1 \\ -(2\cdot\zeta\cdot\omega n\cdot x_1) - (\omega n^2)\cdot x_0 \end{bmatrix} \quad Z := \mathrm{rkfixed}(x, 0, 4, 1000, D)$$

$$t := Z^{\langle 0 \rangle} \qquad x := Z^{\langle 1 \rangle}$$

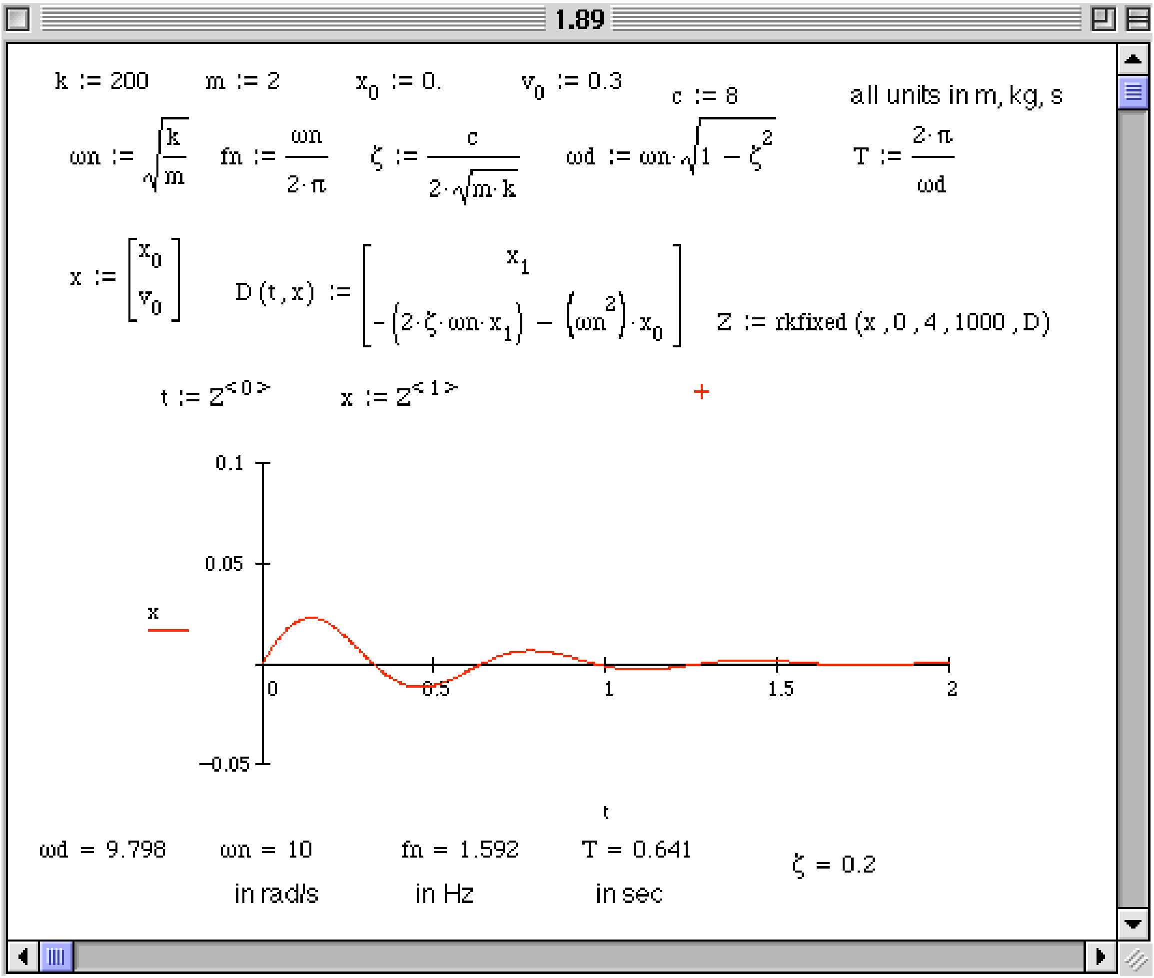


$\omega d = 4.116$ $\omega n = 4.472$ $fn = 0.712$ $T = 1.527$ $\zeta = 0.391$

in rad/s in Hz in sec

1.100* Repeat Example 1.7.1 by using the numerical approach of the previous 5 problems.

Solution: The following Mathcad session can be used to solve this problem by varying the damping for the fixed parameters given in Example 1.7.1.



The other codes or the toolbox may also be used to do this.

1.101* Repeat Example 1.7.1 for the initial conditions $x(0) = 0.01$ m and $v(0) = 1$ mm/s.

Solution: The above Mathcad session can be used to solve this problem by varying the damping for the fixed parameters given in Example 1.7.1. For the given values of initial conditions, the solution to Problem 1.100 also works in this case. Note that if $x(0)$ gets too large, this problem will not have a solution.

Problems and Solutions Section 1.10 (1.102 through 1.114)

1.102 A 2-kg mass connected to a spring of stiffness 10^3 N/m has a dry sliding friction force (F_c) of 3 N. As the mass oscillates, its amplitude decreases 20 cm. How long does this take?

Solution: With m = 2kg, and k = 1000 N/m the natural frequency is just

$$\omega_n = \sqrt{\frac{1000}{2}} = 22.36 \text{ rad/s}$$

From equation (1.101): slope = $\dfrac{-2\mu mg\omega_n}{\pi k} = \dfrac{-2F_c\omega_n}{\pi k} = \dfrac{\Delta x}{\Delta t}$

Solving the last equality for Δt yields:

$$\Delta t = \frac{-\Delta x\pi k}{2f_c\omega_n} = \frac{-(0.20)(\pi)(10^3)}{2(3)(22.36)} = \underline{4.68 \text{ s}}$$

1.103 Consider the system of Figure 1.41 with m = 5 kg and $k = 9 \times 10^3$ N/m with a friction force of magnitude 6 N. If the initial amplitude is 4 cm, determine the amplitude one cycle later as well as the damped frequency.

Solution: Given m = 5 kg, $k = 9\times 10^3$ N/m, f_c = 6 N, $x_0 = 0.04$ m, the amplitude after one cycle is $x_1 = x_0 - \dfrac{4f_c}{k} = 0.04 - \dfrac{(4)(6)}{9\times 10^3} = \underline{0.0373 \text{ m}}$

Note that the damped natural frequency is the same as the natural frequency in the case of Coulomb damping, hence $\omega_n = \sqrt{\dfrac{k}{m}} = \sqrt{\dfrac{9\times 10^3}{5}} = \underline{42.43 \text{ rad/s}}$

1.104* Compute and plot the response of the system of Figure P1.104 for the case where x_0 = 0.1 m, v_0 = 0.1 m/s, μ_κ = 0.05, m = 250 kg, θ = 20° and k =3000 N/m. How long does it take for the vibration to die out?

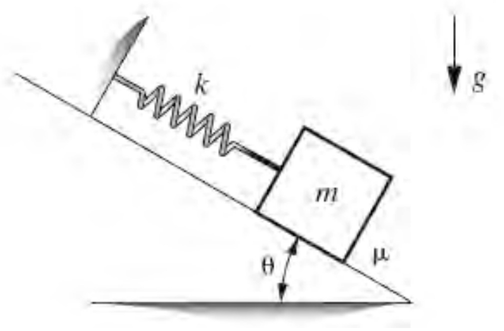
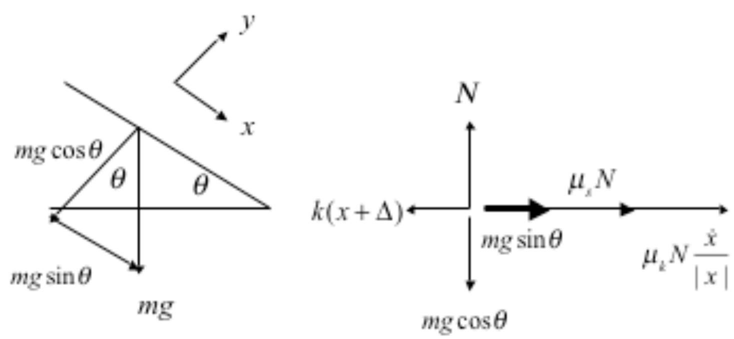


Figure P1.104

Solution: Choose the x y coordinate system to be along the incline and perpendicular to it. Let μ_s denote the static friction coefficient, μ_k the coefficient of kinetic friction and Δ the static deflection of the spring. A drawing indicating the angles and a free-body diagram is given in the figure:



For the static case

$$\sum F_x = 0 \Rightarrow k\Delta = \mu_s N + mg\sin\theta, \text{ and } \sum F_y = 0 \Rightarrow N = mg\cos\theta$$

For the dynamic case

$$\sum F_x = m\ddot{x} = -k(x+\Delta) + \mu_s N + mg\sin\theta - \mu_k N\frac{\dot{x}}{|\dot{x}|}$$

Combining these three equations yields

$$m\ddot{x} + \mu_k mg\cos\theta\frac{\dot{x}}{|\dot{x}|} + kx = 0$$

Note that as the angle θ goes to zero the equation of motion becomes that of a spring mass system with Coulomb friction on a flat surface as it should.

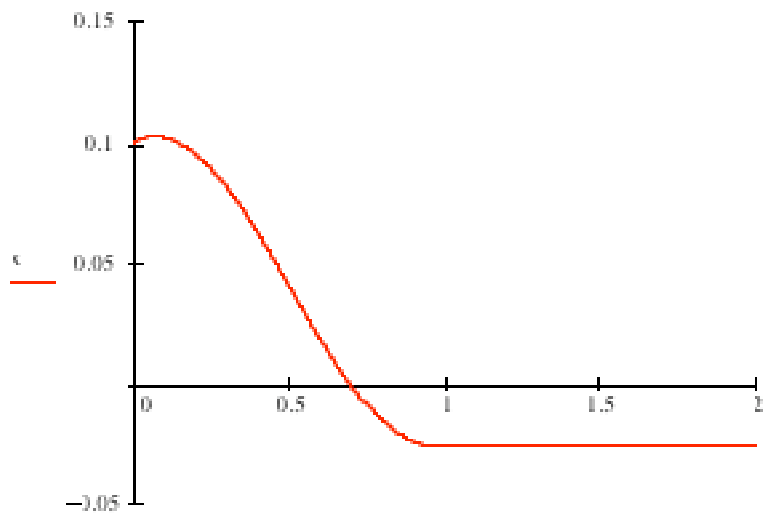
Answer: The oscillation dies out after about 0.9 s. This is illustrated in the following Mathcad code and plot.

$$X := \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix} \qquad k := 3000 \qquad m := 250 \qquad \mu := 0.05$$

$$D(t, X) := \begin{bmatrix} X_1 \\ \dfrac{-k}{m} \cdot X_0 - \cos(20\ \mathrm{deg}) \cdot \mu \cdot g \cdot \dfrac{X_1}{|X_1|} \end{bmatrix}$$

$$Z := \mathrm{rkfixed}(X, 0, 10, 5000, D)$$

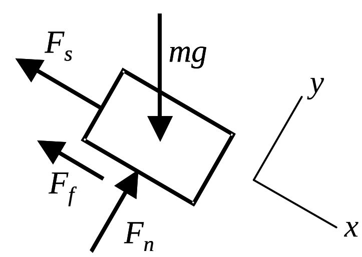
$$t := Z^{<0>} \qquad x := Z^{<1>}$$



Alternate Solution (Courtesy of Prof. Chin An Tan of Wayne State University):

<u>Static Analysis</u>:

In this problem, $x(t)$ is defined as the displacement of the mass from the equilibrium position of the spring-mass system under friction. Thus, the first issue to address is how to determine this equilibrium position, or what is this equilibrium position. In reality, the mass is attached onto an initially unstretched spring on the incline. The free body diagram of the system is as shown. The governing equation of motion is:



$$m\ddot{X} = -k\cancel{X}^{\text{zero initially}} - F_f + mg\sin\theta$$

where $X(t)$ is defined as the displacement measured from the unstretched position of the spring. Note that since the spring is initially unstretched, the spring force $F_s = kX$ is zero initially. If the coefficient of static friction μ_s is sufficiently large, i.e., $\mu_s > \tan(\theta)$, then the mass remains stationary and the spring is unstretched with the mass-spring-friction in equilibrium. Also, in that case, the friction force $F_f \le \mu_s \underbrace{mg\cos\theta}_{F_N}$, not necessarily equal to the maximum static friction. In other words, these situations may hold at equilibrium: (1) the maximum static friction may not be achieved; and (2) there may be no displacement in the spring at all. In this example, $\tan(20^\circ) = 0.364$ and one would expect that μ_s (not given) should be smaller than 0.364 since $\mu_k = 0.05$ (very small). Thus, one would expect the mass to move downward initially (due to weight overcoming the maximum static friction). The mass will then likely oscillate and eventually settle into an equilibrium position with the spring stretched.

Dynamic Analysis:

The equation of motion for this system is:

$$m\ddot{x} = -kx - \mu mg\cos\theta\frac{\dot{x}}{|\dot{x}|}$$

where $x(t)$ is the displacement measured from the equilibrium position. Define $x_1(t) = x(t)$ and $x_2(t) = \dot{x}(t)$. Employing the state-space formulation, we transform the original second-order ODE into a set of two first-order ODEs. The state-space equations (for MATLAB code) are:

$$\frac{d\mathbf{x}}{dt} = \frac{d}{dt}\begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} = \begin{Bmatrix} x_2(t) \\ -\mu g\cos\theta\frac{x_2}{|x_2|} - \frac{kx_1}{m} \end{Bmatrix}$$

MATLAB Code:

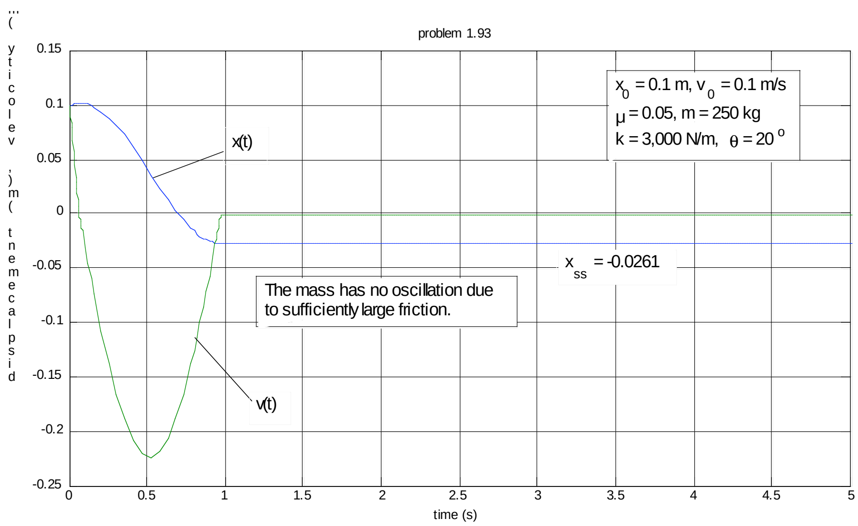
```
x0=[0.1, 0.1];
ts=[0, 5];
[t,x]=ode45('f1_93',ts,x0);
plot(t,x(:,1), t,x(:,2))
title('problem 1.93'); grid on;
xlabel('time (s)');ylabel('displacement (m), velocity (m/s)');

%---------------------------------------------
function xdot = f1_93(t,x)
% computes derivatives for the state-space ODEs
m=250; k=3000; mu=0.05; g=9.81;
angle = 20*pi/180;
xdot(1) = x(2);
xdot(2) = -k/m*x(1) - mu*g*cos(angle)*sign(x(2));
% use the sign function to improve computation time
xdot = [xdot(1); xdot(2)];
```

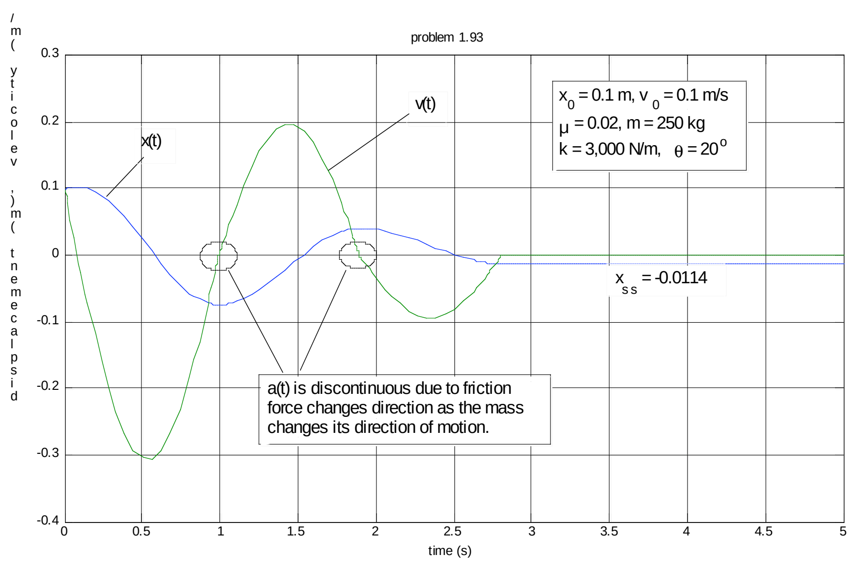
Plots for $\mu = 0.05$ and $\mu = 0.02$ cases are shown. From the $\mu = 0.05$ simulation results, the oscillation dies out after about 0.96 seconds (using `ginput(1)` command to estimate). Note that the acceleration may be discontinuous at $v = 0$ due to the nature of the friction force.

Effects of μ:

Comparing the figures, we see that reducing μ leads to more oscillations (takes longer time to dissipate the energy). Note that since there is a positive initial velocity, the mass is bounded to move down the incline initially. However, if μ is sufficiently large, there may be no oscillation at all and the mass will just come to a stop (as in the case of

$\mu = 0.05$). This is analogous to an overdamped mass-damper-spring system. On the other hand, when μ is very small (say, close to zero), the mass will oscillate for a long time before it comes to a stop.



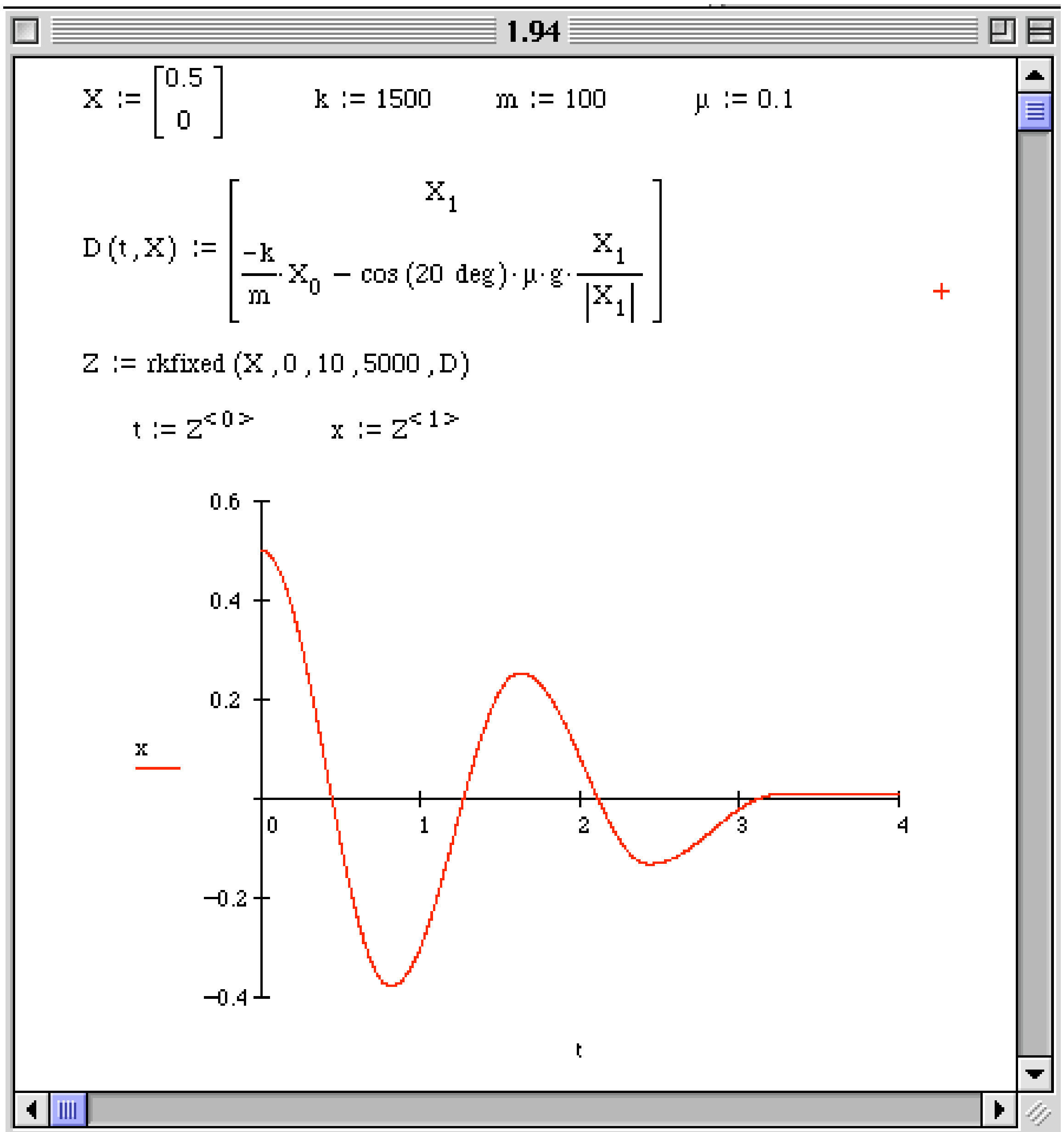


Discussion on the ceasing of motion:

Note that when motion ceases, the mass reaches another state of equilibrium. In both simulation cases, this occurs while the mass is moving upward (negative velocity). Note that the steady-state value of $x(t)$ is very small, suggesting that this is indeed the true equilibrium position, which represents a balance of the spring force, weight component along the incline, and the static friction.

1.105* Compute and plot the response of a system with Coulomb damping of equation (1.90) for the case where x_0 = 0.5 m, v_0 = 0, μ = 0.1, m = 100 kg and k =1500 N/m. How long does it take for the vibration to die out?

Solution: Here the solution is computed in Mathcad using the following code. Any of the codes may be used. The system dies out in about 3.2 sec.



1.106* A mass moves in a fluid against sliding friction as illustrated in Figure P1.106. Model the damping force as a slow fluid (i.e., linear viscous damping) plus Coulomb friction because of the sliding, with the following parameters: m = 250 kg, μ =0.01, c = 25 kg/s and k =3000 N/m . a) Compute and plot the response to the initial conditions: x_0 = 0.1 m, v_0 = 0.1 m/s. b) Compute and plot the response to the initial conditions: x_0 = 0.1 m, v_0 = 1 m/s. How long does it take for the vibration to die out in each case?

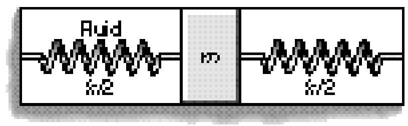
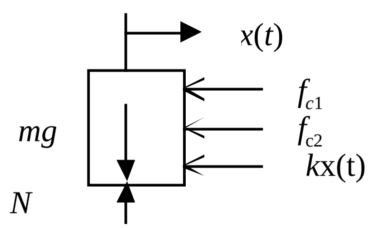


Figure P1.106

Solution: A free-body diagram yields the equation of motion.



$$m\ddot{x}(t) + \mu mg\,\text{sgn}(\dot{x}) + c\dot{x}(t) + kx(t) = 0$$

where the vertical sum of forces gives the magnitude μN = μmg for the Coulomb force as in figure 1.41.

The equation of motion can be solved by using any of the codes mentioned or by using the toolbox. Here a Mathcad session is presented using a fixed order Runge Kutta integration. Note that the oscillations die out after 4.8 seconds for v_0=0.1 m/s for the larger initial velocity of v_0=1 m/s the oscillations go on quite a bit longer ending only after about 13 seconds. While the next problem shows that the viscous damping can be changed to reduce the settling time, this example shows how dependent the response is on the value of the initial conditions. In a linear system the settling time, or time it takes to die out is only dependent on the system parameters, not the initial conditions. This makes design much more difficult for nonlinear systems.

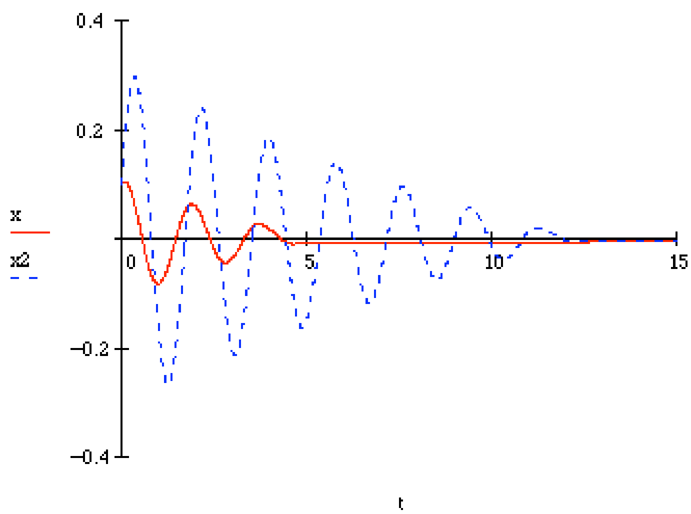
1.95

$c := 25 \quad k := 3000 \quad m := 250 \quad \mu := 0.01 \quad X := \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix} \quad X2 := \begin{bmatrix} 0.1 \\ 1 \end{bmatrix}$

$$D(t,X) := \begin{bmatrix} X_1 \\ \frac{-k}{m} \cdot X_0 - \mu \cdot g \cdot \frac{X_1}{|X_1|} - \frac{c}{m} \cdot X_1 \end{bmatrix} \qquad D2(t,X2) := \begin{bmatrix} X2_1 \\ \frac{-k}{m} \cdot X2_0 - \mu \cdot g \cdot \frac{X2_1}{|X2_1|} - \frac{c}{m} \cdot X2_1 \end{bmatrix}$$

$Z := \mathrm{rkfixed}(X, 0, 15, 3000, D) \qquad Z2 := \mathrm{rkfixed}(X2, 0, 15, 3000, D2)$

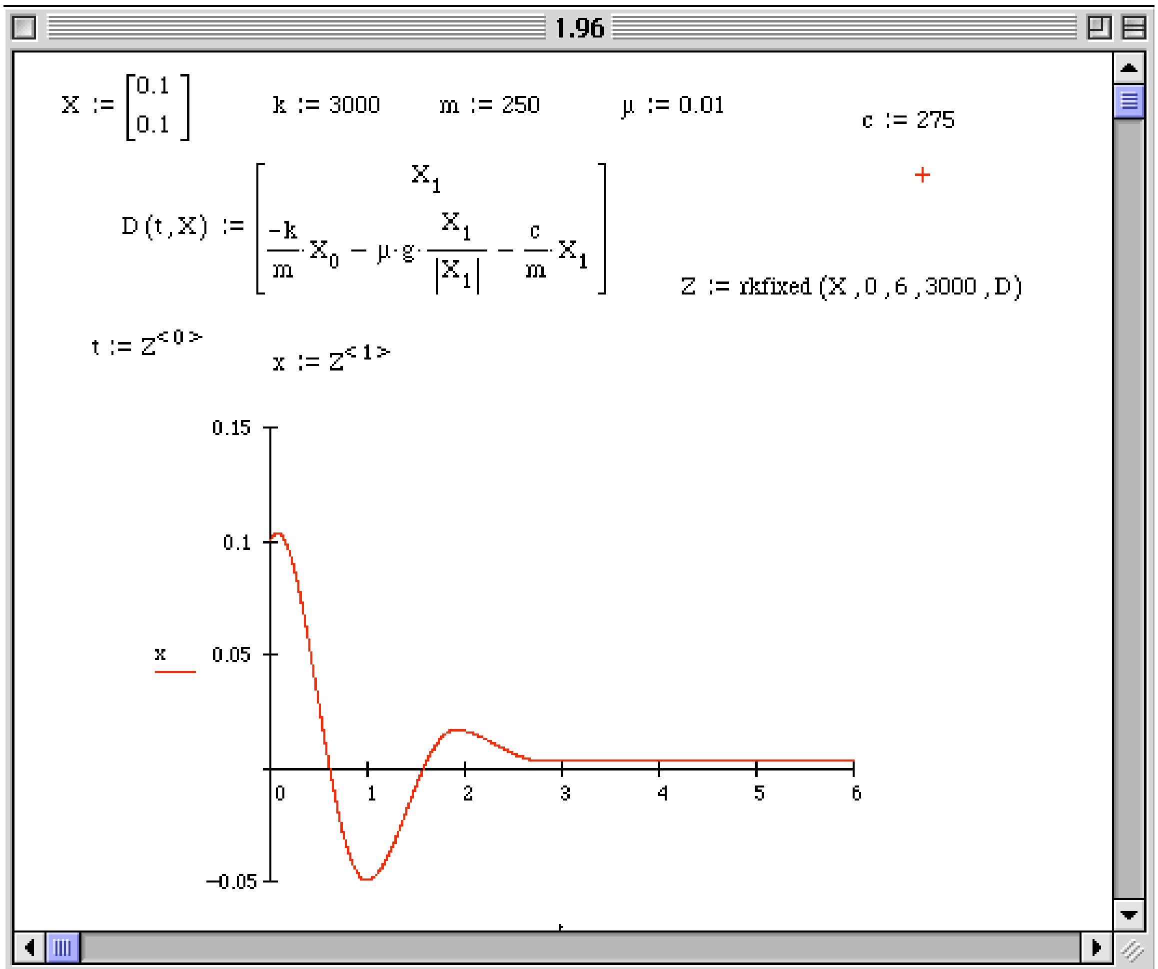
$t := Z^{\langle 0 \rangle} \qquad x := Z^{\langle 1 \rangle} \qquad x2 := Z2^{\langle 1 \rangle}$



1.107* Consider the system of Problem 1.106 part (a), and compute a new damping coefficient, *c*, that will cause the vibration to die out after one oscillation.

Solution: Working in any of the codes, use the simulation from the last problem and change the damping coefficient *c* until the desired response is obtained. A Mathcad solution is given which requires an order of magnitude higher damping coefficient,

<u>c = 275 kg/s</u>



1.108 Compute the equilibrium positions of $\ddot{x} + \omega_n^2 x + \beta x^2 = 0$. How many are there?

Solution: The equation of motion in state space form is

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\omega_n^2 x_1 - \beta x_1^2$$

The equilibrium points are computed from:

$$x_2 = 0$$

$$-\omega_n^2 x_1 - \beta x_1^2 = 0$$

Solving yields the two equilibrium points:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\omega_n^2 / \beta \\ 0 \end{bmatrix}$$

1.109 Compute the equilibrium positions of $\ddot{x} + \omega_n^2 x - \beta^2 x^3 + \gamma x^5 = 0$. How many are there?

Solution: The equation of motion in state space form is

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\omega_n^2 x_1 + \beta^2 x_1^3 - \gamma x_1^5$$

The equilibrium points are computed from:

$$x_2 = 0$$

$$-\omega_n^2 x_1 + \beta^2 x_1^2 - \gamma x_1^5 = 0$$

Solving yields the five equilibrium points (one for each root of the previous equation). The first equilibrium (the linear case) is:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Next divide $-\omega_n^2 x_1 + \beta^2 x_1^2 - \gamma x_1^5 = 0$ by x_1 to obtain:

$$-\omega_n^2 + \beta^2 x_1^2 - \gamma x_1^4 = 0$$

which is quadratic in x_1^2 and has the following roots which define the remaining four equilibrium points: $x_2 = 0$ and

$$\underline{\underline{x_1 = \pm\sqrt{\frac{-\beta^2 + \sqrt{\beta^4 - 4\gamma\omega_n^2}}{-2\gamma}}}}$$

$$\underline{\underline{x_1 = \pm\sqrt{\frac{-\beta^2 - \sqrt{\beta^4 - 4\gamma\omega_n^2}}{-2\gamma}}}}$$

1.110* Consider the pendulum example 1.10.3 with length of 1 m an initial conditions of $\theta_0 = \pi/10$ rad and $\dot{\theta}_0 = 0$. Compare the difference between the response of the linear version of the pendulum equation (i.e. with $\sin(\theta) = \theta$) and the response of the nonlinear version of the pendulum equation by plotting the response of both for four periods.

Solution: First consider the linear solution. Using the formula's given in the text the solution of the linear system is just: $\theta(t) = 0.314\sin(3.132t + \pi/2)$. The following Mathcad code, plots the linear solution on the same plot as a numerical solution of the nonlinear system.

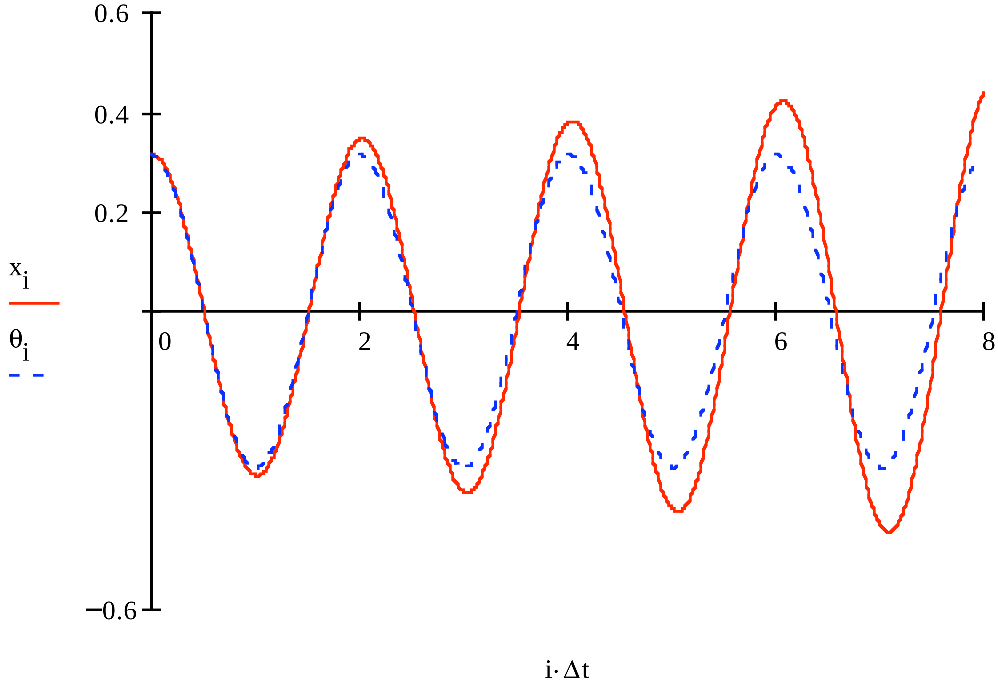
$i := 0 .. 800$

$\Delta t := 0.01$

$$\begin{bmatrix} x_0 \\ v_0 \end{bmatrix} := \begin{bmatrix} \frac{\pi}{10} \\ 0 \end{bmatrix}$$

$$\theta_i := 0.314 \cdot \sin\left(3.132 \cdot \Delta t \cdot i + \frac{\pi}{2}\right)$$

$$\begin{bmatrix} x_{i+1} \\ v_{i+1} \end{bmatrix} := \begin{bmatrix} x_i + v_i \cdot \Delta t \\ v_i - \Delta t \cdot \left(\sin\left(x_i\right)\right) 9.81 \end{bmatrix}$$



Note how the amplitude of the nonlinear system is growing. The difference between the linear and the nonlinear plots are a function of the ration of the linear spring stiffness and the nonlinear coefficient, and of course the size of the initial condition. It is work it to investigate the various possibilities, to learn just when the linear approximation completely fails.

1.111* Repeat Problem 1.110 if the initial displacement is $\theta_0 = \pi/2$ rad.

Solution: The solution in Mathcad is:

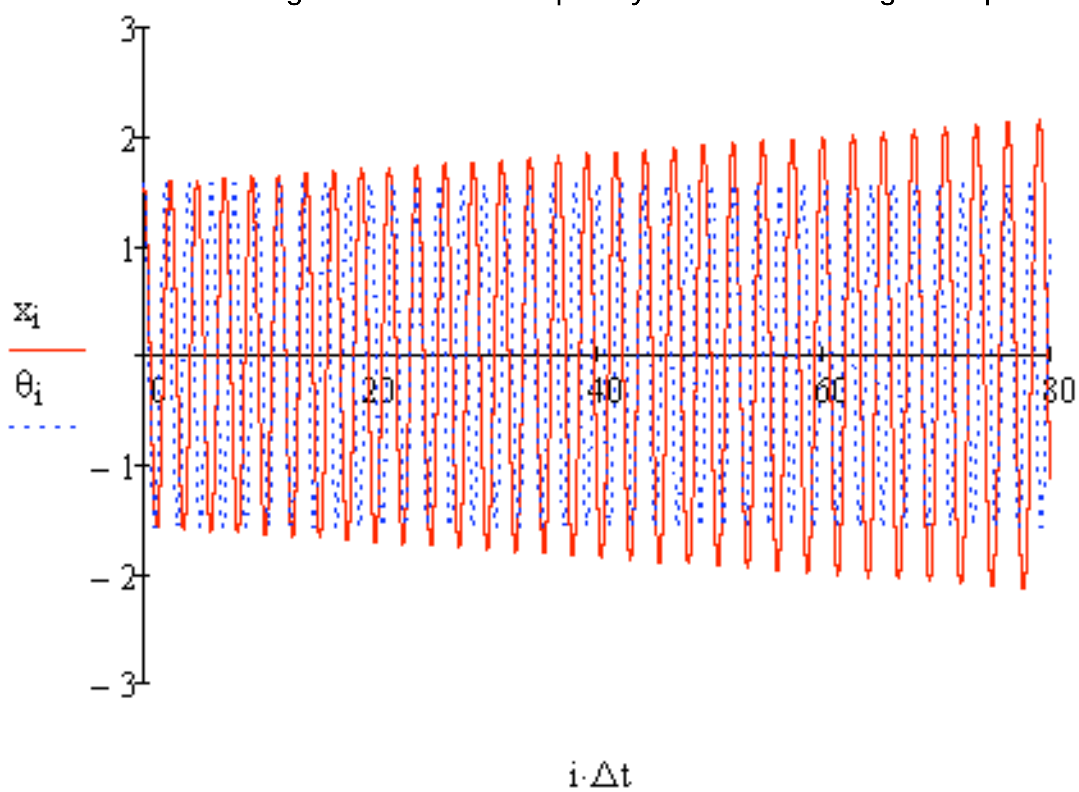
$i := 0..80000$ $\qquad \omega := \sqrt{9.81}$

$\Delta t := 0.001$

$$\begin{pmatrix} x_0 \\ v_0 \end{pmatrix} := \begin{pmatrix} \frac{\pi}{2} \\ 0 \end{pmatrix} \qquad \theta_i := \frac{\pi}{2}\cdot\sin\left(\omega\cdot\Delta t\cdot i + \frac{\pi}{2}\right)$$

$$\begin{pmatrix} x_{i+1} \\ v_{i+1} \end{pmatrix} := \begin{bmatrix} x_i + v_i\cdot\Delta t \\ v_i - \Delta t\cdot(\sin(x_i))\,9.81 \end{bmatrix}$$

Here both solutions oscillate around the "stable" equilibrium, but the nonlinear solution is not oscillating at the natural frequency and is increasing in amplitude.



1.112 If the pendulum of Example 1.10.3 is given an initial condition near the equilibrium position of $\theta_0 = \pi$ rad and $\dot{\theta}_0 = 0$, does it oscillate around this equilibrium?

Solution The pendulum will not oscillate around this equilibrium as it is unstable. Rather it will "wind" around the equilibrium as indicated in the solution to Example 1.10.4.

1.113* Calculate the response of the system of Problem 1.109 for the initial conditions of $x_0 = 0.01$ m, $v_0 = 0$, and a natural frequency of 3 rad/s and for $\beta = 100$, $\gamma = 0$.

Solution: In Mathcad the solution is given using a simple Euler integration as follows:

$\Delta t := 0.01$

$$\begin{bmatrix} x_0 \\ v_0 \end{bmatrix} := \begin{bmatrix} 0.01 \\ 0 \end{bmatrix} \qquad \omega := 3 \qquad A := \frac{1}{\omega}\cdot\sqrt{\omega^2\cdot(x_0)^2}$$

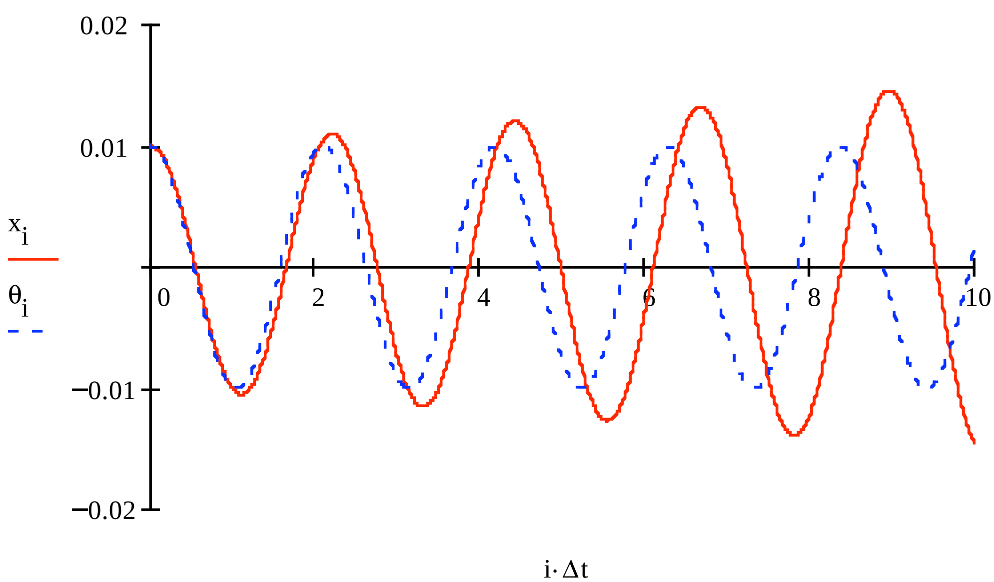
$\beta := 100$

$i := 0..\ 1000$

$$\begin{bmatrix} x_{i+1} \\ v_{i+1} \end{bmatrix} := \begin{bmatrix} x_i + v_i\cdot\Delta t \\ v_i - \Delta t\cdot\left[\omega^2\cdot x_i - \beta^2\cdot(x_i)^3\right] \end{bmatrix}$$

$$\theta_i := A\cdot\sin\left(3\cdot\Delta t\cdot i + \frac{\pi}{2}\right)$$

This is the linear solution $\theta(t)$



The other codes may be used to compute this solution as well.

1.114* Repeat problem 1.113 and plot the response of the linear version of the system (β =0) on the same plot to compare the difference between the linear and nonlinear versions of this equation of motion.

Solution: The solution is computed and plotted in the solution of Problem 1.113. Note that the linear solution starts out very close to the nonlinear solution. The two solutions however diverge. They look similar, but the nonlinear solution is growing in amplitude and period.