Multiple-Degree-of-Freedom Systems

1 Governing Equations of a Two-Degree-of-Freedom System

In previous chapters we have only looked at systems with one changing variable x. In reality situations can hardly ever be expressed by just one variable. To investigate multiple-degree-of-freedom systems, we will first look at two-degree-of-freedom systems. An example of such a system is shown in figure 1.

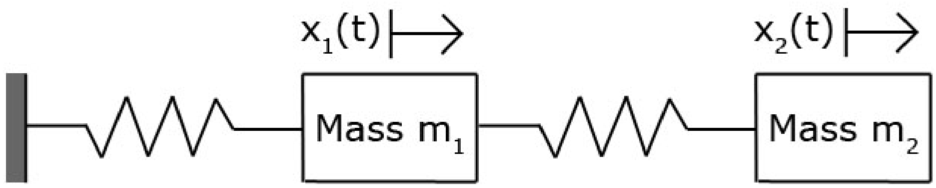

Figure 1: An example of a two-degree-of-freedom system.

When drawing the equations of motion for each mass, the general equations of motion can be derived. These are

$$m_1\ddot{x}_1 = -k_1x_1 + k_2\left(x_2 - x_1\right), \quad (1.1)$$

$$m_2\ddot{x}_2 = -k_2\left(x_2 - x_1\right). \quad (1.2)$$

(We are not considering damping for multiple-degree-of-freedom systems.) When solving this system, four boundary conditions are necessary. These are x_{1_0}, $\dot{x}_{1_0}$, x_{2_0} and $\dot{x}_{2_0}$.

However, writing things like this is a bit annoying. It's better to use vectors and matrices. First let's define the position vector $\mathbf{x}$, the velocity vector $\dot{\mathbf{x}}$ and the acceleration vector $\ddot{\mathbf{x}}$ as

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad \dot{\mathbf{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} \qquad \text{and} \qquad \ddot{\mathbf{x}} = \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix}. \quad (1.3)$$

We can also define the **mass matrix** (also called the **inertia matrix**) for two-degree-of-freedom cases as

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}. \quad (1.4)$$

Finally we also need the **stiffness matrix**. For our example system, this matrix is

$$K = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}. \quad (1.5)$$

Now we can write the system of differential equations as

$$M\ddot{\mathbf{x}} + K\mathbf{x} = \mathbf{0}. \quad (1.6)$$

Note that both M and K are symmetric matrices (meaning that $M^T = M$ and $K^T = K$). M is symmetric because all non-diagonal terms are simply zero. K is symmetric due to Newton's third law.

We now want to find the equation of motion $\mathbf{x}(t)$ for the system of differential equations. To get it, we need to solve equation 1.6. There are multiple ways to do this. We'll discuss two ways.

2 First Method to find the Equation of Motion

The first method we will be discussing is usually the simplest method for hand calculation. It is therefore quite suitable for applying on examinations. Computers, however, don't prefer this method.

Let's suppose our solution has the form $\mathbf{x}(t) = \mathbf{u}e^{i\omega t}$. Filling this in into the differential equation will give

$$\left(K - \omega^2 M\right) \mathbf{u}e^{i\omega t} = \mathbf{0}. \tag{2.1}$$

The exponential can't be zero. Also, if $\mathbf{u} = \mathbf{0}$, we won't have any motion either. So we need to have ω such that the matrix $\left(K - \omega^2 M\right)$ is **singular** (not invertible). In other words, its determinant must be zero. The **characteristic equation** then is

$$\det\left(K - \omega^2 M\right) = 0. \tag{2.2}$$

For our two-degree-of-freedom example system, this results in

$$m_1 m_2 \omega^4 - \left(m_1 k_2 + m_2 k_1 + m_2 k_2\right)\omega^2 + k_1 k_2 = 0. \tag{2.3}$$

From this equation four values of ω will be found, being $\pm\omega_1$ and $\pm\omega_2$. These are the **natural frequencies** of the system. So although a one-degree-of-freedom has only one natural frequency, a two-degree-of-freedom system has 2 natural frequencies. Multiple-degree-of-freedom systems have even more natural frequencies.

The corresponding (nonzero) vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ can now be found using

$$\left(K - M\omega_1^2\right)\mathbf{u_1} = \mathbf{0} \qquad \text{and} \qquad \left(K - M\omega_2^2\right)\mathbf{u_2} = \mathbf{0}. \tag{2.4}$$

Only the direction of the vectors $\mathbf{u}$ can be derived from the above relations. Their magnitudes may be chosen arbitrarily, although they are often **normalized** such that $||\mathbf{u}|| = 1$. The final equation of motion is then given by

$$\mathbf{x}(t) = A_1 \sin\left(\omega_1 t + \phi_1\right)\mathbf{u}_1 + A_2 \sin\left(\omega_2 t + \phi_2\right)\mathbf{u_2}. \tag{2.5}$$

The values of A_1, ϕ_1, A_2 and ϕ_2 now need to be determined from the initial conditions.

3 Second Method to find the Equation of Motion

There is another way to find the equation of motion. Before we discuss this method, we first have to make some definitions. We define the **matrix square root** $M^{1/2}$ of M such that

$$M^{1/2}M^{1/2} = M \qquad \Rightarrow \qquad M^{1/2} = \begin{bmatrix} \sqrt{m_1} & 0 \\ 0 & \sqrt{m_2} \end{bmatrix}. \tag{3.1}$$

This matrix also has an inverse $\left(M^{1/2}\right)^{-1} = M^{-1/2}$. Let's define the vector $\mathbf{q}$ such that

$$\mathbf{x}(t) = M^{-1/2}\mathbf{q}(t). \tag{3.2}$$

Let's assume $\mathbf{q} = \mathbf{v}e^{i\omega t}$, with $\mathbf{v}$ a constant vector. We can now rewrite equation 1.6 to

$$M^{-1/2}KM^{-1/2}\mathbf{v} = \widetilde{K}\mathbf{v} = \omega^2\mathbf{v}, \tag{3.3}$$

where $\widetilde{K} = M^{-1/2}KM^{-1/2}$ is the **mass normalized stiffness**. If we replace ω^2 by λ in the above equation we have exactly the eigenvalue problem from linear algebra. The solutions for λ are then the **eigenvalue**s of the matrix $\widetilde{K}$ and the corresponding vectors $\mathbf{v}$ are the **eigenvectors**.

Since K is symmetric, also $\widetilde{K}$ is symmetric. All the eigenvalues are therefore real numbers and also the eigenvectors are real. Once the eigenvalues λ_1 and λ_2 are known, the natural frequencies ω_1 and ω_2 can easily be found using

$$\omega_1 = \sqrt{\lambda_1} \qquad \text{and} \qquad \omega_2 = \sqrt{\lambda_2}. \tag{3.4}$$

To find the corresponding vectors $\mathbf{u}$, you can use

$$\mathbf{u_1} = M^{-1/2}\mathbf{v_1} \qquad \text{and} \qquad \mathbf{u_2} = M^{-1/2}\mathbf{v_2} \tag{3.5}$$

The equation of motion is then once more given by

$$\mathbf{x}(t) = A_1 \sin(\omega_1 t + \phi_1)\, \mathbf{u}_1 + A_2 \sin(\omega_2 t + \phi_2)\, \mathbf{u_2}. \tag{3.6}$$

4 Modal Analysis

We can also find the equation of motion using **modal analysis**. In the previous paragraph we have found the eigenvectors $\mathbf{v_1}$ and $\mathbf{v_2}$ of the matrix $\widetilde{K}$. These vectors are orthogonal (unless they correspond to the same eigenvalue, in which case they should be made orthogonal). If they have also been normalized (given length 1), then they form an **orthonormal set**. Now let's define the **matrix of eigenvectors** P to consist of these orthonormal eigenvectors. In an equation this is

$$P = \begin{bmatrix} \mathbf{v_1} & \mathbf{v_2} \end{bmatrix}. \tag{4.1}$$

This matrix is an **orthogonal matrix** (as its columns are orthonormal). Such matrices have the convenient property that $P^T P = I$. Also let's define the **matrix of mode shapes** S as

$$S = M^{-1/2}P. \tag{4.2}$$

Furthermore we define the vector $\mathbf{r}(t)$ such that

$$\mathbf{x}(t) = M^{-1/2}\mathbf{q}(t) = M^{-1/2}P\mathbf{r}(t) = S\mathbf{r}(t). \tag{4.3}$$

Using all these definitions, we can rewrite the system of differential equations to

$$\ddot{\mathbf{r}}(t) + \Lambda \mathbf{r}(t) = \mathbf{0}, \tag{4.4}$$

where the matrix Λ is given by

$$\Lambda = P^T \widetilde{K} P = \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix}. \tag{4.5}$$

So we remain with the differential equations

$$\ddot{r}_1 + \omega_1^2 r = 0, \tag{4.6}$$
$$\ddot{r}_2 + \omega_2^2 r = 0. \tag{4.7}$$

The differential equations have been decoupled! They don't depend on each other, and therefore can be solved using simple methods. The two decoupled equations above are called the **modal equations**. Also the coordinate system $\mathbf{r}(t)$ is called the **modal coordinate system**.

To solve the modal equations, we need the initial conditions in the modal coordinate system. Usually we only know the initial conditions $\mathbf{x_0}$ and $\dot{\mathbf{x}}_\mathbf{0}$ in the normal coordinate system. We can transform these to the modal coordinate system using

$$\mathbf{r_0} = S^{-1}\mathbf{x_0} \qquad \text{and} \qquad \dot{\mathbf{r}}_\mathbf{0} = S^{-1}\dot{\mathbf{x}}_\mathbf{0}, \qquad \text{where} \qquad S^{-1} = P^T M^{1/2}. \tag{4.8}$$

Now we can solve for $r_1(t)$ and $r_2(t)$ and thus for $\mathbf{r}(t)$. Once we have found $\mathbf{r}(t)$ we can find the equation of motion $\mathbf{x}(t)$ using

$$\mathbf{x}(t) = S\mathbf{r}(t). \tag{4.9}$$