Harmonic Excitation

1 Introduction to Harmonic Excitation

In the previous chapters, the only force present was the force of the spring. Although we also considered gravity, this was a constant force and thus not very interesting. What will happen if we cause a time-dependent **external force** $F_e(t)$ on the mass? In this case the differential equation for an undamped motion should be rewritten to

$$m\ddot{x} + kx = F_e(t). \tag{1.1}$$

We can get about any motion, depending on the external force. In reality external forces are often harmonic. We therefore assume that

$$F_e(t) = \hat{F}_e \cos \omega t, \tag{1.2}$$

where ω is the **angular frequency of the external force**. To solve this differential equation, we first need to find the homogeneous solution. This solution is already known from previous chapters though. So we focus on the particular solution $x_p(t)$. We assume that it can be written as

$$x_p(t) = \hat{x}_p \cos \omega t. \tag{1.3}$$

Inserting this in the differential equation will give

$$\hat{x}_p = \frac{\hat{F}_e}{m} \frac{1}{(\omega_n^2 - \omega^2)} \qquad \Rightarrow \qquad x_p(t) = \frac{\hat{F}_e}{m} \frac{1}{(\omega_n^2 - \omega^2)} \cos \omega t. \tag{1.4}$$

If we combine this with the general solution to the homogeneous problem, we find that

$$x(t) = \frac{v_0}{\omega_n} \sin \omega_n t + x_0 \cos \omega_n t + \frac{\hat{F}_e}{m} \frac{1}{(\omega_n^2 - \omega^2)} \left(\cos \omega t - \cos \omega_n t \right). \tag{1.5}$$

A very important thing can be noticed from this equation. If $\omega \to \omega_n$, then $x_p(t) \to \infty$ and thus also $x(t) \to \infty$. This phenomenon is called **resonance** and is defined to occur if $\omega = \omega_n$. It is something engineers should definitely prevent.

2 Resonance

When looking at equation 1.5 we can see that it is undefined for $\omega = \omega_n$. What happens if we force a system to vibrate at its natural frequency? To find this out, we set $\omega = \omega_n$. The differential equation now becomes

$$\ddot{x} + \omega_n^2 x(t) = \frac{\hat{F}_e}{m} \cos \omega_n t. \tag{2.1}$$

If we try a solution of the form $x_p(t) = \hat{x}_p \cos \omega_n t$, we will only find the equation $0 = \left(\hat{F}_e / m \right) \cos \omega_n t$. So there are no solutions of the assumed form. Instead, let's try to assume that $x_p(t) = \hat{x}_p t \sin \omega_n t$. We now find that

$$\hat{x}_p = \frac{\hat{F}_e}{2m\omega_n} \qquad \Rightarrow \qquad x_p(t) = \frac{\hat{F}_e}{2m\omega_n} t \sin \omega_n t. \tag{2.2}$$

What we get is a vibration in which the amplitude increases linearly with time. So as the time t increases, also the amplitude of the motion increases. This continues until the system can't sustain the large amplitudes anymore and will fail.

3 Beat Phenomenon

When the external force isn't vibrating at exactly the natural frequency of a system, but only close to it, also interesting things occur. First let's define the two variables $\Delta\omega$ and $\bar{\omega}$ as

$$\Delta\omega = \frac{\omega_n - \omega}{2} \qquad \text{and} \qquad \bar{\omega} = \frac{\omega_n + \omega}{2}. \tag{3.1}$$

Let's once more consider equation 1.5. If we have no initial displacement or velocity ($x_0 = 0$ and $v_0 = 0$), then we can rewrite this equation to

$$2\frac{\hat{F}_e}{m}\frac{1}{(\omega_n^2 - \omega^2)}\sin(\Delta\omega t)\sin(\bar{\omega}t) = 2\frac{\hat{F}_e}{m}\frac{1}{(\omega_n^2 - \omega^2)}\sin\left(\frac{2\pi}{T_1}t\right)\sin\left(\frac{2\pi}{T_2}t\right). \tag{3.2}$$

As $\omega \to \omega_n$ also $\Delta\omega \to 0$ and $\bar{\omega} \to \omega_n$. So it follows that T_1 will become very large, while T_2 is close to the natural frequency of the system. Since T_1 is so large, we can define the amplitude of the vibration as

$$A(t) = 2\frac{\hat{F}_e}{m}\frac{1}{(\omega_n^2 - \omega^2)}\sin\left(\frac{2\pi}{T_1}t\right). \tag{3.3}$$

So we now have a rapid oscillation with a slowly varying amplitude. This phenomenon is called the **beat phenomenon** and one variation of the amplitude is called a **beat**. As the forcing frequency ω goes closer to the natural frequency ω_n, both the amplitude and the period of a beat increase.

4 Harmonic Excitation of Damped Systems

Let's involve damping in our equations. We then get

$$m\ddot{x} + c\dot{x} + kx = \hat{F}_e\cos\omega t \qquad \Leftrightarrow \qquad \ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{\hat{F}_e}{m}\cos\omega t. \tag{4.1}$$

Let's assume our particular solution can be written as

$$x_p(t) = X\cos(\omega t - \theta). \tag{4.2}$$

Inserting this in the differential equation, and solving for X and θ, will eventually give

$$X = \frac{\hat{F}_e}{m}\frac{1}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}} \qquad \text{and} \qquad \theta = \arctan\left(\frac{2\zeta\omega_n\omega}{\omega_n^2 - \omega^2}\right). \tag{4.3}$$

To find the general solution set, add $x_p(t)$ up to the solution of the homogeneous equation and use initial conditions to solve for the coefficients A and ϕ.

Let's define the (dimensionless) **frequency ratio** as

$$r = \frac{\omega}{\omega_n}. \tag{4.4}$$

We can now rewrite X and θ to

$$X = \frac{\hat{F}_e}{k}\frac{1}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}} \qquad \text{and} \qquad \theta = \arctan\left(\frac{2\zeta r}{1 - r^2}\right). \tag{4.5}$$

If $r \to 1$ then X goes to a given maximum value. This maximum value strongly depends on the damping ratio ζ. For large values of ζ, resonance is hardly a problem. However, if ζ is small, resonance can still occur.

5 Sinusoidal Forcing Functions

We have up to know only considered forcing functions involving a cosine. Of course forcing functions can also be expressed using a sine. Let's examine the forcing function

$$F_e(x) = \hat{F}_e \sin \omega t. \tag{5.1}$$

The particular solution to the (damped) differential equation then becomes

$$x_p(t) = X \sin(\omega t - \theta). \tag{5.2}$$

The variables X and θ are still the same as in equation 4.5.

6 Base Excitation

Let's now suppose no external force is acting on the mass. Instead the base on which the spring is connected, is moving by an amount $x_b(t)$, as shown in figure 1.

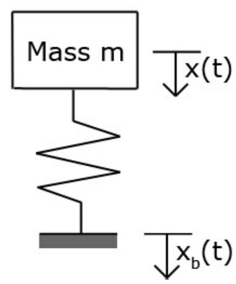

Figure 1: Definition of variables in base excitation.

The elongation of the spring is now not given by just $x(t)$, but by $x(t) - x_b(t)$. Identically, its velocity with respect to the ground is now $\dot{x}(t) - \dot{x}_b(t)$. So this makes the differential equation describing the problem

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 2\zeta\omega_n\dot{x}_b(t) + \omega_n^2 x_b(t). \tag{6.1}$$

Often the base excitation is harmonic, so we assume that

$$x_b(t) = \hat{x}_b \sin \omega_b t. \tag{6.2}$$

This makes the differential equation

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 2\zeta\omega_n\omega_b\hat{x}_b \cos \omega_b t + \omega_n^2\hat{x}_b \sin \omega_b t. \tag{6.3}$$

We have two nonhomogeneous parts. We can therefore find two separate particular solutions for the differential equation (one for each part). If we set $\hat{F}_e/m = 2\zeta\omega_n\omega_b$ (or identically $\hat{F}_e/k = 2\zeta r$), then we have exactly the same problem as we have seen earlier with the cosine forcing function (equation 4.2). If we, on the other hand, set $\hat{F}_e/m = \omega_n^2\hat{x}_b$ (or identically $\hat{F}_e/k = \hat{x}_b$), then we have the same problem as we just saw with the sine forcing function (equation 5.2). Add the two solutions up to get the total particular solution

$$x_p(t) = \frac{2\zeta r\hat{x}_b}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \cos(\omega t - \theta) + \frac{\hat{x}_b}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \sin(\omega t - \theta). \tag{6.4}$$

The value of θ is still the same as it was in equation 4.5.