General Forced Vibrations

1 The Impulse Function

An **impulse excitation** is a force that is applied for a very short duration Δt with respect to the vibration period $T = 2\pi/\omega_n$. It is an example of a **shock loading**. Such an impulse can be mathematically represented by using the **unit impulse function** $\delta(t)$ (also called the **Dirac delta function**), defined such that

$$\delta(t-\tau) = 0 \qquad \text{for } t \neq \tau, \tag{1.1}$$

$$\int_{-\infty}^{\infty} \delta(t-\tau)dt = 1. \tag{1.2}$$

But how does this effect the motion of a system? Let's suppose we have a system with no initial displacement and mass, that is given an impulse $\hat{F}_e$ at time $t = \tau$. The corresponding differential equation is

$$m\ddot{x} + c\dot{x} + kx = F_e\delta(t-\tau). \tag{1.3}$$

This impulse will cause the linear momentum of the mass to change by

$$\hat{F}_e = F_e\Delta t = m\,\Delta v = mv_\tau. \tag{1.4}$$

So this situation is similar to the case where the object simply has an initial velocity of v_τ at time $t = \tau$ (with $x_\tau = 0$). If we apply this, for example, to an underdamped system, we would get the equation of motion

$$x(t) = \hat{F}_e h(t-\tau), \qquad \text{where} \qquad h(t) = \frac{1}{m\omega_d}e^{-\zeta\omega_n t}\sin\omega_d t. \tag{1.5}$$

The function $h(t)$ is now called the **impulse response function**.

2 The Step Function

Another case of a forcing function is the **unit step function** $u(t)$ (also called the **Heaviside step function**, defined such that

$$u(t-\tau) = \begin{cases} 0 & \text{for } t < \tau, \\ 1 & \text{for } t \geq \tau. \end{cases} \tag{2.1}$$

Let's consider the underdamped differential equation

$$m\ddot{x} + c\dot{x} + kx = \hat{F}_e u(t-\tau). \tag{2.2}$$

If $x_0 = 0$ and $v_0 = 0$, it can be shown that

$$x(t) = \frac{\hat{F}_e}{k}\left(1 - \frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\cos\left(\omega_d t - \theta\right)\right), \tag{2.3}$$

where θ is given by

$$\theta = \arctan\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right). \tag{2.4}$$

This solution looks awfully familiar. In fact, it corresponds to a vibration with equilibrium point $x_e = \hat{F}_e/k$ and initial displacement $x_0 = 0$.

3 Replacing a Periodic Forcing Function by a Fourier Series

What if we don't have just an impulse or a step function, but a continuous forcing function $F_e(t)$? In this case we can take the force $F_e(\tau)$ at time τ for a given moment $d\tau$ and replace it by an impulse of magnitude $F_e(\tau)d\tau$. We can then find the impulse response function $h(t-\tau)$ for the time τ. If we do this for all times τ and sum everything up, we will eventually find as particular solution

$$x_p(t) = \int_0^t F_e(\tau)h(t-\tau)d\tau = \int_0^t F_e(t-\tau)h(\tau)d\tau. \tag{3.1}$$

This integral is called the **convolution integral**. It is often difficult to evaluate the integral. If we have a periodic forcing function $F_e(t)$ (with period T and angular frequency $\omega_T = 2\pi/T$), we can apply a trick though. We can replace $F_e(t)$ by a **Fourier series**. To do this, we use

$$F_e(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(n\frac{2\pi}{T}t\right) + b_n \sin\left(n\frac{2\pi}{T}t\right)\right). \tag{3.2}$$

The coefficients a_0, a_n and b_n are given by

$$a_0 = \frac{2}{T}\int_0^T F_e(t)dt, \tag{3.3}$$

$$a_n = \frac{2}{T}\int_0^T F_e(t)\cos\left(n\frac{2\pi}{T}t\right)dt, \tag{3.4}$$

$$b_n = \frac{2}{T}\int_0^T F_e(T)\sin\left(n\frac{2\pi}{T}t\right)dt. \tag{3.5}$$

Now we have a new way to write the forcing function. How we use this will be treated in the next paragraph.

4 Finding the Equation of Motion

When we replace the periodic forcing function $F_e(t)$ by a Fourier Series, we can rewrite the differential equation to

$$m\ddot{x} + c\dot{x} + kx = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(n\omega_T t\right) + b_n \sin\left(n\omega_T t\right)\right). \tag{4.1}$$

We now repeatedly take one element from the right hand side of the equation, solve the equation for that part, and in the end sum everything up. We will then find our particular solution. In an equation this becomes

$$x_p(t) = x_{a_0}(t) + \sum_{n=1}^{\infty} \left(x_{a_n}(t) + x_{b_n}(t)\right). \tag{4.2}$$

The individual solution are then the solutions of the differential equations

$$m\ddot{x}_{a_0} + c\dot{x}_{a_0} + kx_{a_0} = a_0/2, \tag{4.3}$$

$$m\ddot{x}_{a_n} + c\dot{x}_{a_n} + kx_{a_n} = a_n \cos\left(n\omega_T t\right), \tag{4.4}$$

$$m\ddot{x}_{b_n} + c\dot{x}_{b_n} + kx_{b_n} = b_n \sin\left(n\omega_T t\right). \tag{4.5}$$

All these equations are equations we have solved before. For completeness' sake we will give the solutions once more. They are

$$x_{a_0} = \frac{a_0}{2k}, \tag{4.6}$$

$$x_{a_n} = \frac{a_n}{m} X \cos\left(n\omega_T t - \theta_n\right), \tag{4.7}$$

$$x_{b_n} = \frac{b_n}{m} X \sin\left(n\omega_T t - \theta_n\right). \tag{4.8}$$

The variables X and θ_n are defined as

$$X = \frac{1}{\sqrt{\left(\omega_n^2 - \left(n\omega_T\right)^2\right)^2 + \left(2\zeta n\omega_n\omega_T\right)^2}} \qquad \text{and} \qquad \theta_n = \arctan\left(\frac{2\zeta n\omega_n\omega_T}{\omega_n^2 - \left(n\omega_T\right)^2}\right). \tag{4.9}$$

This is how the particular solution is found. Combine this with the specific solution to the problem to find the general solution to the differential equation.

5 Using the Laplace Transform

When solving the differential equation, the **Laplace transform** is often a convenient tool. Let's consider the differential equation

$$m\ddot{x} + c\dot{x} + kx = F_e(x) \qquad \Leftrightarrow \qquad \ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{F_e(x)}{m}. \tag{5.1}$$

Taking the laplace transform, and solving for $X(s)$, will give

$$X(s) = \frac{sx_0 + v_0 + 2\zeta\omega_n x_0}{s^2 + 2\zeta\omega_n s + \omega_n^2} + \frac{1}{m}\frac{\mathcal{F}_e(s)}{s^2 + 2\zeta\omega_n s + \omega_n^2}, \tag{5.2}$$

where $L\left\{F_e(t)\right\} = \mathcal{F}_e(s)$. Often it occurs that $x_0 = 0$ and $v_0 = 0$. The middle term of the above equation then disappears. To find $x(t)$, you apply the inverse Laplace transform. When doing this, you often need to use a Laplace transform table like table 1.

Function $x(t) = L^{-1}\{X(s)\}$	**Laplace Transform** $X(s) = L\{x(t)\}$	**Condition**
e^{-at}	$\frac{1}{s+a}$	
$\sin \omega_n t$	$\frac{a}{s^2+\omega_n^2}$	
$\cos \omega_n t$	$\frac{s}{s^2+\omega_n^2}$	
$\frac{1}{s^2+2\zeta\omega_n s+\omega_n^2}$	$\frac{1}{\omega_d} e^{-\zeta\omega_n t} \sin\left(\omega_d t\right)$	Underdamped Motion ($\zeta < 1$)
$\frac{\omega_n^2}{s}\frac{1}{s^2+2\zeta\omega_n s+\omega_n^2}$	$1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\left(\omega_d t + \arccos\left(\zeta\right)\right)$	Underdamped Motion ($\zeta < 1$)
$e^{-at}x(t)$	$X(s+a)$	
$\delta(t-a)$	e^{-as}	
$u(t-a)x(t)$	$e^{-as}X(s)$	

Table 1: Often used Laplace transforms.