Free Vibrations

1 Introduction to Vibrations

Vibrations are often unwanted phenomena in aerospace engineering. When systems start vibrating at the wrong frequencies, they might fail, which isn't particulary good. In reality all systems are **continuous systems**, meaning that the displacements of parts depend on a lot of factors. To simplify this, the system is often modeled as a **discrete system**. Here the system is split up in parts, which are then evaluated separately.

Two types of vibrations can be distinguished, being **free vibrations** and **forced vibrations**. In free vibrations no energy is exchanged with the environment, while in forced vibrations there is energy exchange. First we will have a look at free vibrations. Forced vibrations will be treated in later chapters.

2 Stiffness of an Axially Loaded Rod

Let's consider an axially loaded rod of negligible mass, having a mass attached to its end. We know that the displacement δ of the mass is given by

$$\delta = \frac{FL}{EA}, \tag{2.1}$$

where F is the (tensional) force in the bar, L is the length of the bar, E is the E-modulus and A is the cross-sectional area. The **stiffness** k is defined as the force needed to reach unit displacement. In an equation this is

$$k = \frac{F}{\delta}. \tag{2.2}$$

So for our axially loaded rod, we will have

$$k = \frac{EA}{L}. \tag{2.3}$$

We can now model the situation. We do this by replacing the bar by a spring with the stiffness k. This is shown in figure 1.

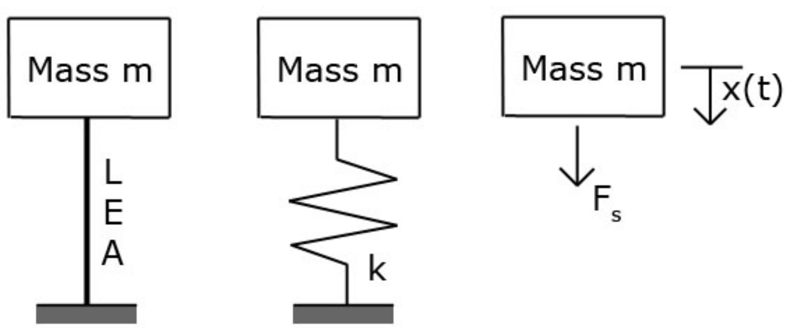

Figure 1: Modeling of an axially loaded rod.

3 Motion of an Axially Loaded Rod

Previously we considered the axially loaded rod and modeled it. Let's turn to figure 1 once more. We would like to know how the system will move, if it is given a certain initial displacement/velocity.

To find this out, we use Newton's second law $F = ma$. The only force acting on the mass is the spring force F_s (we don't consider gravity yet). We know that the spring force varies linearly with the displacement x by the stiffness k. However, if the block moves upward, the spring forces points downward. So there is a negative relation between the two. In an equation this becomes

$$F_s = -kx. \tag{3.1}$$

If we combine this with Newton's second law, we will find that

$$m\ddot{x} = F_s = -kx \qquad \Rightarrow \qquad m\ddot{x} + kx = 0. \tag{3.2}$$

The solution can be found by solving this differential equation. We will get

$$x(t) = c_1 \cos\left(\sqrt{\frac{k}{m}}t\right) + c_2 \sin\left(\sqrt{\frac{k}{m}}t\right). \tag{3.3}$$

So the system will start vibrating with a fixed angular frequency. This frequency, called the **angular eigenfrequency**, is denoted by

$$\omega_n = \sqrt{\frac{k}{m}}. \tag{3.4}$$

From this, the **eigenfrequency** f and **vibration period** T can be derived, according to

$$f = \frac{\omega_n}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \qquad \text{and} \qquad T = \frac{1}{f} = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{m}{k}}. \tag{3.5}$$

However, equation 3.3 isn't very useful. Instead, it is more meaningful to use

$$x(t) = A\sin(\omega_n t + \phi), \tag{3.6}$$

where A is the amplitude (usually taken to be positive) and ϕ is the phase. Both follow from the boundary conditions. If we give the mass an initial displacement x_0 and an initial velocity v_0, then we can find A and ϕ. They will be

$$A = \sqrt{x_0^2 + \left(\frac{v_0}{\omega_0}\right)^2} \qquad \text{and} \qquad \phi = \tan^{-1}\left(\frac{\omega_n x_0}{v_0}\right). \tag{3.7}$$

4 Effects of Gravity

Previously we haven't considered gravity. What happens if we do? In this case the total force acting on the mass will be $F_s + mg$. This would turn the differential equation into

$$m\ddot{x} + kx = mg. \tag{4.1}$$

When solving differential equations, we know we first ought to find the **homogeneous solution** of the differential equation

$$m\ddot{x} + kx = 0. \tag{4.2}$$

We already know the solution for this. After we have found the homogeneous solution, we need to find one **particular solution** $x_p(t)$. Note that the non-homogeneous term mg is just a constant. So the particular solution is probably constant too. It can then be shown that

$$x_p(t) = \frac{mg}{k}. \tag{4.3}$$

This makes the solution for the differential equation

$$x(t) = x_h(t) + x_p(t) = A \sin(\omega_n t + \phi) + \frac{mg}{k}. \tag{4.4}$$

Note that if the amplitude A is zero, then the mass will just have a constant displacement of mg/k. This also follows from statics.

In vibrational engineering the homogeneous solution $x_h(t)$ is sometimes called the **transient solution** $x_{tr}(t)$ and the particular solution $x_p(t)$ is also called the **steady state solution** $x_{ss}(t)$.

5 Motion of a Laterally Loaded Rod

Of course there are more kinds of vibrations then masses on axially loaded rods. Let's consider a laterally loaded rod, as shown in figure 2. The rod has an (area) moment of inertia I.

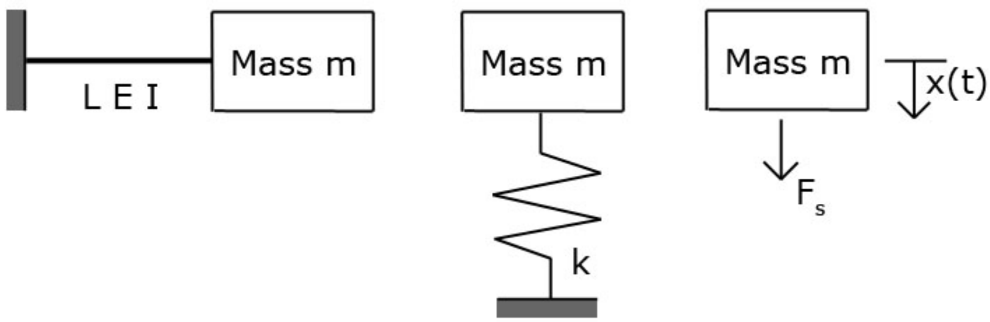

Figure 2: Modeling of a laterally loaded rod.

This time the displacement δ, and thus also the stiffness k and natural frequency ω_n, are given by

$$\delta = \frac{FL^3}{3EI} \qquad \Rightarrow \qquad k = \frac{F}{\delta} = \frac{3EI}{L^3} \qquad \Rightarrow \qquad \omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{3EI}{mL^3}}. \tag{5.1}$$

The rest of the problem is similar to what we have previously discussed.

6 Rotation of a Torsionally Loaded Rod

Now let's consider an other case. We have a disk with (mass) moment of inertia J, connected to a rod with (area) polar moment of inertia I_p, as shown in figure 3.

We will be looking at the angular displacement θ. This depends on the moment M that is acting between the rod and the disk. If this moment is known, then the angular displacement can be found using

$$\theta = \frac{ML}{GI_p}. \tag{6.1}$$

Now we can define the **torsional stiffness** as

$$k = -\frac{M}{\theta} = \frac{GI_p}{L}. \tag{6.2}$$

Note that the torsional stiffness has as unit Nm, while the normal stiffness has as unit N/m.

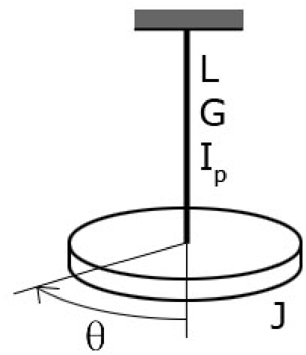

Figure 3: Modeling of a torsionally loaded rod.

Newton's second law for rotations states that $M = J\alpha = J\ddot{\theta}$. Combining this with the torsional stiffness gives us the differential equation

$$J\ddot{\theta} + k\theta = 0. \tag{6.3}$$

We already know the solution to this! It is just

$$\theta(t) = \hat{\theta} \sin(\omega_n t + \phi), \tag{6.4}$$

where $\omega_n = \sqrt{k/J}$ is the angular natural frequency and $\hat{\theta}$ denotes the amplitude of the vibration.

7 Other Cases

We have seen axially loaded rods, laterally loaded rods and torsionally loaded rods. There are, however, infinitely many other types of systems. It is, for example, possible to combine multiple springs in a system. We won't be treating all those combinations, of course. If this is the case, the skills of the engineer come into play.

However, we're not letting you venture into those problems unguided. When face with a more complicated system, just follow the following steps:

- Consider the point of which you want to know the motion.
- Express the force/moment at that point as a function of the (angular) displacement.
- Use Newton's second law to find the differential equation.
- Solve the differential equation to find the equation of motion.

8 Using Energy

In a free vibration (without damping), energy is conserved. You can consider two types of energy in a vibration. These are **kinetic energy** T and **potential energy** U. Let's consider those energies for the axially/laterally loaded rod. The kinetic energy is given by

$$T = \frac{1}{2}m\dot{x}^2. \tag{8.1}$$

The potential energy here consists of spring energy and gravitational energy, and is given by

$$U = \frac{1}{2}kx^2 - mgx, \tag{8.2}$$

A very important rule is the rule of **conservation of energy**. It states that

$$T + U = \text{constant} = E, \tag{8.3}$$

where E is the **vibrational energy**. If the mass passes through the equilibrium point, then T is maximal. If the mass has maximum deflection, then U is maximal.

It all sounds fun, but how can we use this? To use this, we differentiate equation 8.3 with respect to time. What we get is

$$\frac{dT}{dt} + \frac{dU}{dt} = 0. \tag{8.4}$$

If we work this out for an axially/laterally loaded rod, we will get

$$\dot{x}\left(m\ddot{x} + kx - mg\right) = 0. \tag{8.5}$$

Note that $\dot{x}$ can't be zero for all t (or it would be an awfully boring problem). We now remain with

$$m\ddot{x} + kx = mg, \tag{8.6}$$

which is exactly the differential equation we needed to solve the problem.

The method that was just shown is called the **energy method**. When damping occurs, the energy method is slightly more complicated. Now the lost energy also needs to be taken into account. We will not treat this here though.

You may be wondering why we should use energy? Isn't it easier to just use Newton's second law? Well, using Newton's second law is easier for normal one-dimensional problems. However, using energy when solving multi-dimensional problems has various advantages. We will consider multi-dimensional problems in more detail in the latest chapter of this summary.