Damped Motions

1 Introduction to Damping

The free vibrations discussed in the previous chapter don't stop oscillating. This isn't very realistic. So we need to change our model. We therefore apply **viscous damping**. We assume that there is a force acting on the mass in a direction opposite to the motion. This force is also proportional to the motion (fast-moving objects have more friction). So we introduce the **damping force**

$$f_c = -c\dot{x}(t), \tag{1.1}$$

where the factor $c > 0$ is the **damping coefficient**. If we combine this with the previous differential equation, we now get

$$m\ddot{x} + c\dot{x} + kx = 0. \tag{1.2}$$

To solve this differential equation, we should first solve the **characteristic equation**

$$m\lambda^2 + c\lambda + k = 0 \qquad \Rightarrow \qquad \lambda = -\frac{c}{2m} \pm \frac{\sqrt{c^2 - 4km}}{2m}. \tag{1.3}$$

The behaviour of the system now depends on the factor $c^2 - 4km$. Different things occur if this factor is either smaller than zero, equal to zero or bigger than zero. Since this is so important, the **critical damping coefficient** c_{cr} is defined such that

$$c_{cr}^2 - 4km = 0 \qquad \Rightarrow \qquad c_{cr} = 2\sqrt{km} = 2m\omega_n. \tag{1.4}$$

Here ω_n is the natural frequency of the undamped system, also called the **undamped natural frequency**. We can now also define the **damping ratio** as

$$\zeta = \frac{c}{c_{cr}} = \frac{c}{2\sqrt{km}} = \frac{c}{2m\omega_n}. \tag{1.5}$$

Note that since $c > 0$ also $\zeta > 0$. Using ζ, the characteristic equation can be rewritten as

$$\lambda = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}. \tag{1.6}$$

Three cases can now be distinguished, which will be treated in the coming paragraphs.

2 Underdamped Motion

In the **underdamped motion** the damping ratio ζ is smaller than one. The solutions λ_1 and λ_2 of the characteristic equation are now complex conjugates, being

$$\lambda_1 = -\zeta\omega_n - \omega_n\sqrt{1-\zeta^2}i \qquad \text{and} \qquad \lambda_2 = -\zeta\omega_n + \omega_n\sqrt{1-\zeta^2}i. \tag{2.1}$$

Before we write down the solution, we first define the **damped natural frequency** to be

$$\omega_d = \omega_n\sqrt{1-\zeta^2}. \tag{2.2}$$

If we now solve the differential equation, we will find as the general solution

$$x(t) = Ae^{-\zeta\omega_n t}\sin\left(\omega_d t + \phi\right), \tag{2.3}$$

where A is the **initial amplitude**. Note that due to damping, the frequency of the vibration has changed. The values of A and ϕ depend on the initial position x_0 and initial velocity v_0 and can be found using

$$A = \sqrt{x_0^2 + \left(\frac{v_0 + \zeta\omega_n x_0}{\omega_d}\right)^2} \qquad \text{and} \qquad \phi = \tan^{-1}\left(\frac{x_0\omega_d}{v_0 + \zeta\omega_n x_0}\right). \tag{2.4}$$

The underdamped motion results in an oscillation with a decreasing amplitude.

3 Overdamped Motion

In the **overdamped motion** the damping ratio ζ is bigger than one. The roots to the characteristic equation are now two real values, being

$$\lambda_1 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1} \qquad \text{and} \qquad \lambda_2 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}. \tag{3.1}$$

In this case no oscillation occurs. The mass will not even pass the equilibrium position. Instead, it will only converge to it. Before we see how, we first define

$$\omega_c = \omega_n\sqrt{\zeta^2 - 1}. \tag{3.2}$$

The motion of the mass is now described by

$$x(t) = e^{-\zeta\omega_n t}\left(a_1 e^{-\omega_c t} + a_2 e^{\omega_c t}\right). \tag{3.3}$$

The constants a_1 and a_2 once more depend on the initial conditions. They can be found using

$$a_1 = \frac{1}{2}x_0\left(1 - \frac{\zeta\omega_n}{\omega_c}\right) - \frac{v_0}{2\omega_c} \qquad \text{and} \qquad a_2 = \frac{1}{2}x_0\left(1 + \frac{\zeta\omega_n}{\omega_c}\right) + \frac{v_0}{2\omega_c}. \tag{3.4}$$

4 Critically Damped Motion

In the **critically damped motion** we have $\zeta = 1$ and thus $c = c_{cr}$. The roots of the characteristic equation are now

$$\lambda_1 = \lambda_2 = -\omega_n. \tag{4.1}$$

The solution is now given by

$$x(t) = (a_1 + a_2 t)\, e^{-\omega_n t}, \tag{4.2}$$

where the constants a_1 and a_2 are given by

$$a_1 = x_0 \qquad \text{and} \qquad a_2 = v_0 + \omega_n x_0. \tag{4.3}$$

5 Stability

We have, up to now, considered only positive k and c. Of course it is also possible to have a negative k (the force acts in the direction of the displacement or a negative c (the force acts in the direction of motion.

If, for a certain motion, $x \to \infty$, then the motion is **unstable**. Otherwise the motion is **stable**. We will look at the stability of the systems for various combinations of c and k now.

- $k > 0$ - This occurs in normal springs. In case of a deflection, the mass is pulled back to the equilibrium position.
  - For $c = 0$ we are on familiar grounds. The motion is just an undamped vibration. The amplitude is bounded ($x(t) \leq A$ for all t) so we have a stable motion. However, $x(t)$ never converges to zero. So the system is only **marginally stable**.
  - For $c > 0$ we are dealing with a damped motion. It doesn't matter whether the system is underdamped, overdamped or critically damped. In all cases $x(t) \to 0$ as $t \to \infty$, so the system is asymptotically stable. (How x goes to zero does depend on ζ though, but this is irrelevant for the stability.)
  - If $c < 0$, then the amplitude of the motion increases unbounded for increasing t. So the motion is **unstable**. However, we can distinguish two cases.

  * If $c^2 < 4mk$ (thus $\zeta < 1$), then there are still oscillations. In this case we have **flutter instability**.
  * For $c^2 \geq 4mk$ (thus $\zeta \geq 1$) no oscillation occurs. As soon as the mass departs from the equilibrium, it will never return. Now there is **divergent instability**.
- When $k < 0$ the mass gets pushed away from the equilibrium position, independent of the damping coefficient c. For $c > 0$ the motion only occurs slower than for $c < 0$. Since $x(t) \to \infty$ as $t \to \infty$, the motion is **unstable**. To be more precise, there is **divergent instability**, since not a single oscillation occurs.

6 Coulomb Friction

Suppose we have mass, horizontally sliding over a surface, as shown in figure 1.

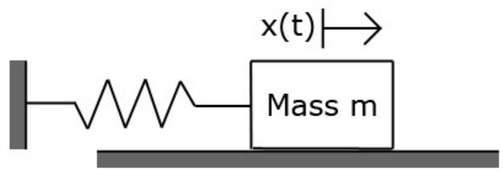

Figure 1: Mass connected to a spring, sliding over a horizontal surface.

The force that acts on the mass depends on whether it is moving, and in which direction, according to

$$f_c(\dot{x}) = \left\{ \begin{matrix} -\mu N & \text{if } \dot{x} > 0 \\ 0 & \text{if } \dot{x} = 0 \\ \mu N & \text{if } \dot{x} < 0 \end{matrix} \right\} = \mu N \left\{ \begin{matrix} -1 & \text{if } \dot{x} > 0 \\ 0 & \text{if } \dot{x} = 0 \\ 1 & \text{if } \dot{x} < 0 \end{matrix} \right\} = -\mu N \text{sgn}(\dot{x}), \tag{6.1}$$

where μ is the **dynamic friction coefficient** and N is the normal force acting on the block. Also $\text{sgn}(\tau)$ is the **signum function**, defined to give 1 when $\tau > 0$, 0 when $\tau = 0$ and -1 if $\tau < 0$. This kind of damping is called **Coulomb damping**. The resulting differential equation is

$$m\ddot{x} + \mu N \text{sgn}(\dot{x}) + kx = 0. \tag{6.2}$$

This is very hard to solve, due to the signum function. It is wiser to examine the problem in steps. Suppose the mass has no initial velocity ($v_0 = 0$), but only an initial displacement δ_0. If the initial displacement is big enough to overcome the friction force ($k\delta_0 > \mu N$), the block will start sliding. After π/ω_n seconds it will have reached a new maximum deflection δ_1. It can be shown that this deflection is

$$\delta_1 = \delta_0 - \frac{2\mu N}{k}. \tag{6.3}$$

If the force is big enough to let the block slide again, it will have another half oscillation of π/ω_n seconds, but its maximum deflection will have decreased again by the same amount. So,

$$\delta_2 = \delta_1 - \frac{2\mu N}{k}. \tag{6.4}$$

This continues until after i half oscillations $k\delta_i \leq \mu N$. The block has been oscillating for $i\pi/\omega_n$ seconds. But now the oscillation has ended and the block will remain at δ_i.