THERMODYNAMICS & FLUIDS - UFMEQU-20-1

FLUIDS – BOOK 2 of 2

Students must read through these notes and work through the various exercises in their own time in parallel with the course of lectures.

CONTENTS

1 Fluid Flow with Friction

In Semester 1, we covered fluid flow through pipes looking at the Continuity Equation and Bernoulli's equation. One of the assumptions we made about the flow is that it was frictionless. In real systems however, friction between the fluid molecules and solid surfaces (like the pipe walls), as well as between the fluid molecules themselves, cause a pressure loss to occur in the fluid.

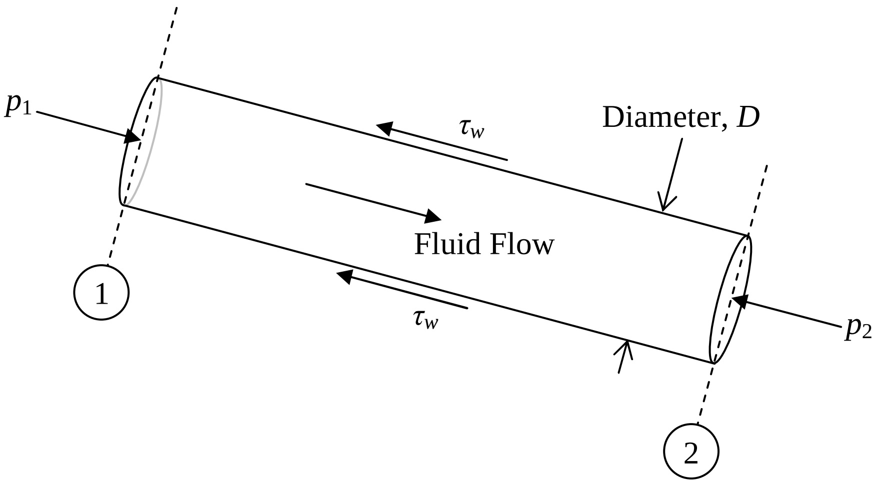


Figure 1: Fluid flowing through a pipe, with wall shear stress, τ_w

Figure 1 illustrates a pipe with fluid flowing through it. While similar to what we previously studied in the first semester, this time, there is **shear stress** occurring between the fluid and the solid pipe wall, represented by $\boldsymbol{\tau_w}$.

Without friction, Bernoulli's equation gives:

$$\text{Total pressure} = p + \frac{1}{2}\rho C^2 + \rho g z = \text{constant}$$

So:

$$p_1 + \frac{1}{2}\rho C_1^2 + \rho g z_1 = p_2 + \frac{1}{2}\rho C_2^2 + \rho g z_2$$

The effect of friction is that shear stresses at the wall of the pipe, τ_w, cause a drop in total pressure along the flow direction of Δp.

The equation becomes:

$$p_1 + \frac{1}{2}\rho C_1^2 + \rho g z_1 = p_2 + \frac{1}{2}\rho C_2^2 + \rho g z_2 + \Delta p \qquad (1)$$

This section of work is about determining $\boldsymbol{\Delta p}$.

The term τ_w represents the shear stress occurring in the fluid at the wall of the pipe. Being a stress, it is defined as a **force divided by an area**. In this case, it is the force required to move the fluid along the pipe wall area:

$$\tau_w = \frac{F}{A_s}$$

where A_s is the internal surface area of the pipe.

The force required to overcome this shear stress is the drop in pressure multiplied by the cross sectional area. This equals the shear stress multiplied by the surface area over which the shear stress is occurring. Mathematically:

$$\Delta p A = \tau_w A_s$$

$$\Delta p\left(\frac{\pi D^2}{4}\right) = \tau_w (\pi D)(L)$$

Rearranging to solve for the shear stresses at the wall of the pipe:

$$\tau_w = \frac{\Delta p\left(\frac{\pi D^2}{4}\right)}{(\pi D)(L)} = \frac{D}{4}\frac{\Delta p}{L} = \frac{R}{2}\frac{\Delta p}{L}$$

where L is the length, D is the diameter and R is the radius.

The problem therefore becomes that of finding out how the shear stress varies with flow. The way in which this is done depends on the nature of the flow.

1.1 Laminar Flow

At low velocities, a few duct flows exhibit the **laminar flow** characteristics, where layers of adjacent fluid slide over each other in an ordered manner, exerting shear forces because of the relative movement. Fluid particle paths (streamlines) are straight, with fluid near stationary solid surfaces (pipe wall) moving more slowly than fluid away from solid surfaces.

The velocity in the flow direction is constant and steady and does not vary with time. The velocity perpendicular to the flow direction is zero at all times. Laminar flow is illustrated in Figure 2(a).

1.2 Turbulent Flow

Turbulent flow is one of the last great mysteries of physics and there is a Nobel Prize waiting for the person who can uncover its mysteries. The unexplained facts are that if the velocity of a laminar flow is gradually increased there comes a point at which the nature of the flow changes. Particle paths become irregular and chaotic, leading to large scale mixing between adjacent layers. The flow becomes very time dependent as sudden bursts of chaotic energy are created within it and then gradually dissipated. It all appears random and unpredictable. Turbulent flow is illustrated in Figure 2(b).

The structure is similar to large-scale turbulence behind a moving lorry or as you see when looking down at the water flow around a supporting pillar of a bridge. There are series of circular motion eddies generated which gradually dissipate in a wake behind the structure. The difference in duct flow is that no physical reason for the turbulence is apparent, as the pipe walls are generally relatively smooth.

1.3 Analysis of Flows

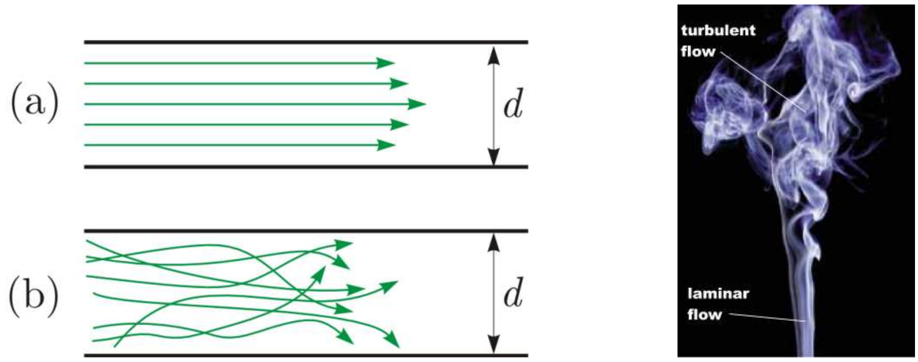


Figure 2: (a) Laminar flow; (b) Turbulent flow. Rising smoke demonstrates the transition between laminar and turbulent flow.

Laminar flow can theoretically be analysed, but turbulent flow cannot, and the approach has to be one of using validated experimental (empirical) results. Whether a flow is laminar or turbulent is determined by the value of the key flow non-dimensional number, the **Reynolds number**, **Re**.

The Reynolds number is defined as:

$$\mathrm{Re} = \frac{\rho c D}{\mu}$$

where ρ is the fluid density, c is the flow velocity, D is the pipe diameter and μ is the dynamic viscosity of the flow. (This term was defined last semester as part of the Dimensional Analysis lecture).

The Reynolds number represents the **ratio of the inertial to the viscous forces within the flow**. The term ρc is the momentum per unit volume, so the higher it is the more likely there is spare energy in the flow for turbulent behaviour. The higher D, the less restrained this excess energy is by the pipe walls, but the higher μ the more likely it is to be damped out.

The **critical Re for pipe geometry is about 2000**. Below this, flow is laminar; above it, turbulence tends to start. The figure is not definitive, as laminar flow can sometimes be maintained up to 3000 where circumstances are favourable. It depends upon the presence or otherwise of turbulence initiators such as bends, fittings and fluid machines. The table below summarises this:

Reynolds Number	*Flow*
Less than approx. 2000	Laminar
2000 – 3000	Critical Region
Above 3000	Partial or fully turbulent

Most flows, especially those in pipes, are turbulent. It is very unusual to achieve laminar pipe flow. The velocity has to be low, the pipe small and the fluid highly viscous. Laminar flow is of greatest value in the analysis of thin films in bearings, shock absorbers and the like.

Note the use of *c* as the symbol for velocity in these notes. A lot of texts use either *u* or *v*, which can be confused with internal energy *u* or specific volume, *v*.

1.4 Shear Stress and Fluid Viscosity

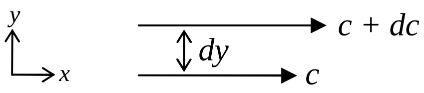


Figure 3: Two streamlines experiencing shear

The shear action is due to relative movement of sliding fluids layers. Figure 3 shows two streamlines in laminar flow, with one streamline travelling faster than the other. If the flow direction is *x* and the direction perpendicular to the flow, *y*, (see coordinate system in Figure 3) then the shear stress of one layer on the other is given by:

$$\tau = \mu \frac{dc}{dy} = \text{viscosity} \times \frac{\text{difference in speed}}{\text{distance between streamlines}} \qquad (2)$$

for a Newtonian fluid where μ is the fluid **dynamic viscosity**.

As viscosity, μ increases, the frictional effects increase. Viscosity does not vary much with pressure, but is highly temperature dependent, decreasing rapidly with temperature for liquids and increasing slowly with temperature for gases.

The SI unit is Pas (Pascal seconds), shear stress being measured in Pa (N/m^2) for a fluid. Sometimes the same units are seen broken down further into kg/ms. The model is analogous to that for solids, the velocity gradient is essentially being a fluid measure of 'strain'.

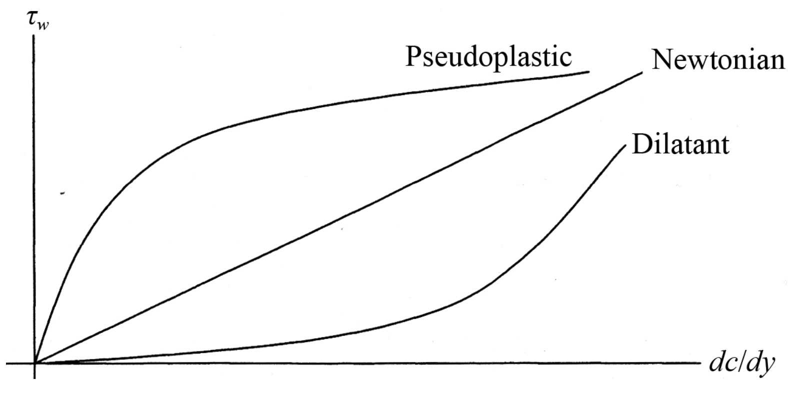


Figure 4: Newtonian and Non-Newtonian Fluids

Most genuine fluids behave in a **Newtonian** manner, but there are many substances which appear to be fluids that contain significant numbers of larger particles. These include milk, blood and ink and exhibit modified behaviour as a result. It can be seen from equation (2) that a Newtonian fluid has a linear relationship between shear stress and the shear rate (*dc*/*dy*). Non-Newtonian fluids do not possess such a linear relationship, as shown in Figure 4.

You may also see viscosity referred to as **kinematic viscosity**, represented by the Greek letter nu, υ and is determined by the following equation:

$$\upsilon = \frac{\mu}{\rho}$$

i.e. the **dynamic viscosity divided by the density**. (Notice this is not the little letter *v*.)

Consequently, you may also see the Reynolds number defined as:

$$\mathrm{Re} = \frac{cD}{\upsilon}$$

1.5 Summary

Bernoulli's Equation including pressure loss due to friction is:

$$p_1 + \frac{1}{2}\rho C_1^2 + \rho g z_1 = p_2 + \frac{1}{2}\rho C_2^2 + \rho g z_2 + \Delta p$$

where the Δp term is the pressure loss due to pipe friction (and other losses)

The Reynolds number is defined as:

$$\mathrm{Re} = \frac{\rho c D}{\mu} = \frac{cD}{\upsilon}$$

and is used to define the state of the flow (either laminar or turbulent). The following table details this:

Reynolds Number	*Flow*
Less than approx. 2000	Laminar
2000 – 3000	Critical Region
Above 3000	Partial or fully turbulent

Dynamic viscosity is represented using μ and the kinematic viscosity is υ. Density is used to relate the two terms, where:

$$\upsilon = \frac{\mu}{\rho}$$

2 Laminar Flow

The analysis of **laminar flow** considers an element of fluid *dx* contained within a pipe, as shown in Figure 5. The pipe radius is *R* and the element radius is *r*. All quantities are assumed to be positive and increase in the flow direction in the length *dx*, and in the *r* direction perpendicular to the flow. If they decrease or are negative, the mathematics will indicate this. Even the shear stress τ is assumed positive in the flow direction and later shown to be negative, as it is essentially friction acting against the flow.

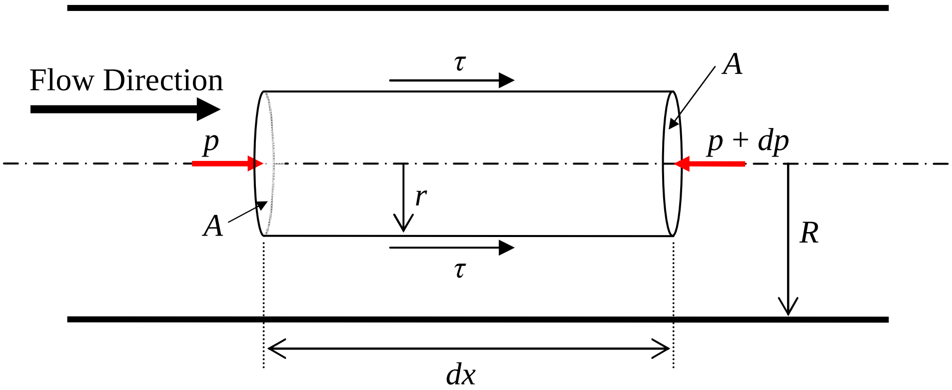


Figure 5: Element of Fluid

In the positive *x* direction, the forces acting are the pressure *p* over area *A* and the shear stresses (which act on the surface area of the element). In the opposite direction, the force acting is the pressure (*p* + *dp*) over area *A*. Therefore, balancing the forces for the element results in:

$$\tau(2\pi r)dx + p\pi r^2 - (p + dp)\pi r^2 = 0$$

Simplifying this equation and solving for τ gives us:

$$\tau(2\pi r)dx + p\pi r^2 = (p + dp)\pi r^2$$

$$2\tau dx = (p + dp)r - pr$$

$$\tau = \frac{pr + dpr - pr}{2dx}$$

$$\tau = \frac{r}{2}\frac{dp}{dx}$$

From the definition of viscosity, equation (2), we can say that:

$$\tau = \frac{r}{2}\frac{dp}{dx} = \mu\frac{dc}{dr}$$

Because there is no flow in the *r* direction, it can be assumed that *p* is also constant across the section. Any change of *p* due to the height change is small, so is neglected.

2.1 Fluid Velocity

The aim of this part of the analysis is to find velocity *c* as a function of the radius from centre-line *r*.

Separating the variables *c* and *r*:

$$\frac{r}{2}\frac{dp}{dx} = \mu\frac{dc}{dr}$$

$$\frac{r}{2\mu}\frac{dp}{dx}dr = dc$$

$$\frac{1}{2\mu}\frac{dp}{dx}rdr = dc$$

Integrating:

$$\int dc = \frac{1}{2\mu}\frac{dp}{dx}\int rdr$$

$$c = \frac{1}{2\mu}\frac{dp}{dx}\frac{r^2}{2} + C_1$$

where C_1 is the constant of integration.

The constant C_1 can be determined by applying the boundary condition that when the radius of the element is the at the pipe wall ($r = R$), the velocity of the flow is zero. This is known as the condition of no slip, where at the boundary with a solid surface the fluid molecules are supposed to be in such intimate contact with the surface that there is no relative movement. Applying this boundary condition, C_1 is determined as:

$$C_1 = -\frac{1}{2\mu}\frac{dp}{dx}\frac{R^2}{2}$$

The overall situation is therefore:

$$c = \frac{1}{2\mu}\frac{dp}{dx}\frac{r^2}{2} - \frac{1}{2\mu}\frac{dp}{dx}\frac{R^2}{2}$$

$$c = \frac{1}{2\mu}\frac{dp}{dx}\frac{1}{2}\left(r^2 - R^2\right)$$

$$c = -\frac{1}{4\mu}\frac{dp}{dx}\left(R^2 - r^2\right) = -\frac{R^2}{4\mu}\frac{dp}{dx}\left(1 - \left(\frac{r}{R}\right)^2\right) \quad (3)$$

This equation gives the **classic parabolic profile with a maximum at the centre** ($r = 0$) and reduction towards the wall which becomes increasing rapid as the wall gets close, as shown in Figure 6.

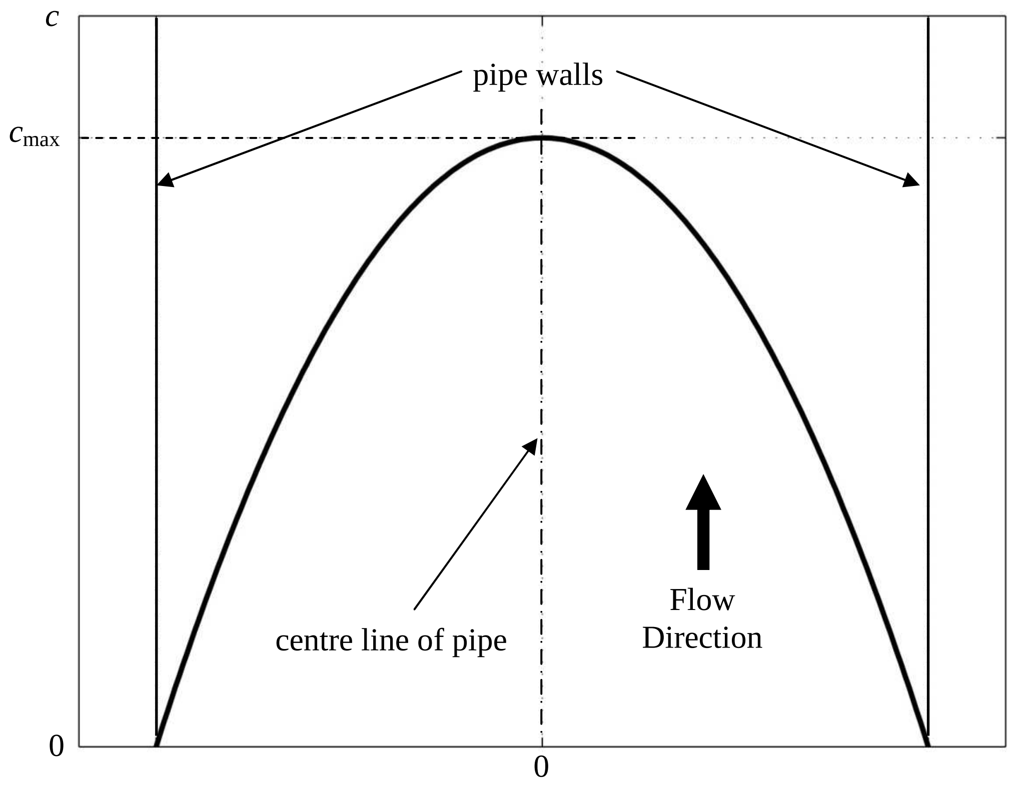


Figure 6: Velocity profile in pipe

The negative sign in equation (3) indicates that for a positive μ, the pressure gradient dp/dx must be negative, i.e. the pressure must decrease in the direction of flow. Note that the maximum velocity at the centre line is given by:

$$c_{\max} = -\frac{R^2}{4\mu}\frac{dp}{dx}\left(1-\left(\frac{0}{R}\right)^2\right) = -\frac{R^2}{4\mu}\frac{dp}{dx}$$

Whilst of interest, knowledge of the velocity profile is not the required end. The required end, is instead, **to determine the pressure drop Δp** in terms of **volumetric flow rate**, $\dot{V}$. To do this, we must first determine volumetric flow rate.

2.2 Volumetric Flow Rate

The volumetric flow rate can found from:

Flow rate = area $\times$ velocity = Ac

with the complication that velocity varies across the section. The approach is to take a **small annular element** between the centre line, shown in Figure 7, and the pipe radius and then add up the flow through all such elements by integrating between r = 0 and r = R.

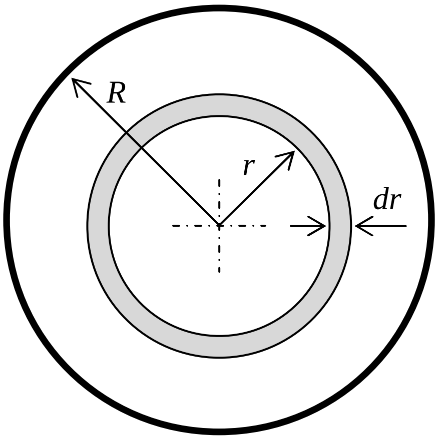


Figure 7: Annular Element of Fluid in cross section of pipe

The area of the element (the shaded area) is:

$$dA = 2\pi r dr$$

Therefore, the volumetric flow rate through the element is:

$$d\dot{V} = cdA = (c)(2\pi r dr)$$

$$d\dot{V} = \left[-\frac{1}{4\mu}\frac{dp}{dx}\left(R^2 - r^2\right)\right](2\pi r dr)$$

$$d\dot{V} = -\frac{\pi}{4\mu}\frac{dp}{dx}\left(R^2 r - r^3\right)dr$$

Integrating between r = 0 and r = R results in:

$$\dot{V} = -\frac{\pi}{4\mu}\frac{dp}{dx}\int_0^R \left(R^2 r - r^3\right)dr$$

$$\dot{V} = -\frac{\pi}{4\mu}\frac{dp}{dx}\left[\frac{R^2 r^2}{2} - \frac{r^4}{4}\right]_0^R$$

$$\dot{V} = -\frac{\pi}{8\mu}\frac{dp}{dx}R^4 \qquad (4)$$

So the volumetric flow rate is proportional to the pressure gradient.

2.3 Pressure Drop, Δp

It is important to note that, along a pipe, the **mass flow rate and the volumetric flow rate are constant** (following the continuity equation, when density is constant). As R and μ are constant, the pressure gradient dp/dx must also be constant.

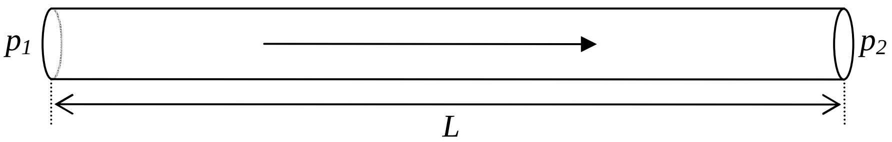


Figure 8: Flow along a pipe

For a horizontal pipe with length L:

$$\frac{dp}{dx} = \frac{-\Delta p}{L} \tag{5}$$

Note that the change in pressure term in the numerator, Δp is negative since the pressure p_1 in Figure 8 is greater than the pressure p_2, (i.e. $p_1 - p_2 = \Delta p$ = positive value). The change in pressure related to dp is $p_2 - p_1 = -\Delta p$.

Rearranging equation (4) above for dp/dx:

$$\dot{V} = -\frac{\pi}{8\mu}\frac{dp}{dx}R^4 \rightarrow \frac{dp}{dx} = -\frac{8\mu}{\pi R^4}\dot{V} \tag{6}$$

Combining equations (5) and (6):

$$\frac{-\Delta p}{L} = -\frac{8\mu}{\pi R^4}\dot{V}$$

Rearranging, an **equation for Δp** can be determined:

$$\boxed{\Delta p = \frac{8\mu L\dot{V}}{\pi R^4}} \tag{7}$$

Equation (7) allows us to determine the **pressure drop in a pipe in which the flow is laminar**. This pressure drop can then be used in the form of Bernoulli's equation defined at the start:

$$p_1 + \frac{1}{2}\rho C_1^2 + \rho g z_1 = p_2 + \frac{1}{2}\rho C_2^2 + \rho g z_2 + \Delta p$$

2.4 Average Velocity

Rearranging equation (7) to solve for the volumetric flow rate results in:

$$\Delta p = \frac{8\mu}{\pi R^4}L\dot{V} \rightarrow \dot{V} = \frac{\pi R^4}{8\mu L}\Delta p$$

Since the volumetric flow rate equals the velocity multiplied by the cross sectional area, it follows that the average velocity is:

$$c_{\text{average}} = \frac{\dot{V}}{A} = \left(\frac{\pi R^4 \Delta p}{8\mu L}\right)\left(\frac{1}{\pi R^2}\right) = \frac{R^2\Delta p}{8\mu L} \tag{8}$$

If we substitute equation (5) above into the equation for the maximum velocity, we have:

$$c_{\max} = -\frac{R^2}{4\mu}\frac{dp}{dx} = \frac{R^2}{4\mu}\frac{\Delta p}{L} \qquad (9)$$

Comparing equations (8) and (9) it is clear to see that:

$$c_{\text{average}} = \frac{R^2 \Delta p}{8\mu L} = \frac{c_{\max}}{2}$$

2.5 Power Consumed

It is also important to be able to relate the pressure drop to the **power consumed** (or equivalently, the power required by a pump). Fluid systems that transport fluid over significant distances are greatly affected by friction. Where possible, the flow is driven by a height difference between the inlet and outlet of the system, but in many cases a pump is required to establish the flow. Analysis of fluid systems is therefore closely related to the analysis of fluid machine performance.

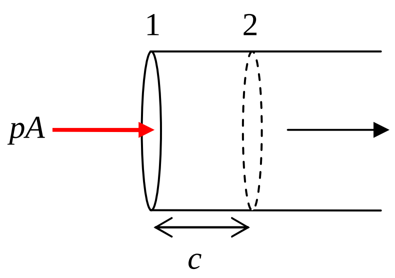


Figure 9: Fluid moves from 1 to 2 under force *pA*

In Figure 9, the fluid moves from section 1 to section 2 in 1 second, under the action of force *pA*. The power applied is the work done because it happens in 1 second, and this can be calculated from force × distance:

$$\text{Power} = \text{Force} \times \text{Distance} = pAc = p\dot{V}$$

This is the power at any point in the flow, so if friction causes a pressure drop Δp, the power loss is:

$$\boxed{\text{Power Loss} = \Delta p\dot{V}}$$

In this analysis, the average velocity has been used, although denoted by c instead of c_{average}. Strictly, this is not correct and the shape of the velocity profile should be taken into account when calculating power or the dynamic pressure. However, it is common not to bother and does not lead to much loss of accuracy.

2.6 Summary

The following itemises the information that you need to solve the problems associated with laminar flow in pipes:

- Bernoulli's equation taking into account a pressure drop:

$$p_1 + \frac{1}{2}\rho C_1^2 + \rho g z_1 = p_2 + \frac{1}{2}\rho C_2^2 + \rho g z_2 + \Delta p$$

- The Reynolds number helps to determine whether a flow is laminar or turbulent:

$$\mathrm{Re} = \frac{\rho c D}{\mu} = \frac{cD}{\upsilon}$$

- The shear stress experienced in the fluid at the wall of a pipe:

$$\tau_w = \frac{F}{A_s} = \frac{D}{4}\frac{\Delta p}{L} = \frac{R}{2}\frac{\Delta p}{L}$$

- The velocity profile is defined as:

$$c = -\frac{R^2}{4\mu}\frac{dp}{dx}\left(1 - \left(\frac{r}{R}\right)^2\right)$$

- The pressure drop associate with fluid friction in a laminar flow in a pipe is derived as:

$$\Delta p = \frac{8\mu L\dot{V}}{\pi R^4}$$

- The power loss (or power required) is:

$$\text{Power Loss} = \Delta p \dot{V}$$

Using these equations, you should be able to solve the problems on the following page.

2.7 Example

Oil of density 900 kg/m^3 and viscosity of 0.17 Pas is pumped through a 75 mm diameter pipe 750 m long at the rate of 2.7 kg/s. If the critical Reynolds number is 2300, show that the critical velocity is not exceeded and calculate the pressure required at the pump and the power required. The pipe is horizontal.

2.8 Laminar Flow in Pipes Exercises

1. Oil of density 850 kg/m^3 is pumped along a horizontal pipe 100 mm diameter and 1300 metres long. The quantity of oil passing through the pipe is 0.01 m^3/s and the pump, which has an efficiency of 60%, takes 5 kW to drive it. Assuming laminar flow (check this) obtain the viscosity and kinematic viscosity of the oil.

   (0.057 Pas; 6.65 × 10^{-5} m^2/s; Re = 1911)

2. The velocity along the centre line of a 150 mm diameter pipe conveying oil under laminar conditions is 3 m/s. The pipe is horizontal and the viscosity of the oil is 0.12 Pas and its density is 900 kg/m^3. Calculate:

   a. The quantity flowing in m^3/s
   b. The shear stress at the pipe wall
   c. The total horizontal force produced on the pipe, which is 500 m long

   (0.0265 m^3/s; 9.6 N/m^2, 2261 N; Re = 1688)

3. Oil of viscosity 0.05 Pas flows through a horizontal pipe 19 mm diameter with a mean velocity of 0.3 m/s. Assuming that the flow is laminar (check this), calculate the pressure drop over 45 m length of pipe. Also calculate the velocity at a point 5 mm from the wall of the pipe, and the shear stress in the oil at this point. Assuming density is 900 kg/m^3.

   (600 mbar; 0.465 m/s; 2.99 N/m^2)

4. Glycerine, with a density of 1260 kg/m^3, viscosity 0.9 Pas is pumped at 20 litres/second through a straight 100 mm diameter pipe, 45 m long inclined at 15° to the horizontal. The gauge pressure at the lower inlet end of the pipe is 5.85 bar. Verify that the flow is laminar, and calculate the pressure (gauge) at the outlet end and the shear stress at the wall.

   (Re = 357; 1.11 bar; 183.3 N/m^2)

5. Oil of density 910 kg/m^3 and viscosity 0.124 Pas is pumped through a 75 mm diameter pipe at 425 litres/min. Show that the flow is laminar and find the power require to pump the oil through 75 m of pipe, which rises 3 m.

   (Re = 883; 790.5 Watts)

3 Turbulent Flow

The basis of the experimental results used in analysing **turbulent flow** lies in Dimensional Analysis, which you have previously studied. The analysis starts by listing the variables that are felt to influence the process, in this case, pipe flow. Most of these can be found from the laminar flow analysis, with the addition of pipe inner surface roughness, which tends to increase the level of turbulence in the flow.

3.1 Shear Stress

These variables are listed in the following table.

Variable	*Description*
τ_w	Wall shear stress (the variable we wish to investigate)
ρ	Density
μ	Viscosity
c	Average velocity
D	Pipe diameter (bore)
ε	Pipe roughness

A relationship of the form below is assumed to represent the functional dependence on the listed variables:

$$\tau_w = \varphi(\rho, \mu, c, D, \varepsilon) \rightarrow \tau_w = K\rho^a \mu^b c^d D^e \varepsilon^f \tag{10}$$

where K is a non-dimensional constant and a, b, d, e and f are unknown exponents to be found by experiment.

Inserting the fundamental dimensions of each variable:

$$\mathrm{MLT}^{-2} = \left(\mathrm{ML}^{-3}\right)^a \left(\mathrm{ML}^{-1}\mathrm{T}^{-1}\right)^b \left(\mathrm{LT}^{-1}\right)^d (\mathrm{L})^e (\mathrm{L})^f \tag{11}$$

Equating powers of each dimension on both sides:

For M: $1 = a + b$ (12a)

For L: $-1 = -3a - b + d + e + f$ (12b)

For T: $-2 = -b - d$ (12c)

We have 3 equations and 5 unknown powers, so we can determine 3 powers in terms of the other two. The rule is to keep the powers that appear most frequently in the equation. Clearly, b is the only power to appear in all equations, so we keep it.

Equation (12a) is thus used to get rid of a and equation (12c) is used to get rid of d. This is to get rid of either e or f using equation (12b). Let us get rid of e:

$$a = 1 - b$$

$$d = 2 - b$$

$$e = 3a + b - d - f - 1 = 3(1-b) + b - (2-b) - f - 1$$

$$e = -b - f$$

Substituting these into the general equation (10) gives:

$$\tau_w = K\rho^a \mu^b c^d D^e \varepsilon^f = K\rho^{1-b}\mu^b c^{2-b} D^{-b-f}\varepsilon^f$$

$$\tau_w = K\rho c^2 \left(\frac{\mu}{\rho c D}\right)^b \left(\frac{\varepsilon}{D}\right)^f$$

Rearranging:

$$\frac{\tau_w}{\frac{1}{2}\rho c^2} = 2K\left(\frac{\rho c D}{\mu}\right)^{-b}\left(\frac{\varepsilon}{D}\right)^f$$

The non-dimensional wall shear stress (see that τ_w is a pressure, divided by dynamic pressure, and is thus non-dimensional) depends on two non-dimensional variables:

The **Reynolds number**: $\frac{\rho c D}{\mu}$

The **relative roughness**: $\frac{\varepsilon}{D}$

The **relative roughness** is a non-dimensional ratio that describes the internal surface of the pipe. Pipes can be made out of a wide variety of materials, and each material has a particular type of surface. These different surfaces can affect the pressure drop in the system, and are hence important to know.

3.2 Pressure Drop, Δp

The nature of the dependence can be investigated by experiment. The general form of the relationship is

$$\frac{\tau_w}{\frac{1}{2}\rho c^2} = F\left(\frac{\rho c D}{\mu}, \frac{\varepsilon}{D}\right) = F \rightarrow \tau_w = F\frac{1}{2}\rho c^2 \qquad (13)$$

where F is a function to be found experimentally, and is a function of the Reynolds number and the relative roughness.

Recall that the shear force experienced at the wall of pipe is:

$$\tau_w = \frac{D}{4}\frac{\Delta p}{L} = \frac{R}{2}\frac{\Delta p}{L} \qquad (14)$$

Substituting equation (13) into equation (14):

$$\tau_w = \frac{D}{4}\frac{\Delta p}{L} = F\frac{1}{2}\rho c^2 \qquad (15)$$

Rearranging equation (15) for Δp gives us:

$$\Delta p = \frac{4FL}{D}\frac{1}{2}\rho c^2$$

If F is known, the pressure drop for a given average velocity of flow rate can be found, as was done by analysis for laminar flow. The variable F is known as the **Fanning Friction Factor**. Often, though not in all textbooks, the factor **4*F* is replaced by *f***, known as the (non-dimensional) friction factor or the **Darcy Friction Factor**. This is the form used from this point onwards.

The final equation to calculate the pressure drop (in N/m^2 or Pa):

$$\Delta p = \frac{fL}{D}\frac{1}{2}\rho c^2 \tag{16}$$

The value found for Δp can then be incorporated into the form of Bernoulli's equation described at the start:

$$p_1 + \frac{1}{2}\rho C_1^2 + \rho g z_1 = p_2 + \frac{1}{2}\rho C_2^2 + \rho g z_2 + \Delta p$$

3.3 Determining the Friction Factor

The experimental results determining the friction factor, f, can be viewed graphically or as equations. The graphical representation is more popular and is known as the Moody Chart, after the person that first presented the results in this way. It is a log/log plot of friction factor against Reynolds number, Re, with different curves for different values of relative roughness. This is shown in Figure 10 below:

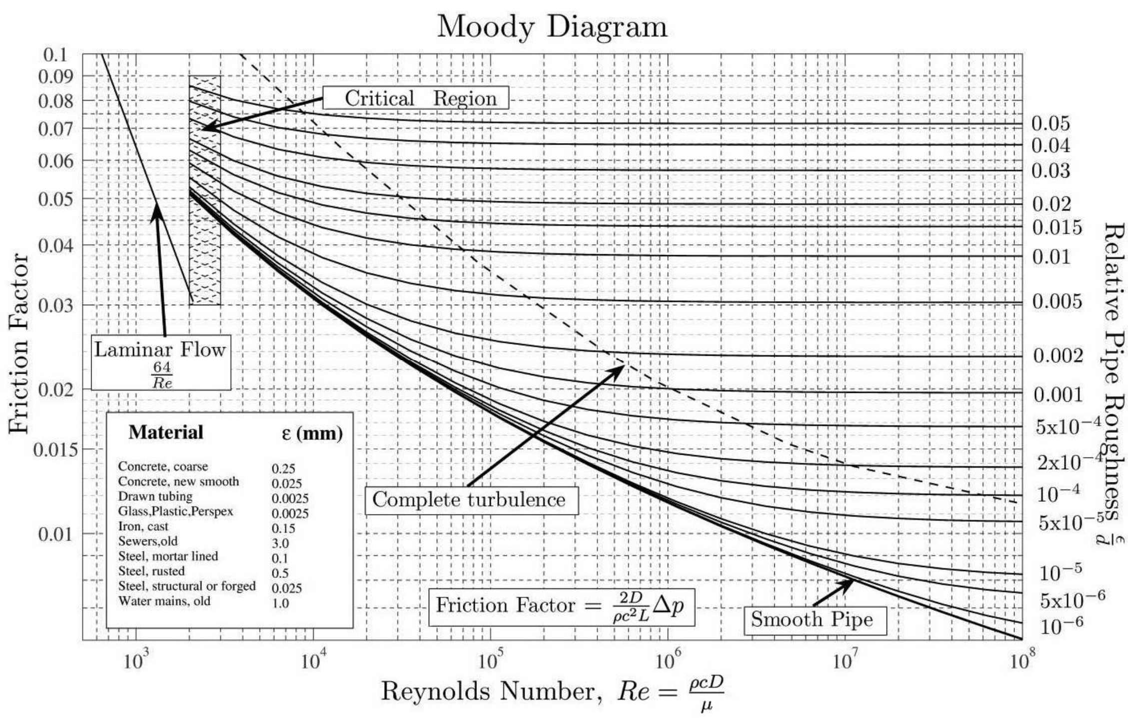


Figure 10: Moody Chart (a larger version of this chart appears on page 23).

There are four zones in the Moody Chart:

1. **Laminar Region**: Using the equation for Δp found in the Laminar Flow analysis section (equation (7)), and matching this up to equation (16), we can do the following analysis to determine the friction factor for the laminar case:
$$\Delta p = \frac{8\mu L\dot{V}}{\pi R^4} = \frac{fL}{D}\frac{1}{2}\rho c^2$$
Replacing R with $D/2$, and $\dot{V}$ with Ac (which in turn equals $\pi D^2 c/4$) and simplifying results in the following:
$$\frac{32\mu Lc}{D^2} = \frac{fL}{D}\frac{1}{2}\rho c^2$$
Solving for the friction factor, f gives:
$$f = \frac{64\mu}{\rho c D} = \frac{64}{\text{Re}}$$
So, knowing this relationship, we can use either equation (7) directly for laminar problems, or equation (16) knowing that the friction factor is 64/Re.
2. **The Critical Region**: The lines here are dotted because we are not really sure whether the flow is laminar or turbulent. None of your problems will rely on determining the pressure drop in this region.
3. **The Transition Region**: In the Moody diagram, this is the area below the dashed line (labelled 'Complete Turbulence'). In this region, the friction factor depends on the Reynolds number and the relative roughness, ε/D. To determine the friction factor, select the relevant relative roughness line, and at the point of the relevant Reynolds number, read across to the left to determine the friction factor.
4. **The Fully Turbulent Region**: In the Moody diagram, this is the area above the dashed line. As can be seen, the lines for relative roughness are flat, meaning that the Reynolds number has no effect on the friction factor. Therefore, the friction factor depends solely on the relative roughness.

Note the table on the chart lists the roughness value (ε) for various materials.

Instead of the chart, some time has been spent (by people who have evidently too much free time) developing equations to determine friction factor from the Reynolds number and the relative roughness factor. These are not derived here, but listed below:

For the turbulent zone in general:
$$f = \frac{1}{\left[-1.8\log\left(\frac{6.9}{\text{Re}} + \left(\frac{1}{3.7}\frac{\varepsilon}{D}\right)^{1.11}\right)\right]^2}$$

For the fully turbulent region:
$$f = \frac{1}{\left[1.14 - 2\log\left(\frac{\varepsilon}{D}\right)\right]^2}$$

Although these can be used as alternatives to the chart, it is **often quicker to choose the value from the Moody chart**, as entering these rather complex equations into a calculator without mistakes can be challenging.

3.4 Solving Problems

There are 3 types of problems that you will come across.

3.4.1 Type 1 – Pressure Drop

This is the pump selection problem. You are generally given the **volumetric flow rate**, $\dot{V}$, and the **pipe diameter**, *D*, so that both the Reynolds number, Re, and the relative roughness, ε/D, are known, so *f* can be read directly from the chart. The pressure drop, Δp, can then be calculated using equation (16) which will provide you with the details of the pump you need to select (that which can provide the necessary pressure to overcome the loss in pressure due to friction; i.e. Δp).

So, in summary, Figure 11 illustrates the process required for a type 1 problem:

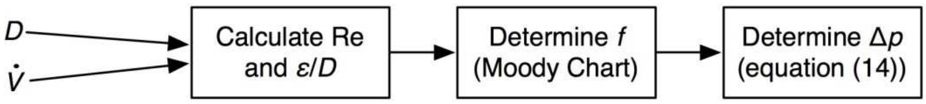


Figure 11: Flow chart for a type 1 problem

3.4.2 Type 2 – Flow Rate

This is where you have the pump and pipe system, and you wish to **find the flow rate,** it will give you, i.e. $\dot{V}$. You are generally given the **pressure drop Δp**, The **diameter, *D***, is also known, so the relative roughness, ε/D, can be determined. The Reynolds number, Re, however is not known since the velocity, *c*, is unknown. This problem is solved by assuming firstly, that the flow is fully turbulent (zone 4) so the friction factor, *f* can be determined from just the relative roughness. You can then use equation (16) to determine *c*. It is useful to rearrange equation 14 to the following:

$$\Delta p = \frac{fL}{D}\frac{1}{2}\rho c^2 \rightarrow c = \sqrt{\frac{2\Delta p D}{\rho f L}} \quad (17)$$

You then compute the Reynolds number from the first value of *c*, and looking at the Moody chart, if the flow is fully turbulent, your calculation is completed and you can now determine the flow rate, $\dot{V}$ (remember: $\dot{V} = Ac$).

If, however, your value of Reynolds number indicates that the flow is not fully turbulent, but transitional, then you have to determine the new friction factor, *f* from the Moody chart, and again use equation (17) to determine *another* new value for *c*, which you can then use to get *another* value for Reynolds number and so on. This iterative process can go until successive values for *c* do not change significantly. **It is seldom necessary to do more than two estimates of *c***. Once *c* is computed the flow rate, $\dot{V}$, may be found (remember: $\dot{V} = Ac$).

So, in summary, Figure 12 illustrates the process required for a type 2 problem.

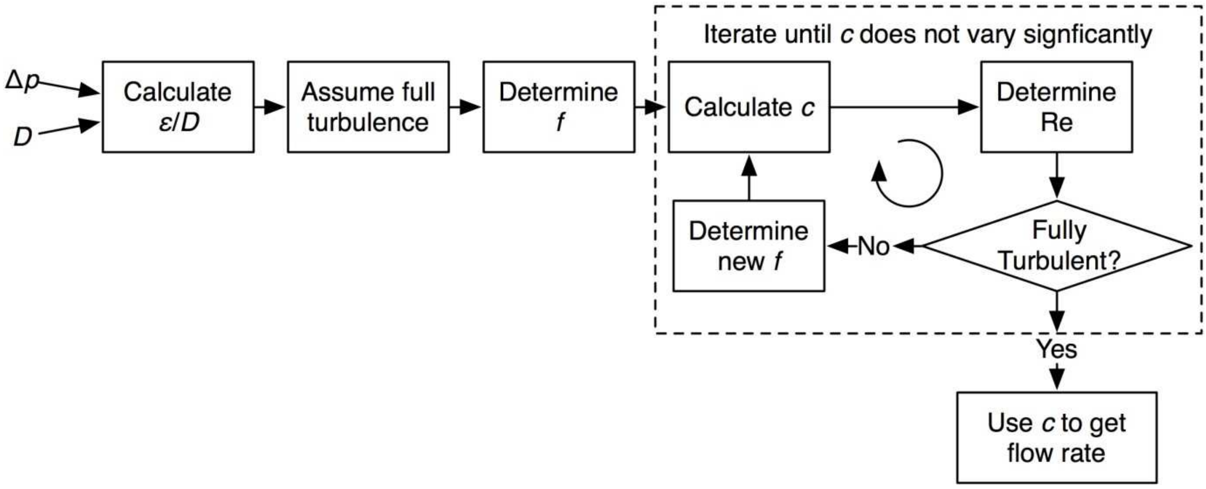


Figure 12: Flow chart for a type 2 problems

3.4.3 Type 3 – Pipe Diameter

This is the pipe design problem, where the pressure available from the pump allows Δp to be known, the required flow is known, but the pipe diameter needed is not. In this case, neither Re nor ε/D is know, so f has to be randomly guessed as being near the centre of the Moody Chart, say $f = 0.03$.

This guess is used to find an estimate of D, from which Re and ε/D are estimated and better guess of f found. This process is then iterated until no significant change of D occurs. For these problems, as velocity c is unknown, because area A is unknown, equation (16) is more usefully express in the form:

$$\Delta p = \frac{fL}{D}\frac{1}{2}\rho c^2 = \frac{fL}{D}\frac{1}{2}\rho\left(\frac{\dot{V}}{A}\right)^2 = \frac{fL}{D}\frac{1}{2}\rho\left(\dot{V}\right)^2\left(\frac{4}{\pi D^2}\right)^2 = \frac{fL}{D^5}\frac{8\rho}{\pi^2}\dot{V}^2 = K\dot{V}^2 \qquad (18)$$

where K is:

$$K = \frac{fL}{D^5}\frac{8\rho}{\pi^2}$$

which is constant if f is constant (fully turbulent condition). Hence, for turbulent flow, the pressure drop is proportional to the square of the flow rate whereas for laminar flow, the drop is directly proportional to the flow rate and the relationship is linear.

Since we are trying to determine D, equation (18) is more usefully written:

$$\boxed{\Delta p = \frac{fL}{D^5}\frac{8\rho}{\pi^2}\dot{V}^2 \rightarrow D = \sqrt[5]{\frac{fL}{\Delta p}\frac{8\rho\dot{V}^2}{\pi^2}}} \qquad (19)$$

So, in summary, Figure 13 illustrates the process required for a type 3 problem:

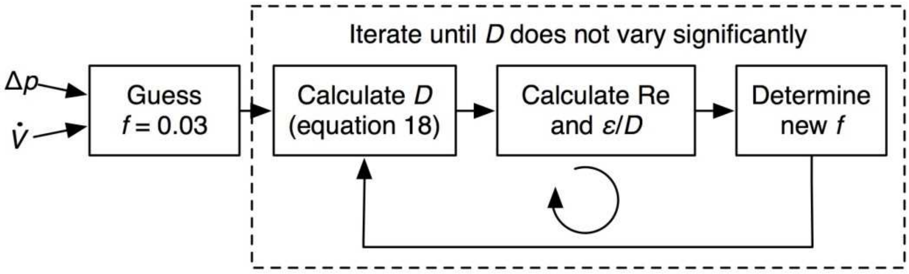


Figure 13: Flow chart for a type 3 problem

3.5 Summary

Remember that we can confirm that a flow is turbulent by checking the Reynolds number. In general, we can safely assume that the flow is turbulent if the Reynolds number is greater than 3000.

- The standard equation to use to determine the pressure drop is:

$$\Delta p = \frac{fL}{D}\frac{1}{2}\rho c^2$$

  where L is length, D is the diameter of the pipe, ρ is the density of the fluid, and c is the average velocity of the flow. The term f is known as the **friction factor**, and is determined from the **Moody Chart**.
- There are four zones of the Moody Chart: Laminar zone, critical region, transitional turbulent, and complete turbulent.
- To read a friction factor from the Moody Chart, you need to know the Reynolds number and the Relative Roughness. If you are dealing with the completely turbulent region, then just having the Relative Roughness is sufficient to determine the friction factor.
- There are three types of problems:
  1. **Type 1: Determine pressure drop**
     - Given: **Pipe Diameter** and **Flow Rate**
     - Determine **Re** and ***ε/D***
     - Determine **friction factor**, f
     - Use standard equation to determine $\boldsymbol{\Delta p}$ (equation 14)
  2. **Type 2: Determine flow rate (*c* is unknown)**
     - Given: **Pipe Diameter** and **Pressure Drop**
     - Determine: ***ε/D***
     - Determine **friction factor** from Moody Chart **assuming full turbulence**
     - Use standard equation to determine value for **flow velocity**, ***c***
     - Determine **Re**
     - Use Moody Chart to determine a **new friction factor** (If full turbulence, this value of c is valid)
     - Use standard equation (in the form shown in equation 17) to determine **new value for flow velocity**, ***c***
     - Iterate until ***c* does not vary significantly**
     - Remember, $\dot{V} = Ac$
  3. **Type 3: Determine Pipe Diameter**

- Given: **Flow rate** and **pressure drop**
- Assume **friction factor, *f* = 0.03**
- Use rearranged form of standard equation (equation 19) to determine **diameter**

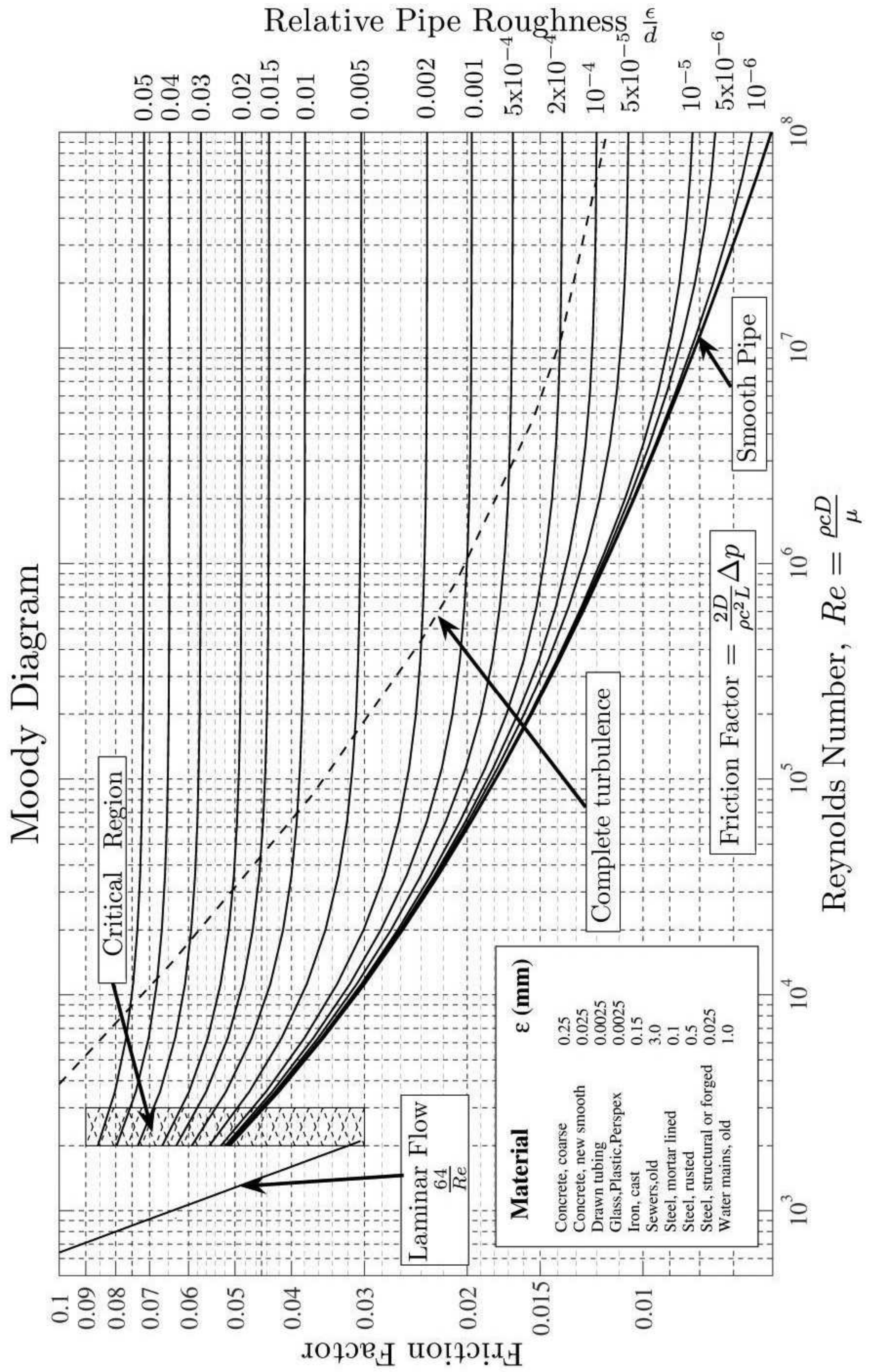
$$\Delta p = \frac{fL}{D^5}\frac{8\rho}{\pi^2}\dot{V}^2 \rightarrow D = \sqrt[5]{\frac{fL}{\Delta p}\frac{8\rho\dot{V}^2}{\pi^2}}$$

- Use value for diameter to determine **Re** and ***ε*/*D***
- Determine **new friction factor, *f***
- Determine **new diameter, *D***
- Iterate until ***D* does not vary significantly**

With the above information, you should be equipped to answer the questions on the following pages.

A full page Moody Chart is given on page 23.

3.6 Moody Chart



3.7 Example

Water flows through a 150 mm diameter pipe for which the relative roughness, ε/D is 0.0002 at a rate of 0.1 m^3/s. Calculate the pressure drop over a 100 m length of pipe. Take μ = 0.001 kg/ms.

3.8 Turbulent Flow in Pipes Exercises

1. Find the power required to pump 10000 tonnes/hour of water ($\mu = 0.8 \times 10^{-3}$ kg/ms) along 3 km of pipe with $\varepsilon = 0.05$ mm. The pipe diameter is 1 m, and the end of the pipe is 20 m above the entry.

   (1134 kW)

2. A pressure difference of 0.0032 bar is required to pump air of density 1.3 kg/m^3 and kinematic viscosity 1.4×10^{-5} m^2/s along a horizontal pipeline 0.75 m diameter, and length 500 m. Given that $\varepsilon = 0.15$ mm, calculate the flow rate of the air.

   (3 m^3/s)

3. The power required to pump 50 litres/s of petrol (density 700 kg/m^3) along a horizontal pipeline of length 1 km is 16 kW. Given that the kinematic viscosity of petrol is 0.5×10^{-6} m^2/s, and that the roughness of the pipe is $\varepsilon = 0.000075$ m, find the size of the pipe used.

   (0.15 m diameter)

4. Oil of density 880 kg/m^3 and kinematic viscosity of 0.37×10^{-5} m^2/s flows at 1515 litres/min through an asphalted cast iron pipe ($\varepsilon = 0.00012$ m) of diameter 152 mm. The pipe is 805 m long and slopes upward at 8° to the horizontal. Determine the pressure difference across the pipe, and the power required. The flow is upward.

   (10.695 bar; 27.0 kW)

5. Water flows at 0.03 m^3/s through a 75 mm diameter pipe of length 100 m, from a constant level reservoir. Calculate the height that the reservoir level must be above the pipe outlet. The discharge is a free jet to atmosphere, the pipe is hydraulically smooth, and $\mu = 0.001$ kg/ms.

   (43.1 m)

6. A pipe of 150 mm diameter and 54 m long is connected to a large tank, the entrance to the pipe being 10 m below the surface level. The lower end of the pipe is 20 m below the upper end, and is joined to a horizontal pipe of diameter 240 mm and length 75 m, which discharges to atmosphere. The roughness, $\varepsilon = 1.2$ mm for both pipes. Assuming that the flow in both pipes is fully turbulent, determine the flow rate. Check your assumptions. The water temperature is 15°C.

   (0.113 m^3/s)

7. Two reservoirs are connected by three cast iron pipes in series. $L_1 = 600$ m, $D_1 = 0.3$ m; $L_2 = 900$ m, $D_2 = 0.4$ m; $L_3 = 1500$ m, $D_3 = 0.45$ m. When the flow rate is 0.11 m^3/s of water, for which $\mu = 0.00114$ kg/s, determine the difference in surface levels.

   (8.13 m)

8. A large tank supplies water to a point 30 m below the level in the tank through a pipe of length 3000 m and diameter 0.6 m. The pipe crosses a ridge whose summit is 9 m above and 300 m distance (along the pipe) from the level in the tank. For the pipe, relative roughness, $\varepsilon/D = 0.006$; $\mu = 0.001$ kg/ms, and $p_{atm} = 1$ bar. Assume that the flow is fully turbulent. Determine:
    - The flow rate, assuming that the discharge is a free jet to atmosphere
    - The depth below the ridge at which the pipe must be laid if the absolute pressure in the pipe is not to fall below 3 m of water
    - Check that the flow is fully turbulent

(541 litres/s; 4.98 m)

4 Minor Losses and Valves

The above analysis allows for the frictional effects of straight lengths of pipe. There are a lot of other components in fluid systems which cause additional losses, often termed minor losses. These include things such as bends, entrances and exits from reservoirs, changes of section, T-junctions and filters. It also includes valves for which losses are not minor but deliberately introduced in order to control the flow rate.

All of these cause additional loss through additional turbulence or swirling imparted to the flow. The devices are usually considered to be of zero length, but their effects are often noticed many pipe diameters of length upstream and downstream of the fitting, particularly downstream. Thus when there are two or more fittings in close proximity, the effects can be complex.

However, it is sufficient simply to add the individual effects to get the overall effect. As some fairly rapid changes in direction take place in the flow, flows through the fittings are considered fully turbulent.

For turbulent flow, the pressure loss due to pipe friction, as defined in equation (16):

$$\Delta p = \frac{fL}{D}\frac{1}{2}\rho c^2$$

For minor losses, we retain the $\frac{fL}{D}$ in the equation, but add a loss factor, *k*, depending on the fitting. Therefore, the loss associated with a component or fitting only is:

$$\Delta p = k\frac{1}{2}\rho c^2 \qquad (20)$$

If there are multiple fittings for a particular system (such as multiple bends, junctions etc.), then these loss factors, k_1, k_2, k_3, … can be summed together to produce a overall loss factor:

$$\text{Effective Loss Factor} = k_e = \sum_{i=1}^{n} k = k_1 + k_2 + \cdots + k_n$$

So, the overall pressure loss through the system becomes:

$$\Delta p = \left(\frac{fL}{D} + k_e\right)\frac{1}{2}\rho c^2 \qquad (21)$$

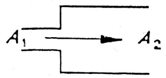
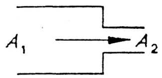
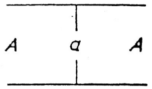
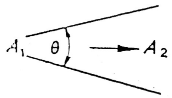
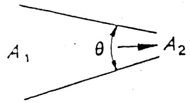
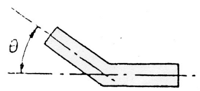
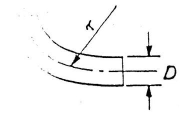
Tables identifying values for *k* for various components are available in many books on fluids. Many are itemised in the following table.

With a valve, the loss coefficient is identified as k_v, and is dealt in the same, way, except that the value of k_v varies according to the valve opening. A graph of k_v against degree of opening is known as the valve characteristic. A list of typical values is provided on page 28.

4.1 Minor Losses Fittings

PRESSURE DROP ACROSS FITTINGS
(turbulent flow of constant density fluid)

where fitting changes velocity $k = \frac{\Delta p}{\frac{1}{2}\rho c^2}$ is based on downstream velocity unless stated otherwise

Fitting	Value k
Abrupt enlargement 	$\left(\frac{A_2}{A_1} - 1\right)^2$ *Note*: for $A_2/A_1 \to \infty$ use $k = 1$ and *upstream* velocity in equation
Abrupt contraction 	$0.45\left(1 - \frac{A_2}{A_1}\right)$
Sharp-edged orifice 	$2.8\left(1 - \frac{a}{A}\right)\left[\left(\frac{A}{a}\right)^2 - 1\right]$ *Note*: 'downstream' velocity is that based on area A
Enlarger 	$\beta\left(\frac{A_2}{A_1} - 1\right)^2$
Reducer 	0.05 for $\theta < 30°$
Mitred Elbow 	
90° bend 	

Enlarger:

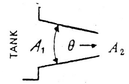
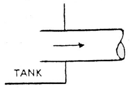
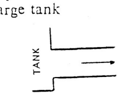
θ	3°	5°–8°	10°	14°	20°	30°	45°	60°	> 90°
β	0.18	0.14	0.16	0.25	0.45	0.7	0.95	1.1	1.0

Mitred Elbow:

θ	90°	60°	45°	30°	15°
q	1.0	0.36	0.18	0.07	0.02

90° bend:

r/D	0.5	0.75	1.0	1.5	2.0	4.1
q	1.0	0.75	0.6	0.47	0.40	0.30

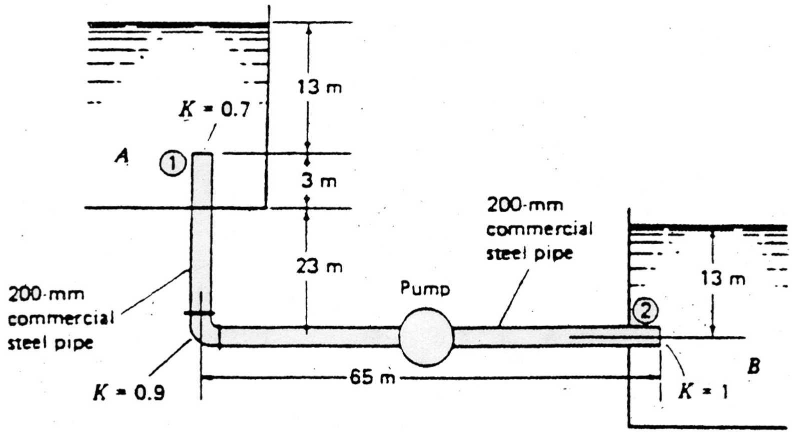
Fitting	Value at : k
Conical exit from a large tank 	$0.45\left(\frac{A_2}{A_1}\right)^2 + 0.05$ for θ less than 30°
Re-entrant (Borda) mouthpiece running full 	0.8
Square exit from a large tank 	0.5

Miscellaneous fittings*

Fitting	k Range	k Mean
90° elbows: mitred, sharp intersections	1.0- 1.5	1.25
sharp bends, $r/D = 0.5$	0.8- 1.2	1.0
castings, $r/D = 1.0$	0.5- 1.0	0.75
Standard tee, 90°, equal areas: from barrel to branch	1.0- 1.4	1.2
from branch to barrel	1.2- 1.8	1.5
through barrel	0.2- 0.3	0.25
(Third leg stagnant in each case)		
Close return bend, 180° $r/D = 1.0$	1.5- 2.5	2.0
Globe valve fully open	5 -15	10
Angle valve, 45° "	3 - 7	5
Flap non-return valve "	1 - 3	2
Gate valve "	0.1- 0.2	0.15
¾ open (by area)	0.6- 1.0	0.8
½ open "	4 - 5	4.5
¼ open "	25 -30	28

4.2 Minor Losses Example

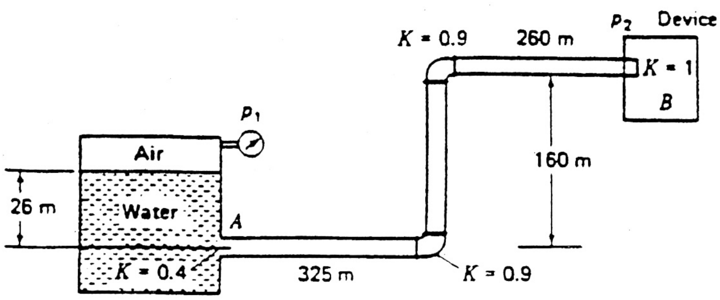
1. The flow rate from A to B is 565 litres/s. Determine the power required from the pump. Take $\nu = 0.113 \times 10^{-5}$ m^2/s. ε for a commercial steel pipe is 0.000045 m.



4.3 Minor Losses in Pipeline Systems Exercises

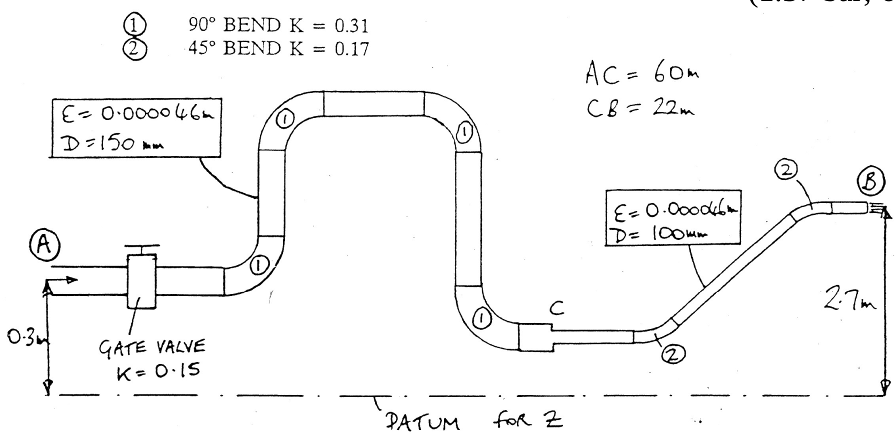
1. What pressure, p_1 is needed to cause 100 litres/s to flow into the device at a pressure p_2 of 0.4 bar (gauge)? The pipe has a diameter 150 mm, $\varepsilon = 0.046$ mm, and $\nu = 0.114 \times 10^{-5}$ m^3/s.

(26.74 bar)



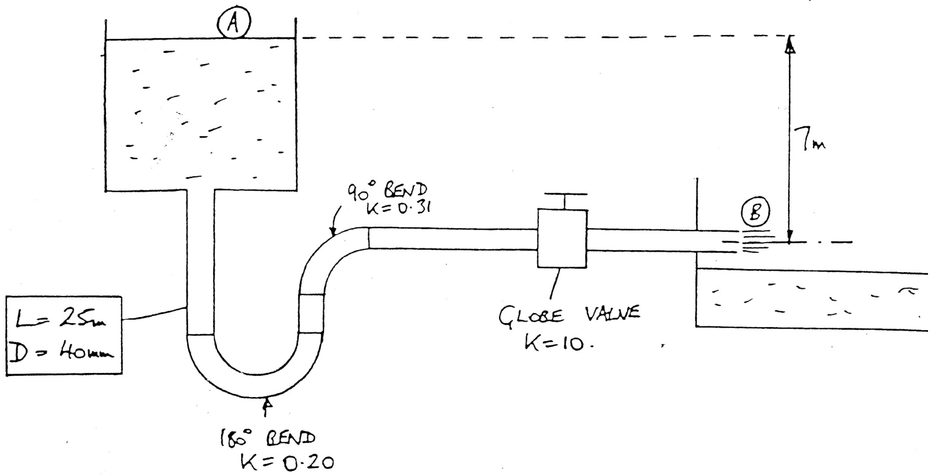
2. In a processing plant, turpentine is pumped from point A to a delivery point at B, where the turpentine emerges as a free jet at atmospheric pressure. The pipeline consists of 60 m of 150 mm diameter pipe, followed by 22 m of 100 mm diameter pipe. $\varepsilon = 0.000046$ m for both pipes. The flow rate is 0.05 m^3/s. For turpentine, $\rho = 870$ kg/m^3 and $\mu = 1.375 \times 10^{-3}$ kg/ms. Determine:
   a. The pressure difference
   b. The power required

(1.37 bar; 6.85 kW)



3. The system shown delivers water from one tank to another. A globe valve (K = 10) controls the flow. The water issues as a free jet at atmospheric pressure at B. The total pipe length is 25 m, and the pipe diameter is 40 mm. The pipe has a roughness ε = 0.00025 m. Take $\nu = 10^{-6}$ m^2/s. Determine the flow rate.

(2.4×10^{-3} m^3/s)



5 Fluid Machines

5.1 Machine characteristics

Fluid machines either take energy from the flow (for example a turbine or a motor) or give energy to the flow (examples are pump, fan or compressor).

Taking the pump example, the simplest way of representing its performance is that it gives a constant pressure rise Δp_p regardless of the volumetric flow rate, $\dot{V}$. The **characteristic** of the pump, which is a plot of Δp_p against $\dot{V}$ then appears as below in Figure 14.

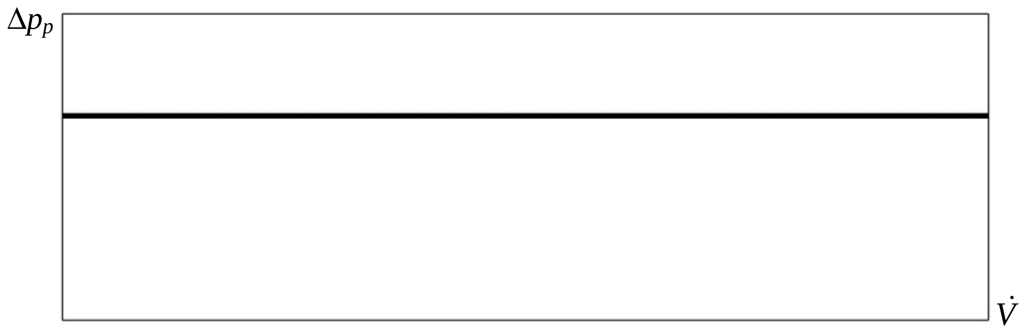


Figure 14: Pump Characteristic: Constant Pressure Rise

This representation is rather unrealistic except over a limited range of flows, the reason being that it implies as $\dot{V}$ increases so does the power given to the flow. Recall the power is volumetric flow rate multiplied by the pressure rise. Therefore:

$$\text{Power to the flow} = \Delta p_p \dot{V} \tag{22}$$

This would eventually become infinite. Whatever drives the pump has a maximum possible power it can deliver.

We can also define efficiency for the machine.

$$\eta = \frac{\text{Power to the flow}}{\text{Power consumption}} = \frac{\Delta p_p \dot{V}}{P} \tag{23}$$

where P is the power consumption. We can rearrange equation (23) solving for Δp_p:

$$\eta = \frac{\Delta p_p \dot{V}}{P} \rightarrow \Delta p_p = \frac{\eta P}{\dot{V}} \tag{24}$$

If η and P are constant, then:

$$\Delta p_p = \frac{\text{constant}}{\dot{V}} \tag{25}$$

The characteristic curve appears as in Figure 15. We can also plot another characteristic curve of η against $\dot{V}$.

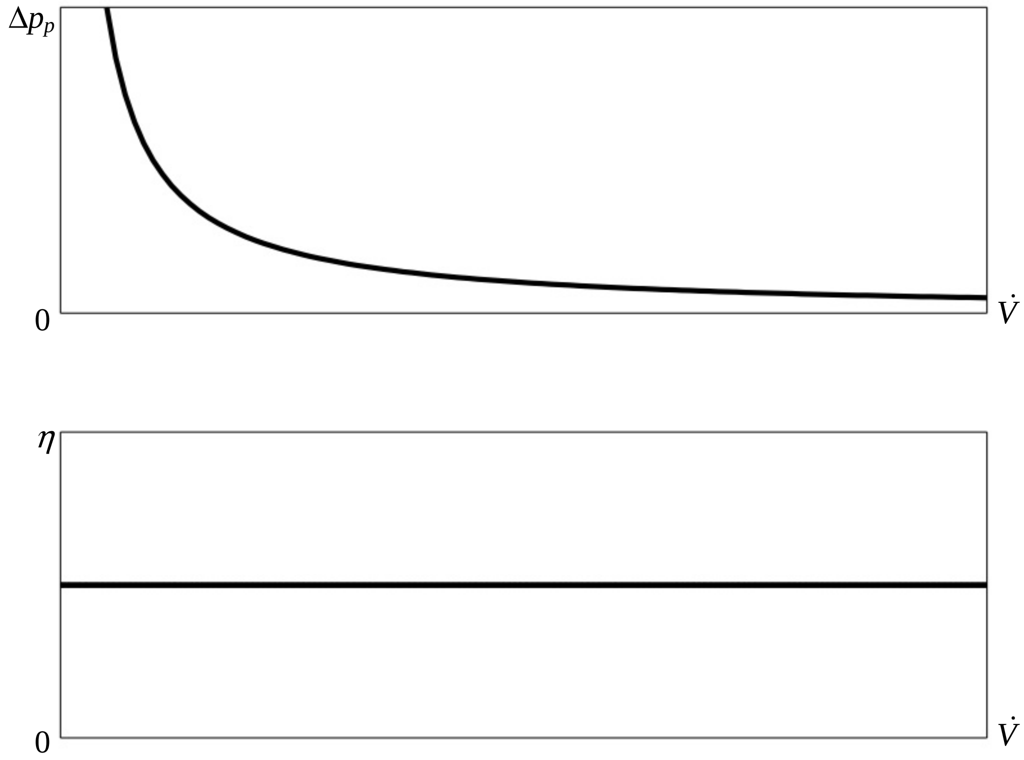


Figure 15: Pressure Rise as a function of flow rate

In practice, η is not constant because the blade shapes are only optimal for a certain combination of angular speed ω and flow rate $\dot{V}$. The power input (power consumed) can usually be varied to produce a variety of angular speeds. This results in a set of characteristics for each machine in Figure 16 overleaf.

The characteristic curves can often be represented as curve-fitted equations such as:

$$\Delta p_p = A_1 - A_2 \dot{V}^2 \qquad (26)$$

where A_1 and A_2 are constants specific to the pump (or machine).

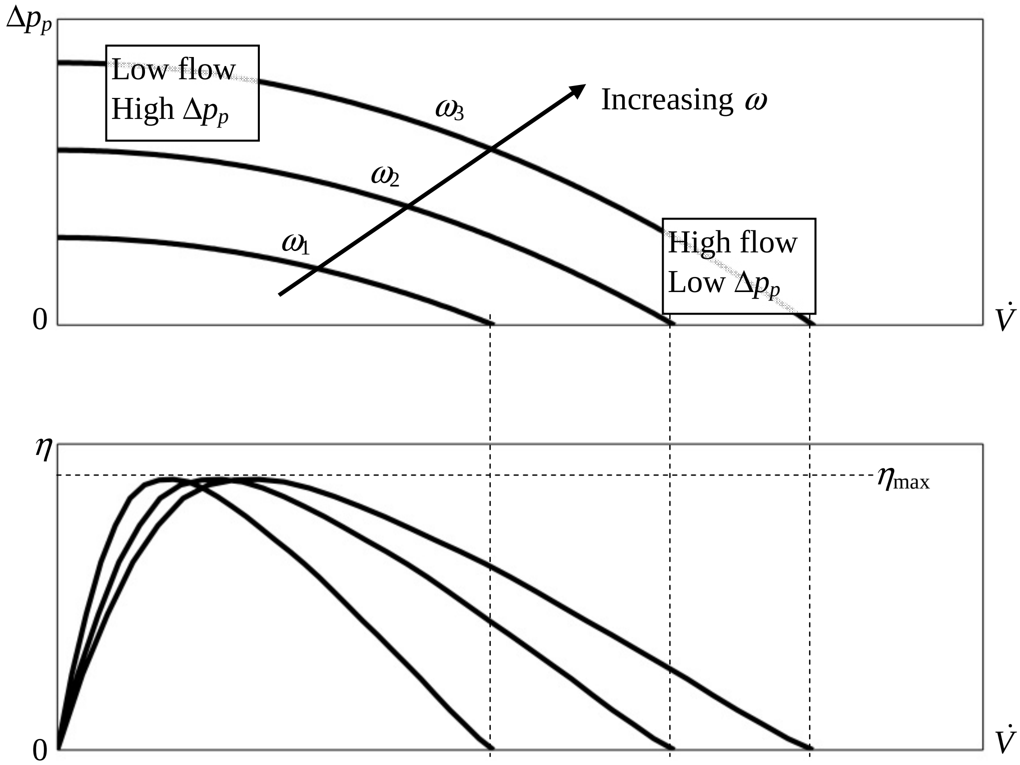


Figure 16: Pressure Rise for different pump angular velocities

5.2 Machine Operation

The operation point on the characteristic depends upon the pipe and system it is attached to. Take the example shown in Figure 17.

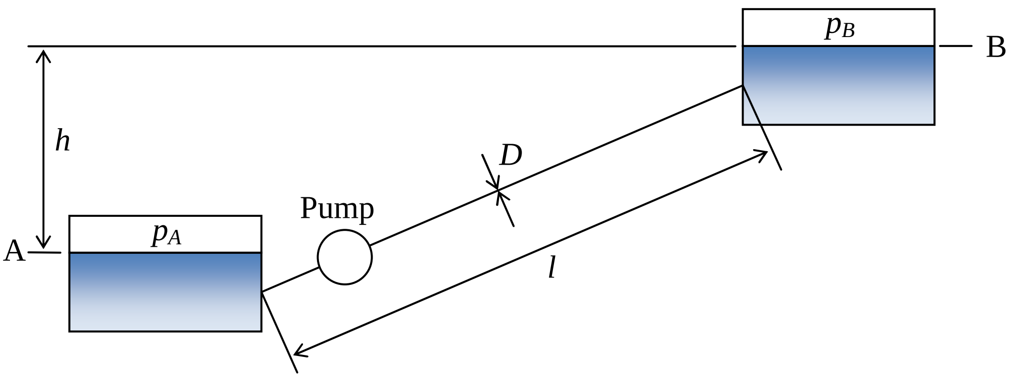


Figure 17: Example Fluid Machine System

The pump has to overcome the head h (equivalent to hydrostatic pressure ρgh) and the pressure difference ($p_A - p_B$) before any flow can occur, the remaining pressure available from the pump being lost to friction.

When dealing with a machine system, Bernoulli's equation has terms representing the pressure provided by the pump as well as the losses due to friction:

$$p_A + \frac{1}{2}\rho c_A^2 + \rho g z_A + \Delta p_p = p_B + \frac{1}{2}\rho c_B^2 + \rho g z_B + \Delta p_L \qquad (27)$$

where Δp_p is the pressure provided by the pump, and Δp_L is the pressure drop due to friction and minor losses in the pipe system. If we take positions **A** and **B** to be the surfaces of the reservoirs A and B, the velocities at these points are negligible compared to the velocity of flow in the pipe, so the dynamic pressure terms are neglected. Rearranging equation (27), we get the following:

$$p_A + \frac{1}{2}\rho c_A^2 + \rho g z_A + \Delta p_p = p_B + \frac{1}{2}\rho c_B^2 + \rho g z_B + \Delta p_L$$

$$p_A + \rho g z_A + \Delta p_p = p_B + \rho g z_B + \Delta p_L$$

$$\Delta p_p = (p_B - p_A) + \rho g (z_B - z_A) + \Delta p_L$$

$$\Delta p_p = (p_B - p_A) + \rho g h + \Delta p_L$$

We can replace Δp_L with the following (see section 3.4.3 on page 20—the section on Type 3 turbulent flow problems—for the derivation of this equation):

$$\Delta p_L = \frac{fL}{D}\frac{1}{2}\rho c^2 = \frac{8 fL\rho}{\pi^2 D^5}\dot{V}^2$$

Substituting this in, the equation is now:

$$\Delta p_p = (p_B - p_A) + \rho g h + \frac{8 fL\rho}{\pi^2 D^5}\dot{V}^2 \qquad (28)$$

We can see that this equation has the same form as equation (26), in that there is a term unrelated to the volumetric flow rate: $(p_A - p_B) + \rho g h$ which is known as the **static lift** and a **flow dependent term**, $\frac{8 fL\rho}{\pi^2 D^5}$. As such, we can rewrite equation (28) as:

$$\boxed{\Delta p_p = C_1 + C_2\dot{V}^2} \qquad (29)$$

where C_1 and C_2 are the static lift and the flow dependent term respectively. Note the similarity of this equation with equation (26). C_1 is a constant, and C_2 is a constant if f is constant, which is true for fully turbulent flow.

Equation (26) is known as the **pump characteristic**: $\Delta p_p = A_1 - A_2\dot{V}^2$

Equation (28) is known as the **pipe characteristic**: $\Delta p_p = C_1 + C_2\dot{V}^2$

The **operating point** is found where these two equations result in the same value. Thus:

$$\boxed{A_1 - A_2\dot{V}^2 = C_1 + C_2\dot{V}^2} \qquad (30)$$

Alternatively, plotting these equations graphically, as shown in Figure 18, the operating point is the point where **the pump characteristic and the pipe characteristic intersect**. The flow rate can then be determined by inspection.

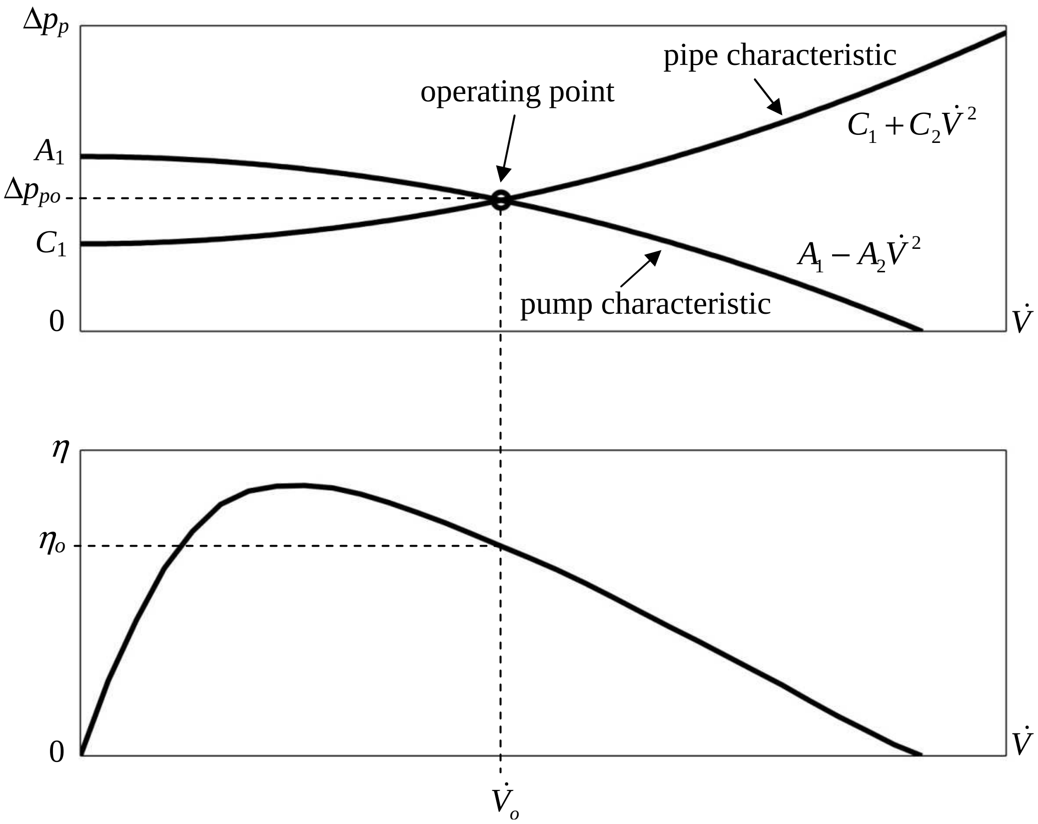


Figure 18: Graphical technique to determine operating point

The subscript *o* in the labels attached to Figure 18 indicate the operating points.

The efficiency characteristic (as shown in the lower plot in Figure 18) can be used to find the power consumption (and hence running costs) of a particular system. As can be seen, this particular system is not running at peak efficiency.

5.3 Using Head Rise Instead of Pressure

It is common to use characteristics and calculations that use heads (in metres) instead of pressures. When dealing with fluid machines, head, H is defined as the following:

$$H = \frac{p}{\rho g}$$

where p is the pressure, ρ is the density and g is the gravitational acceleration.

This is valid for fluid machines because for any given machine, the head characteristic H_p against $\dot{V}$ is independent of the fluid pumped, provided the fluid remains incompressible. The exact reason for this can only be understood through a study of pump design, which is beyond the scope of this work. All of the analysis can be repeated using H instead of p, the only adjustments being that all the equations are divided through by ρg.

5.4 Summary

- The pump characteristic is given by:

$$\Delta p_p = A_1 - A_2 \dot{V}^2$$

where A_1 and A_2 are machine specific constants.

- The pipe characteristic is given by:

$$\Delta p_p = C_1 + C_2 \dot{V}^2$$

where C_1 and C_2 are static lift and flow dependent term respectively.
  - Static lift is: $(p_A - p_B) + \rho g h$
  - Flow dependent term is: $\dfrac{8 f L \rho}{\pi^2 D^5}$

- When equations (26) and (29) are equated, a value for flow rate for the operating point can be determined. The operating point can be found graphically by locating the flow rate for the point of intersection.

$$A_1 - A_2 \dot{V}^2 = C_1 + C_2 \dot{V}^2$$

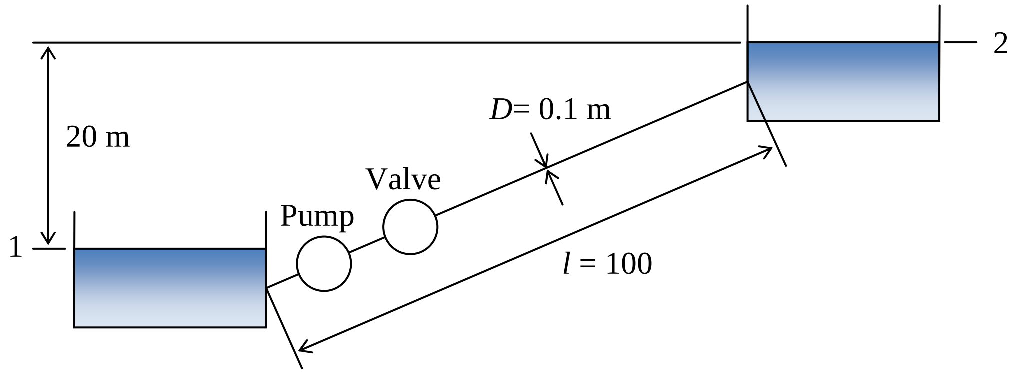
- Efficiency is given by:

$$\eta = \frac{\text{Power to the flow}}{\text{Power consumption}} = \frac{\Delta p_p \dot{V}}{P}$$

- When dealing with head isnetad of pressure, remember to divide all terms by ρg, as:

$$H = \frac{p}{\rho g}$$

5.5 Fluid Machines Example

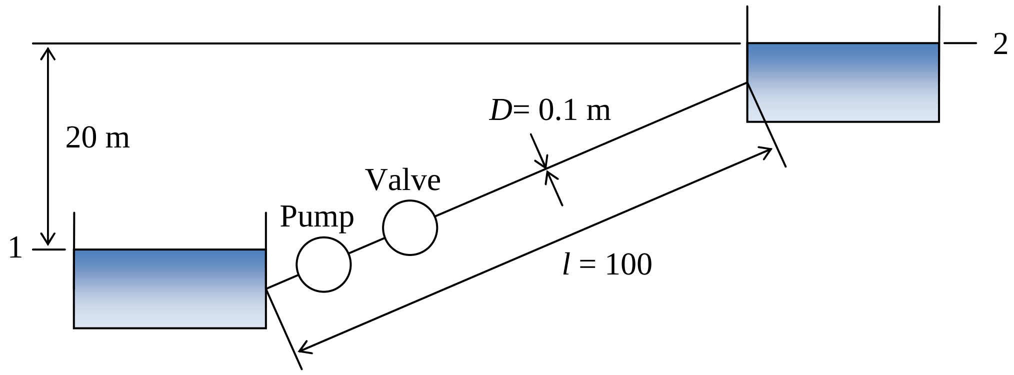


The diagram shows a pumped water system controlled by a valve. The friction factor for the flow in the pipe is $f = 0.025$. The pump characteristic is given by: $H_p = 1000 - 10^5 \dot{V}^2$

The loss through the valve is given by: $\Delta p_v = k_v \frac{1}{2} \rho c^2$ where c is the mean velocity in the pipe. Ignoring all minor losses, calculate the volume flow rate when the valve is wide open and $k_v = 10$.

5.6 Fluid Machines Exercises

1. The diagram shows a pumped system controlled by a valve.



The pump characteristic is given by:

$$H_p = 1000 - 10^5 \dot{V}^2$$

The loss through the valve is given by:

$$\Delta p_v = \frac{k_v \rho C^2}{2}$$

where C is the mean velocity in the pipe.

Ignoring all minor losses, calculate the volume flow rate when the value is wide open and k_v = 10 and when the valve is 50% closed and k_v = 100.

The efficiency of the pump is given by:

$$\eta = -280\dot{V}^2 + 28\dot{V}$$

Calculate the power consumption of the pump and the annual running costs of the system if the pump operates continuously. The valve is fully open. Pump power costs 3p/kWhr.

(0.087 m^3/s; 0.069 m^3/s, 657 kW, £173,000)

2. A centrifugal pump has the following characteristics:

Q (m^3/s)	0	0.1	0.15	0.20	0.25	0.3
ΔH (m)	40	37.5	33	27.5	20	12
η (%)	0	73	82	81	71	48

It is attached to a pipe with the outlet 10 m above the inlet and with equal static pressure. Assume flow is fully turbulent in all conditions.

Find the ΔH_p, power given to the flow and power consumed if the flow through the system is 0.22 m^3/s.

Assuming a friction factor f = 0.02 = constant, and the the pipe is circular section, find the pipe diameter if its length is 100 m.

(0.25 m, 54 kW, 69 kW, 0.222 m)

3. The same pump as in Q2 is used with a different system in which a pipe connects two reservoirs at equal pressure, one of which has its surface 5 m above that of the other. Flow is an upward direction. If the pipe length is 100 m, diameter 0.1 m, roughness of 0.5 mm, find the volume flow rate up the pipe.

A valve in the pipe is shut until the equivalent pipe length of the total system is 280 m. Calculate the reduced flow rate

(0.036 m^3/s, 0.022 m^3/s)

6 Fluid Momentum

The momentum equation for fluids helps us to deal with forces in the fluid flow. For example, we need to use the momentum equation to be able to calculate the force produced by the thrust of a rocket engine, or the force produced by a jet of fluid impinging on a turbine blade.

Consider the steady flow of a fluid along a horizontal pipe, entering a particular section of the pipe at section 1 and leaving at section 2, as shown in Figure 19.

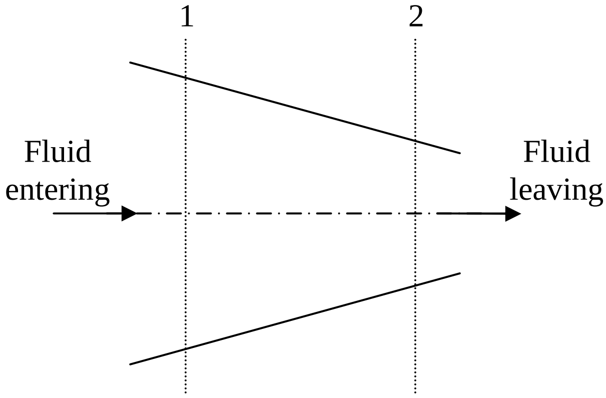


Figure 19: Steady flow in a horizontal pipe

Let the mass flow rate = $\dot{m}$ kg/s
Let the entry velocity = C_1 m/s
Let the exit velocity = C_2 m/s

Then in **1 second**, $\dot{m}$ kg of fluid undergoes an **increase in velocity from C_1 to C_2**. That is, the $\dot{m}$ kg of fluid must experience an acceleration = $(C_2 – C_1)$ m/s/1 s = $(C_2 – C_1)$ m/s^2.

But, in order to undergo an acceleration, the fluid must experience a force according to Newton's Second Law of motion, given by: force = mass × acceleration:

$$F = ma = \dot{m}(C_2 - C_1) \quad (31)$$

Hence the **force acting on the fluid = mass flow rate × change in velocity**.

Note that **the force must act in the same direction as the direction in which the change of velocity occurs**.

Should the flow direction of the fluid change, then velocity components must be used, in say the x and y directions. Considering the flow through the pipe shown in Figure 20.

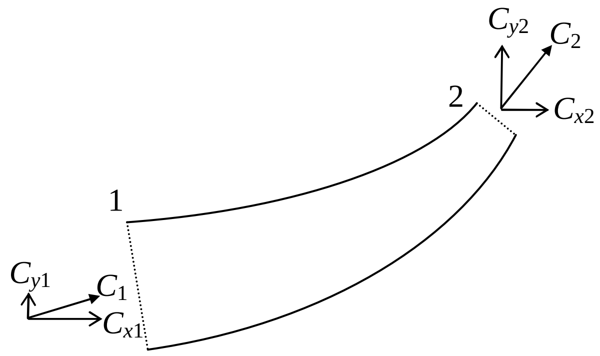


Figure 20: Flow through a curved pipe with velocities shown as components

Let the flow have the following velocity components:

- C_{x1} in the x direction at section 1
- C_{x2} in the x direction at section 2
- C_{y1} in the y direction at section 1
- C_{y2} in the y direction at section 2

Then the **force in the *x* direction** is the mass flow rate multiplied by the change in velocity in the x direction:

$$F_x = \dot{m}\left(C_{x2} - C_{x1}\right) \tag{32}$$

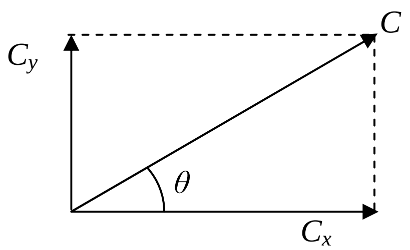
And the force in the y direction is similarly:

$$F_y = \dot{m}\left(C_{y2} - C_{y1}\right) \tag{33}$$

Note that these momentum forces can be produced by pressure differences in the fluid, the weight of the fluid or the force reaction from a solid object.

These forces are called momentum forces because momentum = mass × velocity and obviously then $\dot{m}\left(C_2 - C_1\right)$ is the **rate of change of momentum**, which by Newton's 2nd Law must equal the applied force.

Note that the velocities decomposed into x and y components follow standard trigonometry:



where:

$$C_x = C\cos\theta \quad \text{and} \quad C_y = C\sin\theta$$

6.1 Fluid Momentum Example

A flat plate of mass 10 kg is constrained horizontally but may move vertically without any resistance. A jet of water of 0.2 m diameter impinges at right angles on the underside of the plate. What must be the velocity of the jet if the plate is to remain stationary against the action of gravity?

6.2 Fluid Momentum Exercises

1. In the above problem, if the jet is again vertically upwards but the under-side of the plate is in the form of a cone of 150° included angle, calculate the jet velocity necessary to maintain equilibrium of the plate.

   (2.05 m/s)

2. A jet of water having a velocity of 30 m/s impinges on a series of vanes which divert the water through 120°. If the jet diameter is 0.3 m, calculate the resultant force on the vanes.

   (110.2 kN)

3. A rocket burns its propellant at a rate of 7 kg/s. The exhaust gases leave the rocket at a relative velocity of 1000 m/s. The mass of the rocket is initially 230 kg. Determine:
   a. The rocket thrust.
   b. The initial acceleration of the rocket if it takes off vertically.

   (7 kN; 20.62 m/s^2)

4. A stationary curved vane deflects a 50 mm diameter jet of water through 150°. Because of friction over the surface, the water leaving the vane has only 80% of its original velocity. Determine
   a. The mass flow rate necessary to produce a force of 2000 N on the vane in the direction of the jet.
   b. The force on the vane perpendicular to the jet.

   (48.2 kg/s; 472.9 N)

5. A toy balloon of mass 86 gm is filled with air of density 1.29 kg/m^3. The small filling tube of 6 mm bore is pointed vertically downwards and the balloon is released. Calculate the initial rate at which air escapes if the initial acceleration is 15 m/s^2.

   (8.82 gm/s)

6. A curved plate deflects at 75 mm diameter jet of water, which is initially horizontal, upwards through an angle of 45°. For a jet velocity of 40 m/s, and ignoring friction, calculate the total force acting on the plate, in both magnitude and direction.

   (5410 N at 67.5° to the horizontal)

7. A square plate is hinged about its upper edge, which is horizontal. The length of each side is 0.4 m and the plate weights 200 N. A horizontal jet of water issues from a long horizontal slot 0.3 m by 2 mm and impinges on the plate at a vertical distance of 0.2 m below the hinge. The jet has a mass flow rate of 9 kg/s.
   a. Determine the force which must be applied to the plate at its lower edge in order to keep it vertical.
   b. If the plate is now allowed to swing freely, determine the inclination to the vertical which the plate assumes.

   (67.5 N; 42.6°)

8. A stationary curved vane deflects a 40 mm diameter jet of water through an angle of 120°. Due to friction over the surface of the vane, the water leaves the vane has only 90% of its original velocity. If the jet produces a force 1000 N on the vane in the original direction of the jet, determine:
    a. The jet velocity at entry to the vane
    b. The water mass flow rate
    c. The force produced on the vane in the direction perpendicular to the jet
    d. The magnitude and direction of the resultant force exerted by the water on the vane

(23.43 m/s; 29.44 kg/s; 537.6 N; 1135 N at 28.3°)

9. A stationary curved vane deflects a 30 mm diameter jet of water through an angle of 140°. Due to friction over the surface of the vane, the water leaves the vane has only 95% of its original velocity. If the jet produces a force 500 N on the vane in the original direction of the jet, determine:
    a. The jet velocity at entry to the vane
    b. The water mass flow rate
    c. The force produced on the vane in the direction perpendicular to the jet
    d. The magnitude and direction of the resultant force exerted by the water on the vane

(20.23 m/s; 14.3 kg/s; 176.7 N; 530.3 N at 19.5°)

7 Formulae & Data

7.1 Fluids – General

Hydrostatic pressure $p = \rho g h_p$ where h_p is the pressure head

Note that h_p is sometimes represented by 'z', as in Bernoulli's equation below.

Bernoulli's equation: $$p_1 + \frac{1}{2}\rho c_1^2 + \rho g z_1 = p_2 + \frac{1}{2}\rho c_2^2 + \rho g z_2 + \Delta p$$

Mass flow rate $\dot{m} = \rho_1 A_1 c_1 = \rho_2 A_2 c_2$ or $\dot{m} = \frac{A_1 c_1}{v_1} = \frac{A_2 c_2}{v_2}$

Volume flowing per second, i.e. volumetric flow rate, $\dot{V} = Ac$

7.2 Fluid Flow with Friction

$\mathrm{Re} = \frac{\rho c D}{\mu} = \frac{cD}{\upsilon}$ where μ is dynamic viscosity and υ is kinematic viscosity.

$$\Delta p = \left[f\left(\frac{L}{D}\right) + \sum_{i=1}^{n} k_i \right]\left(\frac{1}{2}\rho c^2\right)$$

Power loss = $\Delta p \dot{V}$

For Laminar Flow $\Delta p = \frac{8\mu L \dot{V}}{\pi R^4}$ and $f = \frac{64}{\mathrm{Re}}$

For Turbulent flow in general: $$f = \frac{1}{\left[-1.8\log\left(\frac{6.9}{\mathrm{Re}} + \left(\frac{1}{3.7}\frac{\varepsilon}{D}\right)^{1.11}\right)\right]^2}$$

For Fully turbulent flow: $$f = \frac{1}{\left[1.14 - 2\log_{10}\left(\frac{\varepsilon}{D}\right)\right]^2}$$