System properties

There are many ways to characterize a system. We can for example divide systems in stable/unstable systems, controllable/uncontrollable systems or observable/unobservable systems. This chapter discusses the meanings of these terms.

1 Stability

1.1 Definitions

Let's consider the time-invariant systems $\dot{\mathbf{x}} = f(\mathbf{x})$. Given an initial point $\mathbf{x_0}$, the system will have a solution $\mathbf{x}(t, \mathbf{x_0})$. Now it's time to make some definitions.

- We say that a vector $\bar{\mathbf{x}}$, which satisfies $\dot{\bar{\mathbf{x}}} = f(\bar{\mathbf{x}}) = \mathbf{0}$, is an **equilibrium point**. If we take $\mathbf{x_0} = \bar{\mathbf{x}}$, then $\mathbf{x}(t, \mathbf{x_0}) = \bar{\mathbf{x}}$.
- An equilibrium point $\bar{\mathbf{x}}$ is called **stable** if, for every boundary $\varepsilon > 0$, there is a distance $\delta > 0$ such that, if $||\mathbf{x_0} - \bar{\mathbf{x}}|| < \delta$, then $||\mathbf{x}(t, \mathbf{x_0}) - \bar{\mathbf{x}}|| < \varepsilon$ for all $t \geq 0$. An equilibrium point that is not stable is termed **unstable**.
- A stable equilibrium point is also called **asymptotically stable** if $\lim_{t\to\infty} ||\mathbf{x}(t, \mathbf{x_0}) - \bar{\mathbf{x}}|| = 0$, given that $||\mathbf{x_0} - \bar{\mathbf{x}}|| < \delta$ for some $\delta > 0$.

1.2 Eigenvalue method

The question is, how do we know if an equilibrium point is stable? There are several methods to find that out. We start with the **eigenvalue method**. This trick works for linear time-invariant systems, with $\dot{\mathbf{x}} = A\mathbf{x}$. These systems have $\bar{\mathbf{x}} = \mathbf{0}$ as equilibrium solution. (There are also other equilibrium solutions if $\det A = 0$, but we won't go into detail on that.)

Let's denote the k eigenvalues of A by $\lambda_1, \lambda_2, \ldots, \lambda_k$. We now have to look at the real parts $\text{Re}\ \lambda_i$ of these eigenvalues.

- The point $\bar{\mathbf{x}} = \mathbf{0}$ is asymptotically stable if and only if $\text{Re}\ \lambda_i < 0$ for all eigenvalues λ_i.
- The point $\bar{\mathbf{x}} = \mathbf{0}$ is stable if $\text{Re}\ \lambda_i \leq 0$ for all eigenvalues λ_i. Also, for the eigenvalues λ_i with $\text{Re}\ \lambda_i = 0$, the algebraic multiplicity m_i must equal the geometric multiplicity g_i.
- In any other case, the point $\bar{\mathbf{x}} = \mathbf{0}$ is unstable. So, either there is an eigenvalue λ_i with $\text{Re}\ \lambda_i > 0$, or there is an eigenvalue λ_i with $\text{Re}\ \lambda_i = 0$ for which $g_i < m_i$.

Now we can ask ourselves, what do we do with a non-linear system $\dot{\mathbf{x}} = f(\mathbf{x})$ with equilibrium point $\bar{\mathbf{x}} = \mathbf{0}$? Well, we simply linearize it. In other words, we write it as $\dot{\mathbf{x}} = A\mathbf{x} + h(\mathbf{x})$, where $h(\mathbf{x})$ only contains higher-order terms of $\mathbf{x}$. The stability of $\bar{\mathbf{x}}$ now depends on the matrix A. In other words, if the eigenvalues of A all have negative real parts, then the equilibrium point $\bar{\mathbf{x}} = \mathbf{0}$ is asymptotically stable. There is one side-note though: $h(\mathbf{x})$ has to be real and continuous for $\mathbf{x}$ near $\bar{\mathbf{x}} = \mathbf{0}$.

1.3 Routh's criterion

To apply the eigenvalue method, we need to find the eigenvalues λ_i. They are usually found, by solving the characteristic polynomial

$$\det(A - \lambda I) = a_0 + a_1\lambda + a_2\lambda^2 + \ldots + a_n\lambda^n. \tag{1.1}$$

Especially for big n, it can be very difficult to solve the above equation. But, in fact, we don't need to know the exact eigenvalues. We only want to know whether they're all in the negative (left) part of the complex plane! And to find this out, we can use **Routh's criterion**. According to Routh's criterion, we first construct the **Routh table**.

$$\begin{array}{cccc} a_n & a_{n-2} & a_{n-4} & \cdots \\ a_{n-1} & a_{n-3} & a_{n-5} & \cdots \\ b_{n-2} & b_{n-4} & b_{n-6} & \cdots \\ c_{n-3} & c_{n-5} & c_{n-7} & \cdots \\ d_{n-4} & d_{n-6} & d_{n-8} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{array} \tag{1.2}$$

There are a few important rules. Any number with a negative coefficient (like for example a_{-1}) is equal to zero. You also continue calculating this pattern, until you wind up with only zeroes. (This happens after at most $n+1$ rows.) For the rest, you can find the coefficients b_i, c_i, d_i, etc. using

$$b_{n-2} = \frac{a_{n-1}a_{n-2} - a_n a_{n-3}}{a_{n-1}}, \qquad b_{n-4} = \frac{a_{n-1}a_{n-4} - a_n a_{n-5}}{a_{n-1}}, \tag{1.3}$$

$$c_{n-3} = \frac{b_{n-2}a_{n-3} - a_{n-1}b_{n-4}}{b_{n-2}}, \qquad c_{n-5} = \frac{b_{n-2}a_{n-5} - a_{n-1}b_{n-6}}{b_{n-2}}. \tag{1.4}$$

Do you see the pattern? To calculate a number x, you always use four numbers. Two of these numbers come from the column just right of x. The other two numbers come from the leftmost column. All four numbers come from the two rows just above x.

We still have one question remaining. What does this tell us about the eigenvalues? Well, according to Routh's criterion, we have to look at the left column. There must be $n+1$ nonzero numbers in this column. Also, all these numbers must have the same sign. Only if this is the case, then the eigenvalues of A all have a negative real part. And this then of course implies that the system is asymptotically stable.

1.4 Lyapunov stability

There is another way in which we can determine whether a system $\dot{\mathbf{x}} = A\mathbf{x}$ is asymptotically stable. For this method, we first have to define some kind of **'generalized' energy** $V(\mathbf{x}(t)) = \mathbf{x}^T P\mathbf{x}$, where P is a positive-definite matrix. (Positive-definite means that $\mathbf{x}^T P\mathbf{x} > 0$ for all vectors $\mathbf{x} \neq \mathbf{0}$. So, V is always positive, unless $\mathbf{x} = \mathbf{0}$.) The derivative of V is

$$\frac{d}{dt}V(\mathbf{x}(t)) = \dot{\mathbf{x}}^T P\mathbf{x} + \mathbf{x}^T P\dot{\mathbf{x}} = \mathbf{x}^T \left(PA + A^T P\right)\mathbf{x} = -\mathbf{x}^{\mathbf{T}} Q\mathbf{x}, \tag{1.5}$$

where Q is defined as

$$Q = -\left(PA + A^T P\right). \tag{1.6}$$

Now let's look at what happens if Q is also positive definite. In this case, we have $dV/dt < 0$ as long as $\mathbf{x} \neq \mathbf{0}$. ($V$ can only decrease.) But we also have $V > 0$ as long as $\mathbf{x} \neq \mathbf{0}$. This implies that $\lim_{t\to\infty} V(\mathbf{x}(t)) = 0$. In other words, V eventually becomes zero, and so does $\mathbf{x}$.

The conclusion? Let's suppose we can find any positive definite matrices P and Q that satisfy the so-called **Lyapunov equation** (1.6). In this case, $\mathbf{x}$ will converge to $\mathbf{0}$ and the system is thus asympotically stable.

1.5 Interval stability

Let's again examine the polynomial

$$p(\lambda) = a_0 + a_1\lambda + a_2\lambda^2 + \ldots + \lambda^n. \tag{1.7}$$

(Note that this time we have divided out a_n. This of course does not change the roots of the equation.) Sometimes, the coefficients a_i are not exactly known. Instead, we only know the interval $[a_i^-, a_i^+]$ in which the coefficient a_i must lie. We usually denote this by the **interval polynomial**

$$p(\lambda, [\mathbf{a}^-, \mathbf{a}^+]) = [a_0^-, a_0^+] + [a_1^-, a_1^+]\lambda + [a_2^-, a_2^+]\lambda^2 + \ldots + [a_{n-1}^-, a_{n-1}^+]\lambda^{n-1} + \lambda^n. \tag{1.8}$$

Associated to this polynomial are four polynomials, called the **Kharitonov polynomials**. They are

$$\begin{aligned} p_{--}(\lambda) &= a_0^- + a_1^-\lambda + a_2^+\lambda^2 + a_3^+\lambda^3 + a_4^-\lambda^4 + a_5^-\lambda^5 + a_6^+\lambda^6 + \ldots + \lambda^n, & (1.9)\\ p_{+-}(\lambda) &= a_0^+ + a_1^-\lambda + a_2^-\lambda^2 + a_3^+\lambda^3 + a_4^+\lambda^4 + a_5^-\lambda^5 + a_6^-\lambda^6 + \ldots + \lambda^n, & (1.10)\\ p_{-+}(\lambda) &= a_0^- + a_1^+\lambda + a_2^+\lambda^2 + a_3^-\lambda^3 + a_4^-\lambda^4 + a_5^+\lambda^5 + a_6^+\lambda^6 + \ldots + \lambda^n, & (1.11)\\ p_{++}(\lambda) &= a_0^+ + a_1^+\lambda + a_2^-\lambda^2 + a_3^-\lambda^3 + a_4^+\lambda^4 + a_5^+\lambda^5 + a_6^-\lambda^6 + \ldots + \lambda^n. & (1.12) \end{aligned}$$

Note that all these polynomials alternately have two pluses and two minuses. The subscripts of p simply indicate the start of the series.

The Karitonov polynomials are important when determining the stability of the system. Let's suppose that all eigenvalues of all these four polynomials p have a negative real part. Only in this case will all eigenvalues of every polynomial $p(\lambda, \mathbf{a})$ (with parameter vector $\mathbf{a} \in [\mathbf{a}^-, \mathbf{a}^+]$) have a negative real part. In other words, if the Karitonov polynomials are asymptotically stable, then every polynomial within the interval is asymptotically stable, and vice versa.

1.6 Input-output stability

Let's examine a system in state-space form. So,

$$\begin{aligned} \dot{\mathbf{x}}(t) &= A(t)\mathbf{x}(t) + B(t)\mathbf{u}(t), & (1.13)\\ \mathbf{y}(t) &= C(t)\mathbf{x}(t) + D(t)\mathbf{u}(t). & (1.14) \end{aligned}$$

Let's suppose that the input $\mathbf{u}(t)$ is bounded. (There is a constant c such that $||\mathbf{u}(t)|| \leq c$ for all t.) If the output $\mathbf{y}(t)$ also remains bounded ($||\mathbf{y}(t)|| \leq k$ for some k), then the system is said to be **BIBO stable**. (BIBO stands for bounded input, bounded output.)

BIBO stability is often also referred to as **external stability**. (It concerns the external parts $\mathbf{u}$ and $\mathbf{y}$ of the system.) On the other hand, the stability of $\dot{\mathbf{x}} = A\mathbf{x}$ is referred to as **internal stability**. A system that is internally stable is always externally stable as well. But the opposite is not always true.

2 Controllability

2.1 Definitions

Next to stability, also controllability is an important aspect of systems. Let's examine a system (A, B, C, D). If both the initial state $\mathbf{x_0}$ and the input $\mathbf{u}(t)$ at any time t are given, then the solution $\mathbf{x}(t, \mathbf{x_0}, \mathbf{u})$ is fully determined.

Now let's suppose that we have a system (A, B, C, D) and a desired state $\mathbf{x_1}$. We say that the system is **controllable** if there is an input $\mathbf{u}(t)$ such that the state of the system $\mathbf{x}(t, \mathbf{x_0}, \mathbf{u})$ equals $\mathbf{x_1}$ at some finite time $t_1 > 0$. (This must hold for every desired state $\mathbf{x_1}$.)

Sometimes it is assumed that $\mathbf{x_1} = \mathbf{0}$ with $\mathbf{x_0} \neq \mathbf{0}$: we want to achieve/keep a zero state. In this case, we are talking about **null-controllability**. On the other hand, sometimes it is assumed that $\mathbf{x_1} \neq \mathbf{0}$ but $\mathbf{x_0} = \mathbf{0}$: we want to reach a certain point $\mathbf{x_1}$ starting from zero. In this case, we are talking about **reachability**. However, it can be shown that the controllability-problem, the null-controllability-problem and the reachability problem are equivalent. (If a system is controllable, it is also reachable and vice verse.) So, we will only consider ourselves with the controllability problem.

2.2 The controllability matrix

An important aspect in controllability is the **controllability matrix** R. It is defined as the $n \times nm$ matrix

$$R = \begin{bmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{bmatrix}. \tag{2.1}$$

Now let's look at the image of R, denoted by im R. (The image of R means the column space of R. It consists of all vectors $\mathbf{b}$ that can be reached by linear combinations of the columns of R: $R\mathbf{a} = \mathbf{b}$.) It can be shown that this image consists of all reachable vectors. In other words, all linear combinations of R are reachable. For this reason, im R is also known as the **reachable subspace** or the **controllable subspace**.

Based on the above fact, we can find something very important. If the matrix R has rank n (meaning that all n rows of R are linearly independent), then the system (A, B) is controllable! This condition is known as the **rank condition** for controllability. (Note that we have denoted the system by (A, B). This is because C and D don't influence the controllability of the state.)

2.3 The modal form

If A is diagonalizable, we can apply a little trick. In this case, we can write $A = TDT^{-1}$. Now, let's define

$$\tilde{A} = D = T^{-1}AT, \qquad \tilde{B} = T^{-1}B, \qquad \tilde{C} = CT, \qquad \text{and} \qquad \tilde{D} = D. \tag{2.2}$$

If we also define $\tilde{\mathbf{x}} = T^{-1}\mathbf{x}$, then we can rewrite the state space representation to

$$\dot{\tilde{\mathbf{x}}} = \tilde{A}\tilde{\mathbf{x}} + \tilde{B}\mathbf{u} \qquad \text{and} \qquad \mathbf{y} = \tilde{C}\tilde{\mathbf{x}} + \tilde{D}\mathbf{u}. \tag{2.3}$$

This system has exactly the same properties as the old (A, B, C, D) system. (Only the state parameters have changed.) But now, we have a matrix A which is diagonal! For diagonal matrices, it's not hard to find the controllability matrix. So, solving this problem further won't be difficult.

2.4 Seperating noncontrollable parts

Let's suppose that we have a noncontrollable matrix. What do we do now? Well, there is a way in which we can separate the noncontrollable part from the controllable part. We'll examine that method now.

First, we find the matrix R. Let's denote the dimension of im R as $k = \text{rank } R < n$. We can find a basis $\{\mathbf{q_1}, \dots, \mathbf{q_k}\}$ of k linearly independent vectors $\mathbf{q_i}$ for the subspace im R. Let's add $(n - k)$ more vectors $\mathbf{q_{k+1}}, \dots, \mathbf{q_n}$ to this basis, such that we have n linearly independent vectors. We now put all these vectors in a matrix. So,

$$T = \begin{bmatrix} \mathbf{q_1} & \mathbf{q_2} & \dots & \mathbf{q_n} \end{bmatrix}. \tag{2.4}$$

Just like in the previous paragraph, we can define $\tilde{A} = T^{-1}AT$ and $\tilde{B} = T^{-1}B$. This again gives us an equivalent system. But we now have

$$\tilde{A} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{bmatrix} \qquad \text{and} \qquad \tilde{B} = \begin{bmatrix} \tilde{B}_1 \\ 0 \end{bmatrix}. \tag{2.5}$$

In this equation, A_{11} has size $k \times k$. (The rest of the matrices are sized accordingly.) The controllability matrix of our new system is

$$\tilde{R} = TR = \begin{bmatrix} \tilde{B} & \tilde{A}\tilde{B} & \tilde{A}^2\tilde{B} & \dots & \tilde{A}^{n-1}\tilde{B} \end{bmatrix} = \begin{bmatrix} \tilde{B}_1 & \tilde{A}_{11}\tilde{B}_1 & \tilde{A}_{11}^2\tilde{B}_1 & \dots & \tilde{A}_{11}^{n-1}\tilde{B}_1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}. \tag{2.6}$$

So, we have split up our system into a controllable part and an uncontrollable part.

3 Observability

3.1 Definitions and theorems

Let's suppose that we have some system (A, B, C, D), of which we do not know the initial state $\mathbf{x_0}$. But we are observing the output $\mathbf{y}(t)$, and also the input $\mathbf{u}(t)$ is known. If, after some time t_1, we are always able to uniquely determine the initial state $\mathbf{x_0}$ of the system, then the system is called **observable**. This must hold for every input function $\mathbf{u}(t)$.

Just like controllability completely depends on A and B, so does observability completely depend on A and C. We therefore usually talk about the system (C, A). (Note that this time A is mentioned last.) Now, let's define the **observability matrix** W as

$$W = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}. \tag{3.1}$$

The size of W is $np \times n$ (with p the height of C). It can be proven that the system (C, A) is observable if and only if rank $W = n$. (This condition is known as the **rank condition** for observability.) In other words, this is the case if all n columns of W are linearly independent.

Let's look at the kernel ker W of W. (The kernel is also known as the null-space: all $\mathbf{x}$ for which $W\mathbf{x} = \mathbf{0}$.) It can be shown that ker W is the **non-observable subspace**. In other words, any vectors $\mathbf{x_0}$ and $\mathbf{x_1}$ in the kernel of W (thus satisfying $W\mathbf{x_0} = W\mathbf{x_1} = \mathbf{0}$) can not be destinguished from each other by just looking at the output.

3.2 Seperating nonobservable parts

Previously, we have separated the noncontrollable part of a system. Now, we will separate the nonobservable part from a nonobservable system. We denote the dimension of ker W as $k = n - \text{rank } W$. We can find a basis $\{\mathbf{q_1}, \ldots, \mathbf{q_k}\}$ of k linearly independent vectors $\mathbf{q_i}$ for the subspace ker W. Let's again add $(n-k)$ more vectors $\mathbf{q_{k+1}}, \ldots, \mathbf{q_n}$ to this basis, such that we again have n linearly independent vectors. Once more, we put all these vectors in a matrix $T = [\mathbf{q_1} \;\; \mathbf{q_2} \;\; \ldots \;\; \mathbf{q_n}]$. This time we have

$$\bar{A} = T^{-1}AT = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ 0 & \bar{A}_{22} \end{bmatrix} \qquad \text{and} \qquad \bar{C} = CT = \begin{bmatrix} 0 & \bar{C}_2 \end{bmatrix}. \tag{3.2}$$

Again, A_{11} has size $k \times k$. (The rest of the matrices are sized accordingly.) The observability matrix of our new system now is

$$\bar{W} = WT = \begin{bmatrix} \bar{C} \\ \bar{C}\bar{A} \\ \bar{C}\bar{A}^2 \\ \vdots \\ \bar{C}\bar{A}^{n-1} \end{bmatrix} = \begin{bmatrix} 0 & \bar{C}_2 \\ 0 & \bar{C}_2\bar{A}_{22} \\ 0 & \bar{C}_2\bar{A}_{22}^2 \\ 0 & \vdots \\ 0 & \bar{C}_2\bar{A}_{22}^{n-1} \end{bmatrix}. \tag{3.3}$$

And we have split up the system into an observable part and a nonobservable part.

3.3 The Hautus test

There are plenty of other ways in which to check if a system is controllable/observable. Although we won't treat all of them, we will treat the **Hautus test**.

- The Hautus test for controllability states that the pair (A, B) is controllable if and only if, for all (possibly) complex s, we have

$$\text{rank} \begin{bmatrix} (sI - A) & B \end{bmatrix} = n. \tag{3.4}$$

- Similarly, the Hautus test for observability states that the pair (C, A) is controllable if and only if, for all (possibly) complex s, we have

$$\text{rank} \begin{bmatrix} (sI - A) \\ C \end{bmatrix} = n. \tag{3.5}$$

It is nice to note that $sI - A$ already has rank n for all s unequal to an eigenvalue λ_i. So, you only have to check the above matrices for values of s equal to one of the eigenvalues of A. That should save us some work. Finally, it is interesting to note that, if the pair (A, B) is controllable, then the pair (B^T, A^T) is observable and vice verse.