Feedback control

When controlling a system, the trick is to provide the right input. But what input will control the system? And how do we find it? That's what we'll look at in this chapter.

1 State feedback

1.1 Definitions

Let's suppose we're controlling a boat. When doing this, we use the output (the current heading) to determine the rudder position. In this case, the output is used to determine the input. When this is the case, we are dealing with a **feedback control** system, also known as a **closed-loop system**. When the input is independent of the output, then we have an **open-loop system**. In this chapter, we will examine closed-loop systems.

The important question is, how do we choose our input? Well, let's suppose that $C = I$ and $D = 0$. Thus, the output $\mathbf{y}(t)$ equals the state $\mathbf{x}(t)$. In this case, we can take $\mathbf{u}(t) = F\mathbf{x}(t)$, with F a matrix. This is called **state feedback**. The equation $\mathbf{u} = F\mathbf{x}$ is called a **control law**. In fact, this specific control law is known as a **static compensator**. (This is because F does not vary with time.)

1.2 Stabilizability

With the static compensator, our system equation turns into $\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u} = (A + BF)\mathbf{x}$. We should choose our F, such that the system $A + BF$ is stable. In other words, the eigenvalues of $A + BF$ should all have a negative real part. We call the system **stabilizable** if there is a matrix F, such that $A + BF$ is stable.

You may wonder, when is a system stabilizable? To find this out, we can use the **Hautus test** for stabilizability. This test states that the system (A, B) is stabilizable if and only if, for all (possibly) complex s with $\text{Re}(s) \geq 0$, we have

$$\text{rank} \begin{bmatrix} (sI - A) & B \end{bmatrix} = n. \tag{1.1}$$

Again, note that the rank of the above matrix always equals n, when s is not an eigenvalue. We thus only need to check the above equation for values $s = \lambda_i$, where λ_i is (in this case) an unstable eigenvalue. (Also note that stable systems are stabilizible by default, since they don't have any unstable eigenvalues. Similarly, controllable systems are also always stabilizable.)

1.3 The pole-assignment theorem

Let's suppose that the system (A, B) is controllable. The **pole-assignment theorem** now states that we can actually choose the poles of $(A + BF)$ ourselves! This implies that the system is surely stabilizable. (We can simply choose poles with negative real parts.) All we need to do is choose F in the right way. But how do we choose F? That is, however, not very easy. But we will describe the process here for the single-input case.

The first step is to put the matrices A and B into so-called **controller (canonical) form**, being

$$\bar{A} = T^{-1}AT = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ -p_0 & -p_1 & \cdots & -p_{n-2} & -p_{n-1} \end{bmatrix} \qquad \text{and} \qquad \bar{B} = T^{-1}B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}. \tag{1.2}$$

In this equation, the coefficients $p_1, \ldots, p_{n-1}$ are the coefficients of the characteristic polynomial of A. In fact, we have

$$\det(\lambda I - A) = \det(\lambda I - \bar{A}) = p(\lambda) = \lambda^n + p_{n-1}\lambda^{n-1} + \ldots + p_1\lambda + p_0. \tag{1.3}$$

The only difficult part here is finding the right transformation matrix T. Luckily, T can be constructed. We simply have $T = [\mathbf{q_1} \;\; \mathbf{q_2} \;\; \ldots \;\; \mathbf{q_n}]$, where the vectors $\mathbf{q_1}, \ldots, \mathbf{q_n}$ are defined as

$$\mathbf{q_n} = B, \tag{1.4}$$

$$\mathbf{q_{n-1}} = AB + p_{n-1}B = A\mathbf{q_n} + p_{n-1}\mathbf{q_n}, \tag{1.5}$$

$$\mathbf{q_{n-2}} = A^2B + p_{n-1}AB + p_{n-2}B = A\mathbf{q_1} + p_{n-2}\mathbf{q_n}, \tag{1.6}$$

$$\vdots \tag{1.7}$$

$$\mathbf{q_1} = A^{n-1}B + p_{n-1}A^{n-2}B + \ldots + p_1B = A\mathbf{q_2} + p_1\mathbf{q_n}. \tag{1.8}$$

Once we have put A and B into the controller form, we can continue with step 2. Let's suppose that we want to have a certain set of poles. These poles then fully determine the characteristic equation of $A + BF$, being

$$\det(\lambda I - (A + BF)) = r(\lambda) = \lambda^n + r_{n-1}\lambda^{n-1} + \ldots + r_1\lambda + r_0. \tag{1.9}$$

Now, all we need to do is define

$$\bar{F} = \begin{bmatrix} p_0 - r_0 & p_1 - r_1 & \ldots & p_{n-1} - r_{n-1}. \end{bmatrix} \tag{1.10}$$

This immediately gives $\bar{A} + \bar{B}\bar{F}$ the required eigenvalues, and thus also $A + BF$. (Linear transformations don't change the eigenvalues of a matrix.) Once we know $\bar{F}$, finding F itself is not difficult. We just use $F = \bar{F}T^{-1}$.

2 The observer

2.1 Detectability

Previously, we have assumed that $C = I$. In other words, the state $\mathbf{x}(t)$ was known. But this is of course not always the case. What do we do if we don't know the state $\mathbf{x}$? One option is to approximate it, by using an **observer**. This observer uses the input $\mathbf{u}(t)$ and the output $\mathbf{y}(t)$ to make an estimate $\hat{\mathbf{x}}$ of the state $\mathbf{x}$. The observer can now be modeled as a separate system. This system has as output the estimated state $\hat{\mathbf{x}}$ of the system. We thus have

$$\dot{\hat{\mathbf{x}}} = P\hat{\mathbf{x}} + Q\mathbf{u} + K\mathbf{y} \qquad \text{and} \qquad \hat{\mathbf{x}} = I\hat{\mathbf{x}}. \tag{2.1}$$

The question remains, what should P, Q and K be? We use two simple rules to derive them. First of all, if we have a correct estimate of the state at some time, then we would like to keep the estimate correct. So, if $\mathbf{x} = \hat{\mathbf{x}}$, then we want to have

$$\frac{d}{dt}(\mathbf{x} - \hat{\mathbf{x}}) = \mathbf{0} = A\mathbf{x} + B\mathbf{u} - P\hat{\mathbf{x}} - Q\mathbf{u} - K\mathbf{y} = (A - KC)\mathbf{x} - P\hat{\mathbf{x}} + (B - Q)\mathbf{u}, \tag{2.2}$$

where we have used $\mathbf{y} = C\mathbf{x}$. The above equation should hold for every $\mathbf{u}(t)$. By using this fact, and by also using $\mathbf{x} = \hat{\mathbf{x}}$, we can find that

$$A - KC = P \qquad \text{and} \qquad B = Q. \tag{2.3}$$

Second, we would like the **error** $\mathbf{e} = \mathbf{x} - \hat{\mathbf{x}}$ to decrease to zero. The derivative of this error is given by

$$\dot{\mathbf{e}} = \frac{d}{dt}(\mathbf{x} - \hat{\mathbf{x}}) = (A - KC)(\mathbf{x} - \hat{\mathbf{x}}) = (A - KC)\,\mathbf{e}. \tag{2.4}$$

This means that, if $A - KC$ is asymptotically stable, then the error will converge to $\mathbf{0}$. The observer will thus be able to estimate the state $\mathbf{x}$. We therefore say that a system is **detectable** if there is a matrix K such that the eigenvalues of $A - KC$ all have negative real parts.

The question remains, when is a system detectable? Again, there is a test. A system is detectable if and only if, for all (possibly) complex s with $\text{Re}(s) \geq 0$, we have

$$\text{rank} \begin{bmatrix} (sI - A) \\ C \end{bmatrix} = n. \tag{2.5}$$

Again, we only need to check the above equation for values $s = \lambda_i$, where λ_i is an unstable eigenvalue. (Also note that stable and observable systems are always detectable.)

2.2 Again the pole-assignment theorem

Let's suppose that (C, A) is observable. In this case, we can again select the poles of $A - KC$. In fact, for every polynomial $w(\lambda) = \lambda^n + w_{n-1}\lambda^{n-1} + \ldots + w_1\lambda + w_0$, there is a K such that $\det(\lambda I - (A - KC)) = w(\lambda)$. Thus, observable systems are always detectable. We just need to select K in the correct way. But how do we find K?

First, we remember that, if (C, A) is observable, then (A^T, C^T) is controllable. The system (A^T, C^T) is thus also stabilizable. There thus is an F such that $\det\left(\lambda I - (A^T + C^T F)\right) = w(\lambda)$. (To find F, use the set of steps discussed earlier in this chapter.) If we now take $K = -F^T$, then there is of course also a K such that

$$\det\left(\lambda I - (A^T - C^T K^T)\right) = \det(\lambda I - (A - KC)) = w(\lambda). \tag{2.6}$$

So, in this way, the correct matrix K can be chosen for the observer.

3 The dynamic compensator

3.1 Combining state feedback with an observer

Let's summarize matters. In the first part of this chapter, we saw how we can stabilize a system, if we know the state. In the second part, we saw how we can estimate the state. What if we use the estimate of the state to stabilize the system? We simply connect the observer to the static compensator. Doing this will give us a **dynamic compensator** (also known as a **dynamic controller**). This dynamic compensator can be seen as a separate system. Its input will be the output $\mathbf{y}$ of the original system, while its output is the input $\mathbf{u}$ to the original system. The dynamic compensator can thus be described by

$$\dot{\hat{\mathbf{x}}} = (A + BF - KC)\hat{\mathbf{x}} + K\mathbf{y}, \tag{3.1}$$

$$\mathbf{u} = F\hat{\mathbf{x}}. \tag{3.2}$$

By connecting the dynamic compensator to the original system, we actually get a **combined system** without any inputs and outputs. This combined system can be described by $\dot{\mathbf{x}}_\mathbf{c} = A_c \mathbf{x}_\mathbf{c}$, where

$$\mathbf{x}_\mathbf{c} = \begin{bmatrix} \mathbf{x} \\ \hat{\mathbf{x}} \end{bmatrix} \qquad \text{and} \qquad A_c = \begin{bmatrix} A & BF \\ KC & A + BF - KC \end{bmatrix}. \tag{3.3}$$

3.2 Stability of the combined system

Let's ask ourselves, is the combined system stable? To answer that question, we need to examine the eigenvalues of the matrix A_c. These are given by $\det(\lambda I - A_c)$. Interestingly enough, it can be shown that

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{e}} \end{bmatrix} = \begin{bmatrix} A + BF & -BF \\ 0 & A - KC \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix}. \tag{3.4}$$

This implies that

$$\det(\lambda I - A_c) = \det(\lambda I - (A + BF)) \cdot \det(\lambda I - (A - KC)). \tag{3.5}$$

In other words, the set of eigenvalues of A_c is the union of the set of eigenvalues of the state feedback $A + BF$ and the set of eigenvalues of the observer $A - KC$. This means that the state feedback and the observer can be designed independently. (This is known as the **separation principle**.) It also means that, if the system is both stabilizable and detectable, then there are F and K such that A_c is stable.