Stress Functions

1 The Airy Stress Function

Previously we have examined general equations. However, solving them can be very hard. So let's look for tools with which we can apply them. In this chapter, we'll be looking at stress functions. The first one to introduce is the Airy stress function.

1.1 Stress state conditions

Before we start defining things, we will make some simplifications. First of all, we assume there are no body forces, so $X = Y = Z = 0$. Second, we will only deal with two-dimensional problems. For that, we have to assume that $\sigma_z = 0$. If this is the case, we have **plane stress** (the stress only occurs in a plane). Together, these two assumptions turn the equilibrium conditions of the previous chapter into

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0 \qquad \text{and} \qquad \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} = 0. \tag{1.1}$$

Next to equilibrium conditions, we also had compatibility conditions. Based on our assumptions, we can simplify those as well. We then get only one equation, being

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma_x + \sigma_y) = 0. \tag{1.2}$$

And finally there were the boundary conditions. Adjusting those will give

$$\overline{X} = \sigma_x l + \tau_{xy} m \qquad \text{and} \qquad \overline{Y} = \sigma_y m + \tau_{xy} l. \tag{1.3}$$

Now we have derived the new conditions for the stress state. Let's see how we can apply them.

1.2 The Airy stress function

It is time to talk about stress functions. A **stress function** is a function from which the stress can be derived at any given point x, y. These stresses then automatically satisfy the equilibrium conditions.

Now let's examine such a stress function. The **Airy stress function** ϕ is defined by

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2}, \qquad \sigma_y = \frac{\partial^2 \phi}{\partial x^2} \qquad \text{and} \qquad \tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}. \tag{1.4}$$

We can insert these stresses in the equilibrium conditions (1.1). We then directly see that they are satisfied for every ϕ! How convenient... However, if we insert the above definitions into the compatibility condition (1.2), we get

$$\frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2\,\partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0. \tag{1.5}$$

This equation is called the **biharmonic equation**. It needs to be satisfied by every valid Airy stress function as well.

1.3 Applying the Airy stress function

Now you may be wondering, how can we apply the Airy stress function? To be honest, that is kind of a problem. Given the loading condition of an object, it's rather difficult to determine a corresponding stress function. On the other hand, if we have a stress function ϕ, it is often possible to find a corresponding loading condition. This idea is called the **inverse method**.

So how do we apply this inverse method? We first have to assume a certain form of ϕ with a number of unknown coefficients $A, B, C, \ldots$. We know ϕ has to satisfy the biharmonic equation (1.5) and the boundary conditions (1.3). From these conditions, the unknown coefficients can (hopefully) be solved. The most difficult step in this process is to choose a form for ϕ. Sadly, that part is beyond the scope of this summary.

1.4 St. Venant's principle

Sometimes a problem occurs when applying the boundary conditions. For example, if the object we are considering is subject to a concentrated (local) force, there will be huge local variations in the stress. It is hard to adjust the boundary conditions to these **local effects**.

In this case, use can be made of **St. Venant's principle**. It states that local variations eventually average out. You just 'cut' the part with local effects out of your object. For the rest of the object, you can then assume loading conditions with which you are able to make calculations.

1.5 Displacements

Let's suppose we have found the stress function ϕ for an object. We can now find the stresses σ_x, σ_y and τ_{xy} at every position in the object. These stresses will thus be functions of x and y.

Using these stresses, we can find the displacements u, v and γ_{xy}. To do this, we first need to adjust the stress-strain relations from the previous chapter to the two-dimensional world. For the direct strain we find

$$\varepsilon_x = \frac{\partial u}{\partial x} = \frac{\sigma_x - \nu\sigma_y}{E} \qquad \text{and} \qquad \varepsilon_y = \frac{\partial v}{\partial y} = \frac{\sigma_y - \nu\sigma_x}{E}. \tag{1.6}$$

So first we can find ε_x and ε_y, as functions of the position x, y. We then integrate those strains to find the displacements u and v. Don't underestimate these integrals. They are often quite difficult, since ε_x and ε_y are functions of both x and y.

After we have found u and v, we can use them to find γ_{xy}. This goes according to

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\tau_{xy}}{G}. \tag{1.7}$$

And now everything is known about the object!

2 The Prandtl stress function

The Airy stress function is quite suitable when a force is applied to a two-dimensional object. Similarly, the Prandtl function is useful when torsion is present. Let's take a look at it.

2.1 Conditions

Let's examine a rod with a constant cross-section. Its axis lies on the z-axis. We can apply a torsion T to both its sides. This torsion T is said to be positive when it is directed counterclockwise about the z-axis (according to the right-hand rule). Since we only apply torsion, we can assume there are no normal (direct) stresses, so $\sigma_x = \sigma_y = \sigma_z = 0$. The same goes for the shear stress τ_{xy}, so $\tau_{xy} = 0$. From this follows that also $\varepsilon_x = \varepsilon_y = \varepsilon_z = \gamma_{xy} = 0$. We also assume no body forces are present.

So most of the stresses are zero. We only have two non-zero stresses left, being τ_{zy} and τ_{zx}. The **Prandtl stress function** ϕ is now defined by

$$\tau_{zy} = -\frac{\partial \phi}{\partial x} \qquad \text{and} \qquad \tau_{zx} = \frac{\partial \phi}{\partial y}. \tag{2.1}$$

It can be shown that τ_{zy} and τ_{zx} only depend on the x and y-coordinates. They don't vary along the z-axis.

We know that ϕ should satisfy the conditions from the first chapter. We can find that ϕ automatically satisfies the equilibrium equations. We can reduce all compatibility equations to one equation, being

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \text{constant}, \tag{2.2}$$

where $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the **two-dimensional Laplace operator**.

Finally there are the boundary conditions. We can derive two things from that. First, we can derive that, along the outer surface of the rod, we have $\partial\phi/ds = 0$. So ϕ is constant along the rod surface. Since this constant doesn't really matter, we usually assume that $\phi = 0$ along the outer surface of the rod.

Second, we can also look at the two rod ends, where the torsion T is being applied. If we sum up the shear stresses in this region, we can find the relation between the torsion T and the function ϕ. This relation states that

$$T = 2 \iint \phi \, dx \, dy. \tag{2.3}$$

2.2 Displacements

With all the conditions we just derived, we often can't find ϕ just yet. We also need to look at the displacements. Let's call θ the **angle of twist** and $d\theta/dz$ the **rate of twist**. It follows that, for the displacements u and v, we have

$$u = -\theta y \qquad \text{and} \qquad v = \theta x. \tag{2.4}$$

Previously we have also seen that $\nabla^2\phi$ is constant. However, we didn't know what constant it was equal to. Now we do. It can be shown that

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -2G\frac{d\theta}{dz}. \tag{2.5}$$

And finally we have all the equations that ϕ must satisfy. That's great! However, we can simplify matters slightly. Let's introduce the **torsion constant** J. It is defined by

$$T = GJ\frac{d\theta}{dz}. \tag{2.6}$$

By the way, the product GJ is called the **torsional rigidity**. From the above two equations, and the relation between T and ϕ, we can find that

$$GJ = -\frac{4G}{\nabla^2\phi} \int \int \phi \, dx \, dy. \tag{2.7}$$

2.3 Finding the Prandtl stress function

We now know all the conditions which ϕ must satisfy. However, finding ϕ is still a bit difficult. Just like for the Airy stress function, we first have to assume a form for ϕ. This form should be such that it satisfies all the above conditions.

The first condition you should pay attention to, is the condition that $\phi = 0$ around the edge. Then we multiply this relation by a constant, to find our stress function. Using the other conditions, we can then find the value of our constant. For example, if our cross-section is a circle, we would have $x^2 + y^2 = R^2$ around the edge. A suitable function for ϕ would then be $\phi = C\left(x^2 + y^2 - R^2\right)$. Find C using the remaining conditions, and you've found ϕ.

2.4 Warping

We know that the rod will twist. But that's not the only way in which it will deform. There is also **warping**, being the displacement of points in the z-direction. To know how an object warps, we have to find an expression for w. For that, we have to use the relations

$$\frac{\partial w}{\partial x} = \frac{\tau_{zx}}{G} + \frac{d\theta}{dz} y \qquad \text{and} \qquad \frac{\partial w}{\partial y} = \frac{\tau_{zy}}{G} - \frac{d\theta}{tz} x. \tag{2.8}$$

Integrating the above expressions should give you w: the displacement in z-direction.

2.5 The membrane analogy

Let's consider the lines along the cross-section for which ϕ is constant. These special lines are called **lines of shear stress** or **shear lines**. You may wonder, why are they special? Well, to see that, let's look at the shear stresses τ_{zx} and τ_{zy} at some point. We find that the resultant shear stress (the sum of τ_{zx} and τ_{zy}) is tangential to the shear line. Furthermore, the magnitude of this stress is equal to $-\partial\phi/\partial n$, where the vector $\mathbf{n}$ is the normal vector of the shear line (pointing outward).

This may be a bit hard to visualize. Luckily, there is a tool that can help you. It's called the **membrane analogy** (also called the **soap film analogy**). Let's suppose we have a membrane (or a soap film) with as shape the cross-section of our rod. We can apply a pressure p to this membrane from below. It then deflects upwards by a distance w. This deflection w now corresponds to our stress function ϕ, so $w(x, y) = \phi(x, y)$. Note that we have $w = 0$ at the edges of our membrane, just like we had $\phi = 0$ at the edges of our rod.

We can also look at the volume beneath our soap bubble. We then find that

$$\text{Volume} = \iint w \, dx \, dy, \qquad \text{which implies that} \qquad T = 2 \times \text{Volume}. \tag{2.9}$$

2.6 Torsion of narrow rectangular strips

Let's examine a narrow rectangular strip. Its height (in y-direction) is s, while its thickness (in x-direction) is t. Normally it is very hard to find the Prandtl stress function ϕ for this rod. However, if t is much smaller than s, we can simplify things. In this case, we can assume that ϕ doesn't vary with y. So we find that

$$\nabla^2\phi = \frac{d^2\phi}{dx^2} = -2G\frac{d\theta}{dz}. \tag{2.10}$$

By integrating this twice, the stress function ϕ can be obtained relatively easily. (Okay, you still have to find the two constants that show up in the integration, but that isn't very hard.) And once the stress function is known, all the other data will follow.