Bending, Shear and Torsion

It is time to examine some basic loads that beams can be subject to. We especially look at thin-walled beams, as they frequently occur in Aerospace Engineering. We can then derive general methods and equations. With those, we can find the stresses that are present in the beam.

1 Bending of Beams

We start by examining bending. This is because we need the bending equations when we examine shear.

1.1 Definitions and conventions for bending

Let's examine a beam of any shape. Just like in the previous chapter, its longitudinal axis lies on the z-axis. Now the beam is subject to a bending moment M. We can dissolve this bending moment M into a component M_x about the x-axis and a component M_y about the y-axis.

Let's discuss the sign convention of these moments. We say a moment M_x is positive, if it causes (positive) tensile stresses in the region $y > 0$. Similarly, M_y is positive, if it causes tensile stresses in the region $x > 0$. We can see that M_x satisfies the right hand rule (it is directed counterclockwise if you look at it from the positive x-direction). However, the moment M_y does not satisfy this rule. If you look at it from the positive y-axis, it is directed clockwise.

When evaluating bending, we will have to use moments of inertia. There are the **moment of inertia about the x-axis** I_{xx}, the **moment of inertia about the y-axis** I_{yy} and the **product of inertia** I_{xy}. They are defined as

$$I_{xx} = \int_A y^2 \, dA, \qquad I_{yy} = \int_A x^2 \, dA \qquad \text{and} \qquad I_{xy} = \int_A xy \, dA. \tag{1.1}$$

1.2 The general bending equation

The bending moments M_x and M_y cause the beam to bend. Now let's look at the cross-section of the beam. Part of the beam is subject to tensile stresses, while the other part is in compression. The line separating these two regions is called the **neutral axis**. It can be shown that this is a straight line. It always goes through the center of gravity of the cross-section.

It would be great to know what stresses are present in the beam. And the good part is, a general equation can be derived for that. What we wind up with is

$$\sigma_z = \left(\frac{M_y I_{xx} - M_x I_{xy}}{I_{xx} I_{yy} - I_{xy}^2} \right) x + \left(\frac{M_x I_{yy} - M_y I_{xy}}{I_{xx} I_{yy} - I_{xy}^2} \right) y = \left(\frac{I_{yy} y - I_{xy} x}{I_{xx} I_{yy} - I_{xy}^2} \right) M_x + \left(\frac{I_{xx} x - I_{xy} y}{I_{xx} I_{yy} - I_{xy}^2} \right) M_y. \tag{1.2}$$

In the above equation, you find two relations for σ_z. As you can see, they are equivalent. You can use either one of them. Which one is the most convenient depends on the circumstances.

If the cross-section of the beam is symmetric about the x-axis or about the y-axis (or both), then we have $I_{xy} = 0$. This simplifies the above equation drastically. We then remain with

$$\sigma_z = \frac{M_x}{I_{xx}} y + \frac{M_y}{I_{yy}} x. \tag{1.3}$$

1.3 Beams of multiple materials

Sometimes beams are made of multiple materials. Different materials generally have different stiffnesses, so also different values of E. How do we take this into account? Well, to do that, we define the **weighted cross-sectional area** A^* as

$$dA^* = \frac{E}{E_{ref}} dA, \qquad \text{which implies that} \qquad A^* = \int_A \frac{E}{E_{ref}} dA. \tag{1.4}$$

Here E_{ref} is just some reference E-modulus. Although it can be anything, it's usually taken to be the E-modulus of one of the present materials. We can now also define the weighted moment of inertias as

$$I_{xx}^* = \int_A y^2 \, dA^*, \qquad I_{yy}^* = \int_A x^2 \, dA^* \quad \text{and} \quad I_{xy}^* = \int_A xy \, dA^*. \tag{1.5}$$

Based on these definitions, we can derive a new expression for σ_z. We now find that

$$\sigma_z = \frac{E}{E_{ref}} \left(\left(\frac{M_y I_{xx}^* - M_x I_{xy}^*}{I_{xx}^* I_{yy}^* - I_{xy}^{*2}} \right) x + \left(\frac{M_x I_{yy}^* - M_y I_{xy}^*}{I_{xx}^* I_{yy}^* - I_{xy}^{*2}} \right) y \right). \tag{1.6}$$

Note that E in the above equation is the E-modulus at the position where you want to know σ_z.

1.4 The neutral axis

We already know that the neutral axis is a straight line that passes through the COG of the cross-section. What we don't know, is its orientation. We define α as the clockwise angle between the x-axis and the neutral axis. (So if the neutral axis is pointing $30°$ upwards, then $\alpha = -30°$.)

Let's find α. We know that for every point on the neutral axis x_{na}, y_{na} we have $\sigma_z = 0$. We can insert this into the previously derived equation for σ_z. We then find that

$$\frac{y_{na}}{x_{na}} = -\frac{M_y I_{xx} - M_x I_{xy}}{M_x I_{yy} - M_y I_{xy}}. \tag{1.7}$$

We can also see that $\tan \alpha = -y_{na}/x_{na}$. It follows that

$$\alpha = \arctan \left(\frac{M_y I_{xx} - M_x I_{xy}}{M_x I_{YY} - M_y I_{xy}} \right). \tag{1.8}$$

We haven't considered the case where the beam consists of multiple materials. However, that case works exactly the same. Just add stars to the Is in the above equation.

2 Shear Forces and Thin-Walled Beams

In aerospace engineering, thin-walled beams often occur. Just think of stringers, stiffeners, or even whole fuselages. How do those beams cope with shear stresses? Let's see if we can find that out.

2.1 Conditions for thin-walled beams

Let's examine a thin-walled beam (a beam with very small thickness). Its cross-section is just a curving line with thickness t. It can be either a closed curve (giving a **closed section beam**) or an open curve (resulting in an **open section beam**).

A shear force S is acting on our beam. We can split this force S up in a part S_x (pointing in the x-direction) and a part S_y (pointing in the y-direction). This shear force causes stresses in the beam. First of all, there is the stress in z-direction σ_z. There are also stresses in x and y-direction. However, this time we don't write them as such. Instead, we only consider the so-called **hoop stress** σ_s. This is the stress in circumferential direction (so the stress along the curve). Similarly, we only deal with one shear stress, being $\tau_{zs} = \tau_{sz} = \tau$. So the only stresses we are considering are σ_z, σ_s and τ.

We're almost ready to examine stresses in the beam. But first we need to make another definition. The **shear flow** q is defined as $q = \tau t$. Now it's time to derive the equilibrium equations for our beam. We find

$$\frac{\partial q}{\partial s} + t\frac{\partial \sigma_z}{\partial z} = 0 \qquad \text{and} \qquad \frac{\partial q}{\partial z} + t\frac{\partial \sigma_s}{\partial s}. \tag{2.1}$$

We can also examine the displacements. The displacement of a point in z-direction is denoted by w. We also have the displacement in circumferential (tangential) direction v_t and the displacement in normal direction v_n. Corresponding to these displacements are the strains ε_z, ε_s and γ. The strain ε_s isn't important, so we ignore that one. The relations for the remaining strains are

$$\varepsilon_z = \frac{\partial w}{\partial z} \qquad \text{and} \qquad \gamma = \frac{\partial w}{\partial s} + \frac{\partial v_t}{\partial z}. \tag{2.2}$$

2.2 Deriving an equation for the shear flow

Let's see if we can find the shear flow q caused by the shear forces S_x and S_y. In equation (2.1) we saw q. However, we also saw $\partial\sigma_z/\partial z$. Let's examine this σ_z a bit closer. What causes it?

The shear force S_x causes a bending moment M_y. Similarly, S_y causes M_x. These bending moments then cause the stress σ_z. From basic mechanics we know that $S_x = \partial M_y/\partial z$ and $S_y = \partial M_x/\partial z$. If we apply this to the bending equation (1.2), we find that

$$\frac{\partial \sigma_z}{\partial z} = \left(\frac{S_x I_{xx} - S_y I_{xy}}{I_{xx}I_{yy} - I_{xy}^2}\right) x + \left(\frac{S_y I_{yy} - S_x I_{xy}}{I_{xx}I_{yy} - I_{xy}^2}\right) y. \tag{2.3}$$

By inserting this relation into the equilibrium equation (2.1), and by integrating, we find that

$$q(s) - q_0 = -\left(\frac{S_x I_{xx} - S_y I_{xy}}{I_{xx}I_{yy} - I_{xy}^2}\right)\int_0^s tx\,ds - \left(\frac{S_y I_{yy} - S_x I_{xy}}{I_{xx}I_{yy} - I_{xy}^2}\right)\int_0^s ty\,ds. \tag{2.4}$$

Here s is the (counterclockwise) distance along the cross-section, from some point 0 with shear flow q_0. This expression for $q(s)$ is quite important, so keep it in mind.

2.3 Finding the shear center

When we apply a shear force S somewhere on the cross-section, then the beam will most likely twist. There is only one point where we can apply S such that the beam does not twist. This point is called the **shear center**, and has coordinates ξ_s, η_s. How do we find the position of this shear center? Let's look at that now. (We will only discuss how to find the x-coordinate ξ_s, since finding η_s goes similar.)

To find ξ_s, we assume a certain position where the shear force S_y applies. If S_x is actually acting on the shear center, then the rate of twist $d\theta/dz$ must be zero. Using this condition, we calculate the shear flow $q(s)$.

We can now evaluate moments about any point. The moment caused by the force S_y should then be equal to the moment caused by the shear flow. From this the position of S_y (and thus also ξ_s) can be derived.

You may be wondering, how do we find the moment caused by the shear flow? To do that, we replace the shear flow by forces. And I'm sure you know how to find the moment caused by a set of forces.

If the cross-section consists of straight lines, we can split it up in parts. For every part i of the cross-section, we can make the integral

$$F_{q_i} = \int_0^{s_i} q(s_i)\, ds_i. \tag{2.5}$$

This F_{q_i} is then the resultant force of the shear flow in part i.

Things are slightly more difficult if the cross-section is curved. Splitting the cross-section up in parts isn't possible anymore. However, we can also find the moment caused by the shear forces directly. Let's suppose we take moments about some point B. We then have

$$M_q = \int_0^{s} q(s) p\, ds, \tag{2.6}$$

where the variable p is the shortest distance between point B and the line tangential to the part ds of the cross-section.

2.4 Shear flow

Let's suppose we have an open section beam. We can now find $q(s)$ quite easily. Since the cross-section is not a closed curve, it must have two edges. At those two edges the shear flow q is zero. We can now apply equation (2.4). If we take one of the edges as point 0, we have $q_0 = 0$. Since we also know the shape of our cross-section, we can solve for $q(s)$. And from this we can find the shear stress τ. Sounds simple, doesn't it?

There is one small addition we have to make though. When you apply a shear stress S to a beam, it can also twist. (Like it does when it is subject to torsion.) Open section beams can't support twist. So to prevent them from twisting, you must apply the shear force S in the shear center. Then the above method works. And luckily, we already know how find this shear center.

2.5 Shear of closed section beams

Now let's look at closed section beams. This time we run into a problem. There isn't any point 0 for which we know the shear flow q_0. To solve this problem, we first rewrite the shear flow $q(s)$ as

$$q(s) = q_b + q_0, \qquad \text{where } q_b = -\left(\frac{S_x I_{xx} - S_y I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\right) \int_0^s tx\, ds - \left(\frac{S_y I_{yy} - S_x I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\right) \int_0^s ty\, ds. \tag{2.7}$$

We now examine the rate of twist $d\theta/dz$. It can be shown that

$$2A\frac{d\theta}{dz} = \oint \frac{q(s)}{Gt} ds = \oint \frac{q_b + q_0}{Gt} ds, \tag{2.8}$$

where A is the area enclosed by the cross-section. The integral $\oint$ means we integrate over the entire curved cross-section. Solving the above equation for q_0 gives

$$q_0 = \frac{2A\frac{d\theta}{dz} - \oint \frac{q_b}{Gt} ds}{\oint \frac{1}{Gt} ds}. \tag{2.9}$$

Often the above integral can be simplified. If the shear force S acts in the shear center, then the rate of twist $d\theta/dz$ is zero. It also often occurs that G or t (or both) are constant. In both cases the above equation simplifies greatly.

3 Torsion and Thin-Walled Beams

Previously we saw that a shear force can cause twist in beams. Torsion causes twist as well. How do thin-walled beams react to pure torsion? Let's find that out.

3.1 The center of twist

Let's suppose we have a thin-walled beam (open section or closed section). We can apply a torsion T to both its sides. The beam will then twist by an angle θ. Every point of the beam will have a displacement u in x-direction and v in y-direction.

Now comes the interesting part. The beam will always twist in such a way, that it appears to be rotating about some point R. This point R is called the **center of twist**. If u, v and θ are known, then its position x_R, y_R can be found using

$$x_R = -\frac{dv/dz}{d\theta/dz} \qquad \text{and} \qquad y_R = \frac{du/dz}{d\theta/dz}. \tag{3.1}$$

There is one surprising thing though. If the beam is only subject to torsion, then the center of twist is equal to the shear center! So if we know the shear center, we also know the center of twist when the beam is subject to torsion.

3.2 Torsion of closed section beams

Now let's look at a closed section beam. Since we only apply torsion, no direct stresses are present. This reduces the equilibrium equations to $\partial q/\partial z = 0$ and $\partial q/\partial s = 0$. This means that q is constant everywhere. It now follows that the torsion T is

$$T = 2Aq, \tag{3.2}$$

with A still the area enclosed by the cross-section. The above equation is known as the **Bredt-Bahto formula**.

What about displacements? Well, it can be shown that both θ, u and v vary linearly with z. So the rate of twist $d\theta/dz$ is constant. And the nice part is, we even got an equation for $d\theta/dz$. This equation is

$$\frac{d\theta}{dz} = \frac{q}{2A} \oint \frac{1}{Gt} ds = \frac{T}{4A^2} \oint \frac{1}{Gt} ds. \tag{3.3}$$

3.3 Warping in closed section beams

When a beam twists, there is usually also warping (meaning $w \neq 0$). It can be shown that the warping w stays constant for different z. However, within a cross-section the warping w is generally not constant. But, calculating it requires a couple of difficult integrations, so we won't elaborate on it further in this summary.

However, there is one important rule you do need to know. Let's define p_R as the shortest distance between the center of twist R, and the line tangential to some part ds of the beam. If we have

$$p_R G t = \text{constant}, \tag{3.4}$$

then the beam does not warp under pure torsion. Such kind of beams are known as **Neuber beams**. Examples are circular beams of constant thickness and triangular beams of constant thickness. But there are plenty more Neuber beams.

3.4 Torsion of open section beams

We have previously said that open section beams can't take torsion. This wasn't entirely true. They can take a bit of torsion. However, the stresses and deformations are, in this case, usually quite big.

When a closed-section beam is subject to torsion, the shear flow can flow all around the cross-section. We saw that in this case the shear flow q was constant. Along the thickness, also the shear stress τ_{zs} was constant. However, this doesn't work for open section beams. Instead, for open section beams, $q = 0$ everywhere. But now the shear stress τ_{zs} varies (linearly) along the thickness of the cross-section.

Let's examine a small piece ds of the cross-section. Let's look at the line in the middle of this piece. We call n the distance from this line. It can now be shown that the shear stress τ_{zs} varies according to

$$\tau_{zs} = \frac{2n}{J}T = 2Gn\frac{d\theta}{dz}. \tag{3.5}$$

The maximum shear stress occurs at maximum n. So this occurs at the edge of the cross-section, where $n = t/2$. By the way, the torsion constant J can be found using

$$J = \frac{1}{3}\int t^3\,ds, \qquad \text{where also} \qquad T = GJ\frac{d\theta}{dz}. \tag{3.6}$$

4 Combined open and closed section beams

Previously we have only considered beams that were either open or closed. But what do we do if a beam has both an open and a closed part?

4.1 Shear

Let's suppose we have a thin-walled beam which is both open and closed. On it is acting a shear force S, acting in the shear center. To find the shear flow q in the beam, we can still use the known equation

$$q(s) - q_0 = -\left(\frac{S_x I_{xx} - S_y I_{xy}}{I_{xx}I_{yy} - I_{xy}^2}\right)\int_0^s tx\,ds - \left(\frac{S_y I_{yy} - S_x I_{xy}}{I_{xx}I_{yy} - I_{xy}^2}\right)\int_0^s ty\,ds. \tag{4.1}$$

However, we will often have to evaluate this equation multiple times, for different parts. And every time, a new value q_0 shows up. But, if we are clever, we choose point 0 such that every time $q_0 = 0$. We can then use the above equation to find the shear force $q(s)$ at every point in the beam.

You might be wondering, how do you know where $q = 0$? Well, we always have $q = 0$ at the end of a cross-section. We can often also deduce points of $q = 0$ from symmetry. Let's look at the line L through which the force S is acting. It often occurs that L is an axis of symmetry of the cross-section. In this case, then every point on L generally also has $q = 0$.

4.2 Torsion

Now suppose we have a thin-walled beam that is subject to torsion. In this case it is often wise to first find the total torsional rigidity GJ_{tot}. To find this, we first need to find the torsional rigidity GJ_i of every sub-part i. We can find this using

$$GJ_i = \frac{4A^2G}{\oint \frac{1}{t}ds} \quad \text{for closed sections, and} \quad GJ_i = \frac{G}{3}\int t^3\,ds \quad \text{for open sections.} \tag{4.2}$$

To find the total torsional rigidity GJ_{tot}, just add up all the separate torsional rigidities GJ_i. The torsional rigidity of closed sections is generally much bigger than that of open sections. So often the value GJ of open sections can be neglected.

The rate of twist now follows from

$$\frac{d\theta}{dz} = \frac{T}{GJ_{tot}}. \tag{4.3}$$

To find the shear flow (and thus also the shear stress) for a closed section i, we can use

$$q = \frac{GJ_i}{2A}\frac{d\theta}{dz}, \tag{4.4}$$

where GJ_i is the torsional rigidity of that closed section i. To find the shear stress in an open section, we still have

$$\tau = \frac{2n}{J}T = 2Gn\frac{d\theta}{dz}. \tag{4.5}$$