Basic Equations

1 Definitions, conventions and basic relations

Before we can start throwing around equations, we have to define some variables and state some conventions. That is what we will be doing in this part.

1.1 Definitions

Let's suppose we have an object, that's subject to forces. There can be two kinds of forces it is subject to. First there are **surface forces** acting on the outside of the object. An example of a surface force is pressure. We can resolve surface forces into three components, along the axes. These components are denoted as $\overline{X}$, $\overline{Y}$ and $\overline{Z}$. There are also **body forces**, acting on every particle of the object. An example is the gravitational force. When we resolve body forces into three components, we write it as X, Y and Z.

The forces acting on the object cause internal stresses. (Stress is force per unit area.) Let's suppose we make a cut through the object and examine a point O on the cut. There is a component of the stress normal to the cut, called the **direct (tensile) stress**. This is denoted by the sign σ. There are also two components of the stress parallel to the cut. These components are called **shear stresses** and are denoted by τ.

1.2 Notation and sign conventions

Now let's discuss some notation and sign conventions. Often direct stresses are examined along the three basic axes. (The x, y and z-axes.) We then say that σ_x is the direct stress along the x-axis, σ_y is the stress along the y-axis and σ_z is the stress along the z axis. If a certain stress is directed away from its related surface, then we define it as a positive stress. Otherwise it is negative.

We can have a similar notation for shear stresses. However, shear stresses don't only have a direction. They also have a plane in which they act. They therefore have two subscript, like τ_{xy}. The x (the first part of the subscript) denotes the plane in which the shear stress acts. In this case it is the plane orthogonal to the x-axis. The y (the second part of the subscript) then denotes the direction of the shear stress.

The sign convention of shear stress is also a bit difficult. We have to examine two arrows for that. First there is the direction of the shear stress itself. Then there is also the normal vector to the plane on which the shear stress is acting. (This normal vector always points outward.) If they both point in a positive direction, or both in a negative direction, then we say that the shear stress is positive. If one points in a positive direction, and the other in a negative direction, then it is negative.

1.3 Basic equations

We can examining the stresses acting on a small part inside an object. By doing so, we can derive a few relations. First, by taking moments, we can derive relations for the shear stresses. These relations are

$$\tau_{xy} = \tau_{yx}, \qquad \tau_{xz} = \tau_{zx} \qquad \text{and} \qquad \tau_{yz} = \tau_{zy}. \tag{1.1}$$

By examining forces in certain directions, we can derive three equilibrium equations, being

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + X = 0, \tag{1.2}$$

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + \frac{\partial \tau_{yx}}{\partial x} + Y = 0, \tag{1.3}$$

$$\frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + Z = 0. \tag{1.4}$$

Instead of examining a particle on the inside of an object, we can also examine a particle on the edge. Now surface forces come into play. We can once more derive three equilibrium equations, being

$$\overline{X} = \sigma_x l + \tau_{yx} m + \tau_{zx} n, \tag{1.5}$$

$$\overline{Y} = \sigma_y m + \tau_{zy} n + \tau_{xy} l, \tag{1.6}$$

$$\overline{Z} = \sigma_z n + \tau_{xz} l + \tau_{yz} m. \tag{1.7}$$

The three parameters l, m and n are **direction cosines**. They are added to the equation to compensate for the direction of the surface. To find their values, examine the normal vector of the surface (still pointing outward). l, m and n are the cosines of the angles which this normal vector makes with respect to the x, y and z axis, respectively.

2 Stresses in different coordinate systems

We don't always evaluate stresses along the x, y and z-axes. We can also examine them in different coordinate systems. What happens when we start shifting coordinate systems?

2.1 Mohr's circle

Let's suppose we know all the stresses in the normal (x, y, z)-coordinate system. When we shift the coordinate system, the normal stresses and the shear stresses change. The way in which this occurs is described by **Mohr's circle**. Mohr stated that if you plot the direct stresses and the shear stresses, you would get a circle. Such a circle is shown in figure 1.

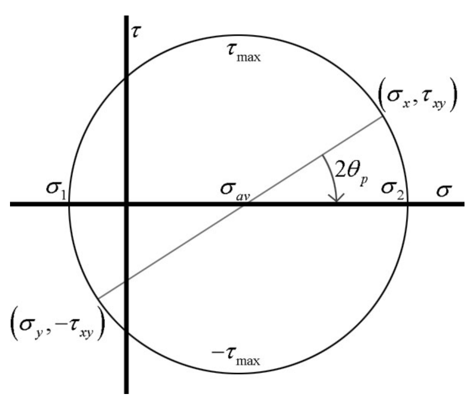

Figure 1: Mohr's Circle

How does this work? Suppose we know the stress in x-direction σ_x, the stress in y-direction σ_y and the shear stress τ_{xy}. Let's draw the points (σ_x, τ_{xy}) and $(\sigma_y, -\tau_{xy})$ in a coordinate system. We then draw a line between them. The point where this line crosses the x-axis denotes the **average stress** σ_{av}. It can be found using

$$\sigma_{av} = \frac{\sigma_x + \sigma_y}{2}. \tag{2.1}$$

The radius of the circle is

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}. \tag{2.2}$$

Now if we rotate our coordinate system by an angle θ, then the line in our circle rotates by an angle 2θ. From this, the new stresses can be found.

2.2 Directions of maximum stress

It would be nice to know when maximum stress occurs. Maximum normal (direct) stress occurs when we rotate our coordinate system over an angle $\theta_{m\sigma}$. $\theta_{m\sigma}$ can be found using

$$\tan 2\theta_{m\sigma} = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}. \tag{2.3}$$

The corresponding stresses are called **principal stresses**. The planes on which they act are the **principal planes**. The **maximum stress** (also called the **major principal stress**) σ_I and the **minimum stress** (also called the **minor principal stress**) σ_{II} can now be found using

$$\sigma_I = \sigma_{av} + R \qquad \text{and} \qquad \sigma_{II} = \sigma_{av} - R. \tag{2.4}$$

Similarly, maximum shear stress occurs when we rotate our coordinate system by an angle $\theta_{m\tau}$, where $\theta_{m\tau}$ now satisfies

$$\tan 2\theta_{m\tau} = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}}. \tag{2.5}$$

This angle will always be 45° bigger or smaller than the angle at which maximum direct stresses occur. (This can also be seen from Mohr's circle.) The corresponding maximum shear stress now is

$$\tau_{max} = R = \frac{\sigma_I - \sigma_{II}}{2}. \tag{2.6}$$

3 Strains

When an object is subject to forces, there will be displacements. These displacements relate to strains. Let's take a look at what kind of strains there are, and how we can find them.

3.1 Strain relations

We generally distinguish two types of strains. The **longitudinal** or **direct strains** (denoted by ε) relate to changes in length. **Shear strains** (denoted by γ) relate to changes in angles.

Let's examine a point O of an object. Due to the deformation of this object, this point O moves. It moves a distance u along the x-axis, a distance v along the y-axis and a distance w along the z-axis. It can now be shown that the direct strains in x, y and z-direction satisfy

$$\varepsilon_x = \frac{\partial u}{\partial x}, \qquad \varepsilon_y = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \varepsilon_z = \frac{\partial w}{\partial z}. \tag{3.1}$$

The orientations of lines passing through point O have also changed. For example, let's consider two lines in the xy-plane that were perpendicular. (There was an angle of $\pi/2$ between them.) Now they aren't perpendicular anymore. Their relative angle now is $\pi/2 - \gamma_{xy}$. This works the same for the xz-plane and the yz-plane. So we have three shear strains γ_{xy}, γ_{xz} and γ_{yz}. If the displacements are small, then it can be shown that

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}, \qquad \gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \qquad \text{and} \qquad \gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}. \tag{3.2}$$

Now we have six kinds of displacements. It seems like a lot of unknowns. Luckily there are relations between them. There are 6 compatibility equations. These equations are

$$\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2}, \qquad 2\frac{\partial^2 \varepsilon_x}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{xz}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right), \tag{3.3}$$

$$\frac{\partial^2 \gamma_{yz}}{\partial y \partial z} = \frac{\partial^2 \varepsilon_y}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial y^2}, \qquad 2\frac{\partial^2 \varepsilon_y}{\partial z \partial x} = \frac{\partial}{\partial y}\left(-\frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{yx}}{\partial z} + \frac{\partial \gamma_{yz}}{\partial x}\right), \tag{3.4}$$

$$\frac{\partial^2 \gamma_{zx}}{\partial z \partial x} = \frac{\partial^2 \varepsilon_z}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial z^2}, \qquad 2\frac{\partial^2 \varepsilon_z}{\partial x \partial y} = \frac{\partial}{\partial z}\left(-\frac{\partial \gamma_{xy}}{\partial z} + \frac{\partial \gamma_{zy}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y}\right). \tag{3.5}$$

3.2 Relations between stress and strain

Currently, we've got quite a couple of equations. But we got even more unknowns. So we need more equations. Where do we get those equations from?

We can try to describe the relationship between stress and strain. For that, we first have to make a few assumptions. First, we assume that the object we're looking at is **homogeneous**. This means that the material properties are the same at every point in the object. We also assume that the object is **isotropic**, meaning that the properties are the same in every direction. It also means that the stress and the strain are proportional.

From these assumptions we can derive that

$$\varepsilon_x = \frac{\sigma_x - \nu(\sigma_y + \sigma_z)}{E}, \qquad \varepsilon_y = \frac{\sigma_y - \nu(\sigma_z + \sigma_x)}{E} \qquad \text{and} \qquad \varepsilon_z = \frac{\sigma_z - \nu(\sigma_x + \sigma_y)}{E}. \tag{3.6}$$

Here ν is the **Poisson ratio**. There are also a relations between the shear stresses and shear strains. These relations are

$$\gamma_{xy} = \frac{\tau_{xy}}{G}, \qquad \gamma_{yz} = \frac{\tau_{yz}}{G} \qquad \text{and} \qquad \gamma_{zx} = \frac{\tau_{zx}}{G}. \tag{3.7}$$

The variable G is called the **modulus of rigidity**. It is related to E and ν according to

$$G = \frac{E}{2(1+\nu)}. \tag{3.8}$$

3.3 Changes of volume

When an object deforms, its volume changes. It would be interesting to know at what rate this happens. If V is the volume of a particle, then the **volumetric strain** e of that particle is

$$e = \frac{\Delta V}{V} = \varepsilon_x + \varepsilon_y + \varepsilon_z = \frac{1-2\nu}{E}(\sigma_x + \sigma_y + \sigma_z). \tag{3.9}$$

If an object is compressed at a constant pressure p, then $\sigma_x = \sigma_y = \sigma_z = -p$. We then have

$$e = -\frac{3(1-2\nu)}{E}p = -\frac{p}{K}, \qquad \text{with } K = \frac{E}{3(1-2\nu)}. \tag{3.10}$$

The constant K is known as the **bulk modulus** or the **modulus of volume expansion**.

3.4 Thermal effects

When an object is heated, it expands. It does this according to

$$\varepsilon = \alpha \, \Delta T, \tag{3.11}$$

where α is the **coefficient of thermal expansion**. If also stresses are involved, then we get new equations for the strains. We simply add $\alpha \, \Delta T$ up to the old equations. We then get

$$\varepsilon_x = \frac{\sigma_x - \nu \left(\sigma_y + \sigma_z\right)}{E} + \alpha \, \Delta T, \tag{3.12}$$

$$\varepsilon_y = \frac{\sigma_y - \nu \left(\sigma_z + \sigma_x\right)}{E} + \alpha \, \Delta T, \tag{3.13}$$

$$\varepsilon_z = \frac{\sigma_z - \nu \left(\sigma_x + \sigma_y\right)}{E} + \alpha \, \Delta T. \tag{3.14}$$