Application of Theory to Aircraft

1 Structural Idealization

It's time to apply some of our theory into practice. Let's look at airplanes. In an airplane are many parts that have a rather complicated shape. Let's find a way to examine them.

1.1 Simplifying a shape

Let's examine an aircraft fuselage. It often consists of a shell with a couple of stringers. Altogether, we have got a complicated shape. We need to make some assumptions and simplifications, such that we can evaluate it.

First we do something about the stringers. We replace them by concentrations of area (so-called **booms**). These booms have the same cross-sectional area as the original stringer.

Now let's look at a piece of fuselage skin with width b and (effective) thickness t_D. The normal stress σ varies along this piece. On the left side is a stress σ_1 and on the right a stress σ_2. We want to replace this piece of skin by two booms at the edges. This should be done, such that the effects are the same. So the two booms should take the same force and the same moment as the piece of skin. From this, we can derive that these two booms have area B_1 (left) and B_2 (right), where

$$B_1 = \frac{t_D b}{6}\left(2 + \frac{\sigma_2}{\sigma_1}\right) \qquad \text{and} \qquad B_2 = \frac{t_D b}{6}\left(2 + \frac{\sigma_1}{\sigma_2}\right). \tag{1.1}$$

Since we have replaced the skin by two booms, the remaining effective thickness t_D of the skin is 0.

Let's take a closer look at the ratio σ_2/σ_1 in the above equation. This ratio depends on the loads which our fuselage is subject to. And thus so do B_1 and B_2. This means that if we load our fuselage differently, our booms will have different areas.

We now make an important assumption. We assume that the booms take all the normal stresses, while the skin takes all the shear stresses. This makes our analysis a lot simpler. To examine normal stresses, we only have to evaluate a set of points with known areas. Also examining shear stress is a bit easier now.

1.2 Normal stress

In our new fuselage, how do we calculate the normal stress? For that, we can still use the general equation we derived for bending. Let's just repeat it. It was

$$\sigma_z = \left(\frac{M_y I_{xx} - M_x I_{xy}}{I_{xx}I_{yy} - I_{xy}^2}\right)x + \left(\frac{M_x I_{yy} - M_y I_{xy}}{I_{xx}I_{yy} - I_{xy}^2}\right)y = \left(\frac{I_{yy}y - I_{xy}x}{I_{xx}I_{yy} - I_{xy}^2}\right)M_x + \left(\frac{I_{xx}x - I_{xy}y}{I_{xx}I_{yy} - I_{xy}^2}\right)M_y. \tag{1.2}$$

Finding the moments of inertia is now quite easy. For the booms $B_1, \ldots, B_n$, just use

$$I_{xx} = \sum_{i=1}^{n} y_i^2 B_i, \qquad I_{yy} = \sum_{i=1}^{n} x_i^2 B_i \quad \text{and} \quad I_{xy} = \sum_{i=1}^{n} x_i y_i B_i. \tag{1.3}$$

1.3 Shear flow

Let's examine a beam subject to shear stresses S_x and S_y. We have assumed that the skin takes all the shear stresses. We stick to this assumption. However, it turns out that the booms do effect the shear stress. Let's suppose we have two pieces of skin with shear flow q_1 and q_2. In between these pieces is a boom with area B_r, coordinates x_r, y_r and direct stress σ_z. It can now be shown that

$$q_2 - q_1 = -\frac{\partial \sigma_z}{\partial z} B_r = -\left(\frac{S_x I_{xx} - S_y I_{xy}}{I_{xx} I_{yy} - I_{xy}^2} \right) B_r x_r - \left(\frac{S_y I_{yy} - S_x I_{xy}}{I_{xx} I_{yy} - I_{xy}^2} \right) B_r y_r. \tag{1.4}$$

From this, we can derive that the shear stress $q(s) = q_b + q_0$ is given by

$$q(s) = -\left(\frac{S_x I_{xx} - S_y I_{xy}}{I_{xx} I_{yy} - I_{xy}^2} \right) \left(\int_0^s t_D x \, ds + \sum_{i=1}^{n} B_r x_r \right) - \left(\frac{S_y I_{yy} - S_x I_{xy}}{I_{xx} I_{yy} - I_{xy}^2} \right) \left(\int_0^s t_D y \, ds + \sum_{i=1}^{n} B_r y_r \right) + q_0. \tag{1.5}$$

The two sums in the above equation sum over all the booms between point 0 and s. Also note that if we have replaced our skin by booms as well, then the remaining effective thickness t_D is zero. This would mean that the integrals in the above equation vanish.

For open section beams, we should take point 0 as some point where $q_0 = 0$, just like we're used to. For closed section beams, we have to find the value for q_0. This can still be done using known methods. Just take moments about some point. The moment caused by the shear stresses should then be equal to the moment caused by the shear force S.

2 Tapered Sections

We have previously always assumed that the cross-section of a beam stays constant for varying z. What happens if it doesn't? Let's find that out.

2.1 Tapered wing spars

Let's consider an I-shaped wing spar, whose height h changes. We assume that the web takes all the shear stress. Similarly, the flanges take all the direct stresses. We thus replace these flanges by two booms with areas B_1 (top) and B_2 (bottom).

When the beam is subject to a shear force S_y (and thus also a bending moment M_x), the flanges will be subject to forces P_1 and P_2. However, only the components in z-direction ($P_{z,1}$ and $P_{z,2}$) counteract the bending moment M_x. This goes according to $P_{z,1} = \sigma_1 B_1$. (σ_1 can be found by using the bending equation.)

But now comes the surprising part, the part of P_1 acting in y-direction (being $P_{y,1}$) effects the shear flow in the web. In fact, the **effective shear force** $S_{y,w}$ acting on the web can be found using

$$S_{y,w} = S_y - P_{y,1} - P_{y,2} = S_y - P_{z,1} \frac{\delta y_1}{\delta z} - P_{z,2} \frac{\delta y_2}{\delta z}. \tag{2.1}$$

Here the parameters y_1 and y_2 denote the y-coordinate of the flanges. When using the above equation, special care should be payed to the direction of the forces $P_{y,1}$ and $P_{y,2}$. Using the effective shear force, the shear flow in the web can be calculated, exactly in the way you are normally used to.

It is sometimes sligthly difficult to see whether the effective shear stress increases, or whether it decreases. There is a rule of thumb for that. If the cross-section is widening, then the effective shear stress is usually lower than the actual shear stress. And similarly, if the cross-section is getting smaller, then the effective shear stress is higher than the actual shear stress.

2.2 General shapes

In the previous paragraph we considered a vertical web, with two booms at the ends. Now let's consider a general (thin-walled) shape, consisting of a skin with booms. Every boom r with coordinates x_r, y_r has an internal force P_r, with components $P_{x,r}$, $P_{y,r}$ and $P_{z,r}$. These relate to each other according to

$$P_{x,r} = P_{z,r}\frac{\delta x_r}{\delta z}, \qquad P_{y,r} = P_{z,r}\frac{\delta y_r}{\delta z} \qquad \text{and also} \qquad P_r = P_{z,r}\frac{\sqrt{\delta x_r^2 + \delta y_r^2 + \delta z_r^2}}{\delta z}. \tag{2.2}$$

The effective shear forces in x and y-direction can now be found using

$$S_{x,w} = S_x - \sum_{r=1}^{n} P_{z,r}\frac{\delta x_r}{\delta z} \qquad \text{and} \qquad S_{y,w} = S_y - \sum_{r=1}^{n} P_{z,r}\frac{\delta y_r}{\delta z}. \tag{2.3}$$

Again, the rest of the analysis goes exactly as you're used to. There's one slight exception though. Suppose you have a closed cross-section. Then at some time you need find a function $q(s) = q_b + q_0$. To find q_0, you can take moments about a certain point. These moments should then be equal to the moment caused by S_x and S_y. However, this time the moments caused by $P_{x,r}$ and $P_{y,r}$ should also be taken into account. Do not forget that.

3 Aircraft Wings

Aircaft wings often have a rather characteristic shape. Examining wings is therefore an art itself — An art we will delve into now.

3.1 The wing shape

Let's consider the cross-section of a wing. It consists of the top and bottom skin of the wing, plus several vertical spars. The wing thus consists of a number N "boxes." Due to the (sometimes large) amount of spars, we have a high amount of redundancy. That is why wings are difficult to analyze.

Soon we will be putting torsion and shear forces on the wing. This causes a certain amount of counter-clockwise (assumed) shear flow q_R in box R. In this case the top of box R has a shear flow q_R, pointed to the left. The bottom has q_R pointed to the right.

But what about the shear stress in the spar to the right of box R? (We call it spar R.) Box R causes a shear flow q_R upward. However, box $R+1$ causes a shear flow q_{R+1} downward. So the shear flow in spar R is $q_R - q_{R+1}$ (upward). In this way the shear flow in every spar can be determined.

3.2 Torsion

Let's subject a wing to a torsion T. The torsion T will be divided over the several boxes. Every box R now supports an amount of torsion T_R, where

$$T_R = 2A_R q_R, \qquad \text{with also} \qquad \sum_{R=1}^{N} T_R = T. \tag{3.1}$$

The area A_R is the area enclosed by box R. There is just one slight problem. In the above equation, we don't know q_R, nor T_R. So we have $N+1$ equations, but $2N$ unknowns. We need more equations.

We now assume that the rate of twist $d\theta/dz$ of the boxes are all equal. We can find the rate of twist of box R using

$$\frac{d\theta}{dz} = \frac{1}{2A_R G}\oint_R \frac{q}{t}ds, \tag{3.2}$$

where we integrate around the entire box. (Note that in this case q is not always q_R. Previously we saw that the shear flow in spar R isn't q_R.) Although we have one extra unknown (being $d\theta/dz$), we have N extra equations. So we can solve our system of $2N+1$ equations.

Sometimes the spars have different shear moduli G. In this case we set a reference modulus G_{ref} and define the **modulus-weighted thickness** t^*, such that

$$t^* = \frac{G}{G_{ref}} t, \qquad \text{after which we use} \qquad \frac{d\theta}{dz} = \frac{1}{2A_R G_{ref}} \oint \frac{q}{t^*} ds. \tag{3.3}$$

3.3 Shear

Now let's apply a shear stress S to our wing. This makes things a bit more complicated. The shear stress $q_R(s)$ in every box is now given by $q_R(s) = q_{b,R} + q_{0,R}$. The value of $q_{b,R}$ around the box can be determined from

$$q_{b,R} = -\left(\frac{S_x I_{xx} - S_y I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\right) \left(\int_0^s t_D x \, ds + \sum_{i=1}^n B_r x_r\right) - \left(\frac{S_y I_{yy} - S_x I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\right) \left(\int_0^s t_D y \, ds + \sum_{i=1}^n B_r y_r\right). \tag{3.4}$$

So $q_{b,R}$ is known. However, $q_{0,R}$ is not. To find it, we once more look at the rate of twist $d\theta/dz$, which is (assumed) equal for all boxes. It is now given by

$$\frac{d\theta}{dz} = \frac{1}{2A_R G} \oint_R \frac{q}{t} ds = \frac{1}{2A_R G} \oint_R \frac{q_{b,R} + q_{0,R}}{t} ds \tag{3.5}$$

This gives us N extra equations, but also one extra unknown. We thus need one more equation. We now look at moments. All the shear flows together cause a moment (about a certain point). This moment must be equal to the moment caused by the shear force S (about that same point).

3.4 Cut-outs in wings

The last subject in this summary is a rather difficult problem. Let's look at a simple wing box, consisting of two spars with two pieces of skin in between. (If we look at the idealized cross-section, we simply see a rectangle, with booms at the corners.) If we look at the 3D wing box, we can split it up in three identical parts. Now we make a cut-out in the middle part (part 2). We remove the entire bottom skin of this part. This severely weakens the structure. We can now ask ourselves, what will happen if the wing box is subjected to loads?

This is, in fact, quite a difficult problem. Many things happen at the same time. In part 1, the shear forces are gradually being transferred (as normal forces) into the spar flanges (the booms). This causes normal forces P in the flanges.

Let's now look at the cross-section between parts 1 and 2. At this cross-section, the torsion has "translated" itself into two shear forces S. These forces are positioned at (and also supported by) the spars. They result in the same moment as the torsion T. Using this fact, you can find the magnitude of S.

The shear forces S cause certain shear flows q in the spars of part 2. These shear flows change the magnitude of the normal forces P in the flanges. By evaluating moments about certain points, the magnitude of these forces P can be determined at certain positions. Once the normal forces P are known, also the shear flows in part 1 can be found.

The above plan of approach might sound a bit short. However, this is a problem of which the solution can't be explained briefly in a summary. For a clear explanation of the problem, you would have to consult a book on this subject.