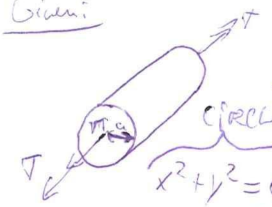
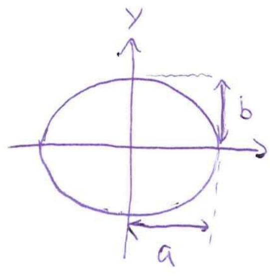
Given:



$x^2+y^2=a^2$ at boundary $\rightarrow$ b.c. $\varphi=0$

$\rightarrow$ choose $\varphi = k(r^2 - a^2)$ with $r^2 = x^2+y^2$

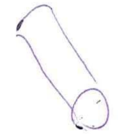
$$\frac{\partial^2 \varphi}{\partial x^2} = 2k \qquad \frac{\partial^2 \varphi}{\partial y^2} = 2k$$



$$\left.\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\right\} \text{ ellipse}$$

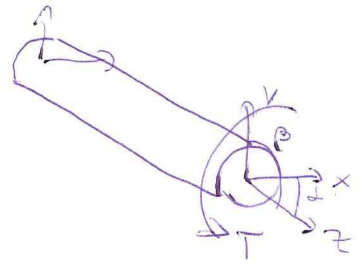
$$\boxed{\varphi = C\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)}$$

$$\frac{\partial^2 \varphi}{\partial x^2} \quad \frac{2C}{a^2} \quad \frac{\partial^2 \varphi}{\partial y^2} = \frac{2C}{b^2}$$



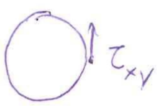
$$\varphi = k\,(x^2+y^2-a^2)$$

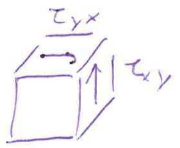
$$T = 2\iint \varphi\,dx\,dy = 2\iint k(x^2+y^2-a^2)\,dx\,dy$$
$$= 2k\Big\{\underbrace{\int x^2\,dx\,dy}_{I_y} + \underbrace{\int y^2\,dx\,dy}_{I_x} - \underbrace{\int a^2\,dx\,dy}_{A}\Big\}$$

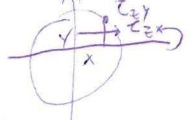


$l = \cos\alpha = 0$
$m = \cos\beta = 0$
$n = \cos\gamma = \cos 0 = 1$

$\bar{X} = \tau_{zx}$
$\bar{Y} = \tau_{zy}$
$\bar{Z} = 0$



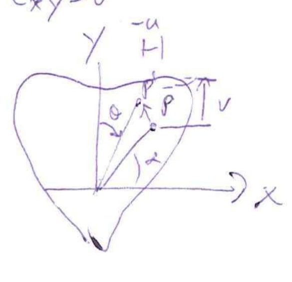




$$S_x = \iint \bar{X}\,dx\,dy = 0$$
$$S_y = \iint \bar{Y}\,dx\,dy = 0$$
$$\mathcal{T}_z = \iint(\tau_{zy}\cdot x - \tau_{zx}\cdot y)\,dx\,dy = \ldots = 2\iint \varphi\,dx\,dy$$

$\sigma_x = \sigma_y = \sigma_z = 0 \rightarrow \varepsilon_x = \varepsilon_y = \varepsilon_z = 0 \qquad \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} = \dfrac{\partial w}{\partial z} = 0$

$\tau_{xy} = 0 \qquad \gamma_{xy} = 0$



$P(r,\alpha) \rightarrow P'(r, \alpha+\theta)$

$u = -r\theta\sin\alpha \qquad v = r\theta\cos\alpha$

$x = r\cos\alpha \qquad y = r\sin\alpha$

subst. Prandtl stress function:

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = -2G\frac{\partial\theta}{\partial z}$$

$$T = GJ\frac{\partial\theta}{\partial z} = GJ = GJ\,\frac{\overbrace{\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}}^{\nabla^2\varphi}}{-2G} = 2\iint \varphi\,dx\,dy$$

$$\boxed{GJ = -\frac{4G}{\nabla^2\varphi}\iint \varphi\,dx\,dy}$$ Torsional Stiffness

$u = -\theta y \qquad v = \theta x$

$$\gamma_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \frac{\tau_{zx}}{G} \qquad \gamma_{zy} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} = \frac{\tau_{zy}}{G}$$

$$\frac{\partial u}{\partial z} = \frac{\partial\theta}{\partial z}y \qquad \frac{\partial v}{\partial z} = \frac{\partial\theta}{\partial z}x$$

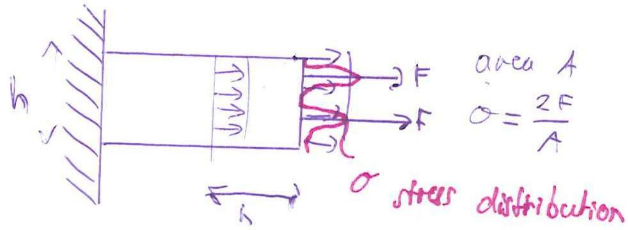
$$\frac{\partial w}{\partial x} = \frac{\tau_{zx}}{G} - \cancel{\frac{\partial u}{\partial z}} + \frac{\partial\theta}{\partial z}y \rightarrow \frac{\partial^2 w}{\partial x\partial y} = \frac{1}{G}\frac{\partial\tau_{zx}}{\partial y} + \frac{\partial\theta}{\partial z}$$

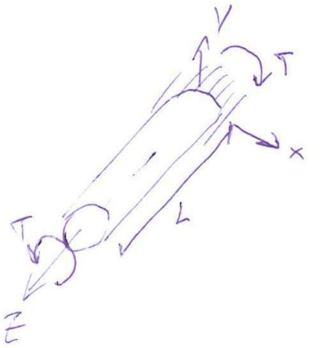
$$\frac{\partial w}{\partial y} = \frac{\tau_{zy}}{G} - \frac{\partial\theta}{\partial z}x \rightarrow \frac{\partial^2 w}{\partial x\partial y} = \frac{1}{G}\frac{\tau_{zy}}{\partial x} - \frac{\partial\theta}{\partial z} -$$

$$0 = \frac{1}{G}\left(\frac{\partial\tau_{zx}}{\partial y} - \frac{\partial\tau_{zy}}{\partial x}\right) + 2\frac{\partial\theta}{\partial z}$$

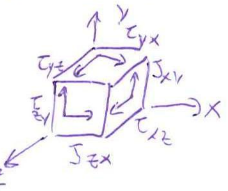
Airy stress function

$$\sigma_x = \frac{\partial^2 \varphi}{\partial y^2} \quad \sigma_y = \frac{\partial^2 \varphi}{\partial x^2} \quad \tau_{xy} = -\frac{\partial^2 \varphi}{\partial x \partial y}$$





$$\sigma_x = \sigma_y = \sigma_z = 0$$



τ_{zy} — normal, positive direction

$$\tau_{xy} = 0$$

$$\frac{\partial \tau_{xz}}{\partial z} = 0$$

ignore body forces

$$\frac{\partial \sigma}{\partial x} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{xz}}{\partial z} + X = 0$$

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial z} + Y = 0 \Rightarrow$$

$$\frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + Z = 0$$

$$\frac{\partial \tau_{xz}}{\partial z} = 0 \rightarrow C$$

$$\frac{\partial \tau_{yz}}{\partial z} = 0 \rightarrow C$$

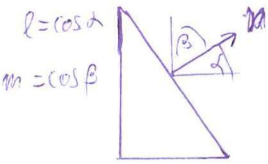
$$\frac{\partial \tau_{zy}}{\partial x} \; \frac{\partial \tau_{zy}}{\partial y} = 0$$

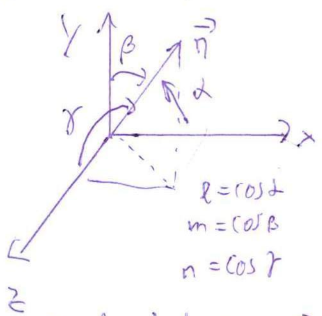
Prandtl Stress function

$$\frac{\partial \varphi}{\partial x} = -\tau_{zy} \qquad \frac{\partial \varphi}{\partial y} = \tau_{zx}$$

Compatibility eq.: $\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = C$

φ should satisfy boundary conditions:

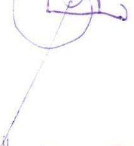




$$\bar{X} = \sigma_x l + \tau_{yx} m + \tau_{zx} n \qquad \bar{X} = 0$$

$$\bar{Y} = \sigma_y m + \tau_{xy} l + \tau_{zy} n \Rightarrow \bar{Y} = 0$$

$$\bar{Z} = \sigma_z n + \tau_{yz} m + \tau_{xz} l \qquad \bar{Z} = \tau_{yz} m + \tau_{xz} l = 0$$



angle between $\vec{n}$ and z-axis is $90°$

$\rightarrow n = \cos 90° = 0$

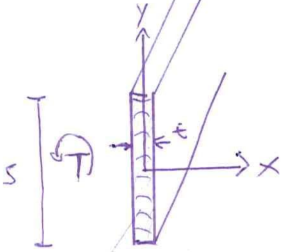
$\rightarrow l = \frac{\partial y}{\partial s}$

$m = -\frac{\partial x}{\partial s}$

in Z and Prandtl $\Rightarrow$ $\frac{\partial \varphi}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial \varphi}{\partial y} \cdot \frac{\partial y}{\partial s} = 0$

$\frac{\partial \varphi}{\partial s} = 0 \rightarrow \varphi =$ constant on the boundary

pick $\varphi = 0$



Soap film analogy

$w = w(x)$ (neglecting ends) $\Rightarrow \varphi = \varphi(x)$

$$\frac{\partial \varphi}{\partial y} = 0 \rightarrow \frac{\partial^2 \varphi}{\partial y^2} = 0$$

$$\frac{\partial^2 \varphi}{\partial x^2} \cancel{\frac{\partial^2 \varphi}{\partial y^2}}^{0} = -2G\frac{d\theta}{dz}$$

$\int\int dx dx \rightarrow$

$$\varphi = -G\frac{d\theta}{dz}x^2 + Bx + C$$

B.C.: $x = \mp \frac{t}{2} \Rightarrow \varphi = 0$

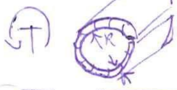
$$\rightarrow \varphi = -G\frac{d\theta}{dz}\left(x^2 - \left(\frac{t}{2}\right)^2\right)$$

$$T = G\int\frac{d\theta}{dz}$$

subst

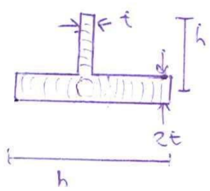
$$GJ = \frac{4G}{\nabla^2\varphi}\int\int \varphi \, dx\, dy \Rightarrow GJ = \frac{1}{3}Gst^3 \Rightarrow J = \frac{1}{3}st^3$$

Exercise/Example:



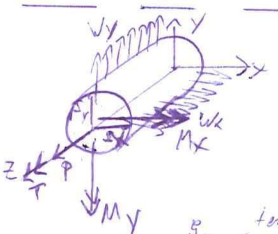
neglecting ends as above

$s = 2\pi R \Rightarrow J = \frac{1}{3}2\pi R t^3$ //

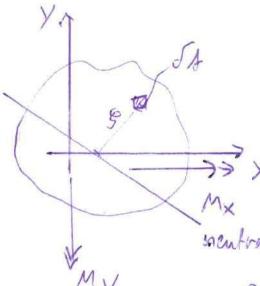


($t \ll h$) Calculate J: - Cut into 2 pieces and add result
- neglect ends and connection

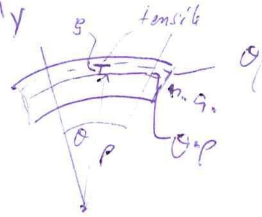
$$J = \frac{1}{3}ht^3 + \frac{1}{3}h(2t)^3 = \frac{9}{3}ht^3$$



Sign Convention
(on Formula sheet)



$$\sigma_z = E\varepsilon_z = E\cdot\frac{\xi}{\rho}$$

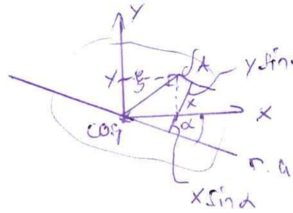


$\theta(\rho + \xi)$

elongation: $\xi\theta$

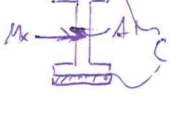
strain: $\varepsilon_z = \frac{\xi\theta}{\rho\theta} = \frac{\xi}{\rho}$

$$\int_A \sigma_z \, dx\, dy = 0 = \frac{E}{\rho}\underbrace{\int \xi \, dx\, dy}_{\text{first moment of inertia } y_{c.g.}}$$



$$\xi = x\sin\alpha + y\cos\alpha$$

$$\sigma_z = \frac{E}{\rho}(x\sin\alpha + y\cos\alpha)$$



$$\boxed{dA^* = \frac{E}{E_1}\cdot dA}$$

$\rightarrow$ if in a composite beam the stiffness of C is 5 times higher that A1 one uses 5x more area...

$$M_x = \int_A \sigma_z y \, dA \quad M_y = \int_A \sigma_z x \, dA$$

with σ_z: $M_x = E_1\int \frac{1}{\rho}(xy\sin\alpha + y^2\cos\alpha)\underbrace{\frac{E}{E_1}dA}_{dA^*} = \frac{E_1}{\rho}\cdot(I_{xy}^*\sin\alpha + I_{xx}^*\cos\alpha)$

$$I_{xx}^* = \int y^2 dA^*, \quad I_{xy}^* = \int xy \, dA^*$$

$$I_{yy}^* = \int x^2 dA^*$$

$$M_y = E_1\int\frac{1}{\rho}(x^2\sin\alpha + xy\cos\alpha)\frac{E}{E_1}dA = \frac{E_1}{\rho}(I_{xx}^*\sin\alpha + I_{xy}^*\cos\alpha)$$

$$\Rightarrow \begin{pmatrix} M_x \\ M_y \end{pmatrix} = \frac{E_1}{\rho}\begin{bmatrix} I_{xy}^* & I_{xx}^* \\ I_{yy}^* & I_{xy}^* \end{bmatrix}\begin{pmatrix} \sin\alpha \\ \cos\alpha \end{pmatrix}$$

inverted:

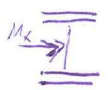
$$\frac{E_1}{\rho}\begin{pmatrix}\sin\alpha\\ \cos\alpha\end{pmatrix} \quad \frac{E_1}{\rho} = \begin{bmatrix} I_{xy}^* & I_{xx}^* \\ I_{yy}^* & I_{xy}^* \end{bmatrix}^{-1} \begin{Bmatrix} M_x \\ M_y \end{Bmatrix} = \frac{1}{I_{xx}^* I_{yy}^* - I_{xy}^{2*}} \begin{bmatrix} -I_{xy}^* & I_{xx}^* \\ I_{yy}^* & -I_{xy}^* \end{bmatrix} \begin{bmatrix} M_x \\ M_y \end{bmatrix}$$

Rewrite σ_z:

$$\sigma_z = \frac{E}{\rho}(x\sin\alpha + y\cos\alpha) = \frac{E}{E_1}\left(\frac{E_1}{\rho}\sin\alpha\cdot x + \frac{E_1}{\rho}\cos\alpha\, y\right)$$

Substitute IV. in σ_z: $$\sigma_z = \frac{E}{E_1}\left[\frac{M_y I_{xx}^* - M_x I_{xy}^*}{I_{xx}^* I_{yy}^* - I_{xy}^{*2}}x + \frac{M_x I_{yy}^* - M_y I_{xy}^*}{I_{xx}^* I_{yy}^* - I_{xy}^{*2}}y\right]$$

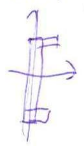
Back to the beam:



1. C.O.G
2. M.o.I
3. $\sigma(x,y)$ in σ_z? using E of the local material

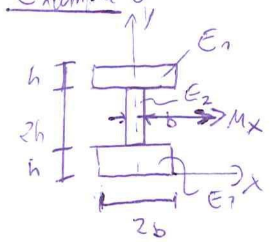
1. C.O.G
2. M.o.I.
3.

For symmetric problems:



$I_{xy} = 0$ → simplified $\sigma_z = \frac{M_y}{I_{yy}}x + \frac{M_x}{I_{xx}}y$

Example ①



$E_1 = 2E_2$

$$\sigma_z = \frac{E}{E_1}\left[\frac{M_y I_{xx}^* - M_x I_{xy}^*}{I_{xx}^* I_{yy}^*}\, x + \frac{M_x I_{yy}^* - M_y I_{xy}^*}{I_{xx}^* I_{yy}^*}\right]$$

$dA^* = \frac{E}{E_1} dA$; let E_1 be reference E_{MOD}

Step 1: Center of gravity:

$$Q_{x_0} = \int y\, dA^* = \underbrace{\frac{E_2}{E_1} 2b}_{area} \cdot \tfrac{7}{2}h + \frac{E_2}{E_1} b \cdot 2h \cdot 2h + \frac{E_1}{E_1} 2bh \tfrac{h}{2} = 10bh^2 \quad [m^3]$$

Steiner: $Q_x = Q_{x_0} - y_0 A^* = 0 \Rightarrow y_0 = \frac{Q_{x_0}}{A^*} \;\wedge\; A^* = 2bh \cdot \frac{E_2}{E_1} + \frac{E_2}{E_1} b \cdot 2h + 2bh \frac{E_1}{E_1} = 5bh$

$$\Rightarrow y_0 = \frac{10bh^2}{5bh} = 2 \cdot h$$

Symmetry $\Rightarrow x_0 = 0$

Step 2: Determine which I_{xx}^*, I_{yy}^* and I_{xy}^*

For symmetric structures: $I_{xy}^* = 0$

$$\rightarrow \sigma_z = \frac{E}{E_1}\frac{M_x}{I_{xx}^*}$$

$$\rightarrow I_{xx}^* = \int y^2 dA^* = 2\frac{E_1}{E_1}\frac{1}{12} 2bh^3 + \frac{E_1}{E_1 12} b(2h)^3 + 2\frac{E_1}{E_1}\overbrace{2b \cdot h\left(\tfrac{3}{2}h\right)^2}^{A \cdot l^2} = \frac{29}{3}bh^3$$

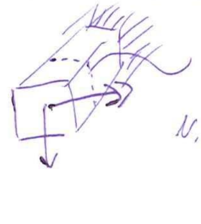
Step 3:

$$\sigma_z = \frac{E}{E_1}\frac{M_x}{\frac{29}{3}bh^3} \cdot y$$

Step 4: $y = h$ $\quad \sigma_z = \frac{M_x \cancel{h}}{\frac{29}{3}bh^{\cancel{3}2}} \quad [E_1]$

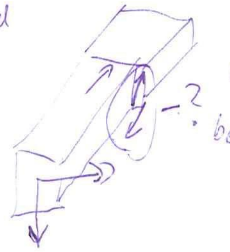
$$\sigma_z = \frac{1}{2}\frac{M_x}{\frac{29}{3}bh^2} \quad [E_2]$$

$\frac{E_2}{E_1}$



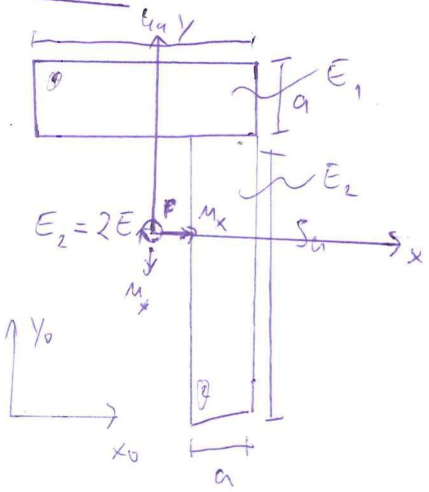
C.S. to be examined

$N, V, M \rightarrow \sigma_z$



σ_z cannot be applied directly!

Example 2



1. Step: C.O.G.

$Q_{x_0} \to y_0$

$Q_{y_0} \to x_0$

2. Step: $I_{xx}^*, I_{yy}^*, I_{xy}^*$?

$I_{xx}^* = \int y^2 dA^* \quad > 0$

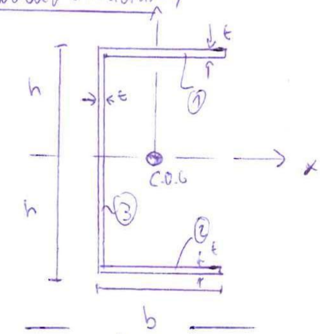
$I_{yy}^* = \int x^2 dA^* \quad > 0$

$I_{xy}^* = \int xy\, dA^* =$

$= \underbrace{\frac{E_1}{E_1} 4a \cdot a}_{\text{Only static of } ①} \cdot \underbrace{(2a - x_0)}_{\text{x-coordinate multiplied by } y_0} \cdot \left(\frac{11}{2}a - y_0\right) + \frac{E_2}{E_1} a \cdot 5a \underbrace{\left(\frac{7}{2}a - x_0\right)}_{\text{x-coord.}} \underbrace{\left(\frac{5}{2}a - y_0\right)}_{\text{y-coordinate}}$

Thinwalled Structures

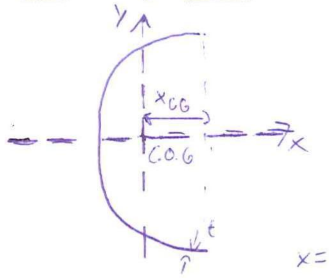
!



thinwalled: $t << b, h$ neglect everything smaller than t^2

$I_{xx} = \int y^2 dA^* = 2 \cdot \frac{1}{12} b t^3 + 2 \cdot h^2 \cdot t b + \frac{1}{12} t (2h)^3$

$= \underbrace{\frac{1}{12} \cdot t (2h)^2}_{3} + \underbrace{2 \cancel{\frac{1}{12} \cdot b t^3}}_{\text{small}} + 2 \cdot b \cdot t (h)^2$



Symmetric $\Rightarrow I_{xy} = 0$; $I_{xx} = ?$; $I_{yy} = ?$

~~$Q_{y_0} = \int x\, dA = \int \ldots \cos\theta \ldots d\theta = \int_{-\pi/2}^{\pi/2} \frac{1}{3} r^3 \cos\theta\, d\theta = \left[\frac{1}{3} r^3 \sin\theta\right]_{-\pi/2}^{\pi/2} = \frac{2}{3} r^3$~~

$Q_{y_0} = \int x\, dA = \int_0^{\pi} -R\sin\theta\, t R\, d\theta = tR^2 [\cos\theta]_0^{\pi} = tR^2(-1-1) = -2tR^2$

$x = -r\sin\theta \quad dA = t r\, d\theta$

$Q_{y_0} = Q + \pi R t \cdot x_{cg} \Rightarrow Q_y = Q_{y_0} - \pi R t x_{cg} \Rightarrow x_{cg} = \frac{Q_{y_0}}{\pi R t} = \frac{-2tR^2}{\pi R t} = -\frac{2R}{\pi}$

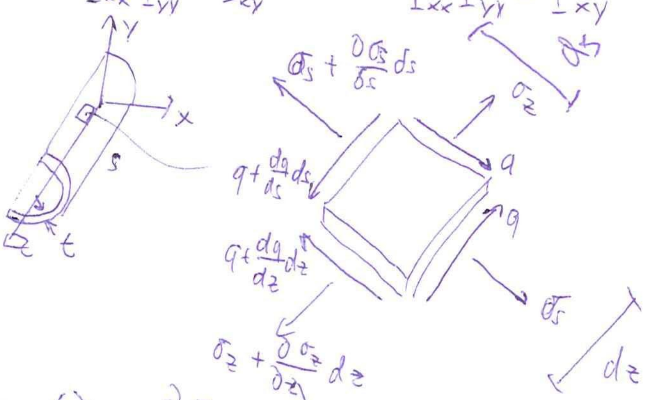
$I_{xx} = \int y^2 dA = \int_0^{\pi} (R\cos\theta)^2 t \cdot R\, d\theta = tR^3 \cdot \int_0^{\pi} \cos^2\theta\, d\theta = tR^3 \int_0^{\pi} (\tfrac{1}{2}(1 + \cos 2\theta))\, d\theta = \tfrac{1}{2} t R^3 \left[\theta + \tfrac{1}{2}\sin 2\theta\right]_0^{\pi} = \tfrac{1}{2} t R^3 \pi$

$I_{y_0 y_0} = \int x^2 dA = \int_0^{\pi} (-r\sin\theta)^2 t R\, d\theta = R^3 t \int_0^{\pi} \sin^2\theta\, d\theta = \frac{\pi}{2} t R^3$

to find I_{yy}: $I_{yy} = I_{y_0 y_0} - \pi R t \cdot y_0^2 = \frac{\pi}{2} t R^3 - \pi R t \cdot \left(+\frac{2R}{\pi}\right)^2$

OR

$$\sigma_z = \frac{M_y I_{yy} - M_x I_{xy}}{I_{xx} I_{yy} - I_{xy}^2} x + \frac{M_x I_{yy} - M_y I_{xy}}{I_{xx} I_{yy} - I_{xy}^2} y \qquad q = \tau \cdot t$$



$$\Sigma F_z: \left(\cancel{\sigma_z} + \frac{\partial \sigma_z}{\partial z} dz\right) t\,ds - \cancel{\sigma_z t\,ds} + \left(\cancel{q} + \frac{\partial q}{\partial s} ds\right) dz - \cancel{q\,dz} = 0$$

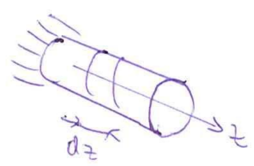
divide by $ds\,dz$: $\boxed{\frac{dq}{ds} + t \frac{\partial \sigma_z}{\partial z} = 0}$ $\boxed{\frac{\partial q}{\partial z} + t \frac{\partial \sigma_s}{\partial s} = 0}$

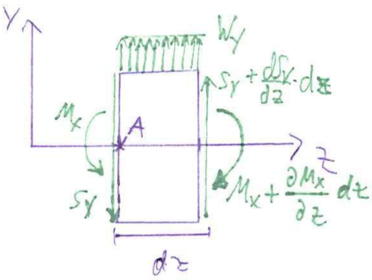
Hoop stress:

$\sigma_z = 0 \Rightarrow \frac{dq}{dz} = 0 \Rightarrow$ q(s) is constant

$$* \quad \frac{d\sigma_z}{dz} = \frac{\frac{\partial M_y}{\partial z} I_{yy} - \frac{\partial M_x}{\partial z} I_{xx}}{I_{xx} I_{yy} - I_{xy}^2} x + \frac{\frac{\partial M_x}{\partial z} I_{yy} - \frac{\partial M_y}{\partial z} I_{xy}}{I_{xx} I_{yy} - I_{xy}^2} y$$

Arbitrary Beam:





$$\Sigma M_A^{\circlearrowright +}: \quad 0 = \cancel{M_x} + \frac{\partial M_x}{\partial z} dz - \cancel{M_x} - \left(S_y + \cancel{\frac{dS_y}{dz} dz}\right) dz \; \cancel{w_y \cdot dz \cdot \tfrac{1}{2} dz}$$

dz^2 → Higher Order term

H.O.T

$$\Rightarrow \boxed{S_y = \frac{\partial M_x}{\partial z}} \quad **$$

$$\Rightarrow \boxed{S_x = \frac{\partial M_y}{\partial z}}$$

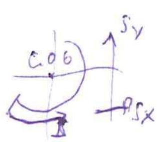
Substituting $*$ in $**$:

$$\frac{dq}{ds} = -\frac{S_x I_{xx} - S_y I_{yy}}{I_{xx} I_{yy} - I_{xy}^2} t x - \frac{S_y I_{yy} - S_x I_{xx}}{I_{xx} I_{yy} - I_{xy}^2} t y$$

$$\int_0^s \frac{dq}{ds} ds = -\frac{S_x I_{xx} - S_y I_{yy}}{I_{xx} I_{yy} - I_{xy}^2} \int_0^s t x\,ds - \frac{S_y I_{yy} - S_x I_{xx}}{I_{xx} I_{yy} - I_{xy}^2} \int_0^s t y\,ds$$

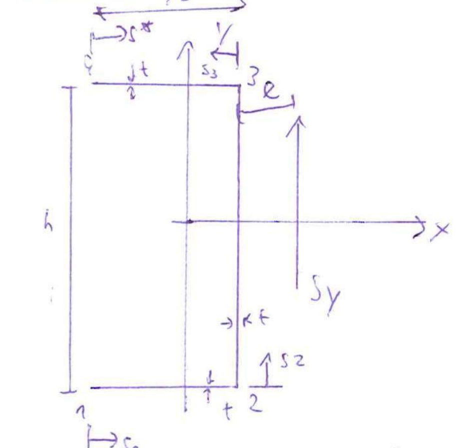
first moments around y and x Axis.

1. C.O.G
2. Calculate I_{xx}, I_{yy}, I_{xy} around C.O.G
3. Calculate first moment of area



Shear Center:
if loads are applied at shear center the bar will not rotate/twist

Exercise:



1. ~~C.O.G.~~ Symmetry and conclusions:

$$\left.\begin{matrix} I_{xy} = 0 \\ S_x = 0 \end{matrix}\right\} \quad q_s q_0 = -\frac{S_y}{I_{xx}} \int_0^s t\, y\, ds$$

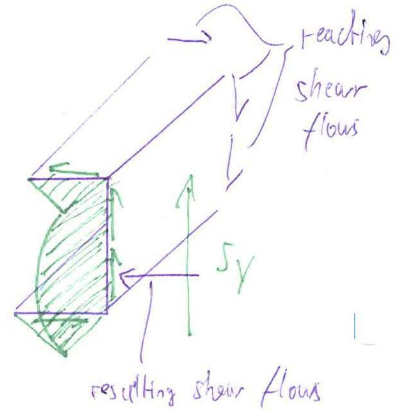
2. C.O.G.

Somewhere on x - Axis as I_{yy} is not needed x_{COG} is not needed

3. Moments of Inertia

$$I_{xx} = \frac{1}{12} h^3 t + 2\, t \cdot \frac{h}{2} \left(\frac{h}{2}\right)^2 = \frac{1}{3} t\, h^3$$

$$q_s \approx q_0 = \frac{S_x I_{xx} - S_y I_{xy}}{I_{xx} I_{yy} - I_{xy}^2} \int_0^s t\, x\, ds - \frac{S_y I_{yy} - S_x I_{xx}}{I_{xx} I_{yy} - I_{xy}^2} \int_0^s t\, y\, ds$$



4

Calculate shear flows from 1 to 4:

$$q_{12}(s_1) = -\frac{S_y}{I_{xx}} \underbrace{t s_1 \left(-\frac{h}{2}\right)}_{\text{Area} \times \text{distance}} = \frac{3}{2} \frac{S_y}{h^2} s_1 \qquad s_1 = \frac{h}{2} \Rightarrow q_2 = \frac{3}{4} \frac{S_y}{h}$$

$$q_{23}(s_2) = q_2 - \frac{S_y}{I_{xx}} \int_0^{s_2} t\left(-\frac{h}{2} + s_2\right) ds_2 = \frac{3}{4} \frac{S_y}{h} - \frac{3}{2} \frac{S_y}{h^2}\left(-h s_2 + s_2^2\right) \Rightarrow q_3 = q_2 = \frac{3}{4} \frac{S_y}{h}$$

$$q_{34}(s_3) = -\frac{S_y}{I_{xx}} t s_1 \left(\frac{h}{2}\right) + q_3$$

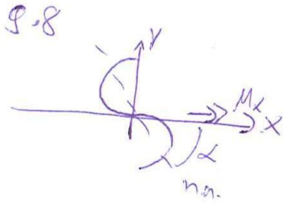
Shear-flows act in direction of S

if one begins at point 4 one will get a negative shear flow

→ acts opposite to s*

with S_y acting upwards it's convenient to begin at 1 (less sign errors)

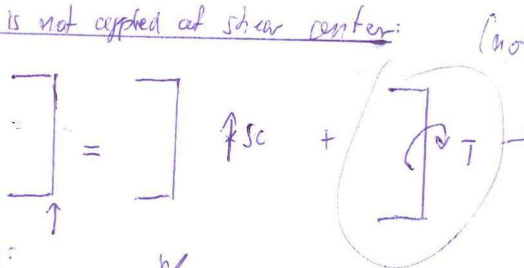
Book 9.8



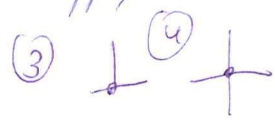
a) Calculate position of neutral axis:

b) $\sigma_{z\,max}$ due to M_x:

If Load is not applied at shear center: (not in this course!)



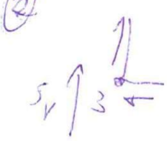
→ bending theory formula doesn't apply to this case!



For above:

$$\Sigma M_3: \quad \underbrace{S_y \cdot e}_{\text{V}\cdot\text{arm}} = \underbrace{\int_0^{h/2} q_{12}(s_1)\, ds_1 \cdot \underset{\text{arm}}{h}}_{\text{moments generated by } q} \Rightarrow e = \frac{3}{16} \cdot h$$

2



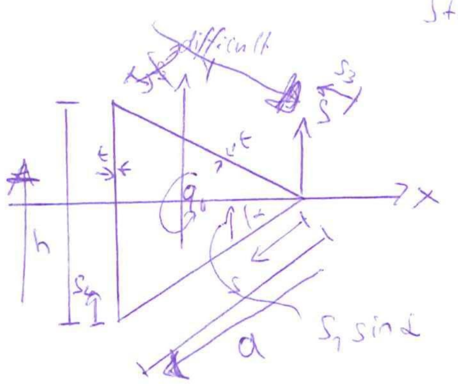
$\Sigma M_2: \quad S_y \cdot e = 0$

$\Rightarrow e = 0$

Moment ⇒ Shear Force · Arm = ∫ shear flow ds · Arm

Structures II

9.05.07



$I_{xy} = 0$

$S_x = 0$

$q_2(s_1) = 0 - \frac{S_y}{I_{xx}}$

$$q_s = -\frac{S_y}{I_{xx}} \int_0^s t y \, ds + q_0 \qquad \sin\alpha = \frac{h}{2d}$$

$$I_{xx} = \int y^2 \, dA = 2 \int_0^a (s_1 \sin\alpha)^2 t \, ds + \frac{1}{12} t h^3$$

calculation $= \frac{1}{12} t h^2 (h + 2a)$

$$q_2(s_1) = 0 - \frac{S_y}{I_{xx}} \int_0^{s_1} t(-s_1 \sin\alpha) \, ds = \frac{3 S_y}{hd(h+2d)} s_1^2$$

$$q_{12}(s_1 = a) = q_2 = \frac{3 S_y a}{h(h+2d)}$$

$$q_{23}(s_2) = q_2 - \frac{S_y}{I_{xx}} \int_0^{s_2} t\left(-\frac{h}{2} + s_2\right) ds = \frac{3 S_y d}{h(h+2d)} - \frac{6 S_y}{h^2(h+2d)} (-h s_2 + s_2^2)$$

$q_{13} \to$ as q_{12}

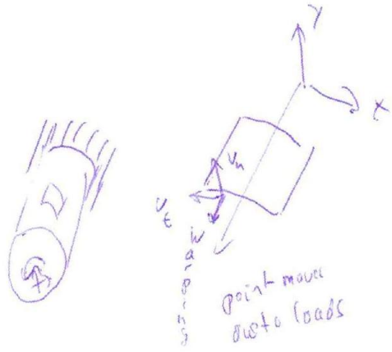
<u>Moment equation:</u>

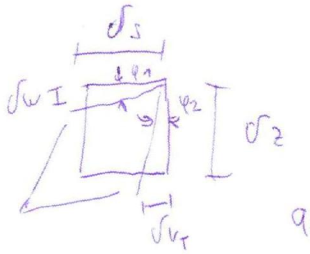
Choose 1 as Point $\to$ 21, 31 neutralize

$$M_1: \quad 0 = d\cos\alpha \int_{s_2=0}^{s_2=h} \left(\frac{3 S_y d}{h(h+2d)} - \frac{6 S_y}{h^2(h+2a)} (-h s_2 + s_2^2) \right) ds_2 - q_0 \times h d \cos\alpha$$

applied loads   arm

$$\Rightarrow q_0 = \frac{h + 3d}{h(h+2a)} S_y$$ $\to$ should be added to each of the shear flows.



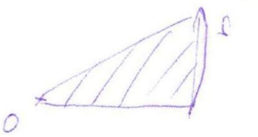


$\gamma = \varphi_1 + \varphi_2$ — small angle approximations

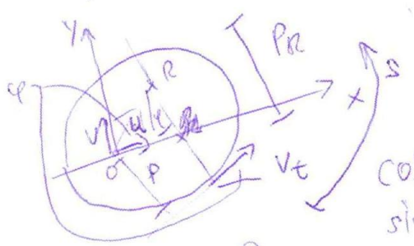
$= \frac{\partial w}{\partial s} + \frac{\partial v_t}{\partial z}$

$q_s = Gt \left(\frac{\partial w}{\partial s} + \frac{\partial v_t}{\partial z} \right)$ *

$q_s = \tau\, t \qquad \tau = G\gamma$



A_{os}



$\cos\varphi\, ds = dx$
$\sin\varphi\, ds = dy$

$v_t = p_R\, \theta$

$= p\theta + u\cos\varphi + v\sin\varphi$

$\frac{dv_t}{dz} = p\frac{d\theta}{\partial z} + \frac{du}{dz}\cos\varphi + \frac{dv}{dz}\sin\varphi$

in * and divide by Gt:

$\int_0^s \frac{q_s}{Gt} = \int_0^s \left(\frac{\partial w}{\partial s} + p\frac{d\theta}{dz} + \frac{du}{dz}\cos\varphi + \frac{dv}{dz}\sin\varphi \right) ds$

$\int_0^s p\,ds = 2A_{os}$

$\rightarrow \int_0^s \frac{q_s}{Gt}\, ds = \int_0^s \frac{\partial w}{\partial s} ds + 2A_{os}\frac{\partial \theta}{\partial s} + \frac{\partial u}{\partial z}\int_0^s dx + \frac{\partial v_t}{\partial z}\int_0^s dy$

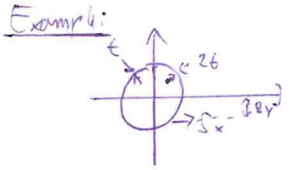
$= w_s - w_0 + 2A_{os}\frac{\partial\theta}{\partial s} + \frac{\partial u}{\partial z}(x - x_0) + \frac{\partial v_t}{\partial z}(y_s -$

if one integrates $\oint$

$\oint_0^s \frac{q_s}{Gt} ds = \boxed{\frac{d\theta}{dz} = \frac{1}{2A}\oint \frac{q_s}{Gt} ds}$ relation between of the single shear flows

Ratio of twist: $\frac{d\theta}{dz} = \frac{1}{2 G A_{encl}} \oint \frac{q\,ds}{t}$

Example:

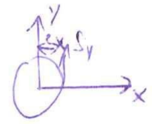


① Determine ~~C.O.G.~~ position of shear center

- Symmetry about $x \Rightarrow x_{sc} = 0$

M_0: $S_x e_y = \ldots$ we already know x_0

→ put S_y in:



What are the q's?

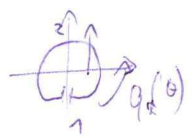
Symmetry $\Rightarrow I_{xy} = 0$

$q = q_0 - \frac{S_y}{I_{xx}} Q_x$

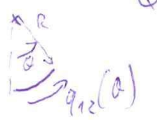
$I_{yy} = \, ? \rightarrow$ not necessary! Saves time, no C.o.g. Y

$I_{xx} = \underset{left}{\tfrac{1}{2}\pi t R^3} + \underset{right}{\tfrac{1}{2}\pi 2t R^3} = \tfrac{3}{2}\pi t R^3$

then: cut $\Rightarrow q = 0$



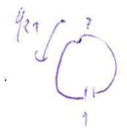
Q_y:



$Q_{x_{12}} = \int y\,dA = \cancel{2tRd\theta dr} \int_0^\theta -R\cos\theta\, 2t\, R\,d\theta =$

$q_{12}(\theta)$ $= -2tR^2 [\sin\theta]_0^\theta = -2tR^2 \sin\theta$

$q_{12}(\theta) = -\frac{S_y}{\frac{3}{2}\pi t R^3} \cdot -2tR^2 \sin\theta = \frac{4}{3} \frac{S_y \sin\theta}{R\pi}$



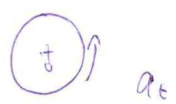
$q_{12}(0) = 0 \qquad q_{12}(\pi) = 0$

$Q_{21_x} = \int_0^\theta y\,dA = \int_0^\theta R\cos\theta\, \cancel{t} R\,d\theta = tR^2 \sin\theta$

$q_{21}(\theta) = -\frac{S_y}{\frac{3}{2}\pi t R^3} \cdot tR^2 \sin\theta = -\frac{2}{3} \frac{S_y \sin\theta}{\pi R}$

Remove the cut!

→ Introduce circular shear flow:



$$d\theta/dz = \frac{1}{2 G A_{encl}} \oint \frac{q\,\overset{ds}{\widetilde{ds}}}{t} = \frac{1}{2 G \pi R^2} \cdot \left\{ \int_0^\pi \frac{q_{12}(\theta) R\,d\theta}{2 \cdot t} + \int_0^\pi \frac{q_{21}(\theta) R\,d\theta}{t} + \int_0^\pi \frac{q_t R\,d\theta}{2t} + \int_0^\pi \frac{q_t R\,d\theta}{t} \right\} = 0 \text{ at S.C.}$$

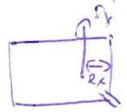
$$= \frac{1}{2 G \pi R^2} \left\{ \underbrace{\int_0^\pi \frac{\frac{4}{3} S_y \sin\theta\, R\,d\theta}{\pi R\, 2t} + -\int_0^\pi \frac{2}{3} \frac{S_y \sin\theta\, R\,d\theta}{\pi R t}}_{\text{cancel out}} + \frac{q_t R}{t}\left(\frac{\pi}{2} + \pi\right) \right\} = 0$$

$\Rightarrow q_t = 0 \rightarrow$ Shear flow will be 0 at point 1 if the force acts at S.C.

$M_0 := S_y e_x = \int_0^\pi \underbrace{q_{12}(\theta)}_{\text{shear flow}} \underbrace{R}_{\text{Arm}} \cdot R\,d\theta + \int_0^\pi q_{21}(\theta) R\, R\,d\theta + \int_0^{2\pi} \cancel{q_t}_0\, R\,d\theta = \int_0^\pi \frac{4}{3} \frac{S_y \sin\theta}{\pi R} R R\,d\theta - \int_0^\pi \frac{2}{3} \frac{S_y \sin\theta}{\pi R} R R\,d\theta$

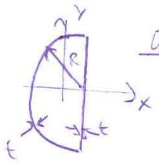
$\Rightarrow e_x = \frac{4}{3} \cdot \frac{R}{\pi}$

Summary of steps: **Position of the Shear Center?**



- apply force in shear center
- make a cut in the section
- calculate C.O.G (if needed)
- Moments of Inertia (if needed)
- Calculate shear flows at cut section
- Close cut introducing a circular shear flow q_t
- Rate of twist formula $\frac{d\theta}{dz} = 0 \rightarrow$ sol. for q_t
- Moment eq. $\rightarrow$ e_x

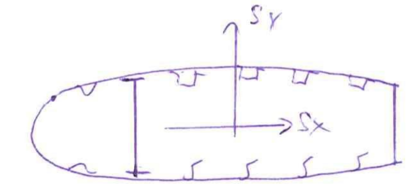
Homework:



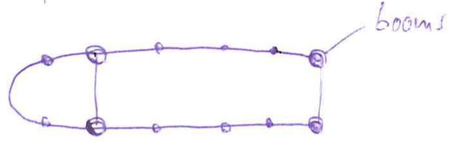
determine S.C.

Structures II

17.05.07



Complex → Idealize

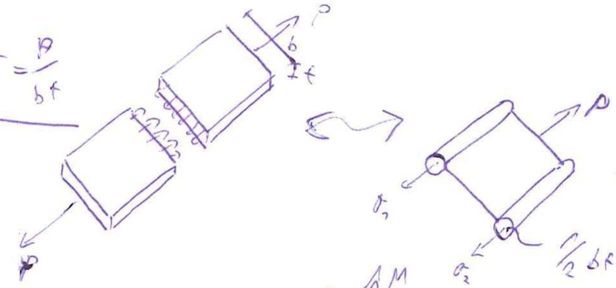


Idealized Structure:

booms: carry normal loads only

plates: carry shear forces only

$\sigma = \frac{P}{bt}$

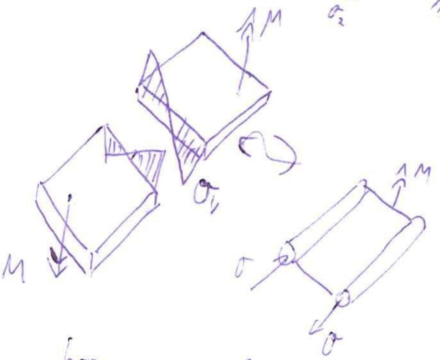


$\sigma_1 = \frac{\frac{1}{2}P}{\frac{1}{2}b \cdot t} = \frac{P}{bt}$

$\frac{1}{2}bt = A$

$I = \frac{1}{12} t b^3$

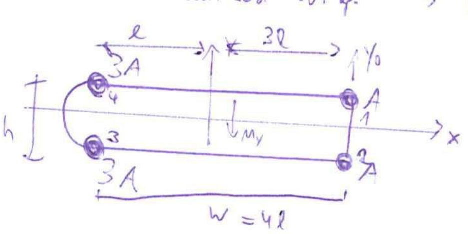
$\sigma_{max} = \frac{M}{I} y_{max}$



boom area $= \frac{1}{6} bt$

$\rightarrow I = 2 \cdot \frac{1}{6} bt \cdot \left(\frac{b}{2}\right)^2 = \frac{1}{12} tb^3$

Exercise: Idealized Wing: → booms only normal forces, plates " shear "



① C.O.G:

symmetry → $y_0 = 0$ → $I_{xy} = 0$

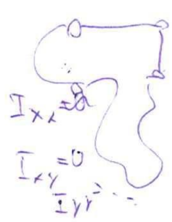
$\sigma_z = \frac{M_y}{I_{yy}} x$ → $I_{xx} = ?$ not necessary!

$I_{yy} = ?$

$Q_{y_0} = 2 \cdot 3A \cdot (-4l) = -24 \cdot Al$

$A_{total} = 8A$ → $x_{c.o.g.} = \frac{Q_{y_0}}{A_{tot}} = -\frac{24}{8} l = -3l$

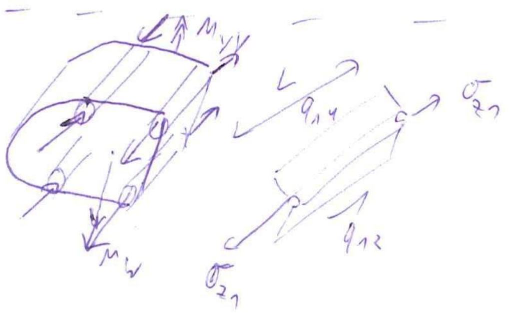
$\Rightarrow I_{yy} = 2 \cdot 3 A l^2 + 2A(3l)^2 = 24 A l^2$



still symmetric as only area of booms is used for c.o.g

$\sigma_{z_1} = \frac{M_y}{24Al^2} \cdot 3l = \frac{M_y}{8Al} = \sigma_{z_2}$ tension

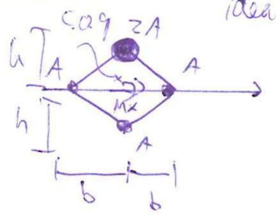
$\sigma_{z_3} = \sigma_{z_4} = \frac{M_y}{24Al^2} \cdot (-l) = -\frac{M_y}{24Al}$ compression



ΣF_{arm}: $0 = \sigma_{z_1} A - \sigma_{z_1} A - q_{12} L - q_{14} L$

$\Rightarrow q_{12} = -q_{14}$

Problem:



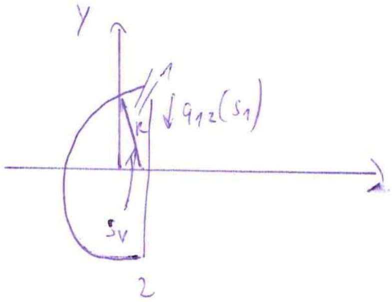
idealized structure

→ symmetry around y → $I_{xy} = 0$

Structures I

16.05.07

(1)



$I_{xy} = 0$

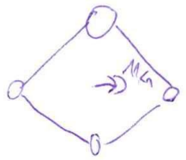
$\underline{q} = q_0 - \frac{S_y}{I_{xx}} Q_x$

$Q_{x_{12}} = \int y \, dA = \int_0^{s_1} (R - s_1) \, t \, ds_1$

$= (R s_1 - \frac{1}{2} s_1^2) t$

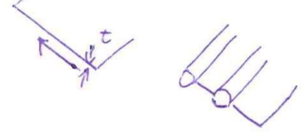
$\underline{q} = 0 - \frac{S_y}{I_{xx}} t (R s_1 - \frac{1}{2} s_1^2)$ see also on Blackboard

(2)



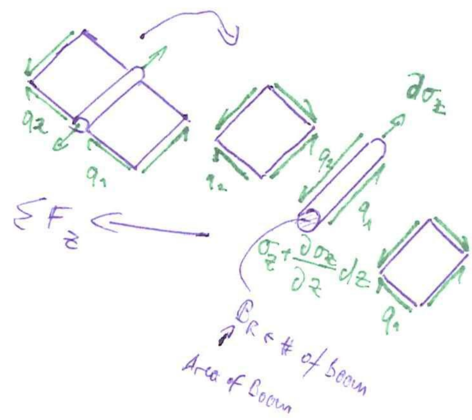
see on Blackboard

From: $q_c = -\left(\frac{S_x I_{xx} - S_y I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\right) \int_0^s t_D x \, ds - \left(\frac{S_y I_{yy} - S_x I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\right) \int_0^s t_D y \, ds + q_0$



↓ normal structure: $t_D = t$

idealized structure: $t_D = 0$ → q is always constant between booms



$\Sigma F_z: \; 0 = \left(\cancel{\sigma_z} + \frac{\partial \sigma_z}{\partial z} \delta z\right) B_r - \cancel{\sigma_z B_r} + q_2 \delta z - q_1 \delta z$

$\Rightarrow q_2 - q_1 = -\frac{\partial \sigma_z}{\partial z} B_r$

to: $\frac{\partial \sigma_z}{\partial z}(-B_r) = q_2 - q_1 = -\left(\frac{S_x I_{xx} - S_y I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\right) B_r x_r - \left(\frac{S_y I_{yy} - S_x I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\right) B_r y_r$

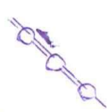
first moments of the booms

One has 2 formulas to calculate q:

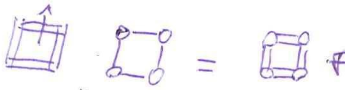
$q_s = -\left(\frac{S_x I_{xx} - S_y I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\right)\left(\int_0^s t_D x \, ds + \sum_{i=1}^{n} B_r x_r\right) - \left(\frac{S_y I_{yy} - S_x I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\right)\left(\int_0^s t_D y \, ds + \sum_{i=1}^{n} B_r y_r\right) + q_r$

Contribution of skin and booms

For Problems such as:

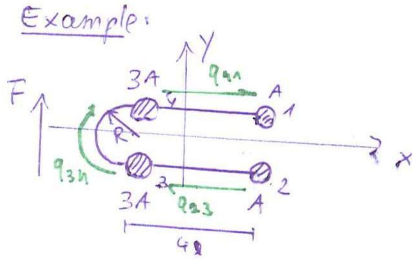


a lot of work...



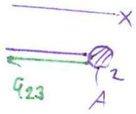
never asked like this at exam (always separately!)

Example:



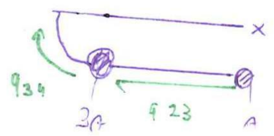
(1) Symmetry: $I_{xy} = 0$

$I_{yy} = 24AR^2$; $I_{xx} = 3A \cdot R^2 + 3AR^2 + AR^2 + A \cdot R^2 = 8AR^2$



$q_{x_{23}} = A(-R)$

$$q_{23} = \frac{S_Y}{I_{xx}} q_{x_{23}} = -\frac{F}{8AR^2}(-AR) = \frac{F}{8R}$$



$q_{x_{34}} = 3A(-R)$

$$q_{34} = q_{23} - \frac{S_Y}{I_{xx}} q_{34} = \frac{F}{8R} - \frac{F}{8AR^2}(-3AR) = \frac{F}{2R}$$

Instead of doing it boom by boom (Σ) one can also ~~summ~~ use the sum of Q's → q_0 is not neccessary:

$$q_{34} = -\frac{S_Y}{I_{xx}}\underbrace{\left(-AR - 3AR\right)}_{\Sigma B_r x_r} = \frac{F}{2R}$$

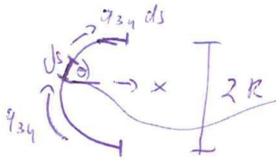
$$q_{41} = -\frac{S_Y}{I_{xx}}\left(3A(+R)\right) + \frac{F}{2R} = \frac{F}{8R}$$

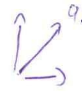
Checking your answer:

$\Sigma F_{horizontal} = 0$ ✓

$\Sigma F_{vertical}: F = q_{34} \cdot \underbrace{2R}_{\text{projected vertical distance of action}}$ ✓

Structures II 16.05.07





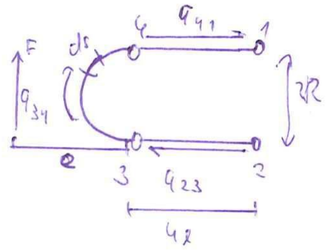
Vertical: $\int_0^{\pi} q_{34}\, ds \cdot \sin\theta \cancel{d\theta} = \int_0^{\pi} q_{34} \sin\theta\, R\, d\theta = q_{34} R \left[-\cos\theta\right]_0^{\pi}$

$= q_{34} 2R$ component of force in vertical direction

$\int_0^{\pi} q_{34}\, ds \cos\theta = q_{34} R (\sin\theta)_0^{\pi} = 0$ $\Rightarrow$ shear flow has no resulting force in horizontal direction

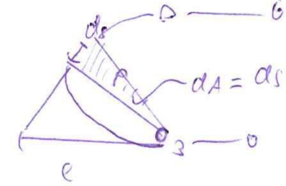
---

Moment equation is used to find the position of the shear center:



$\Sigma M_3^{\curvearrowleft}: \; F \cdot e = q_{41} \cdot \underbrace{\overbrace{4l}^{*}}_{\text{distance}} \cdot \underbrace{2R}_{\text{arm}} + q_{34} \cdot \pi R^2$

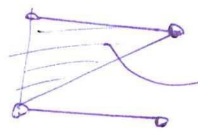
looking at the moment by $q_{34}\, ds$:



$dA = ds \tfrac{1}{2} \rho \Rightarrow \int q_{34} \underbrace{ds \cdot \rho}_{2dA} = q_{34} \int ds\, \rho = q_{34} \pi R^2$

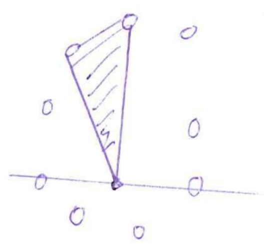
$\Rightarrow$ 

Using an area instead of a moment arm can be used instead of a moment arm:



$\tfrac{1}{2} 4l 2R \rightarrow 2A = 4l \cdot 2R \rightarrow$ see *!

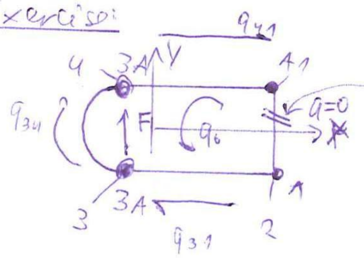
Same for:



← Convenient!

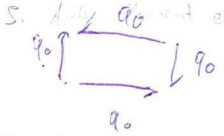
$\Rightarrow e = \tfrac{\pi}{2} R + l$

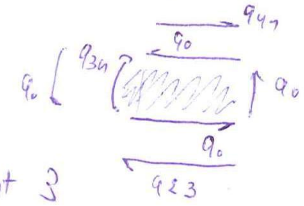
Exercise:



Steps: 1. make a cut at a convenient place
ideally there as $q=0$ and since q is constant between booms one can remove the complete plate (only for idealized structures!)
2. Calculate the shear flows of the cut sections (done earlier)
3. Close cut
4. Introduce circular shear flow q_0
5. solve q_0
6. Add results of step 2 and step 5

4. Introduce q_0



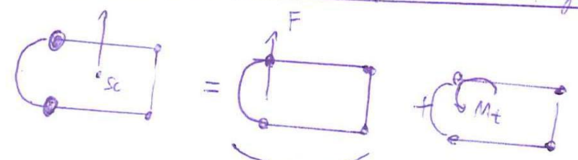


5. Apply Moment equation about point 3

$$\sum \overset{\frown}{M_3} : 0 = q_{34} \underbrace{\pi R^2}_{A} + q_{41} \underbrace{4l\,2R}_{A} - q_0 \left(4l\,eR + \tfrac{1}{2}\pi R^2\right) * 2 \rightarrow q_0 = \dots$$

$$q_0 \cdot 2 \cdot A_{TOTAL} \rightarrow -q_0 \cdot 2R \cdot 4l - q_0\, 4l \cdot 2R - q_0\, \pi R^2 = 2A_{TOTAL} \quad 6. \dots$$

- Find the shear center of the wing box:



Solution is known (all q's)

$M_t = F \cdot e$

$$q_t = \frac{M_t}{2A_{enclosed}} = \frac{Fe}{2A_{enclosed}} \quad *$$

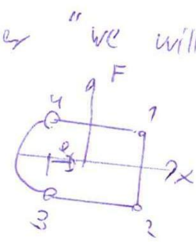
If loads are applied at SC the twist is zero:

$$\Rightarrow \oint q_s ds = 0 = q_{23}\, 4l + q_{34} \cdot \pi R + q_{41} \cdot 4l$$
$$- q_0 \left(2R + 4l + \pi R + 4l\right) \quad \leftarrow \text{from } q_0 \text{ known}$$
$$- q_t \left(2R + 4l + \pi R + 4l\right) \quad \leftarrow \text{from Moment}$$

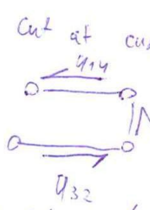
solve $\rightarrow q_t$

Plug into $*$ and solve for e/.

However "we will" place the force directly at the SC. (takes less time!)



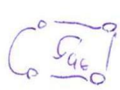
1st: Cut at curved part ($\rightarrow$ is gone)



"simple to solve"

2nd: solve for q_{32}, q_{21}, q_{14}

3rd:



introduce q_t

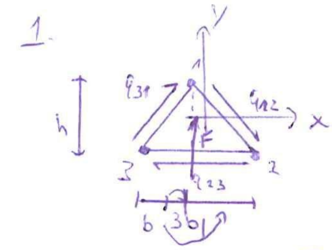
$\rightarrow$ ratio of twist $\oint q ds = q_{32}\, 4l + q_{21} + q_{14}\, 4l + q_t \left(2R + 4l + \pi R + 4l\right) = 0 \rightarrow q_t$/

$\rightarrow \sum M_3 = q_t\, 2 A_{enclosed} + q_{14}\, 4l\, 2R + q_{21} \cdot 2R\, 4l = F \cdot e \rightarrow$ solve for e/

Structures I

21.05.07

1.



①

1. det. c.o.g.
2. I_{xx}, I_{xy}, I_{yy}
3. make a cut → solve q's ⟺
4. close cut → solve q_0

30'

② Test results: *

$\Sigma F^{\uparrow}: F = q_{31} \cdot h - q_{12} h$

$\Sigma F^{\rightarrow}: 0 = q_{31} \cdot b + q_{12} 3b - q_{23} \cdot 4b$

$\Sigma M_1: 0 = q_{23} \cdot 4b \cdot h$

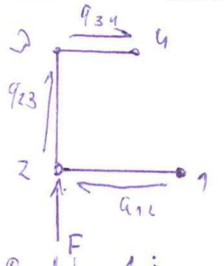
$\left.\begin{matrix} \\ \\ \end{matrix}\right\} q_{31} = \frac{3F}{4h}, \quad q_{12} = \frac{-F}{4h}$

$\rightarrow q_{23} = 0$

10'

when can we use ②*:
- when the structure is idealized
- when there are no more than 3 shear flows (3 eq's)

Therefore always check if a structure is statically determinate!



→ Force is not acting on the shear center → method ① can't be used!

→ use method ② eq. eq.'s

typical 1st exam question!

(!)

Back to 1.:

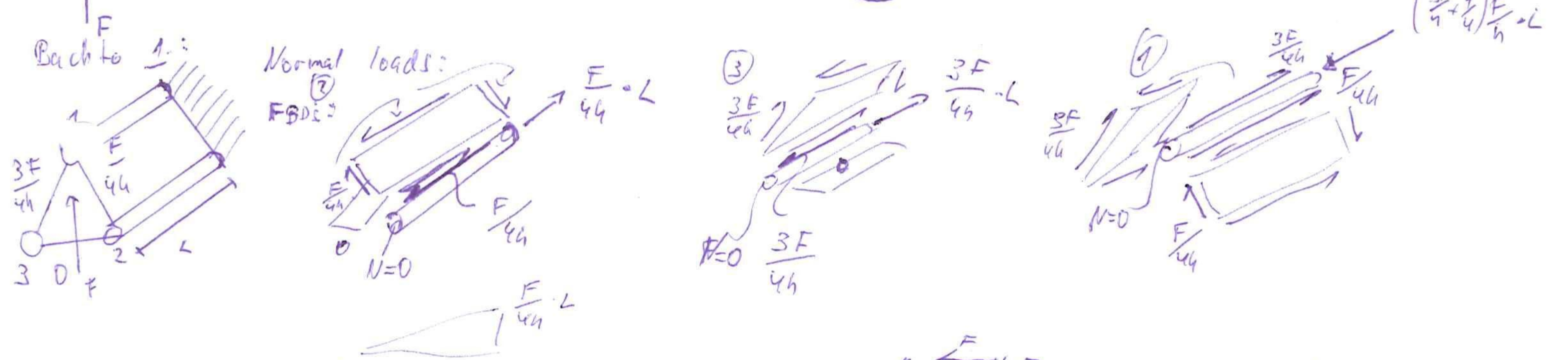
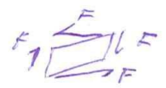


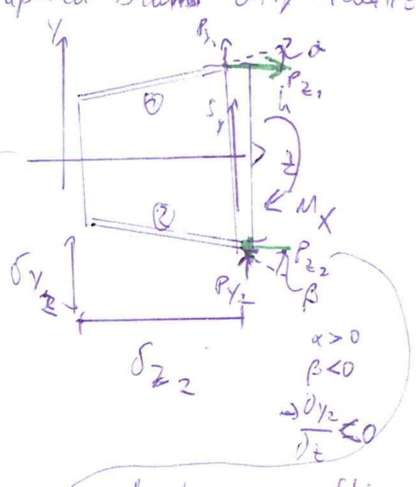
Plate should be in equilibrium ⇒

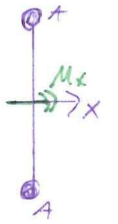


action = reaction

Complicated idealized beams:

- tapered beams only will idealized!





What is M_x doing to the beams?

- Plate do not carry normal stresses

$P_{z_1} = \frac{M_x}{h}$

$F_2 = -\frac{M_x}{h}$

or

$I_{xx} = 2A \cdot \left(\frac{h}{2}\right)^2 = \frac{1}{2}Ah^2$

$\rightarrow \sigma_z = \frac{M_x}{I_{xx}} y \rightarrow \sigma_{z_1} = \frac{M_x}{\frac{1}{2}Ah^2} \cdot \frac{h}{2} = \frac{M_x}{Ah}$

$\Rightarrow P_{z_1} = A\sigma_{z_1} = \frac{M_x}{h}$

$\Rightarrow P_{y_1} = P_{z_1} \cdot \tan\alpha = P_{z_1} \cdot \frac{\delta y_1}{\delta z_1} \rightarrow P_{y_i} = P_{z_i} \frac{\delta y_i}{\delta z_i}$

small angle approximations

$\alpha > 0$

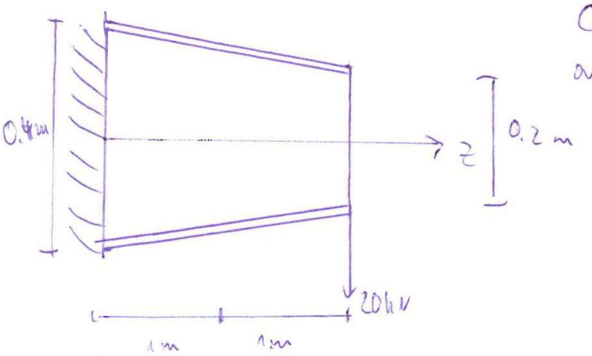
$\beta < 0$

$\rightarrow \frac{\delta y_2}{\delta z} < 0$

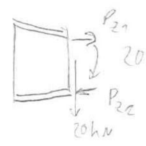
→ direction is this way so that P_{y_1} and P_{y_2} are both pointing upwards!

$S_y = S_{y_{web}} + P_{y_1} + P_{y_2} \rightarrow q = \frac{S_{y_{web}}}{h}$

Exercise:



Calculate the normal forces
and the shear flow at $z = 1m$:



$\rightarrow h = 0.15m$

1st: normal loads
2nd: resulting forces to be carried by plates
3rd: q

$I_{xx} = 2A \cdot \left(\frac{h}{2}\right)^2 = \frac{1}{2} A h^2$

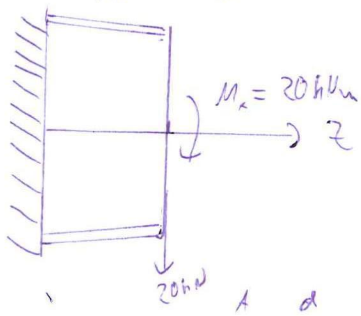
$\sigma_z = \frac{M_x}{I_{xx}} y = \frac{M_x}{Ah} = \frac{20kNm}{400mm^2 \cdot 0.300m} = 167MPa$

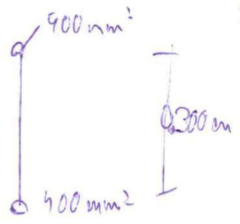
$P_{z_1} = \frac{\sigma_z}{A} \cdot A = \frac{M_x}{h} = -P_{z_2} = 66.67kN$

$P_{z_2} = -\frac{M_x}{h} = -66.67\,kN$

$P_{y_1} = P_{z_1} \cdot \frac{\delta y}{\delta z} = 66.67kN \cdot \frac{0.05}{1.0} = 3.33$

$P_{y_2} = -3.33$





$I_{xx} = 2 \cdot \underbrace{400}_{A} \cdot \underbrace{150}_{d}^2 = 1.8 \cdot 10^7$

$\sigma_z = \frac{M_x}{I_{xx}} \cdot y \Rightarrow \sigma_{z_1} = 166.67MPa \quad \sigma_{z_2} = -166.67MPa$

$\rightarrow P_{z_1} = \sigma_{z_1} \cdot 400mm^2 = 66.67kN \; ; \; P_{z_2} = -66.67kN$

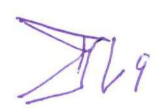
$\rightarrow P_{y_1} = \frac{-50}{1000} \cdot 66.67kN = -3.33kN$

$P_{y_2} = \frac{50}{1000}(-66.67)kN = -3.33kN$

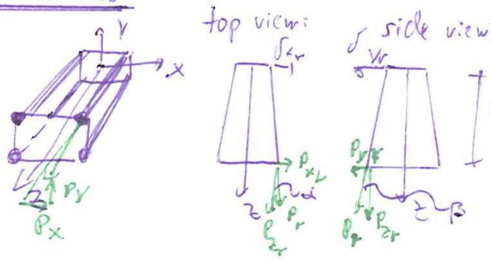
$\Rightarrow S_y = S_{y\,web} + P_{y_1} + P_{y_2} \cancel{= 20kN}$

$\Rightarrow S_{y\,web} = 20000N - 2 \cdot 3.33kN = 13.34kN$

$\Rightarrow q = \frac{S_y}{h} = 44.44\,N/mm$

$\Rightarrow$ 

3D-Structures



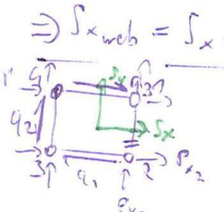
$P_{xr} = \tan\alpha \, P_{zr} = P_{zr} \cdot \frac{\delta x_r}{\delta z}$

$P_{yr} = P_{zr} \frac{\delta y_r}{\delta z}$

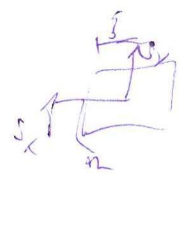
$\Rightarrow P_r = \sqrt{P_{xr}^2 + P_{yr}^2 + P_{zr}^2}$

$S_x = S_{x\,web} + \sum_{r=1}^{n} P_{xr} \qquad S_y = S_{y\,web} + \sum_{r=1}^{n} P_{yr}$

$\Rightarrow S_{x\,web} = S_x - \sum_{r=1}^{n} P_{zr} \frac{\delta x_r}{\delta z_r} \; ; \; S_{y\,web} = S_y - \sum_{r=1}^{n} P_{zr} \frac{\delta y_r}{\delta z}$



1st cut
2nd q_1, q_2, q_3
3rd remove cut and introduce q_0
4th ΣM_3 ~~$S_{x\,web}$ $S_{y\,web}$~~



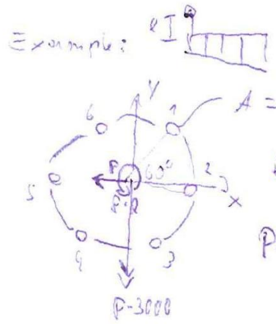
$S_x \eta - S_y \xi = q_3 b h + q_0 2 \cdot h \cdot b + P_{x_1} \cdot h - P_{y_1} \cdot b - P_{y_2} \cdot b + P_{x_4} \cdot h$

5th. $\rightarrow$ solve for q_0

Example: $e I$

Structures I

23.05.07



$A = 500\,mm^2$

$\ell = 1571\,mm$

$P = 3000\,N$

① Normal stress: symmetry $\rightarrow I_{xy} = 0$; $M_x = 0$

$$\sigma_z = \frac{M_y}{I_{yy}} x$$

$$I_{yy} = 2 \cdot 500\,mm^2 \cdot 600\,mm^2 + 4 \cdot 500 \cdot 300^2 = 5.4 \cdot 10^8\,mm^4$$

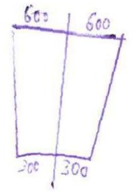
off-sets $A \cdot d^2$ — 5, 2 — 1, 3, 4, 6

1, 3: $x = 300 \rightarrow \sigma_z = 10\,MPa \rightarrow P_z = 5000\,N$

4, 6: $x = -300 \rightarrow \sigma_z = -10\,MPa \rightarrow P_z = -5000\,N$

2: $x = 600 \rightarrow \sigma_z = 20\,MPa \rightarrow P_z = 10000\,N$

5: $x = -600 \rightarrow \sigma_z = -20\,MPa \rightarrow P_z = -10000\,N$

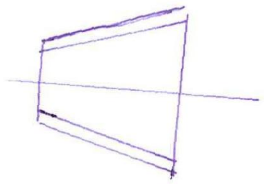


$\ell = 3000\,mm$

①, ③, ④, ⑥: $\delta x_r = 300 \cdot \sin 30° = 150$; $\delta y_r = 300 \cos 30° = 259.8\,mm$

②, ⑤: $\delta x_r = 300\,mm$; $\delta y_r = 0\,mm$

② Normal Forces



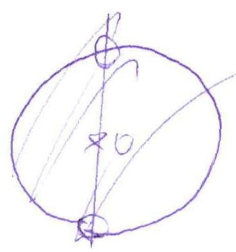
①, ③, ④, ⑥: $|P_{xr}| = |P_{zr}| \cdot \frac{\delta x_r}{\delta z} = \frac{150}{3000} |P_{zr}| = 0.05 |P_{zr}|$

$P = \sqrt{P_{xr}^2 + P_{yr}^2 + P_{zr}^2}$

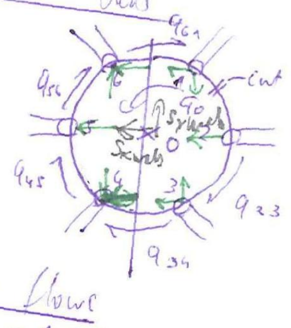
$|P_{yr}| = |P_{zr}| \cdot \frac{\delta y_r}{\delta z} = \frac{259.8}{3000} |P_{zr}| = 0.08 |P_{zr}|$

②, ⑤: $|P_{xr}| = |P_{zr}| \cdot \frac{\delta x_r}{\delta z} = \frac{300}{3000} |P_{zr}| = 0.1 \cdot |P_{zr}|$

$|P_{yr}| = 0$; $\rightarrow \delta y_r = 0$



Shear loads



Vertical components cancel each other out

$\Rightarrow S_{y\,web} = 0$

Horizontal forces: $\rightarrow S_{x\,web}$

~~$\Sigma F_{hor} = 4 \cdot 0.25 P_{zr}$~~

$$S_{x\,web} = P - 4 \cdot 0.05 \overset{5000}{|P_{zr}|} - 2 \cdot 0.1 \cdot \overset{10000}{|P_{zr}|}$$

$= 3000 - \ldots$

Shear flows

For q: 1. Cut

$$q = q_0 - \frac{S_{x\,web}}{I_{yy}} Q_y \rightarrow q_{23} = 0 - \frac{-3000}{5.4 \cdot 10^8} \cdot \underbrace{500 \cdot 600}_{A \cdot d} = 1.667\,N/mm$$

$$q_{34} = 1.6667 + \frac{3000}{5.4 \cdot 10^8} \cdot 500 \cdot 300 = 2.5\,N/mm$$

$q_{45} = 1.667\,N/mm$

$q_{56} = 0$

$q_{61} = -0.833\,N/mm$

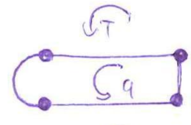
2. closing cut and adding q_0:

XIV

$$\Sigma \overset{\curvearrowleft}{M_0}: P\ell = (-q_{23} - q_{34} - q_{45} - q_{56} - q_{61}) \cdot 2\frac{\pi R^2}{6} + q_0 \cdot 2\pi R^2 + \overset{0}{\text{Resulting Moments}}$$

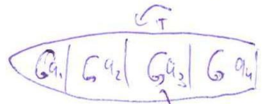
(of the components in the beam)

$\Rightarrow q_0 = 5\,N/mm$



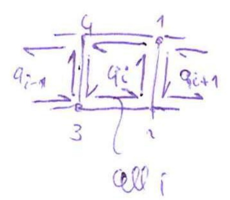
Bredt-Batho:

$T = 2q\,A_{enclosed}$



$T = 2 \cdot \sum_{i=1}^{n} q_i A_{enclosed_i}$; requires $(n-1)$ equations.

$\left(\frac{d\theta}{dz}\right)_{cell\,1} = \left(\frac{d\theta}{dz}\right)_{cell\,2} = \left(\frac{d\theta}{dz}\right)_{cell\,n}$ → leads to the required $(n-1)$-equations

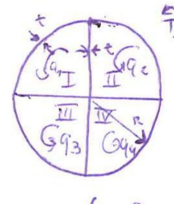


Ratio of twist for cell i:

$$\left(\frac{d\theta}{dz}\right)_{cell\,i} = \frac{1}{2GA_{enclosed}} \oint \frac{q\,ds}{t} = \frac{1}{2GA_{encl.}t}\Big[q_i(l_{12}+l_{23}+l_{34}+l_{41}) - q_{i-1}(l_{34}) - q_{i+1}(l_{12})\Big]$$

in one plate there can only be one shear flow!

Example:



$T = \sum 2 \cdot q_i A_i$, $A_i = \frac{\pi R^2}{4}$ } 1st formula

→ 3 additional equations:

$$\left(\frac{d\theta}{dz}\right)_{cell\,I} = \frac{1}{2GA_I t}\Big[q_I(\tfrac{1}{2}\pi R + 2\cdot R) - q_{II}R - q_{III}R\Big]$$

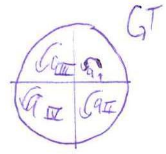
$$\left(\frac{d\theta}{dz}\right)_{cell\,II} = \frac{1}{2GA_{II}t}\Big[q_2(\tfrac{1}{2}\pi R + 2R) - q_I R - q_{IV}R\Big]$$

$$\left(\frac{d\theta}{dz}\right)_{cell\,III} = \frac{1}{2GA_{III}t}\Big[q_3(\tfrac{1}{2}\pi R + 2R) - q_I R - q_{IV}R\Big]$$

$$\left(\frac{d\theta}{dz}\right)_{cell\,IV} = \frac{1}{2GA_{IV}t}\Big[q_4(\tfrac{1}{2}\pi R + 2R) - q_{III}R - q_{II}R\Big]$$

(=) (=) (=)

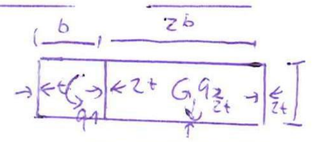
In case of Symmetry (Alternative!):
Suppose one rotates the structure by 90°



Structure and load will be the same

$\Rightarrow \underbrace{q_I = q_{II} = q_{III} = q_{IV}}_{\text{substitute } q \text{ for } q_{I \dots IV}}$

$\Rightarrow T = 4 \cdot 2Aq \Rightarrow q = \frac{T}{8A}$



$T = 10\,kN$, $b = 100\,mm$, $h = 200\,mm$, $t = 1\,mm$

$\boxed{T = 2q_1 bh + 2q_2 2bh}$

$T = 2 \cdot (b \cdot h \cdot q_1 + 2b \cdot h \cdot q_2)$

$\frac{d\theta}{dz} = \frac{1}{2GA_{encl}} \oint \frac{q\,ds}{t}$

$\left(\frac{d\theta}{dz}\right)_1 = \left(\frac{d\theta}{dz}\right)_2 = \frac{1}{\cancel{2Gbht}}\Big[q_1(2b + \tfrac{3}{2}h) - q_2\tfrac{h}{2}\Big] = \frac{1}{4\cancel{Gbht}}\Big[q_2(4b+2h) - q_1 h\Big]$

$\Rightarrow q_1(2b + \tfrac{3}{2}h) - q_2\tfrac{h}{2} = \tfrac{1}{4}(q_2 \cdot 4b + 2h) - q_1 \cdot h$

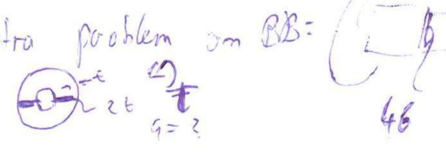
$2q_1 b + \tfrac{3}{2}q_1 h + q_1 h = q_2 b + \tfrac{1}{2}q_2 h + q_2\tfrac{h}{2}$

$\left(\frac{d\theta}{dz}\right)_{cell\,1} = \frac{1}{2Gbh}\left(\frac{q_1}{t}(2b+h) + \frac{q_1}{2t}h - \frac{q_2}{2t}h\right)$; $\left(\frac{d\theta}{dz}\right)_{cell\,2} = \frac{1}{2G2bh}\left(\frac{q_2}{2t}(2b+h+2b+h)\right)$

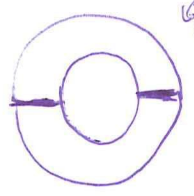
$q_1 = q_2\,\frac{b + \tfrac{1}{2}h + \tfrac{h}{2}}{2b + \tfrac{5}{2}h}$

$\Rightarrow \boxed{\left(\frac{d\theta}{dz}\right)_{cell\,I} = \left(\frac{d\theta}{dz}\right)_{cell\,II}}$ $\Rightarrow q_1 = 53.571\,N/mm$, $q_2 = 98.214\,N/mm$

extra problem on BB:

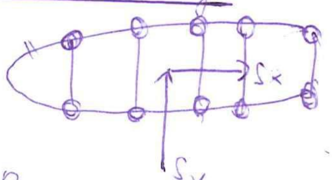


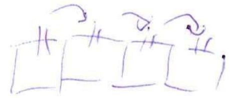
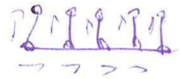
$2t$, $q = ?$



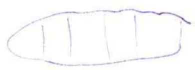
Last week:

Multi Cell Wings



→  →  open section solvable

Remove cuts: → introduce circular shear flow



→ to solve for q_t's use moment equation

→ 5 unknown shear flows

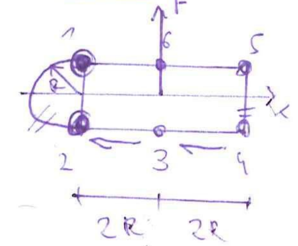
→ 4 extra relations come from the specific twist ratio:

$$\frac{d\theta}{dz} = \frac{1}{2GA_{encl}} \oint \frac{q\,ds}{t} \rightarrow \underbrace{\left(\frac{d\theta}{dz}\right)_1 = \left(\frac{d\theta}{dz}\right)_2}_{1} \underbrace{= \left(\frac{d\theta}{dz}\right)_3}_{2} \underbrace{= \left(\frac{d\theta}{dz}\right)_4}_{3} \underbrace{= \left(\frac{d\theta}{dz}\right)_5}_{4}$$

$\oint \frac{q\,ds}{t} \rightarrow \frac{q \cdot l}{t}$

+ Moment eq. → 5 eq's

Exercise:



$A_1 = A_2 = 2 \cdot A$

$A_3 = A_4 = A_5 = A_6 = A$

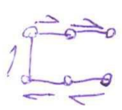
1. 2 cuts

3. calculate shear flow open section

symmetry → $I_{xy} = 0$

Only shear forces in y-direction: $q = q_0 - \frac{F}{I_{xx}} B_r y_r$

$I_{xx} = \int y^2 dA = Area \cdot distance^2 = 2 \cdot R^2 \cdot 2A + 4A \cdot R^2 = 8AR^2$



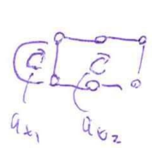
$q_{43} = \cancel{q_0} - \frac{F}{8AR^2} \cdot A \cdot (-R) = \frac{F}{8R}$

$q_{32} = q_{43} - \frac{F}{8AR^2} A(-R) = \frac{F}{4R}$

$q_{21} = \frac{F}{4R} - \frac{F}{8AR^2} \cdot 2A(-R) = \frac{F}{2R}$ → q_{21} is the only shear flow in vertical direction → F acts over q_{21} → $q_{21} = \frac{F}{2R}$

$q_{16} = \frac{F}{4R}$

$q_{65} = q_{43} = \frac{F}{8R}$



3. Close cuts:

$\sum M_{2}$: $-F2R = 2R q_{16} 2R + 2R q_{65} 2R + q_{t_1} \frac{\pi}{2} R^2 + q_{t_2} 2 \cdot 4R \cdot 2R$

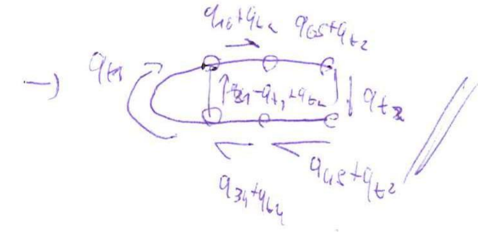
$$\left(\frac{d\theta}{dz}\right)_1 = \frac{1}{2G\frac{\pi}{2}R^2}\left[\frac{q_{21} \cdot 2R}{t} - \frac{q_{t_1}}{t}(2R + \pi R) + \frac{q_{t_2} \cdot 2R}{t}\right]$$

$$\left(\frac{d\theta}{dz}\right)_2 = \frac{1}{2G\,2R \cdot 4R}\left[-\frac{q_{43} \cdot 2R}{t} - \frac{q_{32} \cdot 2R}{t} - \frac{q_{21}2R}{t} - \frac{q_{16}2R}{t} - \frac{q_{65}2R}{t} - \frac{q_{t_2}(2R + 4R + 2R + 4R)}{t} + \frac{q_{t_1} \cdot 2R}{t}\right]$$

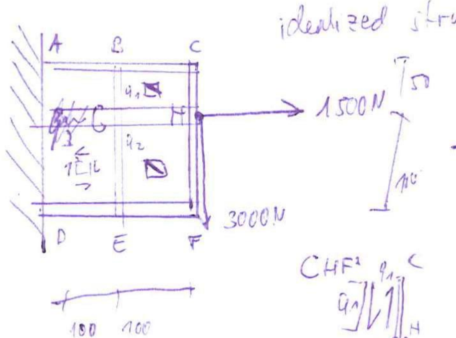
$$\left(\frac{d\theta}{dz}\right)_1 = \left(\frac{d\theta}{dz}\right)_2 \rightarrow q_{t_1}, \rightarrow q_{t_2}$$

Additional Lecture on Wednesday

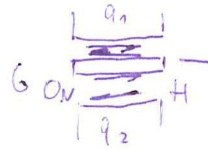
CR-8



idealized structure (plate can only carry V, booms N)



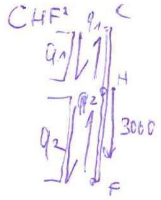
add stiffener



$$\Sigma \vec{F}: 0 = -q_1 \cdot 100 + q_2 100 + 1500 \quad q_1 - q_2 = 15$$

$$q_1 + q_2 = 00$$

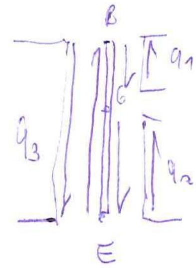
CHF:



$$\Sigma F^{\uparrow}: 0 = q_1 \cdot 50 + q_2 \cdot 100 - 3000$$

$$\Rightarrow q_2 = 15 \text{ N/mm} \Rightarrow q_1 = 30 \text{ N/mm}$$

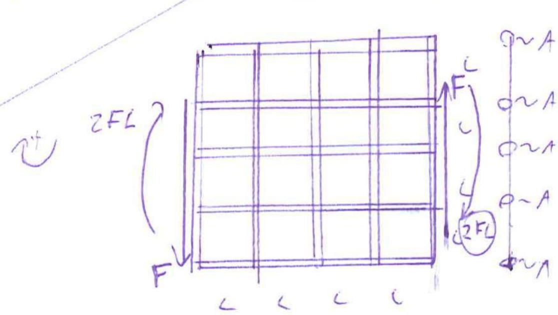
q_3:



$$\rightarrow \Sigma F^{\uparrow}: \quad q_3 \cdot 150 - q_1 \cdot 50 - q_2 \cdot 100 = 0$$

$$\Rightarrow q_3 = 20 \text{ N/mm}$$

Exercise:

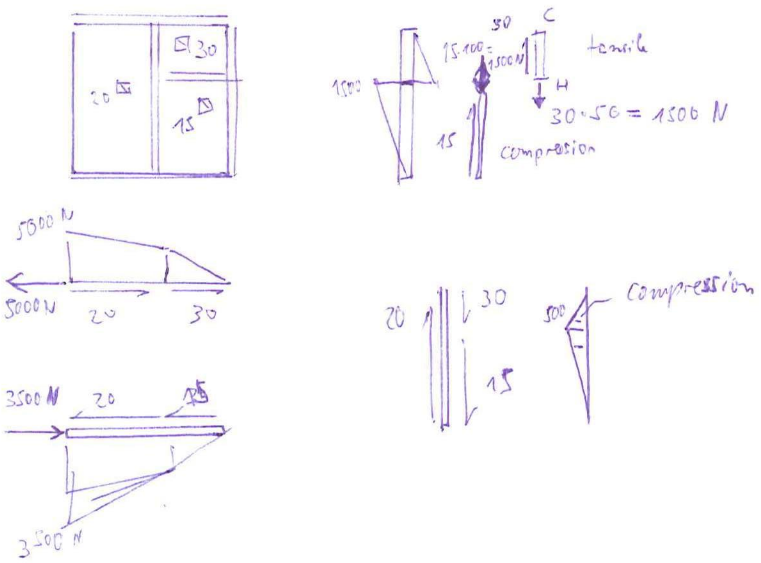


- Symmetric cross section $I_{xy} = 0$
- → only F is acting on it
- → shear flow

$q = \ldots$

For normal loads see ←

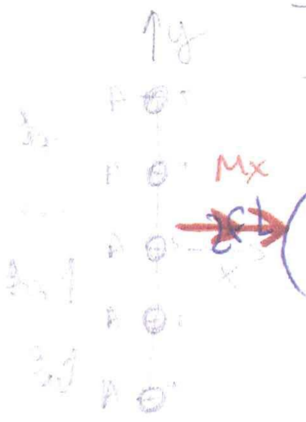
Normal loads:

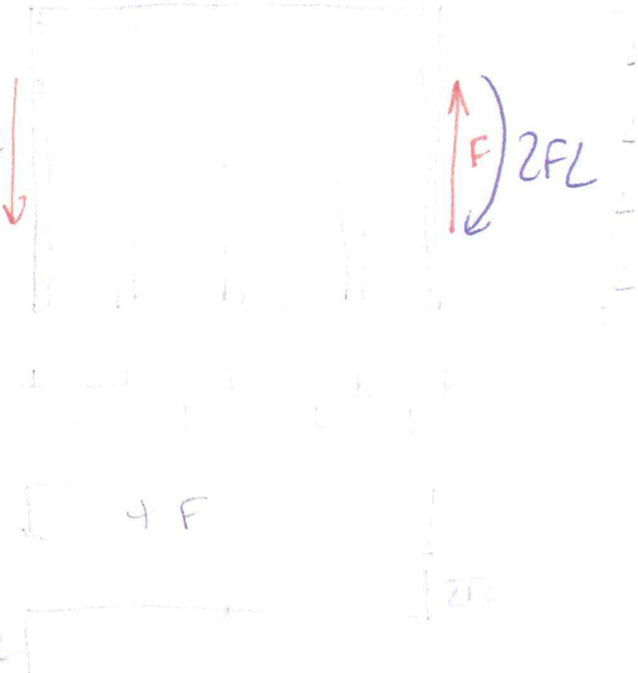


STRUCTURAL ANALYSIS I — 04.06.07

SOLUTION OF EXAMPLE

- 16x ...
- beams only normal forces





$q = q_0 - \frac{F}{I_{xx}} Q_x$

$I_{xx} = A(\dots)^2 + AL^2 + AL^2 + A(2L)^2 = 10AL^2$

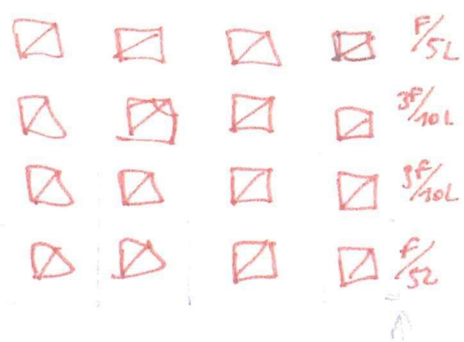
$q_{12} = 0 - \frac{F}{10AL^2} A(-2L) = \frac{F}{5L}$

$q_{23} = \frac{F}{5L} - \frac{F}{10AL^2} A(-L) = \frac{3F}{10L}$

$q_{34} = q_{23} = \frac{3F}{10L}$

$q_{45} = \frac{F}{5L}$

$\frac{\;}{F}$



$\sigma_z = \frac{M_x}{I_{xx}} y = \frac{2FL}{10AL^2} y$

$\sigma_1 = \frac{2FL}{10AL^2}(-2L) = \frac{-2F}{5A}$; $N_1 = -\frac{2}{5}F$

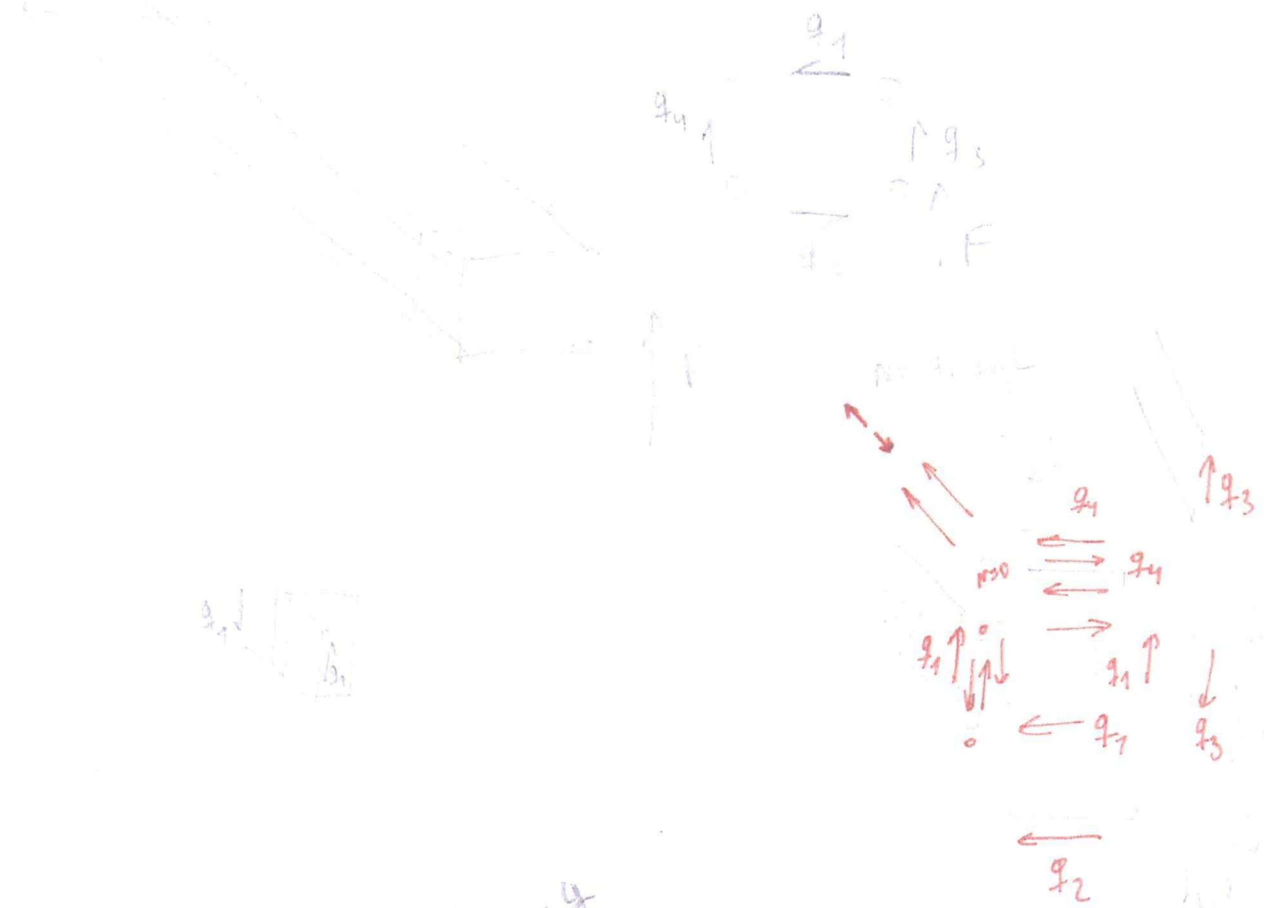
$\sigma_2 = \dots = \frac{-F}{5A}$; $N_2 = \frac{-1}{5}F$

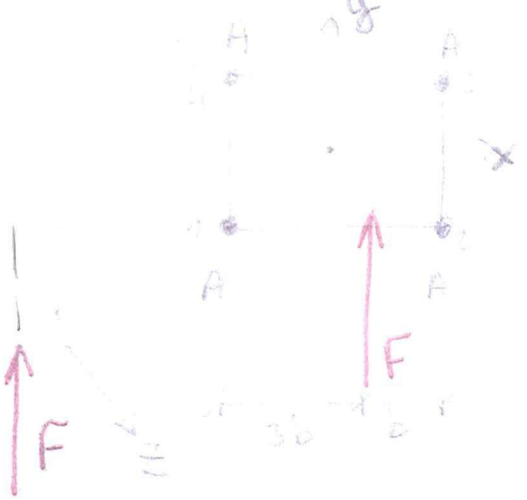
$\sigma_3 = 0$; $N_3 = 0$

$\sigma_4 = -\sigma_2$; $N_4 = \frac{1}{5}F$

$\sigma_5 = -\sigma_1$; $N_5 = \frac{2}{5}F$

3-DIMENSIONAL ANALYSIS



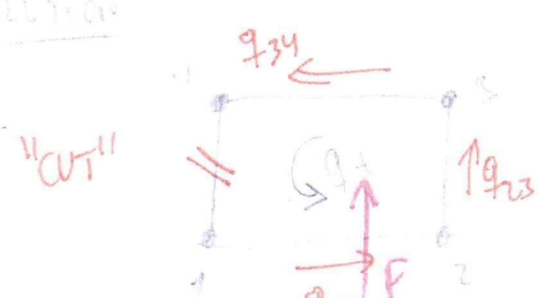


$Q_x = \ldots$

$\frac{-S_x}{I_{xx}} B_r x_r$

$-\frac{F}{Ah^2} b \cdot 3$

SOLUTION



$q_{12} = -\frac{F}{Ah^2} A\left(-\frac{h}{2}\right) = \frac{F}{2h}$

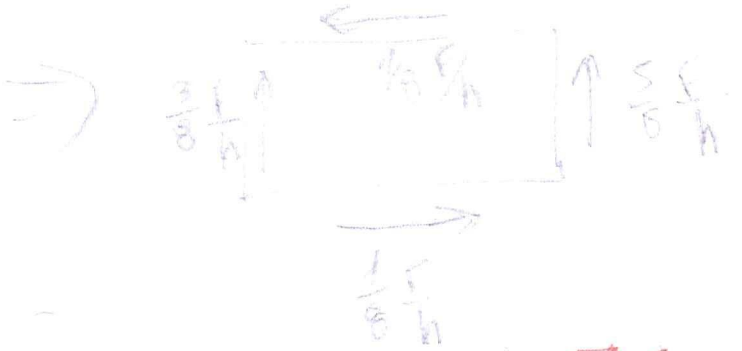
$q_{23} = \frac{F}{2h} - \frac{F}{Ah^2} A\left(-\frac{h}{2}\right) = \frac{F}{h}$

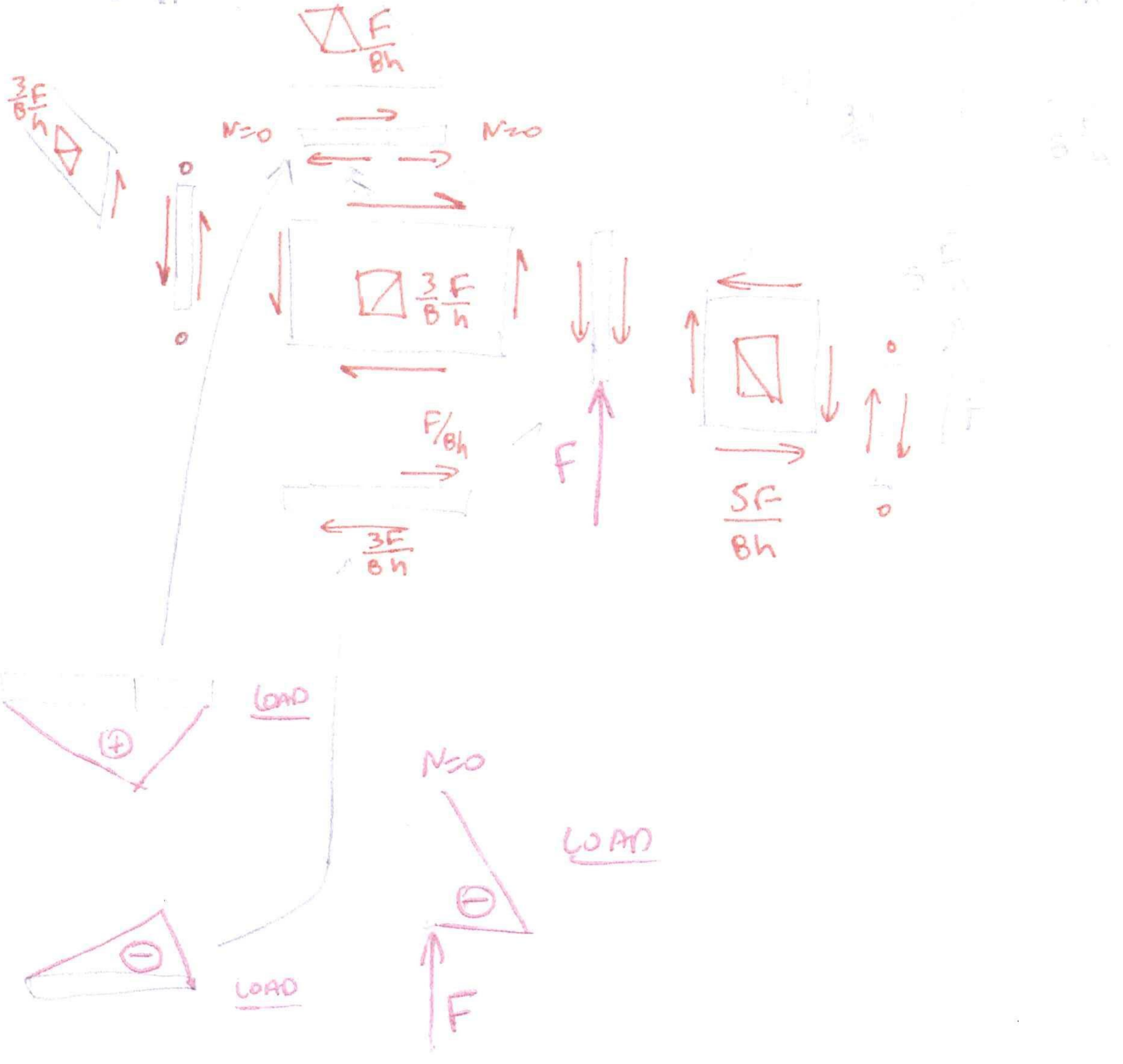
$q_{n} = \frac{F}{2h}$

-3-

$$M_z = -Fb = q_{41}\,4ah + q_2\,2A_{enclosed}$$

$$q_2 = -\frac{5}{8}\frac{F}{h}$$

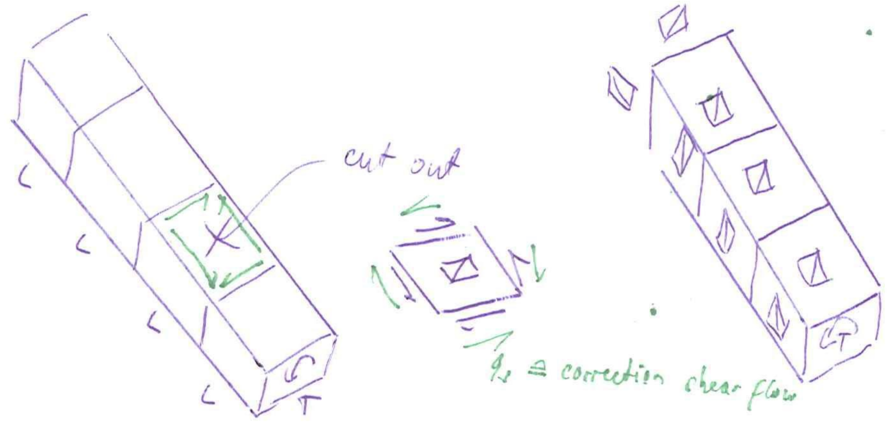




Structures I 06.06.07

Making a hole in the previously discussed problems (→ exam)!

E.g.: Wing



1st look at the structure without hole

2nd make hole and introduce correction q_s

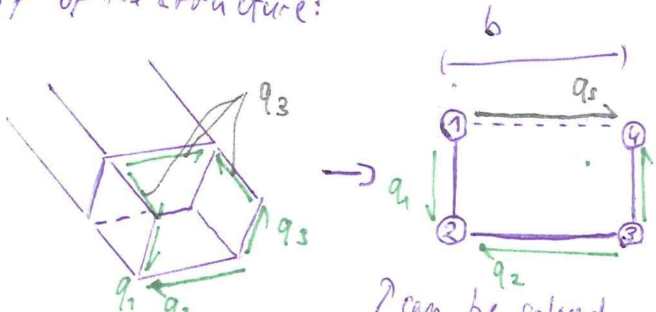
3rd see what happens to the structure with the applied q_s → calculate

Assuming: Disruption of structure is only "felt" one cell ~~before~~ in front and behind the hole/cutout!

Strategy for calculation:

- look at one half of the structure:

(Note: Choose q_1, q_2, q_3 such that the directions lead to positive results and therefore q_4, q_5, q_6, q_7 in opposite direction)



? can be solved

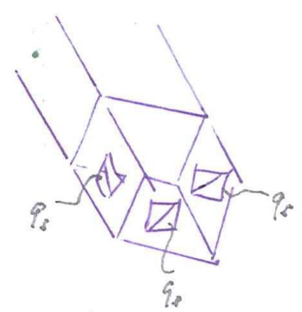
$\Rightarrow \sum F_{hor}\ 0 = q_s b - q_2 b \Rightarrow q_2 = q_s$

$\sum M_{②}: 0 = -q_s b \cdot h + q_3 \cdot h \cdot b \Rightarrow q_3 = q_s$

$\sum F_{ver}: 0 = -q_1 h + q_3 \cdot h \Rightarrow q_1 = q_3 = q_s$

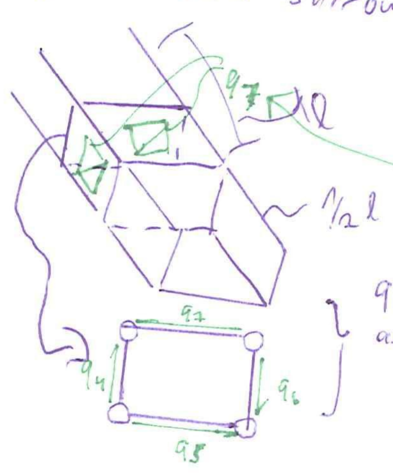
$\left.\right\}\ q_1 = q_2 = q_3 = q_s$

→



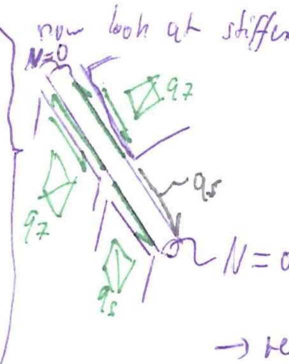
⇒ Shear flows at cut out are known

→ proceed with surrounding structure:



$q_4 = q_5 = q_6 = q_7$ as ?, etc. are the same

now look at stiffener ①:



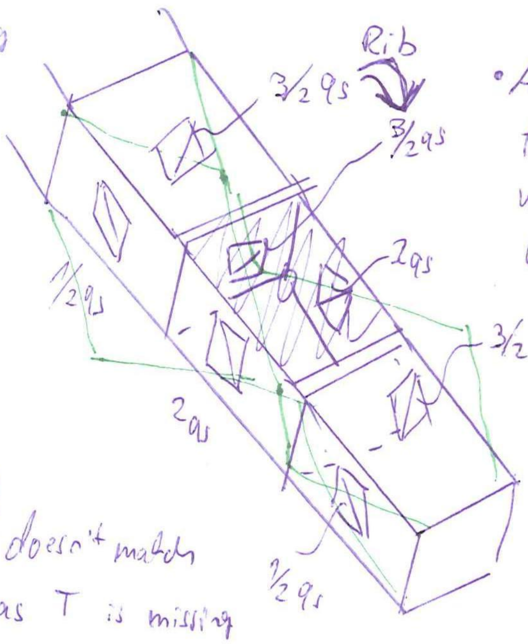
$N = 0$ (symmetry)

$\sum F_{①}\ 0 = -q_7 \cdot L - q_7 \cdot L + q_s \cdot \frac{1}{2} L + q_s \frac{1}{2} L$

$\Rightarrow q_7 = \frac{1}{2} q_s$

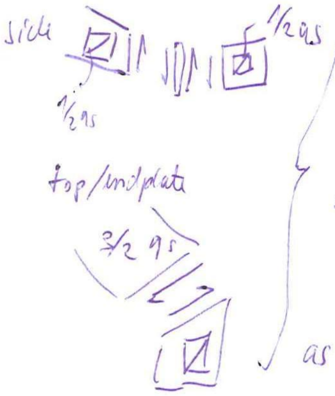
→ reinsert and one has the shear flows

▨ ~ panel missing



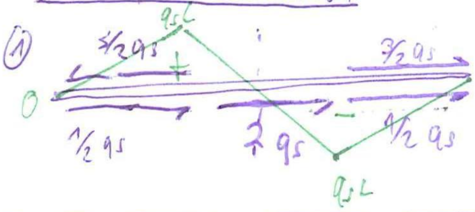
- Adding the shear flows from the undisturbed beam with the disturbed/corrected beam leads to the final result!
- Rib loading follows from gaps (One side of rib plate is loaded $q = \frac{3}{2} q_s$
  → other side missing $q = 0$
  ⇒ rib takes load. $q = \frac{3}{2} q_s$
  → result fits side wall differences $2 q_s - \frac{1}{2} q_s = \frac{3}{2} q_s$)

endplate:



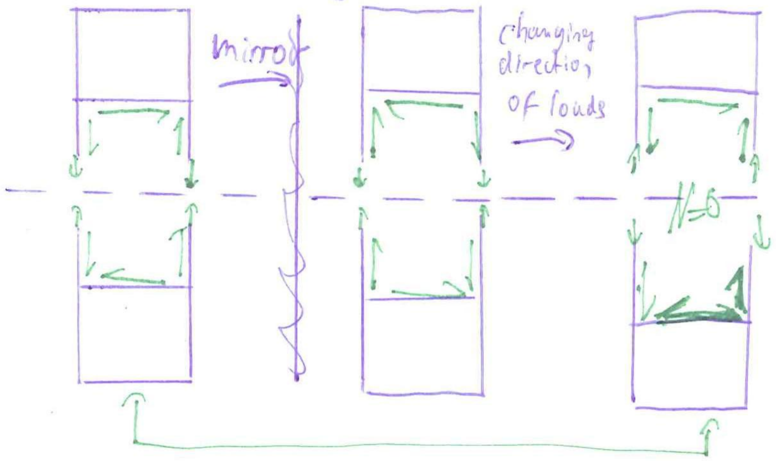
doesn't match as T is missing
⇒ skip calculation of ? endrib as not enough is known...

Normal force distribution:



Beam 1

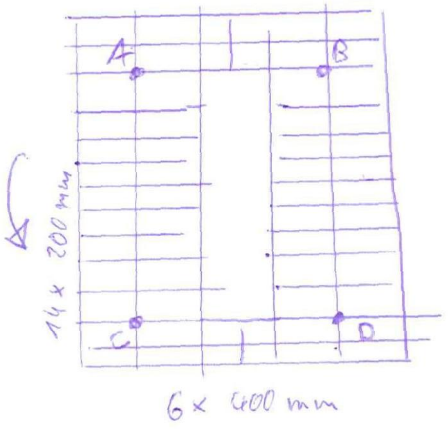
Proof for symmetry argument (→ only half of the structure is calculated):

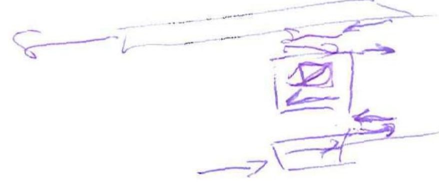


06.06.07

Structures II

Example (not finished → ...)





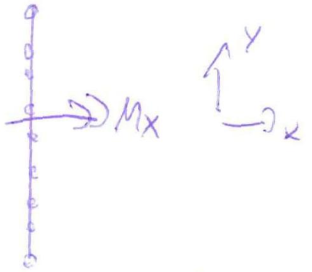
$M = 2.24 \cdot 10^8$ Nmm

1. What are the shear flows in the plates of the undisturbed section?

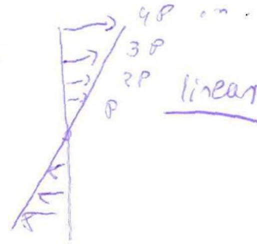
   0 → because there are no shear loads (formula)

2. N:

   $\sigma = \frac{M_x}{I_{xx}} y \ldots$



→ N:

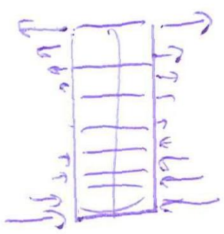


linear

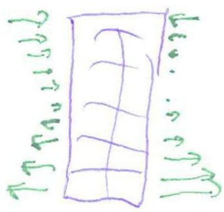
3rd: remove 20 plates and relevant stiffeners

- Assume disruption cannot be "felt" outside of ABCD

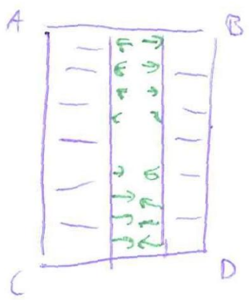
Cutout:



Introduce disruption:



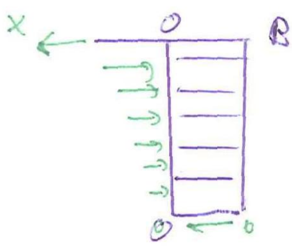
→ put disruption on the rest of the structure:



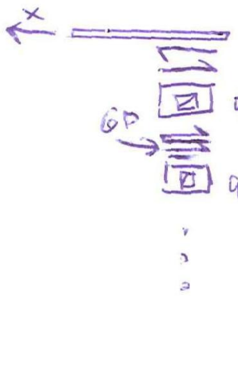
Symmetric → only calculate one half!

point symmetry

→ calculate 1/4



↳ symmetry and point symmetrical loading



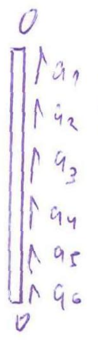
$x + q_1 \cdot 400 = 0$

$\Rightarrow q_1 = \frac{-x}{400}$

$6P + q_1 \cdot 400 - q_2 \cdot 400 = 0$

$\Rightarrow q_2$

⋮



$\left(\sum_{i=0}^{n} q_i\right) \cdot 200 = 0$