Solutions to Chapter 27 Problems

S.27.1

The position of the shear centre, S, is given and is also obvious by inspection (see Fig. S.27.1(a)). Initially, then, the swept area, $2A_{R,0}$ (see Section 27.2) is determined as a function of s. In 12, $2A_{R,0} = 2sd/2 = sd$. Hence, at 2, $2A_{R,0} = d^2$. In 23, $2A_{R,0} = 2(s/2)(d/2) + d^2 = sd/2 + d^2$. Therefore at 3, $2A_{R,0} = 3d^2/2$. In 34, $2A_{R,0}$ remains constant since $p = 0$. The remaining distribution follows from antisymmetry and the complete distribution is shown in Fig. S.27.1(b). The centre of gravity of the 'wire' $1'2'3'4'5'6'$ (i.e. $2A'_R$) is found by taking moments about the s axis. Thus

$$2A'_R 5\mathrm{d}t = \mathrm{d}t\left(\frac{d^2}{2} + \frac{5d^2}{4} + \frac{3d^2}{2} + \frac{5d^2}{4} + \frac{d^2}{2}\right)$$

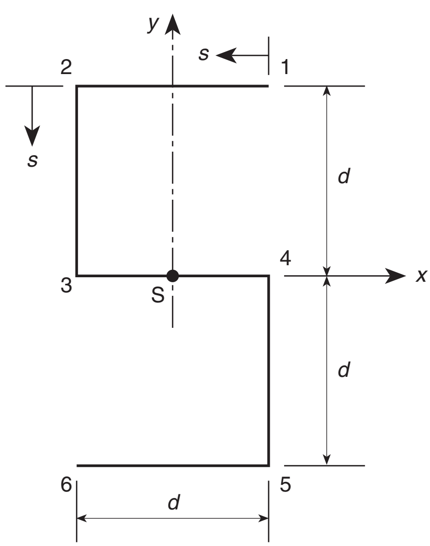

Fig. S.27.1(a)

which gives $2A'_R = d^2$. Therefore, instead of using Eq. (27.9), the moment of inertia of the wire (i.e. Γ_R) may be found directly, i.e.

$$\Gamma_R = 2\mathrm{d}t\frac{(d^2)^2}{3} + 2\mathrm{d}t\frac{\left(\frac{d^2}{2}\right)^2}{3} + \mathrm{d}t\left(\frac{d^2}{2}\right)^2$$

which gives

$$\Gamma_R = \frac{13d^5t}{12}$$

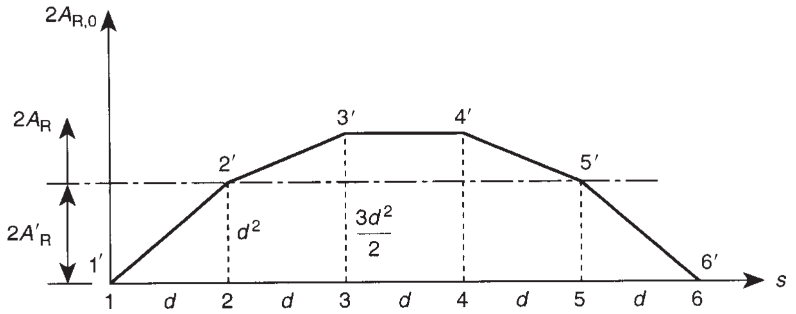

Fig. S.27.1(b)

S.27.2

By inspection the shear centre, S, lies at the mid-point of the wall 34 (Fig. S.27.2(a)). The swept area, $2A_{R,0}$, is then determined as follows. In 12, $2A_{R,0} = (2sa \sin 2\alpha)/2$, i.e. $2A_{R,0} = a^2 \sin 2\alpha$. In 23, $2A_{R,0} = 2 \times \frac{1}{2} sa \sin 2\alpha + a^2 \sin 2\alpha = (sa + a^2) \sin 2\alpha$ and at 3, $2A_{R,0} = 2a^2 \sin 2\alpha$.

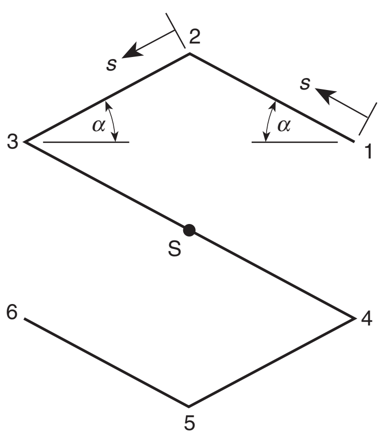

Fig. S.27.2(a)

In 34 there is no contribution to $2A_{R,0}$ since $p = 0$. The remaining distribution follows from anti-symmetry and the complete distribution is shown in Fig. S.27.2(b).

The centre of gravity of the 'wire' $1'2'3'4'5'6'$ (i.e. $2A'_R$) is found by taking moments about the s axis. Thus

$$2A'_R 6at = at(2 \times 2a^2 \sin 2a + 2 \times 2a^2 \sin 2\alpha)$$

i.e.

$$2A'_R = \tfrac{4}{3} a^2 \sin 2\alpha$$

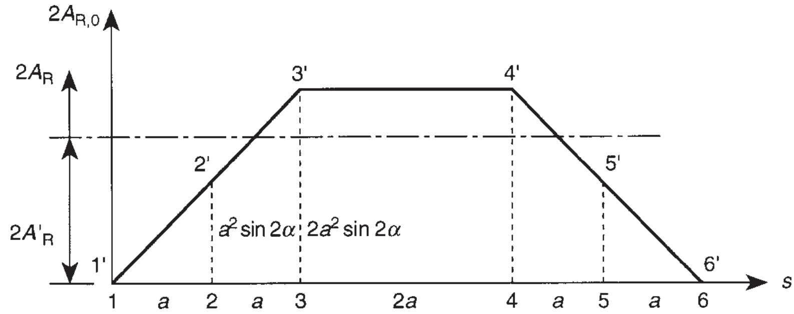

Fig. S.27.2(b)

Then, from Eq. (27.9)

$$\Gamma_R = 2 \times 2at\frac{(2a^2 \sin 2\alpha)^2}{3} + 2at(2a^2 \sin 2\alpha)^2 - \left(\frac{4}{3}a^2 \sin 2\alpha\right)^2 6at$$

which gives

$$\Gamma_R = \tfrac{8}{3}a^5 t \sin^2 2\alpha$$

S.27.3

The shear centre, S, of the section is at a distance $\pi r/3$ above the horizontal through the centers of the semicircular arcs (see P.17.3). Consider the left-hand portion of the section in Fig. S.27.3(a).

$$\begin{aligned}2A_{R,0} &= -2(\text{Area BCS} - \text{Area BSO})\\ &= -2(\text{Area CSF} + \text{Area CFOD} + \text{Area BCD} - \text{Area BSO})\end{aligned}$$

i.e.

$$2A_{R,0} = -2\left[\frac{1}{2}(r\cos\theta_1 + r)\left(\frac{\pi r}{3} - r\sin\theta_1\right) + \frac{1}{2}(2r + r\cos\theta_1)r\sin\theta_1 + \frac{1}{2}r^2\theta_1 - \frac{1}{2}2r\frac{\pi r}{3}\right]$$

i.e.

$$2A_{R,0} = r^2\left(\frac{\pi}{3} - \theta_1 - \sin\theta_1 - \frac{\pi}{3}\cos\theta_1\right) \qquad \text{(i)}$$

When $\theta_1 = \pi$, $2A_{R,0} = -\pi r^2/3$.

Note that in Eq. (i) $A_{R,0}$ is negative for the tangent in the position shown.

Consider now the right-hand portion of the section shown in Fig. S.27.3(b). The swept area $2A_{R,0}$ is given by

$$2A_{R,0} = 2\text{ Area OSJ} - \pi r^2/3$$

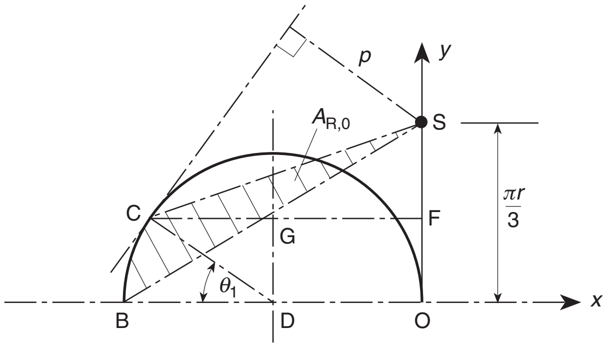

Fig. S.27.3(a)

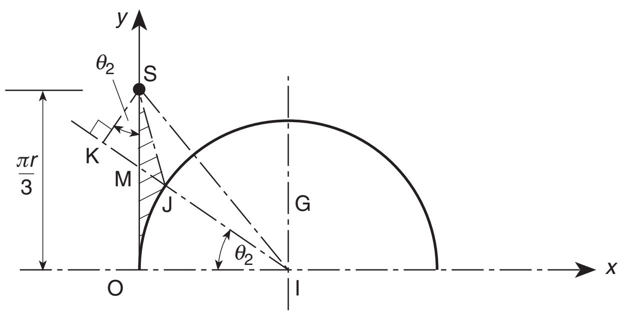

Fig. S.27.3(b)

i.e.

$$2A_{\mathrm{R},0} = 2(\text{Area OSJ} - \text{Area OJI} - \text{Area SJI}) - \pi r^2/3$$

which gives

$$2A_{\mathrm{R},0} = 2\left[\frac{1}{2}r\frac{\pi r}{3} - \frac{1}{2}r^2\theta_2 - \frac{1}{2}r\mathrm{KS}\right] - \frac{\pi r^2}{3} \tag{ii}$$

In Eq. (ii)

$$\mathrm{KS} = \mathrm{MS}\cos\theta_2 = \left(\frac{\pi r}{3} - r\tan\theta_2\right)\cos\theta_2$$

i.e.

$$\mathrm{KS} = \frac{\pi r}{3}\cos\theta_2 - r\sin\theta_2$$

Substituting in Eq. (ii) gives

$$2A_{\mathrm{R},0} = r^2\left(\sin\theta_2 - \theta_2 - \frac{\pi}{3}\cos\theta_2\right) \tag{iii}$$

In Eq. (27.3)

$$\frac{\int_C 2A_{R,0}t\,ds}{\int_C t\,ds} = \frac{1}{2\pi r}\left[\int_0^{\pi} r^3\left(\frac{\pi}{3} - \theta_1 - \sin\theta_1 - \frac{\pi}{3}\cos\theta_1\right)d\theta_1 + \int_0^{\pi} r^3\left(\sin\theta_2 - \theta_2 - \frac{\pi}{3}\cos\theta_2\right)d\theta_2\right]$$

i.e.

$$\frac{\int_C 2A_{R,0}t\,ds}{\int_C t\,ds} = -\frac{\pi r^2}{3}$$

Hence, Eq. (27.3) becomes

$$2A_R = 2A_{R,0} + \frac{\pi r^2}{3}$$

Then

$$\Gamma_R = \int_C (2A_R)^2 t\,ds = \int_0^{\pi} r^4\left(\frac{\pi}{3} - \theta_1 - \sin\theta_1 - \frac{\pi}{3}\cos\theta_1 + \frac{\pi}{3}\right)^2 d\theta_1 + \int_0^{\pi} r^4\left(\sin\theta_2 - \theta_2 - \frac{\pi}{3}\cos\theta_2 + \frac{\pi}{3}\right)^2 d\theta_2$$

which gives

$$\Gamma_R = \pi^2 r^5 t\left(\frac{\pi}{3} - \frac{3}{\pi}\right)$$

S.27.4

The applied loading is equivalent to a shear load, P, through the shear centre (the centre of symmetry) of the beam section together with a torque $T = -Ph/2$. The direct stress distribution at the built-in end of the beam is then, from Eqs (16.21) and (27.1)

$$\sigma = \frac{M_x}{I_{xx}}y - 2A_R E\frac{d^2\theta}{dz^2} \tag{i}$$

In Eq. (i)

$$M_x = Pl \tag{ii}$$

and

$$I_{xx} = 2td^3/12 = td^3/6 \tag{iii}$$

Also $d^2\theta/dz^2$ is obtained from Eq. (27.6), i.e.

$$T = GJ\frac{d\theta}{dz} - E\Gamma_R\frac{d^3\theta}{dz^3}$$

or, rearranging

$$\frac{d^3\theta}{dz^3} - \mu^2 \frac{d\theta}{dz} = -\mu^2 \frac{T}{GJ} \tag{iv}$$

in which $\mu^2 = GJ/E\Gamma_R$. The solution of Eq. (iv) is

$$\frac{d\theta}{dz} = C \cosh \mu z + D \sinh \mu z + \frac{T}{GJ} \tag{v}$$

At the built-in end the warping is zero so that, from Eq. (18.19) $d\theta/dz = 0$ at the built-in end. Thus, from Eq. (v), $C = -T/GJ$. At the free end the direct stress, σ_Γ, is zero so that, from Eq. (27.1), $d^2\theta/dz^2 = 0$ at the free end. Then, from Eq. (v)

$$D = \left(\frac{T}{GJ}\right) \tanh \mu l$$

and Eq. (iii) becomes

$$\frac{d\theta}{dz} = \frac{T}{GJ}\left[1 - \frac{\cosh \mu(l - z)}{\cosh \mu l}\right] \tag{vi}$$

Differentiating Eq. (vi) with respect to z gives

$$\frac{d^2\theta}{dz^2} = \frac{T}{GJ}\mu \frac{\sinh \mu(l - z)}{\cosh \mu l} \tag{vii}$$

Hence, from Eq. (27.1)

$$\sigma_\Gamma = -2A_R E \frac{T}{GJ}\mu \frac{\sinh \mu(l - z)}{\cosh \mu l}$$

which, at the built-in end becomes

$$\sigma_\Gamma = -\sqrt{\frac{E}{GJ\Gamma_R}}\, T 2A_R \tanh \mu l \tag{viii}$$

In Eq. (viii)

$$J = (h + 2d)t^3/3 \quad \text{(see Eq. (18.11))} \tag{ix}$$

The torsion bending constant, Γ_R, is found using the method described in Section 27.2. Thus, referring to Fig. S.27.4(a), in 12, $2A_{R,0} = sh/2$ and at 2, $2A_{R,0} = hd/4$. Also, at 3, $2A_{R,0} = hd/2$. Between 2 and 4, $2A_{R,0}$ remains constant and equal to $hd/4$. At 5, $2A_{R,0} = hd/4 + hd/4 = hd/2$ and at 6, $2A_{R,0} = hd/4 - hd/4 = 0$. The complete distribution is shown in Fig. S.27.4(b). By inspection $2A'_R = hd/4$. Then

$$\Gamma_R = 4t\frac{d}{2}\frac{1}{3}\left(\frac{hd}{4}\right)^2$$

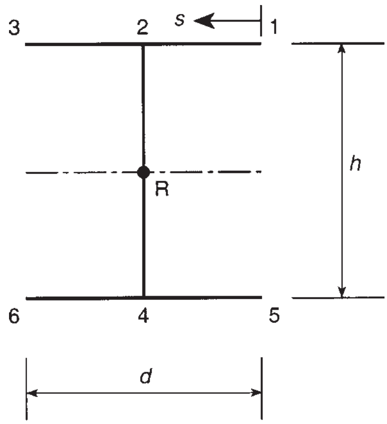

Fig. S.27.4(a)

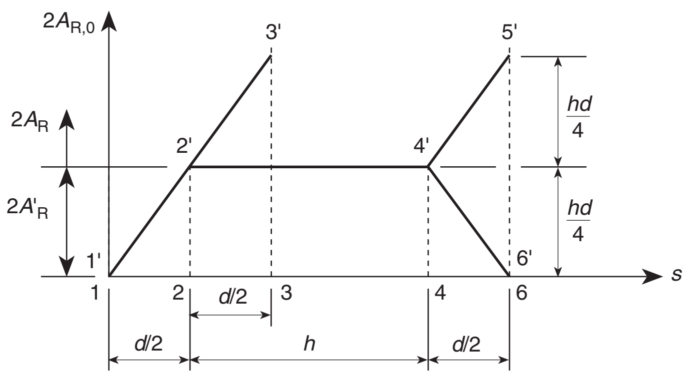

Fig. S.27.4(b)

i.e.

$$\Gamma_R = \frac{td^3h^2}{24} \tag{x}$$

Substituting the given values in Eqs (ii), (iii), (ix) and (x) gives

$$M_x = 200 \times 375 = 75\,000\,\text{N mm}$$

$$I_{xx} = 2.5 \times 37.5^3/6 = 21\,973.0\,\text{mm}^4$$

$$J = (75 + 2 \times 37.5)2.5^3/3 = 781.3\,\text{mm}^4$$

$$\Gamma_R = 2.5 \times 37.5^3 \times 75^2/24 = 3.09 \times 10^7\,\text{mm}^6$$

Then

$$\mu^2 = 781.3/(2.6 \times 3.09 \times 10^7) = 9.72 \times 10^{-6}$$

and

$$\mu = 3.12 \times 10^{-3}$$

Thus from Eqs (i) and (viii)

$$\sigma = 3.41y + 0.064(2A_R) \tag{xi}$$

Then, at 1 where $y = -d/2 = -18.75$ mm and $2A_R = -hd/4 = -703.1\ \text{mm}^2$,

$$\sigma_1 = -108.9\ \text{N/mm}^2 = -\sigma_3$$

Similarly

$$\sigma_5 = -18.9\ \text{N/mm}^2 = -\sigma_6$$

and

$$\sigma_2 = \sigma_4 = \sigma_{24} = 0$$

S.27.5

The rate of twist in each half of the beam is obtained from the solution of Eq. (27.6). Thus, referring to Fig. S.27.5, for BC

$$\frac{\mathrm{d}\theta}{\mathrm{d}z_1} = \frac{T}{8GJ} + A\cosh 2\mu z_1 + B\sinh 2\mu z_1 \tag{i}$$

where $\mu^2 = GJ/E\Gamma$ and for BA

$$\frac{\mathrm{d}\theta}{\mathrm{d}z_2} = \frac{T}{GJ} + C\cosh \mu z_2 + D\sinh \mu z_2 \tag{ii}$$

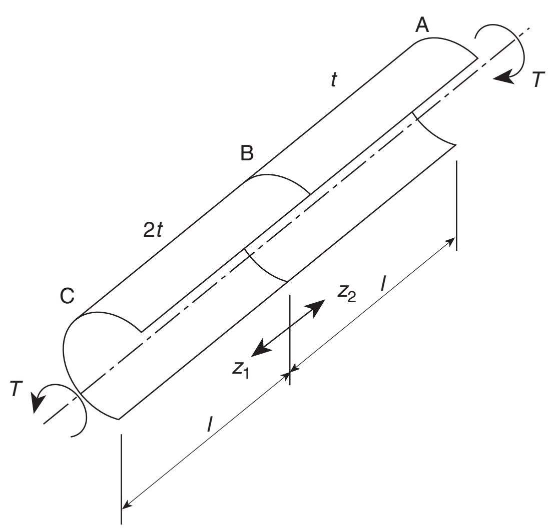

Fig. S.27.5

The boundary conditions are as follows:
When

$$z_1 = z_2 = 0, \quad \mathrm{d}\theta/\mathrm{d}z_1 = \mathrm{d}\theta/\mathrm{d}z_2 \tag{iii}$$

When

$$z_1 = z_2 = l, \quad \mathrm{d}^2\theta/\mathrm{d}z_1^2 = \mathrm{d}^2\theta/\mathrm{d}z_2^2 = 0 \quad \text{(see Eq. (27.1))} \tag{iv}$$

When

$$z_1 = z_2 = 0, \quad 2\mathrm{d}^2\theta/\mathrm{d}z_1^2 = -\mathrm{d}^2\theta/\mathrm{d}z_2^2 \tag{v}$$

(since the loads at B in each half of the section are equal and opposite). From Eqs (i), (ii) and (iv)

$$B = -A \tanh 2\mu l \tag{vi}$$

$$D = -C \tanh \mu l \tag{vii}$$

From Eqs (i)–(iii)

$$\frac{T}{8GJ} + A = \frac{T}{GJ} + C$$

i.e.

$$A - C = \frac{7T}{8GJ} \tag{viii}$$

From Eqs (i), (ii) and (v)

$$D = -4B \tag{ix}$$

Solving Eqs (vi)–(ix) gives

$$B = -\frac{7T \tanh \mu l \tanh 2\mu l}{8GJ(4 \tanh 2\mu l + \tanh \mu l)}$$

$$D = \frac{7T(4 \tanh \mu l \tanh 2\mu l)}{8GJ(4 \tanh 2\mu l + \tanh \mu l)}$$

$$A = \frac{7T \tanh \mu l}{8GJ(4 \tanh 2\mu l + \tanh \mu l)}$$

$$C = -\frac{7T(4 \tanh 2\mu l)}{8GJ(4 \tanh 2\mu l + \tanh \mu l)}$$

Integrating Eq. (i)

$$\theta_1 = \frac{T}{8GJ} z_1 + \frac{A}{2\mu} \sinh 2\mu z_1 + \frac{B}{2\mu} \cosh 2\mu z_1 + F$$

When $z_1 = 0$, $\theta_1 = 0$ so that $F = -B/2\mu$. Integrating Eq. (ii)

$$\theta_2 = \frac{T}{GJ} z_2 + \frac{C}{\mu} \sinh \mu z_2 + \frac{D}{\mu} \cosh \mu z_2 + H$$

When $z_2 = 0$, $\theta_2 = 0$ so that $H = -D/\mu$. Hence, when $z_1 = l$ and $z_2 = l$ the angle of twist of one end of the beam relative to the other is

$$\theta_1 + \theta_2 = \frac{T}{8GJ}(l + 8l) + \frac{7T}{8GJ\mu(4 \tanh 2\mu l + \tanh \mu l)} \times \left[\tfrac{1}{2}(\tanh \mu l \sinh 2\mu l - \tanh \mu l \tanh 2\mu l \cosh 2\mu l - 4 \tanh 2\mu l \sinh \mu l + 4 \tanh \mu l \tanh 2\mu l \cosh \mu l - \tfrac{7}{2}(\tanh \mu l \tanh 2\mu l)\right]$$

which simplifies to

$$\theta_1 + \theta_2 = \frac{Tl}{8GJ}\left[9 - \frac{49 \sinh 2\mu l}{2\mu l(10 \cosh^2 \mu l - 1)}\right]$$

S.27.6

Initially the swept area $2A_{R,0}$ is plotted round the section and is shown in Fig. S.27.6(b).

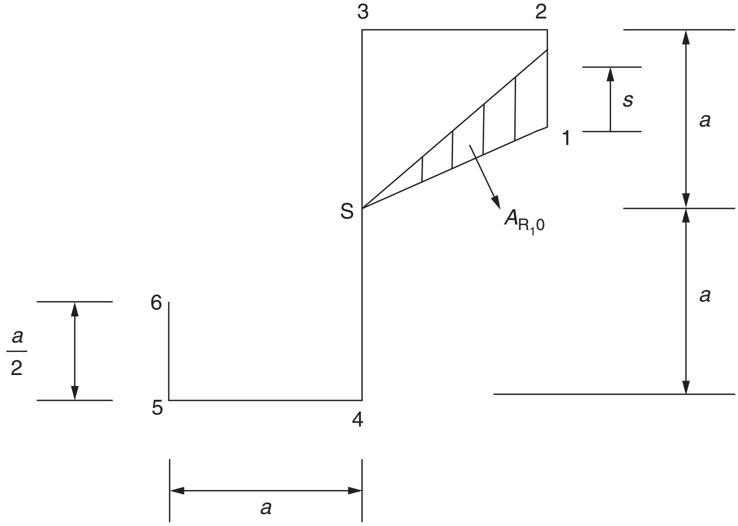

Fig. S.27.6(a)

Then, using the 'wire' analogy and taking moments about the s axis

$$2A'_R 5at = 2\frac{3a}{2}t\left(\frac{3a^2}{4}\right) + 2at\left(\frac{3a^2}{2}\right)$$

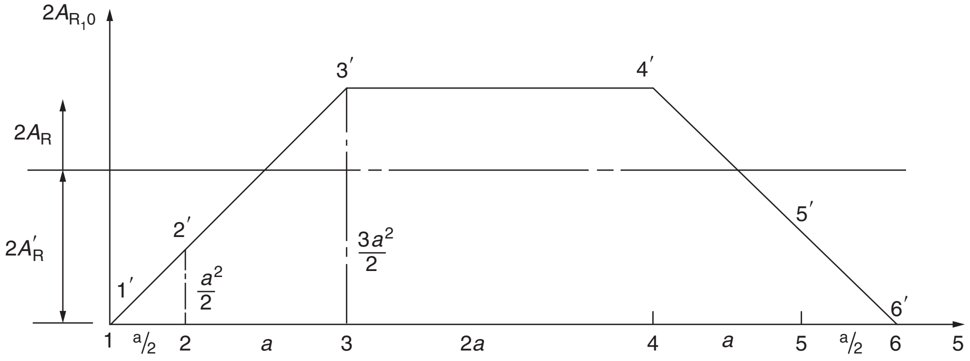

Fig. S.27.6(b)

which gives

$$2A'_R = \frac{21a^2}{20}$$

Then

$$\Gamma_R = 2\frac{3a}{2}t\frac{1}{3}\left(\frac{3a^2}{2}\right)^2 + 2at\left(\frac{3a^2}{2}\right)^2 - 5at\left(\frac{21a^2}{20}\right)^2$$

i.e.

$$\Gamma_R = 1.25a^5 t$$

From Eq. (27.6), i.e.

$$\frac{d^3\theta}{dz^3} - \mu^2\frac{d\theta}{dz} = -\mu^2\frac{T}{GJ}$$

where

$$\mu^2 = \frac{GJ}{E\Gamma_R}$$

$$\frac{d\theta}{dz} = C\cosh\mu z + D\sinh\mu z + \frac{T}{GJ}$$

When $z = 0$, the warping, w, is zero so that $d\theta/dz = 0$ (see Eq. (18.19)), then

$$A = -\frac{T}{GJ}$$

When $z = L$, the direct stress is zero. Therefore, from Eq. (27.1) $d^2\theta/dz^2 = 0$. Therefore

$$B = \frac{T}{GJ}\tanh\mu L$$

so that the rate of twist is

$$\frac{\mathrm{d}\theta}{\mathrm{d}z} = \frac{T}{GJ}\left[1 - \frac{\cosh\mu(L-z)}{\cosh\mu L}\right]$$

and

$$\theta = \frac{T}{GJ}\left[z + \frac{\sinh\mu(L-z)}{\mu\cosh\mu L} + C\right]$$

When $z = 0$, $\theta = 0$ which gives

$$C = -\frac{1}{\mu}\tanh\mu L$$

and

$$\theta = \frac{T}{GJ}\left[z + \frac{\sinh\mu(L-z)}{\mu\cosh\mu L} - \frac{\tanh\mu L}{\mu}\right]$$

At the free end when $z = L$

$$\theta_T = \frac{TL}{GJ}\left(1 - \frac{\tanh\mu L}{\mu L}\right) \tag{i}$$

Inserting the given values in Eq. (i)

$$T = 100 \times 30 = 3000\,\mathrm{N\,mm} \quad J = 5 \times 30 \times \frac{2.0^3}{3} = 400\,\mathrm{mm}^4$$

$$\mu^2 = 2.35 \times 10^{-6} \quad \mu L = 1.53 \quad \theta_T = 6.93^\circ$$

S.27.7

The torsion bending constant is identical to that in S.27.4, i.e.

$$\Gamma_{\mathrm{R}} = \frac{th^2\mathrm{d}^3}{24}$$

The expression for rate of twist is (see S.27.6)

$$\frac{\mathrm{d}\theta}{\mathrm{d}z} = A\cosh\mu z + B\sinh\mu z + \frac{T}{GJ}$$

In AB, $T = 0$ and $\mathrm{d}\theta/\mathrm{d}z = 0$ at $z = 0$ which gives $A = 0$

Therefore, in AB

$$\frac{\mathrm{d}\theta}{\mathrm{d}z} = B\sinh\mu z$$

In BC

$$\frac{\mathrm{d}\theta}{\mathrm{d}z} = [1 - \alpha\cosh\mu(z-L) - \beta\sinh\mu(z-L)] + \beta\sinh\mu z$$

where [ ] is a Macauley bracket
i.e.

$$[\,] = 0 \text{ for } z < L$$
$$= (\,) \text{ ordinary bracket for } z > L$$

For continuity of $\mathrm{d}\theta/\mathrm{d}z$ and $\mathrm{d}^2\theta/\mathrm{d}z^2$ at $z = L$ the Macauley bracket and its first derivative must be zero at $z = L$. Then

$$1 - \alpha = 0 \text{ and } \beta = 0$$

For the complete beam

$$\frac{\mathrm{d}\theta}{\mathrm{d}z} = \frac{T}{GJ}[1 - \cosh \mu(z - L)] + B \sinh \mu z$$

At

$$z = 2L \quad \mathrm{d}^2\theta/\mathrm{d}z^2 = 0 \quad (\sigma_\Gamma = 0 \text{ at } z = 2L).$$

Then

$$\theta = -\frac{T}{GJ}\mu \sinh \mu L + \mu B \cosh 2\mu L$$

which gives

$$B = \frac{T}{GJ}\frac{\sinh \mu L}{\cosh 2\mu L}$$

Then

$$\frac{\mathrm{d}\theta}{\mathrm{d}z} = \frac{T}{GJ}\left\{[1 - \cosh \mu(z - L)] + \frac{\sinh \mu L}{\cosh 2\mu L}\sinh \mu z\right\}$$

Also since $\theta = 0$ at $z = 0$ and the Macauley bracket is zero for $z < L$

$$\theta = \frac{T}{GJ}\left\{\left[z - L - \frac{1}{\mu}\sinh \mu(z - L)\right] + \frac{\sinh \mu L}{\cosh 2\mu L}(\cosh \mu z - 1)\right\}$$

At $z = 2L$

$$\theta_T = \frac{T}{GJ}\left(L - \frac{\sinh \mu L}{\mu L \cosh 2\mu L}\right)$$

S.27.8

The variation of swept area is shown in Fig. S.27.8(b)
Using the 'wire' analogy

$$2A'_R 4at = at\frac{a^2}{2} + 2at\frac{5}{8}a^2 + at\frac{3}{4}a^2$$

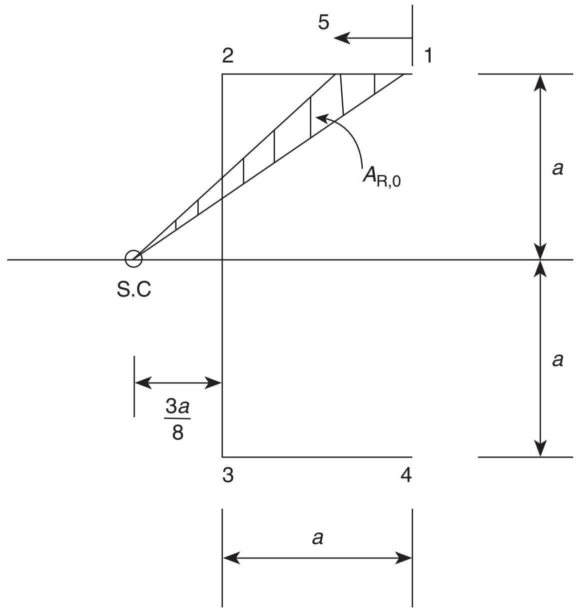

Fig. S.27.8(a)

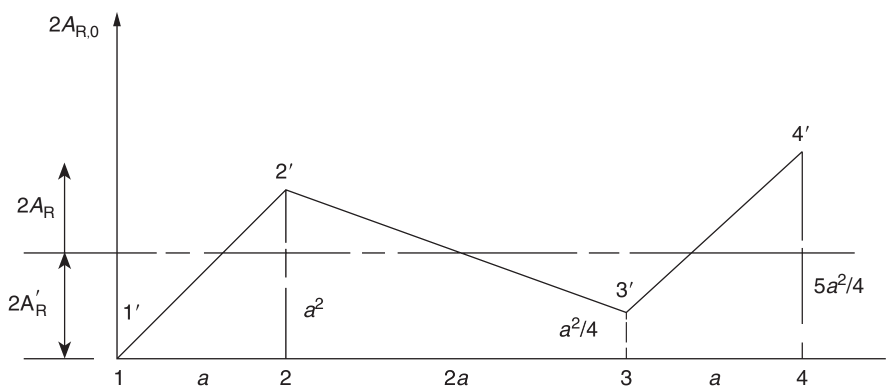

Fig. S.27.8(b)

i.e.

$$2A_R' = \frac{5a^2}{8}$$

Then

$$\Gamma_R = at\frac{1}{3}(a^2)^2 + 2at\left[\frac{1}{3}\left(\frac{3}{8}a^2\right)^2 + \left(\frac{5}{8}a^2\right)^2\right] + at\left[\frac{1}{3}\left(\frac{a^2}{2}\right)^2 + \left(\frac{3}{4}a^2\right)^2\right]$$
$$- 4at\left(\frac{5a^2}{8}\right)^2$$

which gives

$$\Gamma_\mathrm{R} = \frac{7a^5t}{24}$$

The rate of twist is identical to that given by Eq. (vi) in S.27.4, i.e.

$$\frac{\mathrm{d}\theta}{\mathrm{d}z} = \frac{T}{GJ}\left[1 - \frac{\cosh\mu(L-z)}{\cosh\mu L}\right] \tag{i}$$

The direct stress distribution at the built-in end is, from Eq. (ix) of Example 27.1

$$\sigma_\Gamma = -\sqrt{\frac{E}{GJ\Gamma_\mathrm{R}}}T\ 2A_\mathrm{R}\frac{\sinh\mu L}{\cosh\mu L}$$

Evaluating the different constants

$$\Gamma_\mathrm{R} = 9.33\times10^5\,\mathrm{mm}^6 \quad J = 26.67\,\mathrm{mm}^4 \quad T = 1125\,\mathrm{N\,mm}$$
$$\mu^2 = 8.56\times10^{-6} \quad \text{and} \quad \mu L = 1.46$$

Then

$$\sigma_\Gamma = -0.369\ 2A_\mathrm{R}$$

At 2,

$$2A_\mathrm{R} = a^2 - \frac{5a^2}{8} = \frac{3a^2}{8} = \frac{3\times20^2}{8} = 150\,\mathrm{mm}^2$$

so that

$$\sigma_{\Gamma,2} = -55.3\,\mathrm{N/mm}^2$$

The direct stress due to elementary bending theory is, from Eqs (16.21)

$$\sigma_z = \frac{M_x y}{I_{xx}}$$

where

$$M_x = -150\times500 = -75\,000\,\mathrm{N\,mm}$$

and

$$I_{xx} = 2\times1.0\times20\times20^2 + \frac{1.0\times40^3}{12} = 21.3\times10^3\,\mathrm{mm}^4$$

Then

$$\sigma_{z,2} = -\frac{75\,000\times20}{21.3\times10^3} = -70.4\,\mathrm{N/mm}^2$$

The total direct stress at 2 is therefore

$$\sigma_2 = -55.3 - 70.4 = -125.7\,\mathrm{N/mm}^2$$

S.27.9

The torsion bending constant is identical to that in S.27.4, i.e.

$$\Gamma_R = \frac{th^2 d^3}{24}$$

The rate of twist is, from Eq. (27.6)

$$\frac{d\theta}{dz} = A \cosh \mu z + B \sinh \mu z + \frac{wh}{2GJ}(L - z)$$

when $z = 0$, $d\theta/dz = 0$ ($w = 0$ at $z = 0$) which gives

$$A = -\frac{whL}{2GJ}$$

When $z = L$, $d^2\theta/dz^2 = 0$ ($\sigma_\Gamma = 0$ at $z = L$) which gives

$$B = \frac{wh}{2GJ}\left[L \tanh \mu L + \frac{1}{\mu \cosh \mu L}\right]$$

Hence

$$\frac{d\theta}{dz} = \frac{wh}{2GJ}\left[-L \cosh \mu z + \left(\frac{\mu L \sinh \mu L + 1}{\mu \cosh \mu L}\right) \sinh \mu z + L - z\right]$$

Then

$$\sigma_\Gamma = -2A_R E \frac{d^2\theta}{dz^2}$$

is

$$\sigma_\Gamma = -2A_R E \frac{wh}{2GJ}\left[-\mu L \sinh \mu z + \left(\frac{\mu L \sinh \mu L + 1}{\cosh \mu L}\right) \cosh \mu z - 1\right]$$

At the built-in end when $z = 0$

$$\sigma_\Gamma = -2A_R E \frac{wh}{2GJ}\left[\frac{\mu L \sinh \mu L + 1 - \cosh \mu L}{\cosh \mu L}\right]$$

Evaluating the constants

$$\Gamma_R = 1040 \times 10^6\,\text{mm}^6, \quad J = 12\,500\,\text{mm}^4, \quad \mu^2 = 4.0 \times 10^{-6}, \quad \mu L = 3.0.$$

Then

$$\sigma_\Gamma = -0.025(2A_R)$$

The distribution of $2A_R$ is linear round the section so that σ_Γ is also linear.

At 1,

$$2A_R = -\frac{hd}{4} \quad \text{(see S.27.4)}$$

Then

$$\sigma_{\Gamma,1} = +0.025 \times 200 \times \frac{50}{4} = +62.5\,\text{N/mm}^2.$$

From symmetry of the $2A_R$ distribution

$$\sigma_{\Gamma,3} = -\sigma_{\Gamma,1} = -\sigma_{\Gamma,4} = \sigma_{\Gamma,6} = -62.5\,\text{N/mm}^2,$$
$$\sigma_{\Gamma,2} = \sigma_{\Gamma,5} = 0$$

From Eqs (16.21)

$$\sigma_z = \frac{M_y x}{I_{yy}}$$

where

$$M_y = 0.5 \times \frac{1500^2}{2} = 562\,500\,\text{N mm}$$

and

$$I_{yy} = 2 \times 5 \times \frac{50^3}{12} = 104\,200\,\text{mm}^4$$

Then

$$\sigma_z = 5.4\text{y},$$

i.e.

$$\sigma_{z,1} = +135\,\text{N/mm}^2$$

From symmetry

$$\sigma_{\text{z},1} = -\sigma_{\text{z},3} = -\sigma_{\text{z},4} = \sigma_{\text{z},6} = +135\,\text{N/mm}^2$$
$$\sigma_{\text{z},2} = \sigma_{\text{z},5} = 0$$

The complete direct stresses are

$$\sigma_1 = 62.5 + 135 = +197.5\,\text{N/mm}^2 = -\sigma_3$$
$$\sigma_4 = 62.5 - 135 = -72.5\,\text{N/mm}^2 = -\sigma_6$$
$$\sigma_2 = \sigma_5 = 0$$