from which

$$q_{s,0} = -27.0\,\text{N/mm} \quad \text{(clockwise)}$$

Then

$$q_{89} = 27.0\,\text{N/mm}, \quad q_{9\,10} = q_{78} = 22.5\,\text{N/mm}, \quad q_{10\,1} = q_{67} = 11.6\,\text{N/mm},$$
$$q_{21} = q_{65} = 1.9\,\text{N/mm}, \quad q_{32} = q_{54} = 12.8\,\text{N/mm}, \quad q_{43} = 17.3\,\text{N/mm}$$

Solutions to Chapter 23 Problems

S.23.1

The beam section is unsymmetrical and $M_x = -120\,000$ Nm, $M_y = -30\,000$ Nm. Therefore, the direct stresses in the booms are given by Eq. (16.18), i.e.

$$\sigma_z = \left(\frac{M_y I_{xx} - M_x I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\right) x + \left(\frac{M_x I_{yy} - M_y I_{xy}}{I_{xx} I_{yy} - I_{xy}^2}\right) y \tag{i}$$

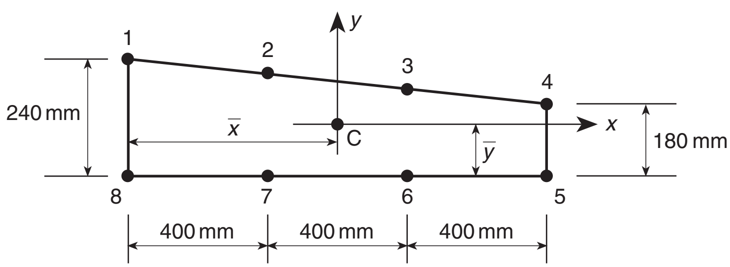

Fig. S.23.1

In Fig. S.23.1 $\bar{x} = 600$ mm by inspection. Also, taking moments of area about the line of the bottom booms

$$(4 \times 1000 + 4 \times 600)\bar{y} = 1000 \times 240 + 1000 \times 180 + 600 \times 220 + 600 \times 200$$

from which

$$\bar{y} = 105\,\text{mm}$$

Then

$$I_{xx} = 2 \times 1000 \times 105^2 + 2 \times 600 \times 105^2 + 1000 \times 135^2 + 1000 \times 75^2 + 600 \times 115^2$$
$$+ 600 \times 95^2 = 72.5 \times 10^6\,\text{mm}^4$$

$$I_{yy} = 4 \times 1000 \times 600^2 + 4 \times 600 \times 200^2 = 1536.0 \times 10^6\,\text{mm}^4$$

$$I_{xy} = 1000[(-600)(135) + (600)(75)] + 600[(-200)(115) + (200)(95)]$$
$$= -38.4 \times 10^6\,\text{mm}^4$$

Table S.23.1

Boom	1	2	3	4	5	6	7	8
x (mm)	−600	−200	200	600	600	200	−200	−600
y (mm)	135	115	95	75	−105	−105	−105	−105
σ_z (N/mm^2)	−190.7	−181.7	−172.8	−163.8	140.0	164.8	189.6	214.4

Note that the sum of the contributions of booms 5, 6, 7 and 8 to I_{xy} is zero. Substituting for M_x, M_y, I_{xx}, etc. in Eq. (i) gives

$$\sigma_z = -0.062x - 1.688y \tag{ii}$$

The solution is completed in Table S.23.1.

S.23.2

From Eq. (23.6) for Cell I

$$\frac{d\theta}{dz} = \frac{1}{2A_I G}[q_I(\delta_{21} + \delta_{16} + \delta_{65} + \delta_{52}) - q_{II}\delta_{52}] \tag{i}$$

and for Cell II

$$\frac{d\theta}{dz} = \frac{1}{2A_{II} G}[-q_I\delta_{52} + q_{II}(\delta_{32} + \delta_{25} + \delta_{54} + \delta_{43})] \tag{ii}$$

In Eqs (i) and (ii)

$$A_I = 7750 + (250 + 600) \times 500/2 = 220\,250\,\text{mm}^2$$

$$A_{II} = 6450 + (150 + 600) \times 920/2 = 351\,450\,\text{mm}^2$$

$$\delta_{21} = (\sqrt{250^2 + 500^2})/1.63 = 343.0 \quad \delta_{16} = 300/2.03 = 147.8$$

$$\delta_{65} = (\sqrt{100^2 + 500^2})/0.92 = 554.2 \quad \delta_{52} = 600/2.54 = 236.2$$

$$\delta_{54} = (\sqrt{250^2 + 920^2})/0.92 = 1036.3 \quad \delta_{43} = 250/0.56 = 446.4$$

$$\delta_{32} = (\sqrt{200^2 + 920^2})/0.92 = 1023.4$$

Substituting these values in Eqs (i) and (ii) gives, for Cell I

$$\frac{d\theta}{dz} = \frac{1}{2 \times 220\,250G}(1281.2q_I - 236.2q_{II}) \tag{iii}$$

and for Cell II

$$\frac{d\theta}{dz} = \frac{1}{2 \times 351\,450G}(-236.2q_I + 2742.3q_{II}) \tag{iv}$$

Equating Eqs (iii) and (iv) gives

$$q_{\mathrm{II}} = 0.73 q_{\mathrm{I}} \tag{v}$$

Then, in Cell I

$$\tau_{\max} = \tau_{65} = \frac{q_{\mathrm{I}}}{0.92} = 1.087 q_{\mathrm{I}}$$

and in Cell II

$$\tau_{\max} = \frac{q_{\mathrm{II}}}{0.56} = 1.304 q_{\mathrm{I}}$$

In the wall 52

$$\tau_{52} = \frac{q_{\mathrm{I}} - q_{\mathrm{II}}}{2.54} = 0.106 q_{\mathrm{I}}$$

Therefore

$$\tau_{\max} = 1.304 q_{\mathrm{I}} = 140\,\mathrm{N/mm^2}$$

which gives

$$q_{\mathrm{I}} = 107.4\,\mathrm{N/mm}$$

and, from Eq. (v)

$$q_{\mathrm{II}} = 78.4\,\mathrm{N/mm}$$

Substituting for q_{I} and q_{II} in Eq. (23.4)

$$T = (2 \times 220\,250 \times 107.4 + 2 \times 351\,450 \times 78.4) \times 10^{-3}$$

i.e.

$$T = 102\,417\,\mathrm{Nm}$$

From Eq. (iii) (or Eq. (iv))

$$\frac{\mathrm{d}\theta}{\mathrm{d}z} = \frac{1}{2 \times 220\,250 \times 26\,600}(1281.2 \times 107.4 - 236.2 \times 78.4)$$

i.e.

$$\frac{\mathrm{d}\theta}{\mathrm{d}z} = 1.02 \times 10^{-5}\,\mathrm{rad/mm}$$

Hence

$$\theta = 1.02 \times 10^{-5} \times 2500 \times \left(\frac{180}{\pi}\right) = 1.46^{\circ}$$

The torsional stiffness is obtained from Eq. (3.12), thus

$$GJ = \frac{T}{(\mathrm{d}\theta/\mathrm{d}z)} = 102\,417 \times 10^{3}/(1.02 \times 10^{-5}) = 10 \times 10^{12}\,\mathrm{Nmm^2/rad}$$

S.23.3

From Eq. (23.6) for Cell I

$$\frac{d\theta}{dz} = \frac{1}{2A_I G}[q_I(\delta_{45^o} + \delta_{45^i}) - q_{II}\delta_{45^i}] \tag{i}$$

For Cell II

$$\frac{d\theta}{dz} = \frac{1}{2A_{II} G}[-q_I\delta_{45^i} + q_{II}(\delta_{34} + \delta_{45^i} + \delta_{56} + \delta_{63}) - q_{III}\delta_{63}] \tag{ii}$$

For Cell III

$$\frac{d\theta}{dz} = \frac{1}{2A_{III} G}[-q_{II}\delta_{63} + q_{III}(\delta_{23} + \delta_{36} + \delta_{67} + \delta_{72}) - q_{IV}\delta_{72}] \tag{iii}$$

For Cell IV

$$\frac{d\theta}{dz} = \frac{1}{2A_{IV} G}[-q_{III}\delta_{72} + q_{IV}(\delta_{27} + \delta_{78} + \delta_{81} + \delta_{12})] \tag{iv}$$

where

$$\delta_{12} = \delta_{78} = 762/0.915 = 832.8 \quad \delta_{23} = \delta_{67} = \delta_{34} = \delta_{56} = 812/0.915 = 887.4$$
$$\delta_{45^i} = 356/1.220 = 291.8 \quad \delta_{45^o} = 1525/0.711 = 2144.9$$
$$\delta_{36} = 406/1.625 = 249.8 \quad \delta_{72} = 356/1.22 = 291.8 \quad \delta_{81} = 254/0.915 = 277.6$$

Substituting these values in Eqs (i)–(iv)

$$\frac{d\theta}{dz} = \frac{1}{2 \times 161\,500G}(2436.7q_I - 291.8q_{II}) \tag{v}$$
$$\frac{d\theta}{dz} = \frac{1}{2 \times 291\,000G}(-291.8q_I + 2316.4q_{II} - 249.8q_{III}) \tag{vi}$$
$$\frac{d\theta}{dz} = \frac{1}{2 \times 291\,000G}(-249.8q_{II} + 2316.4q_{III} - 291.8q_{IV}) \tag{vii}$$
$$\frac{d\theta}{dz} = \frac{1}{2 \times 226\,000G}(-291.8q_{III} + 2235.0q_{IV}) \tag{viii}$$

Also, from Eq. (23.4)

$$T = 2(161\,500q_I + 291\,000q_{II} + 291\,000q_{III} + 226\,000q_{IV}) \tag{ix}$$

Equating Eqs (v) and (vi)

$$q_I - 0.607q_{II} + 0.053q_{III} = 0 \tag{x}$$

Now equating Eqs (v) and (vii)

$$q_I - 0.063q_{II} - 0.528q_{III} + 0.066q_{IV} = 0 \tag{xi}$$

Equating Eqs (v) and (viii)

$$q_{\mathrm{I}} - 0.120q_{\mathrm{II}} + 0.089q_{\mathrm{III}} - 0.655q_{\mathrm{IV}} = 0 \tag{xii}$$

From Eq. (ix)

$$q_{\mathrm{I}} + 1.802q_{\mathrm{II}} + 1.802q_{\mathrm{III}} + 1.399q_{\mathrm{IV}} = 3.096 \times 10^{-6}T \tag{xiii}$$

Subtracting Eq. (xi) from (x)

$$q_{\mathrm{II}} - 1.068q_{\mathrm{III}} + 0.121q_{\mathrm{IV}} = 0 \tag{xiv}$$

Subtracting Eq. (xii) from (x)

$$q_{\mathrm{II}} + 0.074q_{\mathrm{III}} - 1.345q_{\mathrm{IV}} = 0 \tag{xv}$$

Subtracting Eq. (xiii) from (x)

$$q_{\mathrm{II}} + 0.726q_{\mathrm{III}} + 0.581q_{\mathrm{IV}} = 0 \tag{xvi}$$

Now subtracting Eq. (xv) from (xiv)

$$q_{\mathrm{III}} - 1.284q_{\mathrm{IV}} = 0 \tag{xvii}$$

Subtracting Eq. (xvi) from (xiv)

$$q_{\mathrm{III}} + 0.256q_{\mathrm{IV}} = 0.716 \times 10^{-6}T \tag{xviii}$$

Finally, subtracting Eq. (xviii) from (xvii)

$$q_{\mathrm{IV}} = 0.465 \times 10^{-6}T$$

and from Eq. (xvii)

$$q_{\mathrm{III}} = 0.597 \times 10^{-6}T$$

Substituting for q_{III} and q_{IV} in Eq. (viii)

$$\frac{\mathrm{d}\theta}{\mathrm{d}z} = \frac{1.914 \times 10^{-9}T}{G}$$

so that

$$T/(\mathrm{d}\theta/\mathrm{d}z) = 522.5 \times 10^{6}G\,\mathrm{Nmm}^2/\mathrm{rad}$$

S.23.4

In this problem the cells are not connected consecutively so that Eq. (23.6) does not apply. Therefore, from Eq. (23.5) for Cell I

$$\frac{\mathrm{d}\theta}{\mathrm{d}z} = \frac{1}{2A_{\mathrm{I}}G}[q_{\mathrm{I}}(\delta_{12^{\mathrm{U}}} + \delta_{23} + \delta_{34^{\mathrm{U}}} + \delta_{41}) - q_{\mathrm{II}}\delta_{34^{\mathrm{U}}} - q_{\mathrm{III}}(\delta_{23} + \delta_{41})] \tag{i}$$

For Cell II

$$\frac{d\theta}{dz} = \frac{1}{2A_{II}G}[-q_I\delta_{34^U} + q_{II}(\delta_{34^U} + \delta_{34^L}) - q_{III}\delta_{34^L}] \quad \text{(ii)}$$

For Cell III

$$\frac{d\theta}{dz} = \frac{1}{2A_{III}G}[-q_I(\delta_{23} + \delta_{41}) - q_{II}\delta_{34^L} + q_{III}(\delta_{14} + \delta_{43^L} + \delta_{32} + \delta_{21^L})] \quad \text{(iii)}$$

In Eqs (i)–(iii)

$$\delta_{12^U} = 1084/1.220 = 888.5 \quad \delta_{12^L} = 2160/1.625 = 1329.2$$
$$\delta_{14} = \delta_{23} = 127/0.915 = 138.8 \quad \delta_{34^U} = \delta_{34^L} = 797/0.915 = 871.0$$

Substituting these values in Eqs (i)–(iii)

$$\frac{d\theta}{dz} = \frac{1}{2 \times 108\,400G}(2037.1q_I - 871.0q_{II} - 277.6q_{III}) \quad \text{(iv)}$$
$$\frac{d\theta}{dz} = \frac{1}{2 \times 202\,500G}(-871.0q_I + 1742.0q_{II} - 871.0q_{III}) \quad \text{(v)}$$
$$\frac{d\theta}{dz} = \frac{1}{2 \times 528\,000G}(-277.6q_I - 871.0q_{II} + 2477.8q_{III}) \quad \text{(vi)}$$

Also, from Eq. (23.4)

$$565\,000 \times 10^3 = 2(108\,400q_I + 202\,500q_{II} + 528\,000q_{III}) \quad \text{(vii)}$$

Equating Eqs (iv) and (v)

$$q_I - 0.720q_{II} + 0.075q_{III} = 0 \quad \text{(viii)}$$

Equating Eqs (iv) and (vi)

$$q_I - 0.331q_{II} - 0.375q_{III} = 0 \quad \text{(ix)}$$

From Eq. (vii)

$$q_I + 1.868q_{II} + 4.871q_{III} = 260.61 \quad \text{(x)}$$

Now subtracting Eq. (ix) from (viii)

$$q_{II} - 1.157q_{III} = 0 \quad \text{(xi)}$$

Subtracting Eq. (x) from (viii)

$$q_{II} + 1.853q_{III} = 100.70 \quad \text{(xii)}$$

Finally, subtracting Eq. (xii) from (xi)

$$q_{III} = 33.5\,\text{N/mm}$$

Then, from Eq. (xi)

$$q_{\mathrm{II}} = 38.8\,\mathrm{N/mm}$$

and from Eq. (ix)

$$q_{\mathrm{I}} = 25.4\,\mathrm{N/mm}$$

Thus

$$q_{12^{\mathrm{U}}} = 25.4\,\mathrm{N/mm} \quad q_{21^{\mathrm{L}}} = 33.5\,\mathrm{N/mm} \quad q_{14} = q_{32} = 33.5 - 25.4 = 8.1\,\mathrm{N/mm}$$
$$q_{43^{\mathrm{U}}} = 38.8 - 25.4 = 13.4\,\mathrm{N/mm} \quad q_{34^{\mathrm{L}}} = 38.8 - 33.5 = 5.3\,\mathrm{N/mm}$$

S.23.5

In Eq. (23.10) the q_{b} shear flow distribution is given by Eq. (20.6) in which, since the x axis is an axis of symmetry (Fig. S.23.5), $I_{xy} = 0$; also $S_x = 0$. Thus

$$q_{\mathrm{b}} = -\frac{S_y}{I_{xx}} \sum_{r=1}^{n} B_r y_r \tag{i}$$

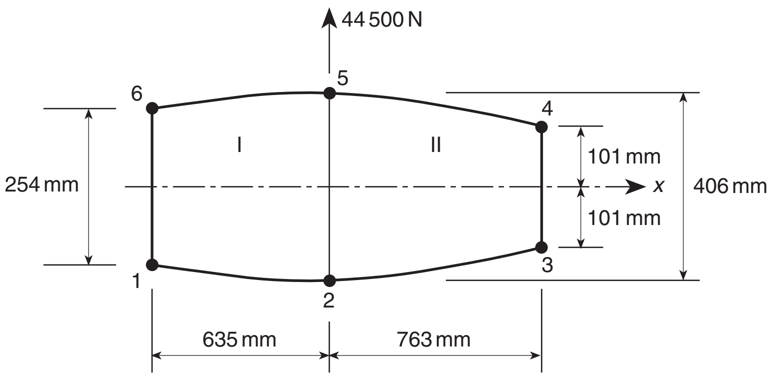

Fig. S.23.5

in which

$$I_{xx} = 2 \times 1290 \times 127^2 + 2 \times 1936 \times 203^2 + 2 \times 645 \times 101^2 = 214.3 \times 10^6\,\mathrm{mm}^4$$

Then Eq. (i) becomes

$$q_{\mathrm{b}} = -\frac{44\,500}{214.3 \times 10^6} \sum_{r=1}^{n} B_r y_r = -2.08 \times 10^{-4} \sum_{r=1}^{n} B_r y_r$$

'Cut' the walls 65 and 54. Then

$$q_{b,65} = q_{b,54} = 0$$
$$q_{b,61} = -2.08 \times 10^{-4} \times 1290 \times 127 = -32.8\,\text{N/mm}$$
$$q_{b,12} = q_{b,23} = 0 \quad \text{(from symmetry)}$$
$$q_{b,25} = -2.08 \times 10^{-4} \times 1936(-203) = 81.7\,\text{N/mm}$$
$$q_{b,34} = -2.08 \times 10^{-4} \times 645(-101) = 13.6\,\text{N/mm}$$

From Eq. (23.10) for Cell I

$$\frac{d\theta}{dz} = \frac{1}{2A_I G}[q_{s,0,I}(\delta_{56} + \delta_{61} + \delta_{12} + \delta_{24}) - q_{s,0,II}\delta_{25} + q_{b,25}\delta_{25} + q_{b,61}\delta_{61}] \qquad \text{(ii)}$$

For Cell II

$$\frac{d\theta}{dz} = \frac{1}{2A_{II} G}[-q_{s,0,I}\delta_{25} + q_{s,0,II}(\delta_{45} + \delta_{52} + \delta_{23} + \delta_{34}) + q_{b,34}\delta_{34} + q_{b,52}\delta_{25}] \qquad \text{(iii)}$$

in which

$$\delta_{56} = \delta_{12} = 647/0.915 = 707.1 \quad \delta_{45} = \delta_{23} = 775/0.559 = 1386.4$$
$$\delta_{61} = 254/1.625 = 156.3 \quad \delta_{52} = 406/2.032 = 199.8 \quad \delta_{34} = 202/1.220 = 165.6$$

Substituting these values in Eqs (ii) and (iii)

$$\frac{d\theta}{dz} = \frac{1}{2 \times 232\,000G}(1770.3q_{s,0,I} - 199.8q_{s,0,II} + 11\,197.0) \qquad \text{(iv)}$$
$$\frac{d\theta}{dz} = \frac{1}{2 \times 258\,000G}(-199.8q_{s,0,I} + 3138.2q_{s,0,II} - 14\,071.5) \qquad \text{(v)}$$

Also, taking moments about the mid-point of the web 25 and from Eq. (23.11) (or Eq. (23.12))

$$0 = 13.6 \times 202 \times 763 - 32.8 \times 254 \times 635 + 2A_I q_{s,0,I} + 2A_{II} q_{s,0,II} \qquad \text{(vi)}$$

Equating Eqs (iv) and (v)

$$q_{s,0,I} - 1.55q_{s,0,II} + 12.23 = 0 \qquad \text{(vii)}$$

From Eq. (vi)

$$q_{s,0,I} + 1.11q_{s,0,II} - 6.88 = 0 \qquad \text{(viii)}$$

Subtracting Eq. (viii) from (vii) gives

$$q_{s,0,II} = 7.2\,\text{N/mm}$$

Then, from Eq. (vii)

$$q_{s,0,I} = -1.1\,\text{N/mm}$$

Thus

$$q_{16} = 32.8 + 1.1 = 33.9\,\text{N/mm} \quad q_{65} = q_{21} = 1.1\,\text{N/mm}$$
$$q_{45} = q_{23} = 7.2\,\text{N/mm} \quad q_{34} = 13.6 + 7.2 = 20.8\,\text{N/mm}$$
$$q_{25} = 81.7 - 1.1 - 7.2 = 73.4\,\text{N/mm}$$

S.23.6

Referring to Fig. P.23.6, the horizontal x axis is an axis of symmetry so that $I_{xy} = 0$. Further, $S_x = 0$ so that, from Eq. (20.6)

$$q_{\text{b}} = -\frac{S_y}{I_{xx}} \sum_{r=1}^{n} B_r y_r \tag{i}$$

in which

$$I_{xx} = 4 \times 1290 \times 153^2 + 4 \times 645 \times 153^2 = 181.2 \times 10^6\,\text{mm}^4$$

Eq. (i) then becomes

$$q_{\text{b}} = -\frac{66\,750}{181.2 \times 10^6} \sum_{r=1}^{n} B_r y_r = -3.68 \times 10^{-4} \sum_{r=1}^{n} B_r y_r$$

Now, 'cutting' Cell I in the wall 45 and Cell II in the wall 12

$$q_{\text{b},45} = 0 = q_{\text{b},12}$$
$$q_{\text{b},43} = -3.68 \times 10^{-4} \times 645 \times 153 = -36.3\,\text{N/mm} = q_{\text{b},65} \quad \text{(from symmetry)}$$
$$q_{\text{b},18} = -3.68 \times 10^{-4} \times 1290 \times 153 = -72.6\,\text{N/mm}$$
$$q_{\text{b},78} = 0 \quad \text{(from symmetry)}$$
$$q_{\text{b},76} = -3.68 \times 10^{-4} \times 645 \times (-153) = 36.3\,\text{N/mm} = q_{\text{b},32} \quad \text{(from symmetry)}$$
$$q_{\text{b},63} = 36.3 + 36.3 - 3.68 \times 10^{-4} \times 1290 \times (-153) = 145.2\,\text{N/mm}$$

The shear load is applied through the shear centre of the section so that the rate of twist of the section, $\mathrm{d}\theta/\mathrm{d}z$, is zero and Eq. (23.10) for Cell I simplifies to

$$0 = \frac{1}{2A_{\text{I}}G_{\text{REF}}}[q_{s,0,\text{I}}(\delta_{34} + \delta_{45} + \delta_{56} + \delta_{63}) - q_{s,0,\text{II}}\delta_{63} + q_{\text{b},63}\delta_{63} + q_{\text{b},34}\delta_{34} + q_{\text{b},56}\delta_{56}] \tag{ii}$$

and for Cell II

$$0 = \frac{1}{2A_{\text{II}}G_{\text{REF}}}[-q_{s,0,\text{I}}\delta_{63} + q_{s,0,\text{II}}(\delta_{12} + \delta_{23} + \delta_{36} + \delta_{67} + \delta_{78} + \delta_{81}) + q_{\text{b},81}\delta_{81} + q_{\text{b},23}\delta_{23} + q_{\text{b},36}\delta_{36} + q_{\text{b},67}\delta_{67}] \tag{iii}$$

in which $G_{\text{REF}} = 24\,200\,\text{N/mm}^2$. Then, from Eq. (23.9)

$$t^*_{34} = t^*_{56} = \frac{20\,700}{24\,200} \times 0.915 = 0.783\,\text{mm}$$

$$t^*_{36} = t^*_{81} = t^*_{45} = \frac{24\,800}{24\,200} \times 1.220 = 1.250\,\text{mm}$$

Thus

$$\delta_{34} = \delta_{56} = 380/0.783 = 485.3$$

$$\delta_{12} = \delta_{23} = \delta_{67} = \delta_{78} = 356/0.915 = 389.1$$

$$\delta_{36} = \delta_{81} = 306/1.250 = 244.8$$

$$\delta_{45} = 610/1.250 = 488.0$$

Eq. (ii) then becomes

$$1703.4q_{s,0,\text{I}} - 244.8q_{s,0,\text{II}} + 70\,777.7 = 0$$

or

$$q_{s,0,\text{I}} - 0.144q_{s,0,\text{II}} + 41.55 = 0 \qquad \text{(iv)}$$

and Eq. (iii) becomes

$$-244.8q_{s,0,\text{I}} + 2046q_{s,0,\text{II}} - 46\,021.1 = 0$$

or

$$q_{s,0,\text{I}} - 8.358q_{s,0,\text{II}} + 188.0 = 0 \qquad \text{(v)}$$

Subtracting Eq. (v) from (iv) gives

$$q_{s,0,\text{II}} = 17.8\,\text{N/mm}$$

Then, from Eq. (v)

$$q_{s,0,\text{I}} = -39.2\,\text{N/mm}$$

The resulting shear flows are then

$$q_{12} = q_{78} = 17.8\,\text{N/mm} \quad q_{32} = q_{76} = 36.3 - 17.8 = 18.5\,\text{N/mm}$$

$$q_{63} = 145.2 - 17.8 - 39.2 = 88.2\,\text{N/mm}$$

$$q_{43} = q_{65} = 39.2 - 36.3 = 2.9\,\text{N/mm} \quad q_{54} = 39.2\,\text{N/mm}$$

$$q_{81} = 72.6 + 17.8 = 90.4\,\text{N/mm}$$

Now taking moments about the mid-point of the web 63

$$66\,750x_{\text{S}} = -2 \times q_{76} \times 356 \times 153 + 2 \times q_{78} \times 356 \times 153 + q_{81} \times 306 \times 712$$
$$- 2 \times q_{43} \times 380 \times 153 - q_{54} \times 2(51\,500 + 153 \times 380) \quad \text{(see Eq. (20.10))}$$

from which

$$x_{\text{S}} = 160.1\,\text{mm}$$

S.23.7

Referring to Fig. P.23.7 the horizontal x axis is an axis of symmetry so that $I_{xy} = 0$ and the shear centre lies on this axis. Further, applying an arbitrary shear load, S_y, through the shear centre then $S_x = 0$ and Eq. (20.6) simplifies to

$$q_{\mathrm{b}} = -\frac{S_y}{I_{xx}} \sum_{r=1}^{n} B_r y_r \tag{i}$$

in which

$$I_{xx} = 2 \times 645 \times 102^2 + 2 \times 1290 \times 152^2 + 2 \times 1935 \times 150^2 = 162.4 \times 10^6\ \mathrm{mm}^4$$

Eq. (i) then becomes

$$q_{\mathrm{b}} = -6.16 \times 10^{-9} S_y \sum_{r=1}^{n} B_r y_r \tag{ii}$$

'Cut' the walls 34° and 23. Then, from Eq. (ii)

$$q_{\mathrm{b},34^{\circ}} = q_{\mathrm{b},23} = 0 = q_{\mathrm{b},45} \quad \text{(from symmetry)}$$

$$q_{\mathrm{b},43^{\mathrm{i}}} = -6.16 \times 10^{-9} S_y \times 1935 \times (-152) = 1.81 \times 10^{-3} S_y\ \mathrm{N/mm}$$

$$q_{\mathrm{b},65} = -6.16 \times 10^{-9} S_y \times 645 \times (-102) = 0.41 \times 10^{-3} S_y\ \mathrm{N/mm} = q_{\mathrm{b},21}$$

(from symmetry)

$$q_{\mathrm{b},52} = 0.41 \times 10^{-3} S_y - 6.16 \times 10^{-9} S_y \times 1290 \times (-152) = 1.62 \times 10^{-3} S_y\ \mathrm{N/mm}$$

Since the shear load, S_y, is applied through the shear centre of the section the rate of twist, $\mathrm{d}\theta/\mathrm{d}z$, is zero. Thus, for Cell I, Eq. (23.10) reduces to

$$0 = q_{s,0,\mathrm{I}}(\delta_{34^{\circ}} + \delta_{34^{\mathrm{i}}}) - q_{s,0,\mathrm{II}}\delta_{34^{\mathrm{i}}} + q_{\mathrm{b},43^{\mathrm{i}}}\delta_{34^{\mathrm{i}}} \tag{iii}$$

and for Cell II

$$0 = -q_{s,0,\mathrm{I}}\delta_{43^{\mathrm{i}}} + q_{s,0,\mathrm{II}}(\delta_{23} + \delta_{34^{\mathrm{i}}} + \delta_{45} + \delta_{52}) + q_{\mathrm{b},52}\delta_{52} - q_{\mathrm{b},43^{\mathrm{i}}}\delta_{43^{\mathrm{i}}} \tag{iv}$$

in which

$$\delta_{34^{\circ}} = 1015/0.559 = 1815.7 \quad \delta_{34^{\mathrm{i}}} = 304/2.030 = 149.8$$

$$\delta_{23} = \delta_{45} = 765/0.915 = 836.1$$

$$\delta_{25} = 304/1.625 = 187.1$$

Thus Eq. (iii) becomes

$$1965.5 q_{s,0,\mathrm{I}} - 149.8 q_{s,0,\mathrm{II}} + 0.271 S_y = 0$$

or

$$q_{s,0,\mathrm{I}} - 0.076 q_{s,0,\mathrm{II}} + 0.138 \times 10^{-3} S_y = 0 \tag{v}$$

and Eq. (iv) becomes

$$-149.8 q_{s,0,\mathrm{I}} + 2009.1 q_{s,0,\mathrm{II}} + 319.64 \times 10^{-4} S_y = 0$$

or

$$q_{s,0,\mathrm{I}} - 13.411 q_{s,0,\mathrm{II}} - 0.213 \times 10^{-3} S_y = 0 \tag{vi}$$

Subtracting Eq. (vi) from (v)

$$13.335 q_{s,0,\mathrm{II}} + 0.351 \times 10^{-3} S_y = 0$$

whence

$$q_{s,0,\mathrm{II}} = -0.026 \times 10^{-3} S_y$$

Then from Eq. (vi)

$$q_{s,0,\mathrm{I}} = -0.139 \times 10^{-3} S_y$$

Now taking moments about the mid-point of the web 43

$$S_y x_s = -2q_{\mathrm{b},21}(508 \times 152 + 50 \times 762) + q_{\mathrm{b},52} \times 304 \times 762 + 2 \times 258\,000 q_{s,0,\mathrm{II}} + 2 \times 93\,000 q_{s,0,\mathrm{I}}$$

from which

$$x_s = 241.4\,\mathrm{mm}$$

S.23.8

The direct stresses in the booms are given by the first of Eqs (16.21) in which, referring to Fig. P.23.8, at the larger cross-section

$$I_{xx} = 2 \times 600 \times 105^2 + 4 \times 800 \times 160^2 = 95.2 \times 10^6\,\mathrm{mm}^4$$

Then, from Eq. (21.8)

$$P_{z,r} = \sigma_{z,r} B_r = \frac{M_x B_r}{I_{xx}} y_r$$

or

$$P_{z,r} = \frac{1800 \times 10^3}{95.2 \times 10^6} B_r y_r = 1.89 \times 10^{-2} B_r y_r \tag{i}$$

The components of boom load in the y and x directions (see Fig. 21.4(a) for the axis system) are found using Eqs. (21.9) and (21.10). Then, choosing the intersection of the web 52 and the horizontal axis of symmetry (the x axis) as the moment centre

Table S.23.8

Boom	$P_{z,r}$ (N)	$\delta y_r/\delta z$	$\delta x_r/\delta z$	$P_{y,r}$ (N)	$P_{x,r}$ (N)	P_r (N)	η_r (mm)	ξ_r (mm)	$P_{y,r}\xi_r$ (N mm)	$P_{x,r}\eta_r$ (N mm)
1	1190.7	0.045	−0.12	53.6	−142.9	1200.4	590	105	31 624	−15 004.5
2	2419.2	0.060	0	145.2	0	2423.6	0	160	0	0
3	2419.2	0.060	0.18	145.2	435.5	2462.4	790	160	−114 708	69 680
4	−2419.2	−0.060	0.18	145.2	−435.5	−2462.4	790	160	−114 708	69 680
5	−2419.2	−0.060	0	145.2	0	−2423.6	0	160	0	0
6	−1190.7	−0.045	−0.12	53.6	142.9	−1200.4	590	105	31 624	−15 004.5

and defining the boom positions in relation to the moment centre as in Fig. 21.5 the moments corresponding to the boom loads are calculated in Table S.23.8. In Table S.23.8 anticlockwise moments about the moment centre are positive, clockwise negative. Also

$$\sum_{r=1}^{n} P_{x,r} = 0$$

$$\sum_{r=1}^{n} P_{y,r} = 688.0\,\text{N}$$

$$\sum_{r=1}^{n} P_{y,r}\xi_r = -166\,168\,\text{N mm}$$

$$\sum_{r=1}^{n} P_{x,r}\eta_r = 109\,351\,\text{N mm}$$

The shear load resisted by the shear stresses in the webs and panels is then

$$S_y = 12\,000 - 688 = 11\,312\,\text{N}$$

'Cut' the walls 12, 23 and 34° in the larger cross-section. Then, from Eq. (20.6) and noting that $I_{xy} = 0$

$$q_{\text{b}} = -\frac{S_y}{I_{xx}} \sum_{r=1}^{n} B_r y_r$$

i.e.

$$q_{\text{b}} = -\frac{11\,312}{95.2 \times 10^6} \sum_{r=1}^{n} B_r y_r = -1.188 \times 10^{-4} \sum_{r=1}^{n} B_r y_r$$

Thus

$$q_{\text{b},12} = q_{\text{b},23} = q_{\text{b},34^\circ} = q_{\text{b},45} = q_{\text{b},56} = 0$$

$$q_{\text{b},61} = -1.188 \times 10^{-4} \times 600 \times (-105) = 7.48\,\text{N/mm}$$

$$q_{\text{b},52} = -1.188 \times 10^{-4} \times 800 \times (-160) = 15.21\,\text{N/mm}$$

$$q_{\text{b},43^{\text{i}}} = -1.188 \times 10^{-4} \times 800 \times (-160) = 15.21\,\text{N/mm}$$

From Eq. (23.10) for Cell I

$$\frac{d\theta}{dz} = \frac{1}{2A_{I}G}[q_{s,0,I}(\delta_{34^{\circ}} + \delta_{34^{i}}) - q_{s,0,II}\delta_{34^{i}} + q_{b,43^{i}}\delta_{43^{i}}] \tag{ii}$$

For Cell II

$$\frac{d\theta}{dz} = \frac{1}{2A_{II}G}[-q_{s,0,I}\delta_{34^{i}} + q_{s,0,II}(\delta_{23} + \delta_{34^{i}} + \delta_{45} + \delta_{52}) - q_{s,0,III}\delta_{52} + q_{b,52}\delta_{52} - q_{b,43^{i}}\delta_{43^{i}}] \tag{iii}$$

For Cell III

$$\frac{d\theta}{dz} = \frac{1}{2A_{III}G}[-q_{s,0,II}\delta_{52} + q_{s,0,III}(\delta_{12} + \delta_{25} + \delta_{56} + \delta_{61}) + q_{b,61}\delta_{61} - q_{b,52}\delta_{52}] \tag{iv}$$

in which

$$\delta_{12} = \delta_{56} = 600/1.0 = 600 \quad \delta_{23} = \delta_{45} = 800/1.0 = 800$$
$$\delta_{34^{\circ}} = 1200/0.6 = 2000 \quad \delta_{34^{i}} = 320/2.0 = 160 \quad \delta_{52} = 320/2.0 = 160$$
$$\delta_{61} = 210/1.5 = 140$$

Substituting these values in Eqs (ii)–(iv)

$$\frac{d\theta}{dz} = \frac{1}{2 \times 100\,000G}(2160q_{s,0,I} - 160q_{s,0,II} + 2433.6) \tag{v}$$
$$\frac{d\theta}{dz} = \frac{1}{2 \times 260\,000G}(-160q_{s,0,I} + 1920q_{s,0,II} - 160q_{s,0,III}) \tag{vi}$$
$$\frac{d\theta}{dz} = \frac{1}{2 \times 180\,000G}(-160q_{s,0,II} + 1500q_{s,0,III} - 1384.8) \tag{vii}$$

Also, taking moments about the mid-point of web 52, i.e. the moment centre (see Eq. (23.13))

$$0 = q_{b,61} \times 210 \times 590 - q_{b,43^{i}} \times 320 \times 790 + 2A_{I}q_{s,0,I} + 2A_{II}q_{s,0,II} + 2A_{III}q_{s,0,III} + \sum_{r=1}^{n} P_{x,r}\eta_r + \sum_{r=1}^{n} P_{y,r}\xi_r \tag{viii}$$

Substituting the appropriate values in Eq. (viii) and simplifying gives

$$q_{s,0,I} + 2.6q_{s,0,II} + 1.8q_{s,0,III} - 14.88 = 0 \tag{ix}$$

Equating Eqs (v) and (vi)

$$q_{s,0,I} - 0.404q_{s,0,II} + 0.028q_{s,0,III} + 1.095 = 0 \tag{x}$$

Equating Eqs (v) and (vii)

$$q_{s,0,\mathrm{I}} - 0.033q_{s,0,\mathrm{II}} - 0.386q_{s,0,\mathrm{III}} + 1.483 = 0 \quad \text{(xi)}$$

Now subtracting Eq. (x) from (ix)

$$q_{s,0,\mathrm{II}} + 0.590q_{s,0,\mathrm{III}} - 5.318 = 0 \quad \text{(xii)}$$

and subtracting Eq. (xi) from (ix)

$$q_{s,0,\mathrm{II}} + 0.830q_{s,0,\mathrm{III}} - 6.215 = 0 \quad \text{(xiii)}$$

Finally, subtracting Eq. (xiii) from (xii) gives

$$q_{s,0,\mathrm{III}} = 3.74\,\mathrm{N/mm}$$

Then, from Eq. (xiii)

$$q_{s,0,\mathrm{II}} = 3.11\,\mathrm{N/mm}$$

and from Eq. (ix)

$$q_{s,0,\mathrm{I}} = 0.06\,\mathrm{N/mm}$$

The complete shear flow distribution is then

$$q_{12} = q_{56} = 3.74\,\mathrm{N/mm} \quad q_{32} = q_{45} = 3.11\,\mathrm{N/mm}$$
$$q_{34^{\circ}} = 0.06\,\mathrm{N/mm} \quad q_{43^{\mathrm{i}}} = 12.16\,\mathrm{N/mm}$$
$$q_{52} = 14.58\,\mathrm{N/mm} \quad q_{61} = 11.22\,\mathrm{N/mm}$$

S.23.9

Consider first the flange loads and shear flows produced by the shear load acting through the shear centre of the wing box. Referring to Fig. S.23.9(a), in bay ① the shear load is resisted by the shear flows q_1 in the spar webs. Then

$$q_1 = \frac{2000}{2 \times 200} = 5\,\mathrm{N/mm}$$

Similarly in bay ②

$$q_2 = \frac{2000}{2 \times 200} = 5\,\mathrm{N/mm}$$

From symmetry the bending moment produced by the shear load will produce equal but opposite loads in the top and bottom flanges. These flange loads will increase with bending moment, i.e. linearly, from zero at the free end to

$$\pm\frac{2000 \times 1000}{2 \times 200} = \pm 5000\,\mathrm{N}$$

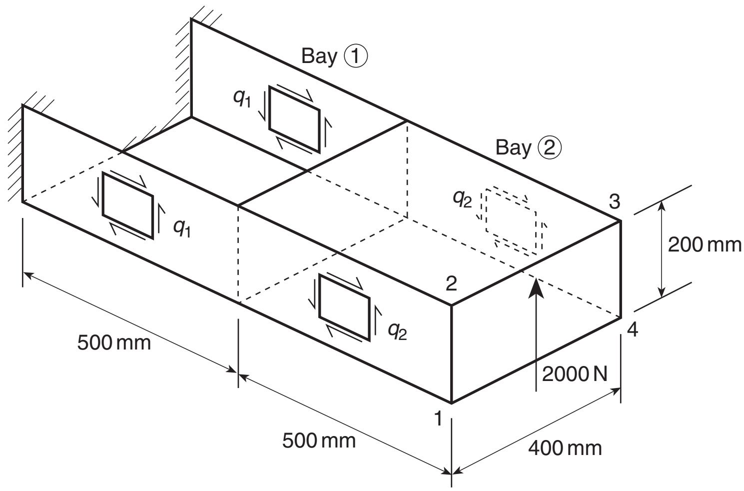

Fig. S.23.9(a)

at the built-in end. Then, at the built-in end

$$P_1 = P_4 = -P_2 = -P_3 = 5000\,\text{N}$$

Alternatively, the flange loads may be determined by considering the equilibrium of a single flange subjected to the flange load and the shear flows in the adjacent spar webs.

Now consider the action of the applied torque in Fig. S.23.9(b). In bay ① the torque is resisted by differential bending of the spar webs. Thus

$$q_1 \times 200 \times 400 = 1000 \times 10^3$$

which gives

$$q_1 = 12.5\,\text{N/mm}$$

The differential bending of the spar webs in bay ① induces flange loads as shown in Fig. S.23.9(c). For equilibrium of flange 1

$$2P_1 = 500q_1 = 500 \times 12.5$$

so that

$$P_1 = 3125\,\text{N}$$

Now considering the equilibrium of flange 1 in bay ②

$$P_1 + q_2 \times 500 - q_3 \times 500 = 0$$

whence

$$q_2 - q_3 = -6.25 \qquad \text{(i)}$$

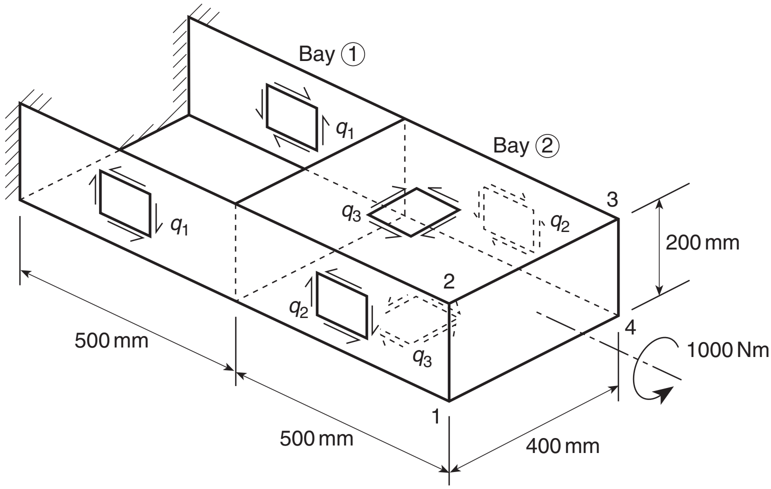

Fig. S.23.9(b)

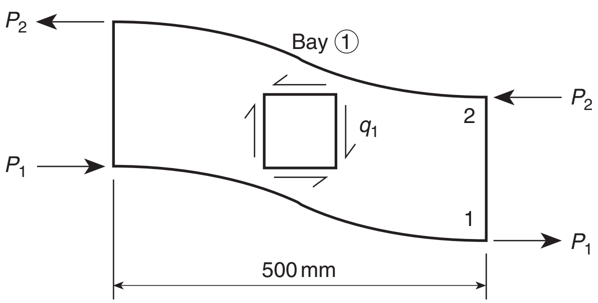

Fig. S.23.9(c)

Also, the resultant of the shear flows in the spar webs and skin panels in bay ② is equivalent to the applied torque. Thus

$$2 \times 2 \times \frac{1}{2} \times 200 \times 200 q_2 + 2 \times 2 \times \frac{1}{2} \times 400 \times 100 q_3 = 1000 \times 10^3$$

i.e.

$$q_2 + q_3 = 12.5 \qquad \text{(ii)}$$

Adding Eqs (i) and (ii) gives

$$q_2 = 3.125\,\text{N/mm}$$

whence

$$q_3 = 9.375\,\text{N/mm}$$

The shear flows due to the combined action of the shear and torsional loads are then as follows:

Bay ①

$$\text{Spar webs:}\quad q = 12.5 - 5 = 7.5\,\text{N/mm}$$

Bay ②

$$\text{Spar webs:}\quad q = 5 - 3.125 = 1.875\,\text{N/mm}$$

$$\text{Skin panels:}\quad q = 9.375\,\text{N/mm}$$

The flange loads are:

Bay ①

$$\text{At the built-in end:}\quad P_1 = 5000 - 3125 = 1875\,\text{N}\quad\text{(tension)}$$

$$\text{At the central rib:}\quad P_1 = 2500 + 3125 = 5625\,\text{N}\quad\text{(tension)}$$

Bay ②

$$\text{At the central rib:}\quad P_1 = 3625\,\text{N}\quad\text{(tension)}$$

$$\text{At the free end:}\quad P_1 = 0$$

Finally the shear flows on the central rib are:

$$\text{On the horizontal edges:}\quad q = 9.375\,\text{N/mm}$$

$$\text{On the vertical edges:}\quad q = 7.5 + 1.875 = 9.375\,\text{N/mm}$$

Solutions to Chapter 24 Problems

S.24.1

From the overall equilibrium of the beam in Fig. S.24.1(a)

$$R_F = 4\,\text{kN}\quad R_D = 2\,\text{kN}$$

The shear load in the panel ABEF is therefore 4 kN and the shear flow q is given by

$$q_1 = \frac{4 \times 10^3}{1000} = 4\,\text{N/mm}$$

Similarly

$$q_2 = \frac{2 \times 10^3}{1000} = 2\,\text{N/mm}$$

Considering the vertical equilibrium of the length h of the stiffener BE in Fig. S.24.1(b)

$$P_{EB} + (q_1 + q_2)h = 6 \times 10^3$$