Solutions to Chapter 16 Problems

S.16.1

From Section 16.2.2 the components of the bending moment about the x and y axes are, respectively

$$M_x = 3000 \times 10^3 \cos 30^\circ = 2.6 \times 10^6 \text{ N mm}$$

$$M_y = 3000 \times 10^3 \sin 30^\circ = 1.5 \times 10^6 \text{ N mm}$$

The direct stress distribution is given by Eq. (16.18) so that, initially, the position of the centroid of area, C, must be found. Referring to Fig. S.16.1 and taking moments of area about the edge BC

$$(100 \times 10 + 115 \times 10)\,\bar{x} = 100 \times 10 \times 50 + 115 \times 10 \times 5$$

i.e.

$$\bar{x} = 25.9 \text{ mm}$$

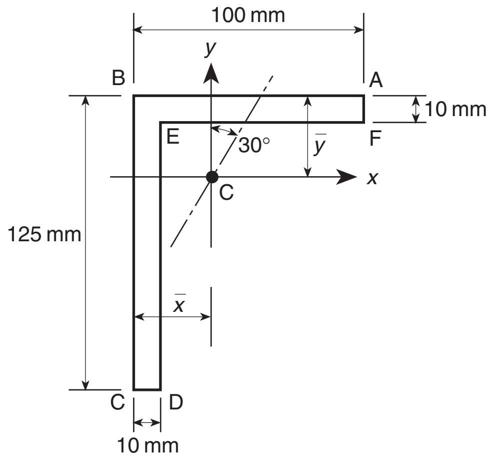

Fig. S.16.1

Now taking moments of area about AB

$$(100 \times 10 + 115 \times 10)\,\bar{y} = 100 \times 10 \times 5 + 115 \times 10 \times 67.5$$

from which

$$\bar{y} = 38.4 \text{ mm}$$

The second moments of area are then

$$I_{xx} = \frac{100 \times 10^3}{12} + 100 \times 10 \times 33.4^2 + \frac{10 \times 115^3}{12} + 10 \times 115 \times 29.1^2$$
$$= 3.37 \times 10^6\,\text{mm}^4$$

$$I_{yy} = \frac{10 \times 100^3}{12} + 10 \times 100 \times 24.1^2 + \frac{115 \times 10^3}{12} + 115 \times 10 \times 20.9^2$$
$$= 1.93 \times 10^6\,\text{mm}^4$$

$$I_{xy} = 100 \times 10 \times 33.4 \times 24.1 + 115 \times 10(-20.9)(-29.1)$$
$$= 1.50 \times 10^6\,\text{mm}^4$$

Substituting for M_x, M_y, I_{xx}, I_{yy} and I_{xy} in Eq. (16.18) gives

$$\sigma_z = 0.27x + 0.65y \qquad \text{(i)}$$

Since the coefficients of x and y in Eq. (i) have the same sign the maximum value of direct stress will occur in either the first or third quadrants. Then

$$\sigma_{z(\text{A})} = 0.27 \times 74.1 + 0.65 \times 38.4 = 45.0\,\text{N/mm}^2 \quad \text{(tension)}$$

$$\sigma_{z(\text{C})} = 0.27 \times (-25.9) + 0.65 \times (-86.6) = -63.3\,\text{N/mm}^2 \quad \text{(compression)}$$

The maximum direct stress therefore occurs at C and is 63.3 N/mm^2 compression.

S.16.2

The bending moments half-way along the beam are

$$M_x = -800 \times 1000 = -800\,000\,\text{N mm} \quad M_y = 400 \times 1000 = 400\,000\,\text{N mm}$$

By inspection the centroid of area (Fig. S.16.2) is midway between the flanges. Its distance $\bar{x}$ from the vertical web is given by

$$(40 \times 2 + 100 \times 2 + 80 \times 1)\,\bar{x} = 40 \times 2 \times 20 + 80 \times 1 \times 40$$

i.e.

$$\bar{x} = 13.33\,\text{mm}$$

The second moments of area of the cross-section are calculated using the approximations for thin-walled sections described in Section 16.4.5. Then

$$I_{xx} = 40 \times 2 \times 50^2 + 80 \times 1 \times 50^2 + \frac{2 \times 100^3}{12} = 5.67 \times 10^5\,\text{mm}^4$$

$$I_{yy} = 100 \times 2 \times 13.33^2 + \frac{2 \times 40^3}{12} + 2 \times 40 \times 6.67^2 + \frac{1 \times 80^3}{12}$$
$$+ 1 \times 80 \times 26.67^2$$
$$= 1.49 \times 10^5\,\text{mm}^4$$

$$I_{xy} = 40 \times 2(6.67)(50) + 80 \times 1(26.67)(-50) = -0.8 \times 10^5\,\text{mm}^4$$

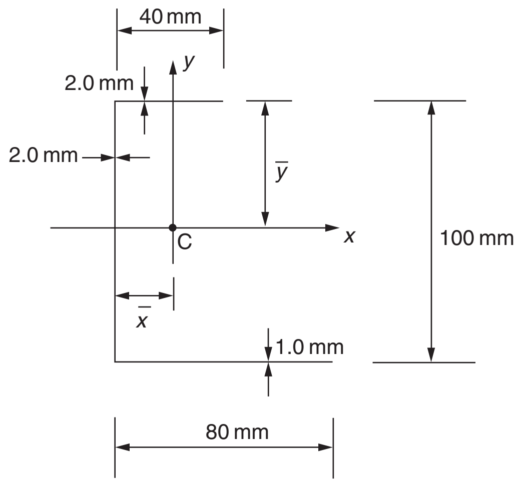

Fig. S.16.2

The denominator in Eq. (16.18) is then $(5.67 \times 1.49 - 0.8^2) \times 10^{10} = 7.81 \times 10^{10}$. From Eq. (16.18)

$$\sigma = \left(\frac{400\,000 \times 5.67 \times 10^5 - 800\,000 \times 0.8 \times 10^5}{7.81 \times 10^{10}}\right)x + \left(\frac{-800\,000 \times 1.49 \times 10^5 + 400\,000 \times 0.8 \times 10^5}{7.81 \times 10^{10}}\right)y$$

i.e.

$$\sigma = 2.08x - 1.12y$$

and at the point A where $x = 66.67$ mm, $y = -50$ mm

$$\sigma(\text{A}) = 194.7\,\text{N/mm}^2 \text{ (tension)}$$

S.16.3

Initially, the section properties are determined. By inspection the centroid of area, C, is a horizontal distance $2a$ from the point 2. Now referring to Fig. S.16.3 and taking moments of area about the flange 23

$$(5a + 4a)t\bar{y} = 5at(3a/2)$$

from which

$$\bar{y} = 5a/6$$

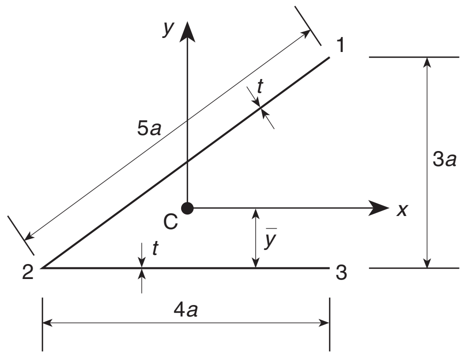

Fig. S.16.3

From Section 16.4.5

$$I_{xx} = 4at(5a/6)^2 + (5a)^3t(3/5)^2/12 + 5at(2a/3)^2 = 105a^3t/12$$
$$I_{yy} = t(4a)^3/12 + (5a)^3t(4/5)^2/12 = 12a^3t$$
$$I_{xy} = t(5a)^3(3/5)\,(4/5)/12 = 5a^3t$$

From Fig. P.16.3 the maximum bending moment occurs at the mid-span section in a horizontal plane about the y axis. Thus

$$M_x = 0 \quad M_y(\text{max}) = wl^2/8$$

Substituting these values and the values of I_{xx}, I_{yy} and I_{xy} in Eq. (16.18)

$$\sigma_z = \frac{wl^2}{8a^3t}\left(\frac{7}{64}x - \frac{1}{16}y\right) \tag{i}$$

From Eq. (i) it can be seen that σ_z varies linearly along each flange. Thus

$$\text{At 1 where } x = 2a \quad y = \frac{13a}{6} \quad \sigma_{z,1} = \frac{wl^2}{96a^2t}$$
$$\text{At 2 where } x = -2a \quad y = \frac{-5a}{6} \quad \sigma_{z,2} = \frac{-wl^2}{48a^2t}$$
$$\text{At 3 where } x = 2a \quad y = \frac{-5a}{6} \quad \sigma_{z,3} = \frac{13wl^2}{384a^2t}$$

Therefore, the maximum stress occurs at 3 and is $13wl^2/384a^2t$.

S.16.4

Referring to Fig. S.16.4.

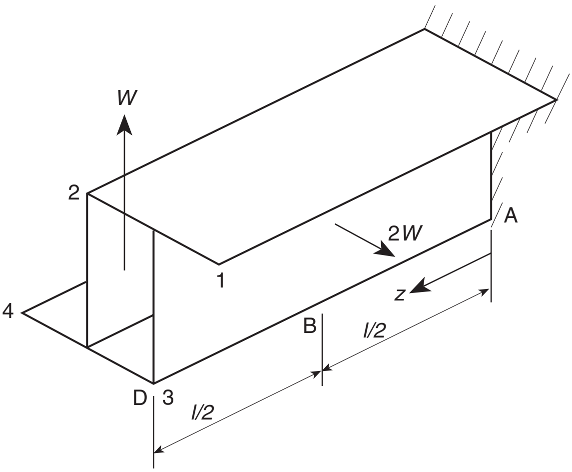

Fig. S.16.4

In DB

$$M_x = -W(l - z) \tag{i}$$

$$M_y = 0$$

In BA

$$M_x = -W(l - z) \tag{ii}$$

$$M_y = -2W\left(\frac{l}{2} - z\right) \tag{iii}$$

Now referring to Fig. P.16.4 the centroid of area, C, of the beam cross-section is at the centre of antisymmetry. Then

$$I_{xx} = 2\left[td\left(\frac{d}{2}\right)^2 + \frac{td^3}{12}\right] = \frac{2td^3}{3}$$

$$I_{yy} = 2\left[td\left(\frac{d}{4}\right)^2 + \frac{td^3}{12} + td\left(\frac{d}{4}\right)^2\right] = \frac{5td^3}{12}$$

$$I_{xy} = td\left(\frac{d}{4}\right)\left(\frac{d}{2}\right) + td\left(-\frac{d}{4}\right)\left(-\frac{d}{2}\right) = \frac{td^3}{4}$$

Substituting for I_{xx}, I_{yy} and I_{xy} in Eq. (16.18) gives

$$\sigma_z = \frac{1}{td^3}[(3.10M_y - 1.16M_x)x + (1.94M_x - 1.16M_y)y] \tag{iv}$$

Along the edge l, $x = 3d/4$, $y = d/2$. Equation (iv) then becomes

$$\sigma_{z,1} = \frac{1}{td^2}(1.75M_y + 0.1M_x) \tag{v}$$

Along the edge 2, $x = -d/4$, $y = d/2$. Equation (iv) then becomes

$$\sigma_{z,2} = \frac{1}{td^2}(-1.36M_y + 1.26M_x) \tag{vi}$$

From Eqs (i)–(iii), (v) and (vi)

In DB

$$\sigma_{z,1} = -\frac{0.1W}{td^2}(1-z) \quad \text{whence } \sigma_{z,1}(\mathrm{B}) = -\frac{0.05Wl}{td^2}$$
$$\sigma_{z,2} = -\frac{1.26W}{td^2}(1-z) \quad \text{whence } \sigma_{z,2}(\mathrm{B}) = -\frac{0.63Wl}{td^2}$$

In BA

$$\sigma_{z,1} = \frac{W}{td^2}(3.6z - 1.85l) \quad \text{whence } \sigma_{z,1}(\mathrm{A}) = -\frac{1.85Wl}{td^2}$$
$$\sigma_{z,2} = \frac{W}{td^2}(-1.46z + 0.1l) \quad \text{whence } \sigma_{z,2}(\mathrm{A}) = \frac{0.1Wl}{td^2}$$

S.16.5

By inspection the centroid of the section is at the mid-point of the web. Then

$$I_{xx} = \left(\frac{h}{2}\right)(2t)h^2 + ht\,h^2 + \frac{2t(2h)^3}{12} = \frac{10h^3t}{3}$$
$$I_{yy} = \frac{2t(h/2)^3}{3} + \frac{th^3}{3} = \frac{5h^3t}{12}$$
$$I_{xy} = 2t\left(\frac{h}{2}\right)\left(-\frac{h}{4}\right)(h) + ht\left(\frac{h}{2}\right)(-h) = -\frac{3h^3t}{4}$$

Since $M_y = 0$, Eq. (16.18) reduces to

$$\sigma_z = \frac{-M_xI_{xy}}{I_{xx}I_{yy} - I_{xy}^2}x + \frac{M_xI_{yy}}{I_{xx}I_{yy} - I_{xy}^2}y \tag{i}$$

Substituting in Eq. (i) for I_{xx}, etc.

$$\sigma_z = +\frac{M_x}{h^3t}\left[\frac{3/4}{(10/3)(5/12) - (3/4)^2}x + \frac{5/12}{(10/3)(5/12) - (3/4)^2}y\right]$$

$$\text{i.e.} \quad \sigma_z = \frac{M_x}{h^3t}(0.91x + 0.50y) \tag{ii}$$

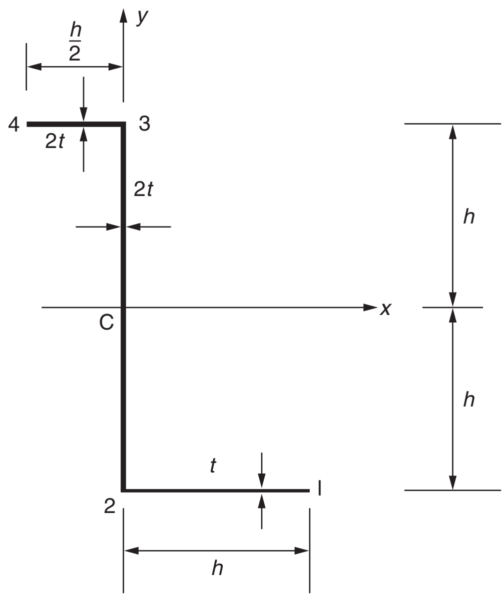

Fig. S.16.5

Between 1 and 2, $y = -h$ and σ_z is linear. Then

$$\sigma_{z,1} = \frac{M_x}{h^3 t}(0.91 \times h - 0.5h) = \frac{0.41}{h^2 t} M_x$$

$$\sigma_{z,2} = \frac{M_x}{h^3 t}(0.91 \times 0 - 0.5h) = -\frac{0.5}{h^2 t} M_x$$

Between 2 and 3, $x = 0$ and σ_z is linear. Then

$$\sigma_{z,2} = -\frac{0.5}{h^2 t} M_x$$

$$\sigma_{z,3} = \frac{M_x}{h^3 t}(0.91 \times 0 + 0.5h) = \frac{0.5}{h^2 t} M_x$$

$$\sigma_{z,4} = \frac{M_x}{h^3 t}\left(-0.91 \times \frac{h}{2} + 0.5h\right) = \frac{0.04}{h^2 t} M_x$$

S.16.6

The centroid of the section is at the centre of the inclined web. Then,

$$I_{xx} = 2\,ta(a \sin 60^\circ)^2 + \frac{t(2a)^3 \sin^2 60^\circ}{12} = 2a^3 t$$

$$I_{yy} = 2 \times \frac{ta^3}{12} + \frac{t(2a)^3 \cos^2 60^\circ}{12} = \frac{a^3 t}{3}$$

$$I_{xy} = \frac{t(2a)^3 \sin 60^\circ \cos 60^\circ}{12} = \frac{\sqrt{3} a^3 t}{6}$$

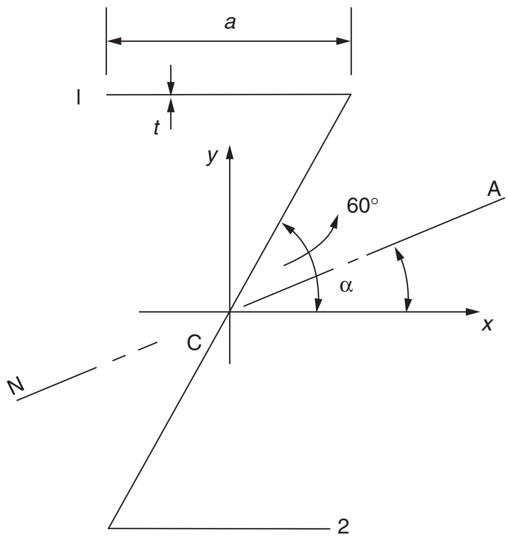

Fig. S.16.6

Substituting in Eq. (16.18) and simplifying ($M_y = 0$)

$$\sigma_z = \frac{M_x}{a^3 t}\left(\frac{4}{7}y - \frac{2\sqrt{3}}{7}x\right) \tag{i}$$

On the neutral axis, $\sigma_z = 0$. Therefore, from Eq. (i)

$$y = \frac{\sqrt{3}}{2}x$$

and

$$\tan\alpha = \frac{\sqrt{3}}{2}$$

so that

$$\alpha = 40.9°$$

The greatest stress will occur at points furthest from the neutral axis, i.e. at points 1 and 2. Then, from Eq. (i) at 1,

$$\sigma_{z,\max} = \frac{M_x}{a^3 t}\left(\frac{4}{7} \times \frac{a\sqrt{3}}{2} + \frac{2\sqrt{3}}{7} \times \frac{a}{2}\right)$$

$$\text{i.e.} \quad \sigma_{z,\max} = \frac{3\sqrt{3}M_x}{7a^2 t} = \frac{0.74}{ta^2}M_x$$

S.16.7

Referring to Fig. P.16.7, at the built-in end of the beam

$$M_x = 50 \times 100 - 50 \times 200 = -5000\,\text{N mm}$$

$$M_y = 80 \times 200 = 16\,000\,\text{N mm}$$

and at the half-way section

$$M_x = -50 \times 100 = -5000\,\text{N mm}$$
$$M_y = 80 \times 100 = 8000\,\text{N mm}$$

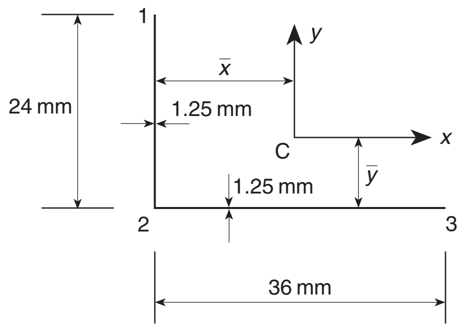

Fig. S.16.7

Now referring to Fig. S.16.7 and taking moments of areas about 12

$$(24 \times 1.25 + 36 \times 1.25)\,\bar{x} = 36 \times 1.25 \times 18$$

which gives

$$\bar{x} = 10.8\,\text{mm}$$

Taking moments of areas about 23

$$(24 \times 1.25 + 36 \times 1.25)\,\bar{y} = 24 \times 1.25 \times 12$$

which gives

$$\bar{y} = 4.8\,\text{mm}$$

Then

$$I_{xx} = \frac{1.25 \times 24^3}{12} + 1.25 \times 24 \times 7.2^2 + 1.25 \times 36 \times 4.8^2 = 4032\,\text{mm}^4$$
$$I_{yy} = 1.25 \times 24 \times 10.8^2 + \frac{1.25 \times 36^3}{12} + 1.25 \times 36 \times 7.2^2 = 10\,692\,\text{mm}^4$$
$$I_{xy} = 1.25 \times 24 \times (-10.8)(7.2) + 1.25 \times 36 \times (7.2)(-4.8) = -3888\,\text{mm}^4$$

Substituting for I_{xx}, I_{yy} and I_{xy} in Eq. (16.18) gives

$$\sigma_z = (1.44M_y + 1.39M_x) \times 10^{-4}x + (3.82M_x + 1.39M_y) \times 10^{-4}y \tag{i}$$

Thus, at the built-in end Eq. (i) becomes

$$\sigma_z = 1.61x + 0.31y \tag{ii}$$

whence $\sigma_{z,1} = -11.4\,\text{N/mm}^2$, $\sigma_{z,2} = -18.9\,\text{N/mm}^2$, $\sigma_{z,3} = 39.1\,\text{N/mm}^2$. At the half-way section Eq. (i) becomes

$$\sigma_z = 0.46x - 0.80y \tag{iii}$$

whence $\sigma_{z,1} = -20.3\,\text{N/mm}^2$, $\sigma_{z,2} = -1.1\,\text{N/mm}^2$, $\sigma_{z,3} = -15.4\,\text{N/mm}^2$.

S.16.8

The section properties are, from Fig. S.16.8

$$I_{xx} = 2\int_0^{\pi} t(r - r\cos\theta)^2 r\,\mathrm{d}\theta = 3\pi t r^3$$

$$I_{yy} = 2\int_0^{\pi} t(r\sin\theta)^2 r\,\mathrm{d}\theta = \pi t r^3$$

$$I_{xy} = 2\int_0^{\pi} t(-r\sin\theta)(r - r\cos\theta) r\,\mathrm{d}\theta = -4tr^3$$

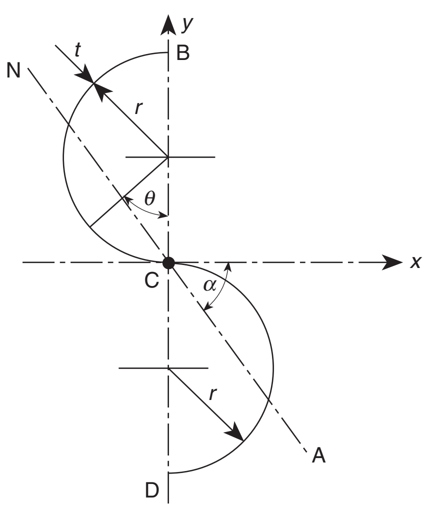

Fig. S.16.8

Since $M_y = 0$, Eq. (16.22) reduces to

$$\tan\alpha = -\frac{I_{xy}}{I_{yy}} = \frac{4tr^3}{\pi t r^3}$$

i.e.

$$\alpha = 51.9^\circ$$

Substituting for $M_x = 3.5 \times 10^3$ N mm and $M_y = 0$, Eq. (16.18) becomes

$$\sigma_z = \frac{10^3}{tr^3}(1.029x + 0.808y) \tag{i}$$

The maximum value of direct stress will occur at a point a perpendicular distance furthest from the neutral axis, i.e. by inspection at B or D. Thus

$$\sigma_z(\max) = \frac{10^3}{0.64 \times 5^3}(0.808 \times 2 \times 5)$$

i.e.

$$\sigma_z(\max) = 101.0\,\text{N/mm}^2$$

Alternatively Eq. (i) may be written

$$\sigma_z = \frac{10^3}{tr^3}[1.029(-r\sin\theta) + 0.808(r - r\cos\theta)]$$

or

$$\sigma_z = \frac{808}{tr^2}(1 - \cos\theta - 1.27\sin\theta) \tag{ii}$$

The expression in brackets has its greatest value when $\theta = \pi$, i.e. at B (or D).

S.16.9

The beam is as shown in Fig. S.16.9

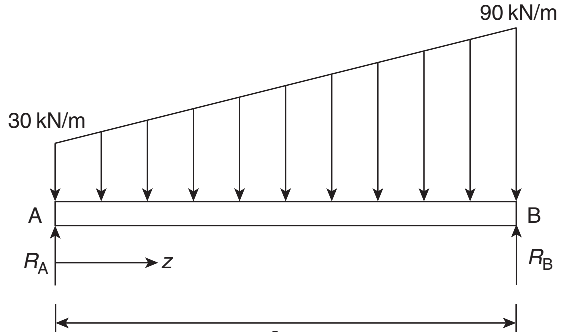

Fig. S.16.9

Taking moments about B

$$R_A \times 6 - \frac{30 \times 6^2}{2} - \frac{60}{2} \times 6 \times \frac{6}{3} = 0$$

which gives

$$R_A = 150\,\text{kN}$$

The bending moment at any section a distance z from A is then

$$M = -150z + \frac{30z^2}{2} + (90-30)\left(\frac{z}{6}\right)\left(\frac{z}{2}\right)\left(\frac{z}{3}\right)$$

i.e.

$$M = -150z + 15z^2 + \frac{5z^3}{3}$$

Substituting in the second of Eqs (16.33)

$$EI\left(\frac{d^2v}{dz^2}\right) = 150z - 15z^2 - \frac{5z^3}{3}$$

$$EI\left(\frac{dv}{dz}\right) = 75z^2 - 5z^3 - \frac{5z^4}{12} + C_1$$

$$EIv = 25z^3 - \frac{5z^4}{4} - \frac{z^5}{12} + C_1z + C_2$$

When $x = 0$, $v = 0$ so that $C_2 = 0$ and when $z = 6\,\text{m}$, $v = 0$. Then

$$0 = 25 \times 6^3 - \frac{5 \times 6^4}{4} - \frac{6^5}{12} + 6C_1$$

from which

$$C_1 = -522$$

and the deflected shape of the beam is given by

$$EIv = 25z^3 - \frac{5z^4}{4} - \frac{z^5}{12} - 522z$$

The deflection at the mid-span point is then

$$EIv_{\text{mid-span}} = 25 \times 3^3 - \frac{5 \times 3^4}{4} - \frac{3^5}{12} - 522 \times 3 = -1012.5\,\text{kN m}^3$$

Therefore

$$v_{\text{mid-span}} = \frac{-1012.5 \times 10^{12}}{120 \times 10^6 \times 206\,000} = -41.0\,\text{mm} \quad \text{(downwards)}$$

S.16.10

Take the origin of x at the free end of the cantilever. The load intensity at any section a distance z from the free end is wz/L. The bending moment at this section is given by

$$M_z = \left(\frac{z}{2}\right)\left(\frac{wz}{L}\right)\left(\frac{z}{3}\right) = \frac{wz^3}{6L}$$

Substituting in Eqs (16.32)

$$EI\left(\frac{d^2v}{dz^2}\right) = \frac{-wz^3}{6L}$$

$$EI\left(\frac{dv}{dz}\right) = \frac{-wz^4}{24L} + C_1$$

$$EIv = \frac{-wz^5}{120L} + C_1z + C_2$$

When $z = L$, $(dv/dz) = 0$ so that $C_1 = wL^3/24$. When $z = L$, $v = 0$, i.e. $C_2 = -wL^4/30$. The deflected shape of the beam is then

$$EIv = -\left(\frac{w}{120L}\right)(z^5 - 5zL^4 + 4L^5)$$

At the free end where $z = 0$

$$v = -\frac{wL^4}{30EI}$$

S.16.11

The uniformly distributed load is extended from D to F and an upward uniformly distributed load of the same intensity applied over DF so that the overall loading is unchanged (see Fig. S.16.11).

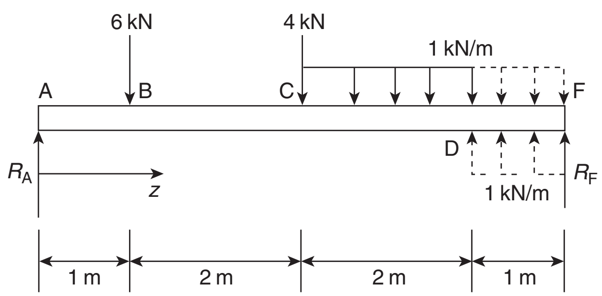

Fig. S.16.11

The support reaction at A is given by

$$R_A \times 6 - 6 \times 5 - 4 \times 3 - 1 \times 2 \times 2 = 0$$

Then

$$R_{\mathrm{A}} = 7.7\,\mathrm{kN}$$

Using Macauley's method, the bending moment in the bay DF is

$$M = -7.7z + 6[z-1] + 4[z-3] + \frac{1[z-3]^2}{2} - \frac{1[z-5]^2}{2}$$

Substituting in Eqs (16.33)

$$EI\left(\frac{\mathrm{d}^2v}{\mathrm{d}z^2}\right) = 7.7z - 6[z-1] - 4[z-3] - \frac{[z-3]^2}{2} + \frac{[z-5]^2}{2}$$

$$EI\left(\frac{\mathrm{d}v}{\mathrm{d}z}\right) = \frac{7.7z^2}{2} - 3[z-1]^2 - 2[z-3]^2 - \frac{[z-3]^3}{6} - \frac{[z-5]^3}{6} + C_1$$

$$EIv = \frac{7.7z^3}{6} - [z-1]^3 - \frac{2[z-3]^3}{3} - \frac{[z-3]^4}{24} - \frac{[z-5]^4}{24} + C_1z + C_2$$

When $z = 0$, $v = 0$ so that $C_2 = 0$. Also when $z = 6\,\mathrm{m}$, $v = 0$. Then

$$0 = \frac{7.7 \times 6^3}{6} - 5^3 - \frac{2 \times 3^3}{3} - \frac{3^4}{24} - \frac{1^4}{24} + 6C_1$$

which gives

$$C_1 = -21.8$$

Guess that the maximum deflection lies between B and C. If this is the case the slope of the beam will change sign from B to C.
At B

$$EI\left(\frac{\mathrm{d}v}{\mathrm{d}z}\right) = \frac{7.7 \times 1^2}{2} - 21.8 \text{ which is clearly negative}$$

At C

$$EI\left(\frac{\mathrm{d}v}{\mathrm{d}z}\right) = \frac{7.7 \times 3^2}{2} - 3 \times 2^2 - 21.8 = +0.85$$

The maximum deflection therefore occurs between B and C at a section of the beam where the slope is zero.
i.e.

$$0 = \frac{7.7z^2}{2} - 3[z-1]^2 - 21.8$$

Simplifying

$$z^2 + 7.06z - 29.2 = 0$$

Solving

$$z = 2.9\,\mathrm{m}$$

The maximum deflection is then

$$EI v_{\max} = \frac{7.7 \times 2.9^3}{6} - 1.9^3 - 21.8 \times 2.9 = -38.8$$

i.e.

$$v_{\max} = \frac{-38.8}{EI} \quad \text{(downwards)}$$

S.16.12

Taking moments about D

$$R_A \times 4 + 100 - 100 \times 2 \times 1 + 200 \times 3 = 0$$

from which

$$R_A = -125\,\text{N}$$

Resolving vertically

$$R_B - 125 - 100 \times 2 - 200 = 0$$

Therefore

$$R_B = 525\,\text{N}$$

The bending moment at a section a distance z from A in the bay DF is given by

$$M = +125z - 100[z-1]^0 + \frac{100[z-2]^2}{2} - 525[z-4] - \frac{100[z-4]^2}{2}$$

in which the uniformly distributed load has been extended from D to F and an upward uniformly distributed load of the same intensity applied from D to F.

Substituting in Eqs (16.33)

$$EI\left(\frac{d^2v}{dz^2}\right) = -125z + 100[z-1]^0 - 50[z-2]^2 + 525[z-4] + 50[z-4]^2$$

$$EI\left(\frac{dv}{dz}\right) = \frac{-125z^2}{2} + 100[z-1]^1 - \frac{50[z-2]^3}{3} + \frac{525[z-4]^2}{2} + \frac{50[z-4]^3}{3} + C_1$$

$$EIv = \frac{-125z^3}{6} + 50[z-1]^2 - \frac{50[z-2]^4}{12} + \frac{525[z-4]^3}{6} + \frac{50[z-4]^4}{12} + C_1 z + C_2$$

When $z = 0$, $v = 0$ so that $C_2 = 0$ and when $z = 4$ m, $v = 0$ which gives $C_1 = 237.5$. The deflection curve of the beam is then

$$v = \frac{1}{EI}\left(\frac{-125z^3}{6} + 50[z-1]^2 - \frac{50[z-2]^4}{12} + \frac{525[z-4]^3}{12} + \frac{50[z-4]^4}{12} + 237.5z\right)$$

S.16.13

From Eqs (16.30) the horizontal component of deflection, u, is given by

$$u'' = \frac{M_x I_{xy} - M_y I_{xx}}{E(I_{xx} I_{yy} - I_{xy}^2)} \tag{i}$$

in which, for the span BD, referring to Fig. P.16.13, $M_x = -R_D z$, $M_y = 0$, where R_D is the vertical reaction at the support at D. Taking moments about B

$$R_D 2l + Wl = 0$$

so that

$$R_D = -W/2 \quad \text{(downward)}$$

Eq. (i) then becomes

$$u'' = \frac{W I_{xy}}{2E(I_{xx} I_{yy} - I_{xy}^2)} z \tag{ii}$$

From Fig. P.16.13

$$I_{xx} = \frac{t(2a)^3}{12} + 2at(a)^2 + 2\left[\frac{t(a/2)^3}{12} + t\frac{a}{2}\left(\frac{3a}{4}\right)^2\right] = \frac{13a^3 t}{4}$$

$$I_{yy} = \frac{t(2a)^3}{12} + 2\frac{a}{2}t(a)^2 = \frac{5a^3 t}{3}$$

$$I_{xy} = \frac{a}{2}t(-a)\left(\frac{3a}{4}\right) + at\left(-\frac{a}{2}\right)(a) + \frac{a}{2}t(a)\left(-\frac{3a}{4}\right) + at\left(\frac{a}{2}\right)(-a) = -\frac{7a^3 t}{4}$$

Equation (ii) then becomes

$$u'' = -\frac{42W}{113Ea^3 t} \tag{iii}$$

Integrating Eq. (iii) with respect to z

$$u' = -\frac{21W}{113Ea^3 t} z^2 + A$$

and

$$u = -\frac{7W}{113Ea^3 t} z^3 + Az + B \tag{iv}$$

When $z = 0$, $u = 0$ so that $B = 0$. Also $u = 0$ when $z = 2l$ which gives

$$A = -\frac{28Wl^2}{113Ea^3 t}$$

Then

$$u = \frac{7W}{113Ea^3t}(-z^3 + 4l^2z) \tag{v}$$

At the mid-span point where $z = l$, Eq. (v) gives

$$u = \frac{0.186Wl^3}{Ea^3t}$$

Similarly

$$v = \frac{0.177Wl^3}{Ea^3t}$$

S.16.14

(a) From Eqs (16.30)

$$u'' = \frac{M_xI_{xy} - M_yI_{xx}}{E(I_{xx}I_{yy} - I_{xy}^2)} \tag{i}$$

Referring to Fig. P.16.14

$$M_x = -\frac{w}{2}(l - z)^2 \tag{ii}$$

and

$$M_y = -T(l - z) \tag{iii}$$

in which T is the tension in the link. Substituting for M_x and M_y from Eqs (ii) and (iii) in Eq. (i).

$$u'' = -\frac{1}{E(I_{xx}I_{yy} - I_{xy}^2)}\left[w\frac{I_{xy}}{2}(l - z)^2 - TI_{xx}(l - z)\right]$$

Then

$$u' = -\frac{1}{E(I_{xx}I_{yy} - I_{xy}^2)}\left[w\frac{I_{xy}}{2}\left(l^2z - lz^2 + \frac{z^3}{3}\right) - TI_{xx}\left(lz - \frac{z^2}{2}\right) + A\right]$$

When $z = 0$, $u' = 0$ so that $A = 0$. Hence

$$u = -\frac{1}{E(I_{xx}I_{yy} - I_{xy}^2)}\left[w\frac{I_{xy}}{2}\left(l^2\frac{z^2}{2} - l\frac{z^3}{3} + \frac{z^4}{12}\right) - TI_{xx}\left(l\frac{z^2}{2} - \frac{z^3}{6}\right) + B\right]$$

When $z = 0$, $u = 0$ so that $B = 0$. Hence

$$u = -\frac{1}{E(I_{xx}I_{yy} - I_{xy}^2)}\left[w\frac{I_{xy}}{2}\left(l^2\frac{z^2}{2} - l\frac{z^3}{3} + \frac{z^4}{12}\right) - TI_{xx}\left(l\frac{z^2}{2} - \frac{z^3}{6}\right)\right] \tag{iv}$$

Since the link prevents horizontal movement of the free end of the beam, $u = 0$ when $z = l$. Hence, from Eq. (iv)

$$w\frac{I_{xy}}{2}\left(\frac{l^4}{2} - \frac{l^4}{3} + \frac{l^4}{12}\right) - TI_{xx}\left(\frac{l^3}{2} - \frac{l^3}{6}\right) = 0$$

whence

$$T = \frac{3wlI_{xy}}{8I_{xx}}$$

(b) From Eqs (16.30)

$$v'' = \frac{M_x I_{yy} - M_y I_{xy}}{E(I_{xx}I_{yy} - I_{xy}^2)} \tag{v}$$

The equation for v may be deduced from Eq. (iv) by comparing Eqs (v) and (i). Thus

$$v = \frac{1}{E(I_{xx}I_{yy} - I_{xy}^2)}\left[w\frac{I_{yy}}{2}\left(l^2\frac{z^2}{2} - l\frac{z^3}{3} + \frac{z^4}{12}\right) - TI_{xy}\left(l\frac{z^2}{2} - \frac{z^3}{6}\right)\right] \tag{vi}$$

At the free end of the beam where $z = l$

$$v_{\mathrm{FE}} = \frac{1}{E(I_{xx}I_{yy} - I_{xy}^2)}\left(\frac{wI_{yy}l^4}{8} - TI_{xy}\frac{l^3}{3}\right)$$

which becomes, since $T = 3wlI_{xy}/8I_{xx}$

$$v_{\mathrm{FE}} = \frac{wl^4}{8EI_{xx}}$$

S.16.15

The beam is allowed to deflect in the horizontal direction at B so that the support reaction, R_{B}, at B is vertical. Then, from Eq. (5.12), the total complementary energy, C, of the beam is given by

$$C = \int_L \int_0^M \mathrm{d}\theta\, \mathrm{d}M - R_{\mathrm{B}}\Delta_{\mathrm{B}} - W\Delta_{\mathrm{C}} \tag{i}$$

From the principle of the stationary value of the total complementary energy of the beam and noting that $\Delta_{\mathrm{B}} = 0$

$$\frac{\partial C}{\partial R_{\mathrm{B}}} = \int_L \mathrm{d}\theta \frac{\partial M}{\partial R_{\mathrm{B}}} = 0$$

Thus

$$\frac{\partial C}{\partial R_{\mathrm{B}}} = \int_L \frac{M}{EI}\frac{\partial M}{\partial R_{\mathrm{B}}}\mathrm{d}z = 0 \tag{ii}$$

In CB

$$M = W(2l - z) \quad \text{and} \quad \partial M/\partial R_B = 0$$

In BA

$$M = W(2l - z) - R_B(l - z) \quad \text{and} \quad \partial M/\partial R_B = -(l - z)$$

Substituting in Eq. (ii)

$$\int_0^l [W(2l - z) - R_B(l - z)](l - z)\mathrm{d}z = 0$$

from which

$$R_B = \frac{5W}{2}$$

Then

$$M_C = 0 \quad M_B = Wl \quad M_A = -Wl/2$$

and the bending moment diagram is as shown in Fig. S.16.15.

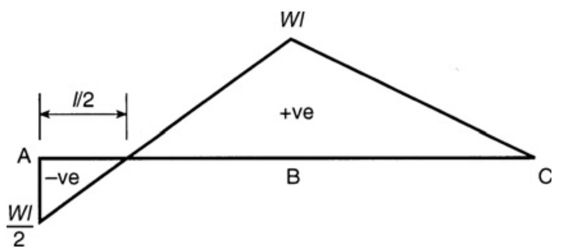

Fig. S.16.15

S.16.16

From Eq. (16.50) and Fig. P.16.4

$$N_T = E\alpha(4T_0\,dt + 2 \times 2T_0\,dt + T_0\,dt)$$

i.e.

$$N_T = 9E\alpha\,dt\,T_0$$

From Eq. (16.50)

$$M_{xT} = E\alpha\left[4T_0\,dt\left(\frac{d}{2}\right) + 2 \times 2T_0\,dt(0) + T_0\,dt\left(-\frac{d}{2}\right)\right]$$

i.e.

$$M_{xT} = \frac{3E\alpha\,d^2t\,T_0}{2}$$

From Eq. (16.52)

$$M_{yT} = E\alpha\left[4T_0\,dt\left(\frac{d}{4}\right) + 2T_0\,dt\left(\frac{d}{4}\right) + 2T_0\,dt\left(-\frac{d}{4}\right) + T_0\,dt\left(-\frac{d}{4}\right)\right]$$

i.e.

$$M_{yT} = \frac{3E\alpha\,d^2 t\,T_0}{4}$$

S.16.17

Taking moments of areas about the upper flange

$$(at + 2at)\bar{y} = 2at\,a$$

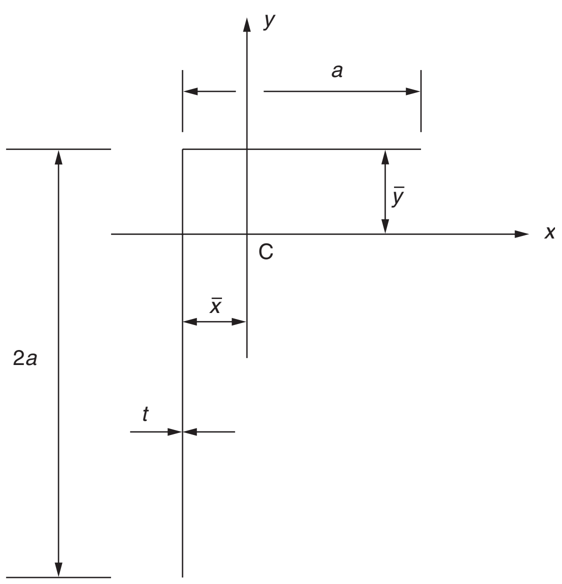

Fig. S.16.17

which gives

$$\bar{y} = \frac{2}{3}a$$

Now taking moments of areas about the vertical web

$$3at\bar{x} = at\frac{a}{2}$$

so that

$$\bar{x} = \frac{a}{6}$$

From Eq. (16.53)

$$N_T = \int_A E\alpha \frac{T_0 y}{2a} t\,\mathrm{d}s = \frac{E\alpha\,T_0 t}{2a} \int_A y\,\mathrm{d}s$$

But $t\int_A y\,\mathrm{d}s$ is the first moment of area of the section about the centroidal axis Cx, i.e. $\int_A y\,\mathrm{d}s = 0$. Therefore

$$N_T = 0$$

From Eq. (16.54)

$$M_{xT} = \int_A E\alpha \frac{T_0}{2a} ty^2\,\mathrm{d}s = \frac{E\alpha T_0}{2a} \int_A ty^2\,\mathrm{d}s$$

But

$$\int_A ty^2\,\mathrm{d}s = I_{xx} = at\left(\frac{2}{3}a\right)^2 + t\frac{(2a)^3}{3} + 2at\left(\frac{a}{3}\right)^2$$

i.e.

$$I_{xx} = \frac{10a^3 t}{3}$$

Therefore

$$M_{xT} = \frac{5E\alpha\,a^2 t\,T_0}{3}$$

From Eq. (16.55)

$$M_{yT} = \int_A E\alpha \frac{T_0}{2a} txy\,\mathrm{d}s = \frac{E\alpha T_0}{2a} \int_A txy\,\mathrm{d}s$$

But

$$\int_A txy\,\mathrm{d}s = I_{xy} = at\left(\frac{a}{3}\right)\left(\frac{2}{3}a\right) + 2at\left(-\frac{a}{6}\right)\left(-\frac{a}{3}\right)$$

i.e.

$$I_{xy} = \frac{a^3 t}{3}$$

Then

$$M_{yT} = \frac{E\alpha\,a^2 t\,T_0}{6}$$