Taking moments of areas about the skin

$$[(19.0 + 2 \times 31.8 + 2 \times 9.5) \times 0.9 + 48 \times 1.6]\bar{y} = 19 \times 0.9 \times 31.8 + 2 \times 31.8 \times 0.9 \times 15.9$$

from which $\bar{y} = 8.6\,\text{mm}$.

Then

$$I_{xx} = 19.0 \times 0.9 \times 23.2^2 + 2\left(\frac{0.9 \times 31.8^3}{12} + 0.9 \times 31.8 \times 7.3^2\right) + 2 \times 9.5 \times 0.9 \times 8.6^2 + 48 \times 1.6 \times 8.6^2$$

i.e.

$$I_{xx} = 24\,022.7\,\text{mm}^4$$

From Eq. (8.5)

$$\sigma = \frac{\pi^2 \times 69\,000 \times 24\,022.7}{168.2\,L^2}$$

Therefore

$$L^2 = \frac{\pi^2 \times 69\,000 \times 24\,022.7}{168.2 \times 200.1}$$

i.e.

$$L = 697\,\text{mm}$$

say

$$L = 700\,\text{mm}$$

Solutions to Chapter 10 Problems

S.10.1

Referring to Fig. S.10.1(a), with unit load at D (1), $R_C = 2$. Then

$$M_1 = 1z \quad (0 \le z \le l)$$
$$M_1 = 1z - R_C(z - l) = 2l - z \quad (l \le z \le 2l)$$
$$M_1 = -1(z - 2l) \quad (2l \le z \le 3l)$$
$$M_2 = 0 \quad (0 \le z \le 2l)$$
$$M_2 = 1(z - 2l) \quad (2l \le z \le 3l)$$

Hence, from the first of Eqs (5.21)

$$\delta_{11} = \frac{1}{EI}\int_0^l M_1^2\,dz + \frac{1}{EI}\int_l^{2l} M_1^2\,dz + \frac{1}{EI}\int_{2l}^{3l} M_1^2\,dz$$

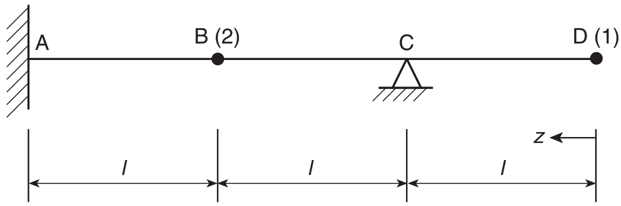

Fig. S.10.1(a)

Substituting for M_1 from the above

$$\delta_{11} = \frac{1}{EI}\left[\int_0^l z^2\,dz + \int_l^{2l} (2l - z)^2\,dz + \int_{2l}^{3l} (z - 2l)^2\,dz\right]$$

which gives

$$\delta_{11} = \frac{l^3}{EI}$$

Also

$$\delta_{22} = \frac{1}{EI}\int_{2l}^{3l} (z - 2l)^2\,dz$$

from which

$$\delta_{22} = \frac{l^3}{3EI}$$

and

$$\delta_{12} = \delta_{21} = \frac{1}{EI}\int_{2l}^{3l} -(z - 2l)^2\,dz$$

i.e.

$$\delta_{12} = \delta_{21} = -\frac{l^3}{3EI}$$

From Eqs (10.5) the equations of motion are

$$m\ddot{v}_1\delta_{11} + 2m\ddot{v}_2\delta_{12} + v_1 = 0 \qquad \text{(i)}$$

$$m\ddot{v}_1\delta_{21} + 2m\ddot{v}_2\delta_{22} + v_2 = 0 \qquad \text{(ii)}$$

Assuming simple harmonic motion, i.e. $v = v_0 \sin \omega t$ and substituting for δ_{11}, δ_{12} and δ_{22}, Eqs (i) and (ii) become

$$-3\lambda\omega^2 v_1 + 2\lambda\omega^2 v_2 + v_1 = 0$$

$$\lambda\omega^2 v_1 - 2\lambda\omega^2 v_2 + v_2 = 0$$

in which $\lambda = ml^3/3EI$ or, rearranging

$$(1 - 3\lambda\omega^2)v_1 + 2\lambda\omega^2 v_2 = 0 \quad \text{(iii)}$$

$$\lambda\omega^2 v_1 + (1 - 2\lambda\omega^2)v_2 = 0 \quad \text{(iv)}$$

From Eq. (10.7) and Eqs (iii) and (iv)

$$\begin{vmatrix} (1 - 3\lambda\omega^2) & 2\lambda\omega^2 \\ \lambda\omega^2 & (1 - 2\lambda\omega^2) \end{vmatrix} = 0$$

from which

$$(1 - 3\lambda\omega^2)(1 - 2\lambda\omega^2) - 2(\lambda\omega^2)^2 = 0$$

or

$$4(\lambda\omega^2)^2 - 5\lambda\omega^2 + 1 = 0$$

i.e.

$$(4\lambda\omega^2 - 1)(\lambda\omega^2 - 1) = 0 \quad \text{(v)}$$

Hence

$$\lambda\omega^2 = \tfrac{1}{4} \quad \text{or} \quad 1$$

so that

$$\omega^2 = \frac{3EI}{4ml^3} \quad \text{or} \quad \omega^2 = \frac{3EI}{ml^3}$$

Hence

$$\omega_1 = \sqrt{\frac{3EI}{4ml^3}} \quad \omega_2 = \sqrt{\frac{3EI}{ml^3}}$$

The frequencies of vibration are then

$$f_1 = \frac{\omega_1}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{3EI}{4ml^3}} \quad f_2 = \frac{1}{2\pi}\sqrt{\frac{3EI}{ml^3}}$$

From Eq. (iii)

$$\frac{v_1}{v_2} = -\frac{2\lambda\omega^2}{1 - 3\lambda\omega^2} \quad \text{(vi)}$$

When $\omega = \omega_1$, v_1/v_2 is negative and when $\omega = \omega_2$, v_1/v_2 is positive. The modes of vibration are therefore as shown in Fig. S.10.1(b) and (c).

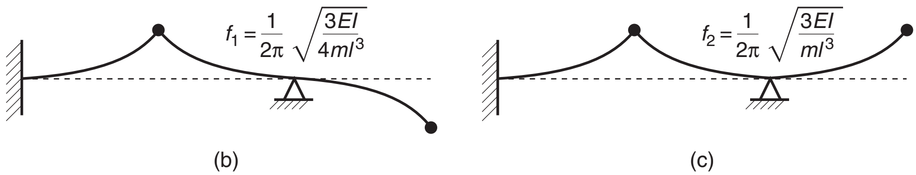

Fig. S.10.1(b) and (c)

S.10.2

Referring to Fig. S.10.2

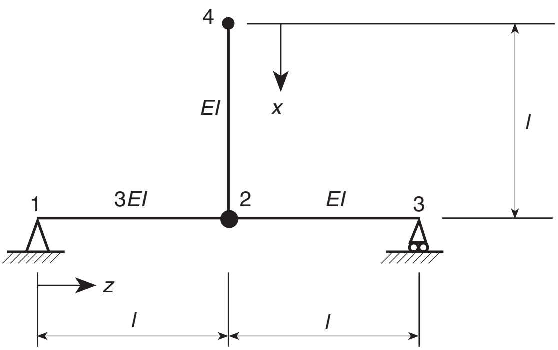

Fig. S.10.2

$$
\begin{aligned}
M_2 &= -\tfrac{1}{2}z && (0 \le z \le l)\\
M_2 &= -\tfrac{1}{2}(2l - z) && (l \le z \le 2l)\\
M_2 &= 0 && (0 \le x \le l)\\
M_4 &= 1x && (0 \le x \le l)\\
M_4 &= \tfrac{1}{2}z && (0 \le z \le l)\\
M_4 &= -\tfrac{1}{2}(2l - z) && (l \le z \le 2l)
\end{aligned}
$$

Then from the first of Eqs (5.21)

$$
\delta_{22} = \frac{1}{3EI}\int_0^l \frac{z^2}{4}\,\mathrm{d}z + \frac{1}{EI}\int_l^{2l} \frac{(2l - z)^2}{4}\,\mathrm{d}z
$$

which gives

$$
\delta_{22} = \frac{l^3}{9EI}
$$

Also

$$\delta_{44} = \frac{1}{EI}\int_0^l x^2\,\mathrm{d}x + \frac{1}{3EI}\int_0^l \frac{z^2}{4}\,\mathrm{d}z + \frac{1}{EI}\int_l^{2l} \frac{(2l-z)^2}{4}\,\mathrm{d}z$$

from which

$$\delta_{44} = \frac{4l^3}{9EI}$$

and

$$\delta_{42} = \delta_{24} = -\frac{1}{3EI}\int_0^l \frac{z^2}{4}\,\mathrm{d}z + \frac{1}{EI}\int_l^{2l} \frac{(2l-z)^2}{4}\,\mathrm{d}z$$

Thus

$$\delta_{42} = \delta_{24} = \frac{l^3}{18EI}$$

From Eqs (10.5) the equations of motion are

$$m\ddot{v}_4\delta_{44} + 2m\ddot{v}_2\delta_{42} + v_4 = 0 \qquad \text{(i)}$$

$$m\ddot{v}_4\delta_{24} + 2m\ddot{v}_2\delta_{22} + v_2 = 0 \qquad \text{(ii)}$$

Assuming simple harmonic motion, i.e. $v = v_0 \sin\omega t$ and substituting for δ_{44}, δ_{42} and δ_{22}, Eqs (i) and (ii) become

$$-8\lambda\omega^2 v_4 - 2\lambda\omega^2 v_2 + v_4 = 0 \qquad \text{(iii)}$$

$$-\lambda\omega^2 v_4 - 4\lambda\omega^2 v_2 + v_2 = 0 \qquad \text{(iv)}$$

in which $\lambda = ml^3/18EI$. Then, from Eq. (10.7)

$$\begin{vmatrix} (1-8\lambda\omega^2) & -2\lambda\omega^2 \\ -\lambda\omega^2 & (1-4\lambda\omega^2) \end{vmatrix} = 0$$

which gives

$$(1-8\lambda\omega^2)(1-4\lambda\omega^2) - 2(\lambda\omega^2)^2 = 0$$

i.e.

$$30(\lambda\omega^2)^2 - 12\lambda\omega^2 + 1 = 0 \qquad \text{(v)}$$

Solving Eq. (v)

$$\lambda\omega^2 = 0.118 \quad \text{or} \quad \lambda\omega^2 = 0.282$$

Hence

$$\omega^2 = 0.118 \times \frac{18EI}{ml^3} \quad \text{or} \quad \omega^2 = 0.282 \times \frac{18EI}{ml^3}$$

Then, since $f = \omega/2\pi$, the natural frequencies of vibration are

$$f_1 = \frac{1}{2\pi}\sqrt{\frac{2.13EI}{ml^3}} \quad f_2 = \frac{1}{2\pi}\sqrt{\frac{5.08EI}{ml^3}}$$

S.10.3

The second moment of area, I, of the tube cross-section is given by

$$I = \frac{\pi}{64}(D^4 - d^4)$$

in which D and d are the outer and inner diameters respectively. Now,

$$D = 25 + 1.25 = 26.25\,\text{mm} \quad d = 25 - 1.25 = 23.75\,\text{mm}$$

Thus

$$I = \frac{\pi}{64}(26.25^4 - 23.75^4) = 7689.1\,\text{mm}^4$$

The polar second moment of area, J, for a circular section is $2I$, i.e. $J = 15\,378.2\,\text{mm}^4$. From Eqs (5.21)

$$\delta_{ij} = \int_L \frac{M_i M_j}{EI}\,\mathrm{d}s + \int_L \frac{T_i T_j}{GJ}\,\mathrm{d}s \qquad \text{(i)}$$

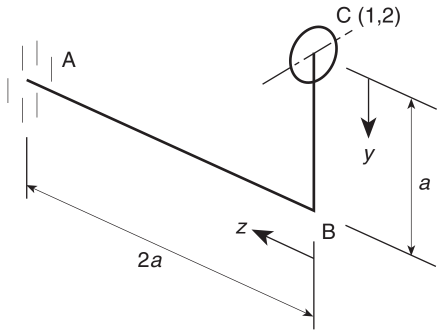

Fig. S.10.3(a)

Then, referring to Fig. S.10.3(a)

$$M_1 = 1y \quad (0 \le y \le a)$$
$$M_1 = 1z \quad (0 \le z \le 2a)$$
$$T_1 = 0 \quad (0 \le y \le a)$$
$$T_1 = 1a \quad (0 \le z \le 2a)$$
$$M_2 = 1 \quad (0 \le y \le a)$$
$$T_2 = 1 \quad (0 \le z \le 2a)$$

Thus, from Eq. (i)

$$\delta_{11} = \int_0^a \frac{y^2}{EI}\,\mathrm{d}y + \int_0^{2a} \frac{z^2}{EI}\,\mathrm{d}z + \int_0^{2a} \frac{a^2}{GJ}\,\mathrm{d}z$$

which gives

$$\delta_{11} = a^3\left(\frac{3}{EI} + \frac{2}{GJ}\right) = 250^3\left(\frac{3}{70\,000 \times 7689.1} + \frac{2}{28\,000 \times 15\,378.2}\right)$$

i.e.

$$\delta_{11} = 0.16$$

Also

$$\delta_{22} = \int_0^a \frac{1^2}{EI}\,\mathrm{d}y + \int_0^{2a} \frac{1^2}{GJ}\,\mathrm{d}z$$

i.e.

$$\delta_{22} = a\left(\frac{1}{EI} + \frac{2}{GJ}\right) = 250\left(\frac{1}{70\,000 \times 7689.1} + \frac{2}{28\,000 \times 15\,378.2}\right)$$

which gives

$$\delta_{22} = 1.63 \times 10^{-6}$$

Finally

$$\delta_{12} = \delta_{21} = \int_0^a \frac{y}{EI}\,\mathrm{d}y + \int_0^{2a} \frac{a}{GJ}\,\mathrm{d}z$$

so that

$$\delta_{12} = \delta_{21} = a^2\left(\frac{1}{2EI} + \frac{2}{GJ}\right) = 250^2\left(\frac{1}{2 \times 70\,000 \times 7689.1} + \frac{2}{28\,000 \times 15\,378.2}\right)$$

Thus

$$\delta_{12} = \delta_{21} = 3.48 \times 10^{-4}$$

The equations of motion are then, from Eqs (10.5)

$$m\ddot{v}\delta_{11} + mr^2\ddot{\theta}\delta_{12} + v = 0 \qquad \text{(ii)}$$

$$m\ddot{v}\delta_{21} + mr^2\ddot{\theta}\delta_{22} + \theta = 0 \qquad \text{(iii)}$$

Assuming simple harmonic motion, i.e. $v = v_0 \sin\omega t$ and $\theta = \theta_0 \sin\omega t$, Eqs (i) and (ii) may be written

$$-m\delta_{11}\omega^2 v - mr^2\delta_{12}\omega^2\theta + v = 0$$

$$-m\delta_{21}\omega^2 v - mr^2\delta_{22}\omega^2\theta + \theta = 0$$

Substituting for m, r and δ_{11}, etc.

$$-20 \times 0.16\omega^2 v - 20 \times 62.5^2 \times 3.48 \times 10^{-4}\omega^2\theta + v = 0$$

$$-20 \times 3.48 \times 10^{-4}\omega^2 v - 20 \times 62.5^2 \times 1.63 \times 10^{-6}\omega^2\theta + \theta = 0$$

which simplify to

$$v(1 - 3.2\omega^2) - 27.2\omega^2\theta = 0 \tag{iv}$$

$$-0.007\omega^2 v + \theta(1 - 0.127\omega^2) = 0 \tag{v}$$

Hence, from Eqs (10.7)

$$\begin{vmatrix} (1 - 3.2\omega^2) & -27.2\omega^2 \\ -0.007\omega^2 & (1 - 0.127\omega^2) \end{vmatrix} = 0$$

which gives

$$(1 - 3.2\omega^2)(1 - 0.127\omega^2) - 0.19\omega^4 = 0$$

or

$$\omega^4 - 15.4\omega^2 + 4.63 = 0 \tag{vi}$$

Solving Eq. (vi) gives

$$\omega^2 = 15.1 \quad \text{or} \quad 0.31$$

Hence the natural frequencies are

$$f = 0.62\,\text{Hz} \quad \text{and} \quad 0.09\,\text{Hz}$$

From Eq. (iv)

$$\frac{v}{\theta} = \frac{27.2\omega^2}{1 - 3.2\omega^2}$$

Thus, when $\omega^2 = 15.1$, v/θ is negative and when $\omega^2 = 0.31$, v/θ is positive. The modes of vibration are then as shown in Figs S.10.3(b) and (c).

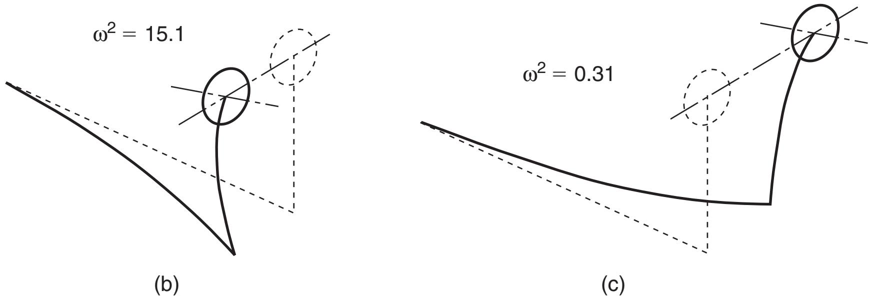

Fig. S.10.3(b) and (c)

S.10.4

Choosing the origin for z at the free end of the tube

$$\begin{aligned} M_1 &= z, \quad S_1 = 1 \quad \text{and} \quad T_1 = 0 \\ M_2 &= z, \quad S_2 = 1 \quad \text{and} \quad T_2 = 2a \end{aligned}$$

in which the point 1 is at the axis of the tube and point 2 at the free end of the rigid bar.
From Eqs (5.21) and (20.19)

$$\delta_{ij} = \int_0^L \frac{M_i M_j}{EI}\,\mathrm{d}z + \int_0^L \frac{T_i T_j}{GJ}\,\mathrm{d}z + \int_0^L \left(\oint \frac{q_i q_j}{Gt}\,\mathrm{d}s \right) \mathrm{d}z \qquad \text{(i)}$$

in which q_i and q_j are obtained from Eq. (17.15) in which $S_{y,i} = S_{y,j} = 1$, $S_x = 0$ and $I_{xy} = 0$. Thus

$$q_i = q_j = -\frac{1}{I_{xx}} \int_0^s ty\,\mathrm{d}s + q_{s,0}$$

'Cutting' the tube at its lowest point in its vertical plane of symmetry gives $q_{s,0} = 0$. Then, referring to Fig. S.10.4

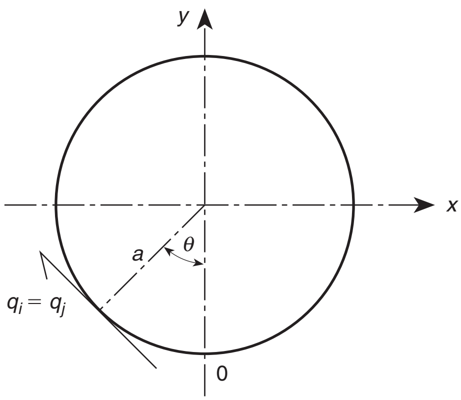

Fig. S.10.4

$$q_i = q_j = \frac{1}{I_{xx}} \int_0^\theta ta\cos\theta a\,\mathrm{d}\theta$$

i.e.

$$q_i = q_j = \frac{a^2 t \sin\theta}{I_{xx}}$$

From Fig. 16.33, $I_{xx} = \pi a^3 t$. Hence $q_i = q_j = \sin\theta/\pi a$ and

$$\oint \frac{q_i q_j}{Gt}\,\mathrm{d}s = 2\int_0^\pi \frac{\sin^2\theta}{G\pi^2 a^2 t} a\,\mathrm{d}\theta = \frac{1}{G\pi a t}$$

Also in Eq. (i) the torsion constant J is obtained from Eq. (18.4), i.e.

$$J = \frac{4A^2}{\oint \mathrm{d}s/t} = \frac{4(\pi a^2)^2}{2\pi a/t} = 2\pi a^3 t$$

Therefore from Eq. (i)

$$\delta_{11} = \int_0^L \frac{z^2}{EI}\,\mathrm{d}z + \int_0^L \frac{1}{G\pi at}\,\mathrm{d}z = \frac{L^3}{3EI} + \frac{L}{G\pi at} \qquad \text{(ii)}$$

Putting $\lambda = 3Ea^2/GL^2$, Eq. (ii) becomes

$$\delta_{11} = \frac{L^3}{3EI}(1+\lambda)$$

Also

$$\delta_{22} = \int_0^L \frac{z^2}{EI}\,\mathrm{d}z + \int_0^L \frac{4a^2}{G2\pi a^3 t}\,\mathrm{d}z + \int_0^L \frac{1}{G\pi at}\,\mathrm{d}z$$

which gives

$$\delta_{22} = \frac{L^3}{3EI}(1+3\lambda)$$

Finally

$$\delta_{12} = \delta_{21} = \int_0^L \frac{z^2}{EI}\,\mathrm{d}z + \int_0^L \frac{1}{G\pi at}\,\mathrm{d}z$$

i.e.

$$\delta_{12} = \delta_{21} = \frac{L^3}{3EI}(1+\lambda)$$

From Eqs (10.5) the equations of motion are

$$m\ddot{v}_1\delta_{11} + m\ddot{v}_2\delta_{12} + v_1 = 0 \qquad \text{(iii)}$$

$$m\ddot{v}_1\delta_{21} + m\ddot{v}_2\delta_{22} + v_2 = 0 \qquad \text{(iv)}$$

Assuming simple harmonic motion, i.e. $v = v_0 \sin \omega t$, Eqs (iii) and (iv) become

$$-m\delta_{11}\omega^2 v_1 - m\delta_{12}\omega^2 v_2 + v_1 = 0$$

$$-m\delta_{21}\omega^2 v_1 - m\delta_{22}\omega^2 v_2 + v_2 = 0$$

Substituting for δ_{11}, δ_{22} and δ_{12} and writing $\mu = L^3/3EI$ gives

$$v_1[1 - m\omega^2\mu(1+\lambda)] - m\omega^2\mu(1+\lambda)v_2 = 0$$

$$-m\omega^2\mu(1+\lambda)v_1 + v_2[1 - m\omega^2\mu(1+3\lambda)] = 0$$

Hence, from Eqs (10.7)

$$\begin{vmatrix} [1 - m\omega^2\mu(1+\lambda)] & -m\omega^2\mu(1+\lambda) \\ -m\omega^2\mu(1+\lambda) & [1 - m\omega^2\mu(1+3\lambda)] \end{vmatrix} = 0$$

Then

$$[1 - m\omega^2\mu(1+\lambda)][1 - m\omega^2\mu(1+3\lambda)] - m^2\omega^4\mu^2(1+\lambda)^2 = 0$$

which simplifies to

$$\frac{1}{\omega^4} - \frac{1}{\omega^2}2\,m\mu(1+2\lambda) + 2m^2\mu^2\lambda(1+\lambda) = 0$$

Solving gives

$$\frac{1}{\omega^2} = m\mu(1+2\lambda) \pm m\mu(1+2\lambda+2\lambda^2)^{1/2}$$

i.e.

$$\frac{1}{\omega^2} = \frac{mL^3}{3E\pi a^3 t}[1 + 2\lambda \pm (1+2\lambda+2\lambda^2)^{1/2}]$$

S.10.5

Choosing the origin for z at the free end of the beam

$$M_1 = z, \quad S_1 = 1$$

Also, from Eqs (5.21) and Eq. (20.19)

$$\delta_{ij} = \int_0^L \frac{M_i M_j}{EI}\,\mathrm{d}z + \int_0^L \left(\oint \frac{q_i q_j}{Gt}\,\mathrm{d}s \right) \mathrm{d}z \tag{i}$$

in which q_i and q_j are obtained from Eq. (20.11) and in which $S_{y,i} = S_{y,j} = 1$, $S_x = 0$, $I_{xy} = 0$ and $t_D = 0$. Thus

$$q_i = q_j = -\frac{1}{I_{xx}} \sum_{r=1}^{n} B_r y_r + q_{s,0}$$

where I_{xx} is given by (see Fig. S.10.5)

$$I_{xx} = 2 \times 970 \times 100^2 + 2 \times 970 \times 150^2 = 6.305 \times 10^7\,\mathrm{mm}^4$$

Thus

$$q_{b,i} = q_{b,j} = -\frac{1}{6.305 \times 10^7} \sum_{r=1}^{n} B_r y_r$$

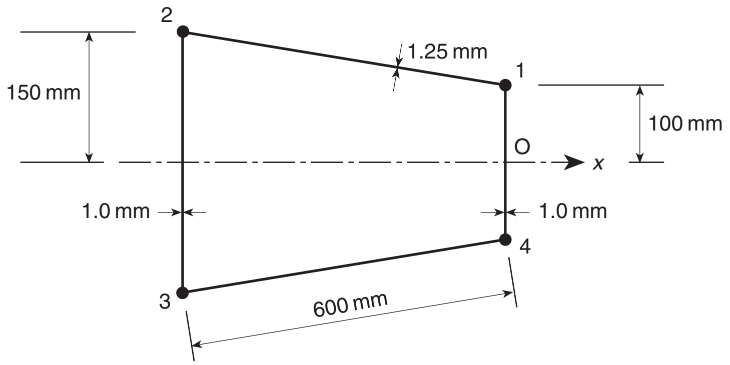

Fig. S.10.5

Hence, cutting the tube at O,

$$q_{b,O1} = 0$$

$$q_{b,12} = -\frac{970 \times 100}{6.305 \times 10^7} = -0.0015\,\text{N/mm}$$

$$q_{b,23} = -0.0015 - \frac{970 \times 150}{6.305 \times 10^7} = -0.0038\,\text{N/mm}$$

Then, from Eq. (17.27)

$$q_{s,0} = -\frac{2}{2(100/1.0 + 600/1.25 + 150/1.0)}\left(-\frac{0.0015 \times 600}{1.25} - \frac{0.0038 \times 150}{1.0}\right)$$

i.e.

$$q_{s,0} = 0.0018\,\text{N/mm}$$

Therefore

$$q_{i,O1} = q_{j,O1} = 0.0018\,\text{N/mm}$$

$$q_{i,12} = q_{j,12} = -0.0015 + 0.0018 = 0.0003\,\text{N/mm}$$

$$q_{i,23} = q_{j,23} = -0.0038 + 0.0018 = -0.002\,\text{N/mm}$$

Then

$$\oint \frac{q_i q_j}{Gt}\,\text{d}s = \frac{2}{26\,500}\left(\frac{0.0018^2 \times 100}{1.0} + \frac{0.0003^2 \times 600}{1.25} + \frac{0.002^2 \times 150}{1.0}\right)$$
$$= 7.3 \times 10^{-8}$$

Hence

$$\delta_{11} = \int_0^{1525} \frac{z^2}{EI}\,\text{d}z + \int_0^{1525} 7.3 \times 10^{-8}\,\text{d}z$$

i.e.

$$\delta_{11} = \frac{1525^3}{3 \times 70\,000 \times 6.305 \times 10^7} + 7.3 \times 10^{-8} \times 1525 = 3.79 \times 10^{-4}$$

For flexural vibrations in a vertical plane the equation of motion is, from Eqs (10.5)

$$m\ddot{v}_1\delta_{11} + v_1 = 0$$

Assuming simple harmonic motion, i.e. $v = v_0 \sin \omega t$ Eq. (ii) becomes

$$-m\delta_{11}\omega^2 v_1 + v_1 = 0$$

i.e.

$$\omega^2 = \frac{1}{m\delta_{11}} = \frac{9.81 \times 10^3}{4450 \times 3.79 \times 10^{-4}} = 5816.6$$

Hence

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{5816.6} = 12.1\,\text{Hz}$$

S.10.6

Assume a deflected shape given by

$$V = \cos\frac{2\pi z}{l} - 1 \tag{i}$$

where z is measured from the left-hand end of the beam. Eq. (i) satisfies the boundary conditions of $V = 0$ at $z = 0$ and $z = l$ and also $\mathrm{d}V/\mathrm{d}z = 0$ at $z = 0$ and $z = l$. From Eq. (i)

$$\frac{\mathrm{d}V}{\mathrm{d}z} = -\frac{2\pi}{l}\sin\frac{2\pi z}{l}$$

and

$$\frac{\mathrm{d}^2V}{\mathrm{d}z^2} = -\frac{4\pi^2}{l^2}\cos\frac{2\pi z}{l}$$

Substituting these expressions in Eq. (10.22)

$$\omega^2 = \frac{2\left[\int_0^{l/4} 4EI\left(\frac{4\pi^2}{l^2}\right)^2\cos^2\left(\frac{2\pi z}{l}\right)\mathrm{d}z + \int_{l/4}^{l/2} EI\left(\frac{4\pi^2}{l^2}\right)^2\cos^2\left(\frac{2\pi z}{l}\right)\mathrm{d}z\right]}{2\left[\int_0^{l/4} 2m\left(\cos\frac{2\pi z}{l} - 1\right)^2\mathrm{d}z + \int_{l/4}^{l/2} m\left(\cos\frac{2\pi z}{l} - 1\right)^2\mathrm{d}z\right] + 2\frac{1}{2}ml(-1)^2 + \frac{1}{4}ml(2)^2}$$

which simplifies to

$$\omega^2 = \frac{EI\left(\dfrac{4\pi^2}{l^2}\right)^2\left[\displaystyle\int_0^{l/4} 4\cos^2\left(\frac{2\pi z}{l}\right)\mathrm{d}s + \int_{l/4}^{l/2}\cos^2\left(\frac{2\pi z}{l}\right)\mathrm{d}z\right]}{m\left[\displaystyle\int_0^{l/4} 2\left(\cos\frac{2\pi z}{l} - 1\right)^2\mathrm{d}z + \int_{l/4}^{l/2}\left(\cos\frac{2\pi z}{l} - 1\right)^2\mathrm{d}z + l\right]} \qquad \text{(ii)}$$

Now

$$\int_0^{l/4}\cos^2\frac{2\pi z}{l}\,\mathrm{d}z = \frac{1}{2}\left(z + \frac{l}{4\pi}\sin\frac{4\pi z}{l}\right)_0^{l/4} = \frac{l}{8}$$

$$\int_{l/4}^{l/2}\cos^2\frac{2\pi z}{l}\,\mathrm{d}z = \frac{1}{2}\left(z + \frac{l}{4\pi}\sin\frac{4\pi z}{l}\right)_{l/4}^{l/2} = \frac{l}{8}$$

$$\int_0^{l/4}\left(\cos\frac{2\pi z}{l} - 1\right)^2\mathrm{d}z = \int_0^{l/4}\left[\frac{1}{2}\left(1 + \cos\frac{4\pi z}{l}\right) - 2\cos\frac{2\pi z}{l} + 1\right]\mathrm{d}z$$

$$= \left[\frac{1}{2}\left(z + \frac{l}{4\pi}\sin\frac{4\pi z}{l}\right) - \frac{l}{\pi}\sin\frac{2\pi z}{l} + z\right]_0^{l/4} = \frac{3l}{8} - \frac{l}{\pi}$$

Similarly

$$\int_{l/4}^{l/2}\left(\cos\frac{2\pi z}{l} - 1\right)^2\mathrm{d}z = \frac{3l}{8} + \frac{l}{\pi}$$

Substituting these values in Eq. (ii)

$$\omega^2 = \frac{EI\left(\dfrac{4\pi^2}{l^2}\right)^2\left(\dfrac{4l}{8} + \dfrac{l}{8}\right)}{m\left[2\left(\dfrac{3l}{8} - \dfrac{l}{\pi}\right) + \dfrac{3l}{8} + \dfrac{l}{\pi} + l\right]}$$

i.e.

$$\omega^2 = 539.2\frac{EI}{ml^4}$$

Then

$$f = \frac{\omega}{2\pi} = 3.7\sqrt{\frac{EI}{ml^4}}$$

The accuracy of the solution may be improved by assuming a series for the deflected shape, i.e.

$$V(z) = \sum_{s=1}^{n} B_s V_s(z) \quad \text{(Eq. (10.23))}$$