Solutions to Chapter 9 Problems

S.9.1

Assuming that the elastic deflection, w, of the plate is of the same form as the initial curvature, then

$$w = A \sin \frac{\pi x}{a} \sin \frac{\pi y}{a}$$

Hence, from Eq. (7.36) in which $m = n = 1$, $a = b$ and $N_x = \sigma t$

$$w = \frac{\delta \sigma t}{(4\pi^2 D/a^2) - \sigma t} \sin \frac{\pi x}{a} \sin \frac{\pi y}{a} \tag{i}$$

The deflection, w_C, at the centre of the plate where $x = a/2, y = a/2$ is, from Eq. (i)

$$w_C = \frac{\delta \sigma t}{(4\pi^2 D/a^2) - \sigma t} \tag{ii}$$

When $\sigma t \to 4\pi^2 D/a, w \to \infty$ and $\sigma t \to N_{x,CR}$, the buckling load of the plate. Eq. (ii) may then be written

$$w_C = \frac{\delta \sigma t}{N_{x,CR} - \sigma t} = \frac{\delta \sigma t / N_{x,CR}}{1 - \sigma t / N_{x,CR}}$$

from which

$$w_C = N_{x,CR} \frac{w_C}{\sigma t} - \delta \tag{iii}$$

Therefore, from Eq. (iii), a graph of w_C against $w_C/\sigma t$ will be a straight line of slope $N_{x,CR}$ and intercept δ, i.e. a Southwell plot.

S.9.2

The total potential energy of the plate is given by Eq. (9.1), i.e.

$$U + V = \frac{1}{2} \int_0^l \int_0^b \left[D \left\{ \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right)^2 - 2(1 - \nu) \left[\frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \, \partial y} \right)^2 \right] \right\} - N_x \left(\frac{\partial w}{\partial x} \right)^2 \right] \mathrm{d}x \, \mathrm{d}y \tag{i}$$

in which

$$w = a_{11} \sin \frac{m\pi x}{l} \sin^2 \frac{\pi y}{b} \tag{ii}$$

and

$$N_x = \sigma t$$

From Eq. (ii)

$$\frac{\partial w}{\partial x} = a_{11}\frac{m\pi}{l}\cos\frac{m\pi x}{l}\sin^2\frac{\pi y}{b}$$

$$\frac{\partial^2 w}{\partial x^2} = -a_{11}\frac{m^2\pi^2}{l^2}\sin\frac{m\pi x}{l}\sin^2\frac{\pi y}{b}$$

$$\frac{\partial^2 w}{\partial y^2} = a_{11}\frac{2\pi^2}{b^2}\sin\frac{m\pi x}{l}\cos\frac{2\pi y}{b}$$

$$\frac{\partial^2 w}{\partial x\,\partial y} = a_{11}\frac{m\pi^2}{bl}\cos\frac{m\pi x}{l}\sin\frac{2\pi y}{b}$$

Substituting these expressions in Eq. (i) and integrating gives

$$U + V = \frac{D}{2}a_{11}^2\pi^4\left(\frac{3m^4 b}{16l^3} + \frac{m^3}{2lb} + \frac{l}{b^3}\right) - \frac{3\sigma t a_{11}^2 m^2\pi^2 b}{32l}$$

The total potential energy of the plate has a stationary value in the neutral equilibrium of its buckled state, i.e. when $\sigma = \sigma_{\mathrm{CR}}$. Thus

$$\frac{\partial(U+V)}{\partial a_{11}} = Da_{11}\pi^4\left(\frac{3m^4 b}{16l^3} + \frac{m^2}{2lb} + \frac{l}{b^3}\right) - \frac{3\sigma_{\mathrm{CR}} t a_{11} m^2\pi^2 b}{16l} = 0$$

whence

$$\sigma_{\mathrm{CR}} = \frac{16l\pi^2 D}{3tm^2 b}\left(\frac{3m^4 b}{16l^3} + \frac{m^2}{2lb} + \frac{l}{b^3}\right) \qquad \text{(iii)}$$

When $l = 2b$, Eq. (iii) gives

$$\sigma_{\mathrm{CR}} = \frac{32\pi^2 D}{3tb^2}\left(\frac{3m^2}{128} + \frac{1}{4} + \frac{2}{m^2}\right) \qquad \text{(iv)}$$

σ_{CR} will be a minimum when $\mathrm{d}\sigma_{\mathrm{CR}}/\mathrm{d}m = 0$, i.e. when

$$\frac{6m}{128} - \frac{4}{m^3} = 0$$

or

$$m^4 = \frac{4 \times 128}{6}$$

from which

$$m = 3.04$$

i.e.

$$m = 3$$

Substituting this value of m in Eq. (iv)

$$\sigma_{CR} = \frac{71.9D}{tb^2}$$

whence

$$\sigma_{CR} = \frac{6E}{(1-\nu^2)}\left(\frac{t}{b}\right)^2$$

S.9.3

(a) The length, l, of the panel is appreciably greater than the dimension b so that failure will occur due to buckling rather than yielding. The modes of buckling will then be those described in Section 9.5.

(1) *Buckling as a column of length l*

Consider a stiffener and an associated portion of sheet as shown in Fig. S.9.3. The critical stress, σ_{CR}, is given by Eq. (8.8), i.e.

$$\sigma_{\mathrm{CR}} = \frac{\pi^2 E}{(l/r)^2} \tag{i}$$

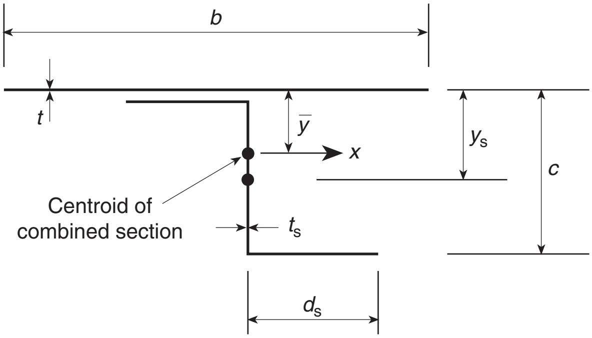

Fig. S.9.3

In Eq. (i) r is the radius of gyration of the combined section. Thus, $r = \sqrt{I_x/A}$, where A and I_x are the cross-sectional area and the second moment of area of the combined section respectively. From Fig. S.9.3

$$A = bt + t_s(2d + c) = bt + A_s \tag{ii}$$

Also

$$(bt + A_s)\bar{y} = A_s y_s$$

so that

$$\bar{y} = \frac{A_s y_s}{bt + A_s}$$

Then

$$I_x = bt(\bar{y})^2 + 2dt_s\left(\frac{c}{2}\right)^2 + \frac{t_s c^3}{12} + A_s(\bar{y} - y_s)^2$$

or

$$I_x = bt(\bar{y})^2 + t_s\frac{c^2}{2}\left(d + \frac{c}{6}\right) + A_s(\bar{y} - y_s)^2 \qquad \text{(iii)}$$

The radius of gyration follows from Eqs (ii) and (iii) and hence the critical stress from Eq. (i).

(2) *Buckling of the sheet between stiffeners*
The sheet may buckle as a long plate of length, l, and width, b, which is simply supported on all four edges. The buckling stress is then given by Eq. (9.7), i.e.

$$\sigma_{\text{CR}} = \frac{\eta k \pi^2 E}{12(1 - \nu^2)}\left(\frac{t}{b}\right)^2 \qquad \text{(iv)}$$

Since l is very much greater than b, k is equal to 4 (from Fig. 9.2). Therefore, assuming that buckling takes place in the elastic range ($\eta = 1$), Eq. (iv) becomes

$$\sigma_{\text{CR}} = \frac{4\pi^2 E}{12(1 - \nu^2)}\left(\frac{t}{b}\right)^2 \qquad \text{(v)}$$

(3) *Buckling of stiffener flange*
The stiffener flange may buckle as a long plate simply supported on three edges with one edge free. In this case $k = 0.43$ (see Fig. 9.3(a)) and, assuming elastic buckling (i.e. $\eta = 1$)

$$\sigma_{\text{CR}} = \frac{0.43\pi^2 E}{12(1 - \nu^2)}\left(\frac{t_s}{d_s}\right)^2 \qquad \text{(vi)}$$

(b) A suitable test would be a panel buckling test.

S.9.4

(a) Consider, initially, the buckling of the panel as a pin-ended column. For a section comprising a width of sheet and associated stiffener as shown in Fig. S.9.4,

$$A = 120 \times 3 + 30 \times 3.5 = 465\,\text{mm}^2$$

Then

$$465\bar{y} = 30 \times 3.5 \times 15 + 120 \times 3 \times 1.5$$

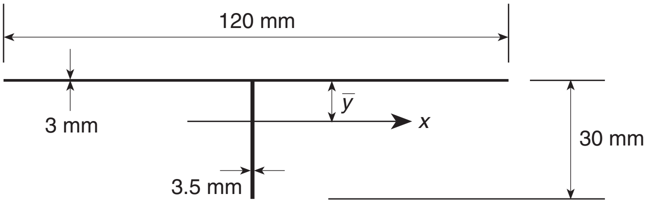

Fig. S.9.4

i.e.

$$\bar{y} = 4.5\,\text{mm}$$

Then

$$I_x = 120 \times 3 \times 4.5^2 + \frac{120 \times 3^3}{12} + \frac{3.5 \times 4.5^3}{3} + \frac{3.5 \times 25.5^3}{3}$$

i.e.

$$I_x = 27\,011\,\text{mm}^4$$

Hence

$$r = \sqrt{\frac{27\,011}{465}} = 7.62\,\text{mm}$$

From Eq. (8.8)

$$\sigma_{\text{CR}} = \frac{\pi^2 \times 70\,000}{(500/7.62)^2}$$

i.e.

$$\sigma_{\text{CR}} = 160.5\,\text{N/mm}^2$$

From Section 9.5 the equivalent skin thickness is

$$\bar{t} = \frac{30 \times 3.5}{120} + 3 = 3.875\,\text{mm}$$

Overall buckling of the panel will occur when

$$N_{x,\text{CR}} = \sigma_{\text{CR}}\bar{t} = 160.5 \times 3.875 = 621.9\,\text{N/mm} \qquad \text{(i)}$$

Buckling of the sheet will occur when, from Eq. (9.6)

$$\sigma_{\text{CR}} = 3.62E\left(\frac{t}{b}\right)^2 = 3.62 \times 70\,000\left(\frac{3}{120}\right)^2$$

i.e.

$$\sigma_{\text{CR}} = 158.4\,\text{N/mm}^2$$

Hence

$$N_{x,\mathrm{CR}} = 158.4 \times 3.875 = 613.8\,\mathrm{N/mm} \tag{ii}$$

Buckling of the stiffener will occur when, from Eq. (9.6)

$$\sigma_{\mathrm{CR}} = 0.385E\left(\frac{t}{b}\right)^2 = 0.385 \times 70\,000\left(\frac{3.5}{30}\right)^2$$

i.e.

$$\sigma_{\mathrm{CR}} = 366.8\,\mathrm{N/mm^2}$$

whence

$$N_{x,\mathrm{CR}} = 366.8 \times 3.875 = 1421.4\,\mathrm{N/mm} \tag{iii}$$

By comparison of Eqs (i), (ii) and (iii) the onset of buckling will occur when

$$N_{x,\mathrm{CR}} = 613.8\,\mathrm{N/mm}$$

(b) Since the stress in the sheet increases parabolically after reaching its critical value then

$$\sigma = CN_x^2 \tag{iv}$$

where C is some constant. From Eq. (iv)

$$\sigma_{\mathrm{CR}} = CN_{x,\mathrm{CR}}^2 \tag{v}$$

so that, combining Eqs (iv) and (v)

$$\frac{\sigma}{\sigma_{\mathrm{CR}}} = \left(\frac{N_x}{N_{x,\mathrm{CR}}}\right)^2 \tag{vi}$$

Suppose that $\sigma = \sigma_{\mathrm{F}}$, the failure stress, i.e. $\sigma_{\mathrm{F}} = 300\,\mathrm{N/mm^2}$. Then, from Eq. (vi)

$$N_{x,\mathrm{F}} = \sqrt{\frac{\sigma_{\mathrm{F}}}{\sigma_{\mathrm{CR}}}}N_{x,\mathrm{CR}}$$

or

$$N_{x,\mathrm{F}} = \sqrt{\frac{300}{158.4}} \times 613.8$$

i.e.

$$N_{x,\mathrm{F}} = 844.7\,\mathrm{N/mm}$$

S.9.5

The beam may be regarded as two cantilevers each of length 1.2 m, built-in at the mid-span section and carrying loads at their free ends of 5 kN. The analysis of a complete tension field beam in Section 9.7.1 therefore applies directly. From Eq. (9.29)

$$\tan^4 \alpha = \frac{1 + 1.5 \times 350/2 \times 300}{1 + 1.5 \times 300/280} = 0.7192$$

hence

$$\alpha = 42.6^\circ$$

From Eq. (9.19)

$$F_{\mathrm{T}} = \frac{5 \times 1.2 \times 10^3}{350} + \frac{5}{2 \tan 42.6^\circ}$$

i.e.

$$F_{\mathrm{T}} = 19.9\,\mathrm{kN}$$

From Eq. (9.23)

$$P = \frac{5 \times 300 \tan 42.6^\circ}{350}$$

i.e.

$$P = 3.9\,\mathrm{kN}$$

S.9.6

(i) The shear stress buckling coefficient for the web is given as $K = 7.70[1 + 0.75(b/d)^2]$. Thus Eq. (9.33) may be rewritten as

$$\tau_{\mathrm{CR}} = KE\left(\frac{t}{b}\right)^2 = 7.70\left[1 + 0.75\left(\frac{b}{d}\right)^2\right]E\left(\frac{t}{b}\right)^2$$

Hence

$$\tau_{\mathrm{CR}} = 7.70\left[1 + 0.75\left(\frac{250}{725}\right)^2\right] \times 70\,000\left(\frac{t}{250}\right)^2$$

i.e.

$$\tau_{\mathrm{CR}} = 9.39t^2 \tag{i}$$

The actual shear stress in the web, τ, is

$$\tau = \frac{100\,000}{750t} = \frac{133.3}{t} \tag{ii}$$

Two conditions occur, firstly

$$\tau \leq 165\,\text{N/mm}^2$$

so that, from Eq. (ii) $t = 0.81$ mm and secondly

$$\tau \leq 15\tau_{\text{CR}}$$

so that, from Eqs (i) and (ii)

$$15 \times 9.39t^2 = \frac{133.3}{t}$$

whence

$$t = 0.98\,\text{mm}$$

Therefore, from the range of standard thicknesses

$$t = 1.2\,\text{mm}$$

(ii) For $t = 1.2$ mm, τ_{CR} is obtained from Eq. (i) and is

$$\tau_{\text{CR}} = 13.5\,\text{N/mm}^2$$

and, from Eq. (ii), $\tau = 111.1\,\text{N/mm}^2$. Thus, $\tau/\tau_{\text{CR}} = 8.23$ and, from the table, the diagonal tension factor, k, is equal to 0.41.

The stiffener end load follows from Eq. (9.35) and is

$$Q_{\text{s}} = \sigma_{\text{s}} A_{\text{s}} = \frac{A_{\text{s}} k\tau \tan\alpha}{(A_{\text{s}}/tb) + 0.5(1-k)}$$

i.e.

$$Q_{\text{s}} = \frac{A_{\text{s}} \times 0.41 \times 111.1 \tan 40^\circ}{(A_{\text{s}}/1.2 \times 250) + 0.5(1 - 0.41)} = \frac{130A_{\text{s}}}{1 + 0.0113A_{\text{s}}}$$

The maximum secondary bending moment in the flanges is obtained from Eq. (9.25) multiplied by k, thus

$$\text{maximum secondary bending moment} = \frac{kWb^2 \tan\alpha}{12d}$$

i.e.

$$\text{maximum secondary bending moment} = \frac{0.41 \times 100\,000 \times 250^2 \times \tan 40^\circ}{12 \times 750}$$
$$= 238\,910\,\text{N/mm}$$

S.9.7

Stringer local instability:

The buckling stress will be less for the 31.8 mm side than for the 19.0 mm side. Then, from Eq. (9.6)

$$\sigma_{CR} = KE\left(\frac{t}{b}\right)^2 = 3.62 \times 69\,000\left(\frac{0.9}{31.8}\right)^2$$

i.e.

$$\sigma_{CR} = 200.1\,\text{N/mm}^2$$

Skin buckling:

Referring to Fig. P.9.7(a)

$$KE\left(\frac{t}{b}\right)^2 = 200.1$$

Then

$$b^2 = \frac{3.62 \times 69\,000 \times 1.6^2}{200.1}$$

i.e.

$$b = 56.5\,\text{mm}$$

Panel strut instability:

Consider stringer and skin as a strut. Add to stringer a length of skin equal to the lesser of $30t$ or b.

$$b = 56.5\,\text{mm}, \quad 30t = 30 \times 1.6 = 48.0\,\text{mm}$$

The section is then as shown in Fig. S.9.7

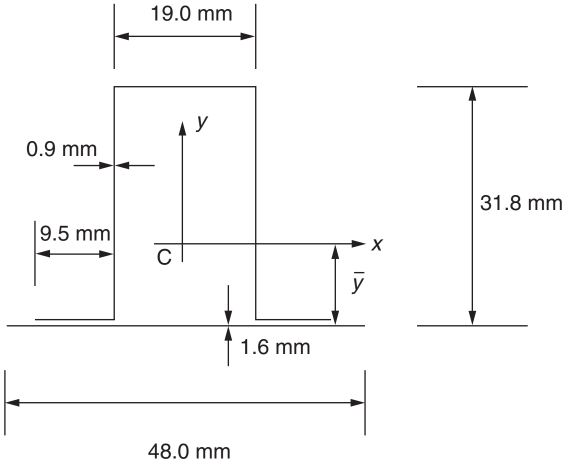

Fig. S.9.7

Taking moments of areas about the skin

$$[(19.0 + 2 \times 31.8 + 2 \times 9.5) \times 0.9 + 48 \times 1.6]\bar{y} = 19 \times 0.9 \times 31.8 + 2 \times 31.8 \times 0.9 \times 15.9$$

from which $\bar{y} = 8.6\,\text{mm}$.

Then

$$I_{xx} = 19.0 \times 0.9 \times 23.2^2 + 2\left(\frac{0.9 \times 31.8^3}{12} + 0.9 \times 31.8 \times 7.3^2\right) + 2 \times 9.5 \times 0.9 \times 8.6^2 + 48 \times 1.6 \times 8.6^2$$

i.e.

$$I_{xx} = 24\,022.7\,\text{mm}^4$$

From Eq. (8.5)

$$\sigma = \frac{\pi^2 \times 69\,000 \times 24\,022.7}{168.2\,L^2}$$

Therefore

$$L^2 = \frac{\pi^2 \times 69\,000 \times 24\,022.7}{168.2 \times 200.1}$$

i.e.

$$L = 697\,\text{mm}$$

say

$$L = 700\,\text{mm}$$

Solutions to Chapter 10 Problems

S.10.1

Referring to Fig. S.10.1(a), with unit load at D (1), $R_C = 2$. Then

$$M_1 = 1z \quad (0 \leq z \leq l)$$

$$M_1 = 1z - R_C(z - l) = 2l - z \quad (l \leq z \leq 2l)$$

$$M_1 = -1(z - 2l) \quad (2l \leq z \leq 3l)$$

$$M_2 = 0 \quad (0 \leq z \leq 2l)$$

$$M_2 = 1(z - 2l) \quad (2l \leq z \leq 3l)$$

Hence, from the first of Eqs (5.21)

$$\delta_{11} = \frac{1}{EI}\int_0^l M_1^2\,\mathrm{d}z + \frac{1}{EI}\int_l^{2l} M_1^2\,\mathrm{d}z + \frac{1}{EI}\int_{2l}^{3l} M_1^2\,\mathrm{d}z$$