## S.7.12

From Eq. (7.36) the deflection of the plate from its initial curved position is

$$w_1 = B_{11} \sin \frac{\pi x}{a} \sin \frac{\pi y}{b}$$

in which

$$B_{11} = \frac{A_{11} N_x}{\dfrac{\pi^2 D}{a^2}\left(1 + \dfrac{a^2}{b^2}\right)^2 - N_x}$$

The total deflection, $w$, of the plate is given by

$$w = w_1 + w_0$$

i.e.

$$w = \left[ \frac{A_{11} N_x}{\dfrac{\pi^2 D}{a^2}\left(1 + \dfrac{a^2}{b^2}\right)^2 - N_x} + A_{11} \right] \sin \frac{\pi x}{a} \sin \frac{\pi y}{b}$$

i.e.

$$w = \frac{A_{11}}{1 - \dfrac{N_x a^2}{\pi^2 D} \Big/ \left(1 + \dfrac{a^2}{b^2}\right)^2} \sin \frac{\pi x}{a} \sin \frac{\pi y}{b}$$

# Solutions to Chapter 8 Problems

## S.8.1

The forces on the bar AB are shown in Fig. S.8.1 where

$$M_B = K \left( \frac{dv}{dz} \right)_B \tag{i}$$

and $P$ is the buckling load.

From Eq. (8.1)

$$EI \frac{d^2 v}{dz^2} = -Pv \tag{ii}$$

The solution of Eq. (ii) is

$$v = A \cos \mu z + B \sin \mu z \tag{iii}$$

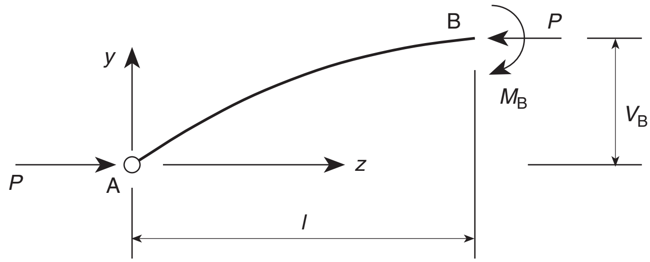


**Fig. S.8.1**

where $\mu^2 = P/EI$.

When $z = 0$, $v = 0$ so that, from Eq. (iii), $A = 0$. Hence

$$v = B \sin \mu z \tag{iv}$$

Then

$$\frac{\mathrm{d}v}{\mathrm{d}z} = \mu B \cos \mu z$$

and when $z = l$, $\mathrm{d}v/\mathrm{d}z = M_B/K$ from Eq. (i). Thus

$$B = \frac{M_B}{\mu K \cos \mu l}$$

and Eq. (iv) becomes

$$v = \frac{M_B}{\mu K \cos \mu l} \sin \mu z \tag{v}$$

Also, when $z = l$, $Pv_B = M_B$ from equilibrium. Hence, substituting in Eq. (v) for $M_B$

$$v_B = \frac{Pv_B}{\mu K \cos \mu l} \sin \mu l$$

from which

$$P = \frac{\mu K}{\tan \mu l} \tag{vi}$$

(a) When $K \to \infty$, $\tan \mu l \to \infty$ and $\mu l \to \pi/2$, i.e.

$$\sqrt{\frac{P}{EI}} l \to \frac{\pi}{2}$$

from which

$$P \to \frac{\pi^2 EI}{4l^2}$$

which is the Euler buckling load of a pin-ended column of length $2l$.

(b) When $EI \to \infty$, $\tan \mu l \to \mu l$ and Eq. (vi) becomes $P = K/l$ and the bars remain straight.

## S.8.2

Suppose that the buckling load of the column is $P$. Then from Eq. (8.1) and referring to Fig. S.8.2, in AB

$$EI\frac{d^2v}{dz^2} = -Pv \tag{i}$$

and in BC

$$4EI\frac{d^2v}{dz^2} = -Pv \tag{ii}$$

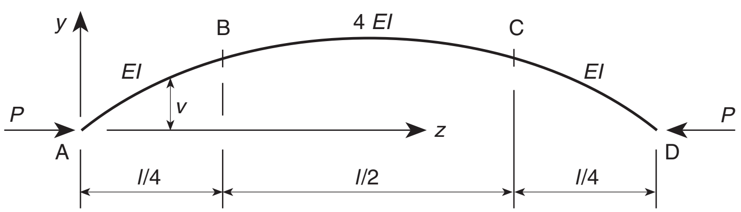


**Fig. S.8.2**

The solutions of Eqs (i) and (ii) are, respectively

$$v_{AB} = A\cos\mu z + B\sin\mu z \tag{iii}$$

$$v_{BC} = C\cos\frac{\mu}{2}z + D\sin\frac{\mu}{2}z \tag{iv}$$

in which

$$\mu^2 = \frac{P}{EI}$$

When $z = 0$, $v_{AB} = 0$ so that, from Eq. (iii), $A = 0$. Thus

$$v_{AB} = B\sin\mu z \tag{v}$$

Also, when $z = l/2$, $(dv/dz)_{BC} = 0$. Hence, from Eq. (iv)

$$0 = -\frac{\mu}{2}C\sin\frac{\mu l}{4} + \frac{\mu}{2}D\cos\frac{\mu l}{4}$$

whence

$$D = C\tan\frac{\mu l}{4}$$

Then

$$v_{BC} = C\left(\cos\frac{\mu}{2}z + \tan\frac{\mu l}{4}\sin\frac{\mu}{2}z\right) \tag{vi}$$

When $z = l/4$, $v_{AB} = v_{BC}$ so that, from Eqs (v) and (vi)

$$B \sin \frac{\mu l}{4} = C \left( \cos \frac{\mu l}{8} + \tan \frac{\mu l}{4} \sin \frac{\mu l}{8} \right)$$

which simplifies to

$$B \sin \frac{\mu l}{4} = C \sec \frac{\mu l}{4} \cos \frac{\mu l}{8} \tag{vii}$$

Further, when $z = l/4$, $(\mathrm{d}v/\mathrm{d}z)_{AB} = (\mathrm{d}v/\mathrm{d}z)_{BC}$. Again from Eqs (v) and (vi)

$$\mu B \cos \frac{\mu l}{4} = C \left( -\frac{\mu}{2} \sin \frac{\mu l}{8} + \frac{\mu}{2} \tan \frac{\mu l}{4} \cos \frac{\mu l}{8} \right)$$

from which

$$B \cos \frac{\mu l}{4} = \frac{C}{2} \sec \frac{\mu l}{4} \sin \frac{\mu l}{8} \tag{viii}$$

Dividing Eq. (vii) by Eq. (viii) gives

$$\tan \frac{\mu l}{4} = 2 \Big/ \tan \frac{\mu l}{8}$$

or

$$\tan \frac{\mu l}{4} \tan \frac{\mu l}{8} = 2$$

Hence

$$\frac{2 \tan^2 \mu l/8}{1 - \tan^2 \mu l/8} = 2$$

from which

$$\tan \frac{\mu l}{8} = \frac{1}{\sqrt{2}}$$

and

$$\frac{\mu l}{8} = 35.26^\circ = 0.615 \,\text{rad}$$

i.e.

$$\sqrt{\frac{P}{EI}} \frac{l}{8} = 0.615$$

so that

$$P = \frac{24.2EI}{l^2}$$

## S.8.3

With the spring in position the forces acting on the column in its buckled state are shown in Fig. S.8.3. Thus, from Eq. (8.1)

$$EI\frac{d^2v}{dz^2} = 4P(\delta - v) - k\delta(l - z) \tag{i}$$

The solution of Eq. (i) is

$$v = A\cos\mu z + B\sin\mu z + \frac{\delta}{4P}[4P + k(z - l)] \tag{ii}$$

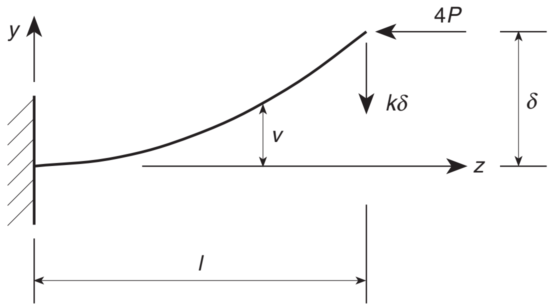


**Fig. S.8.3**

where

$$\mu^2 = \frac{4P}{EI}$$

When $z = 0$, $v = 0$, hence, from Eq. (ii)

$$0 = A + \frac{\delta}{4P}(4P - kl)$$

from which

$$A = \frac{\delta(kl - 4P)}{4P}$$

Also when $z = 0$, $dv/dz = 0$ so that, from Eq. (ii)

$$0 = \mu B + \frac{\delta k}{4P}$$

and

$$B = \frac{-\delta k}{4P\mu}$$

Eq. (ii) then becomes

$$v = \frac{\delta}{4P}\left[(kl - 4P)\cos\mu z - \frac{k}{\mu}\sin\mu z + 4P + k(z - l)\right] \qquad \text{(iii)}$$

When $z = l$, $v = \delta$. Substituting in Eq. (iii) gives

$$\delta = \frac{\delta}{4P}\left[(kl - 4P)\cos\mu l - \frac{k}{\mu}\sin\mu l + 4P\right]$$

from which

$$k = \frac{4P\mu}{\mu l - \tan\mu l}$$

## S.8.4

The compressive load $P$ will cause the column to be displaced from its initial curved position to that shown in Fig. S.8.4. Then, from Eq. (8.1) and noting that the bending moment at any point in the column is proportional to the change in curvature produced (see Eq. (8.22))

$$EI\frac{d^2 v}{dz^2} - EI\frac{d^2 v_0}{dz^2} = -Pv \qquad \text{(i)}$$

Now

$$v_0 = a\frac{4z}{l^2}(l - z)$$

so that

$$\frac{d^2 v_0}{dz^2} = -\frac{8a}{l^2}$$

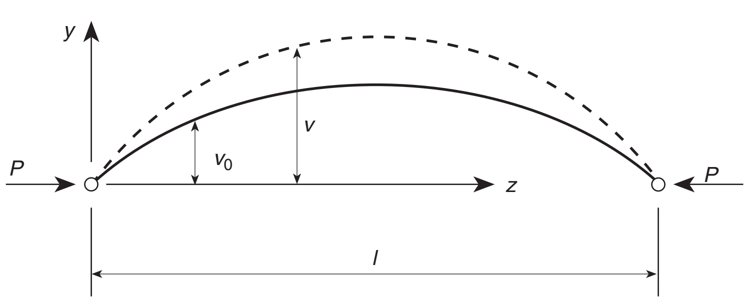


**Fig. S.8.4**

and Eq. (i) becomes

$$\frac{d^2 v}{dz^2} + \frac{P}{EI}v = -\frac{8a}{l^2} \qquad \text{(ii)}$$

The solution of Eq. (ii) is

$$v = A\cos\lambda z + B\sin\lambda z - 8a/(\lambda l)^2 \tag{iii}$$

where $\lambda^2 = P/EI$.

When $z = 0$, $v = 0$ so that $A = 8a/(\lambda l)^2$. When $z = l/2$, $\mathrm{d}v/\mathrm{d}z = 0$. Thus, from Eq. (iii)

$$0 = -\lambda A \sin\frac{\lambda l}{2} + \lambda B \cos\frac{\lambda l}{2}$$

whence

$$B = \frac{8a}{(\lambda l)^2}\tan\frac{\lambda l}{2}$$

Eq. (iii) then becomes

$$v = \frac{8a}{(\lambda l)^2}\left(\cos\lambda z + \tan\frac{\lambda l}{2}\sin\lambda z - 1\right) \tag{iv}$$

The maximum bending moment occurs when $v$ is a maximum at $z = l/2$. Then, from Eq. (iv)

$$M(\max) = -Pv_{\max} = -\frac{8aP}{(\lambda l)^2}\left(\cos\frac{\lambda l}{2} + \tan\frac{\lambda l}{2}\sin\frac{\lambda l}{2} - 1\right)$$

from which

$$M(\max) = -\frac{8aP}{(\lambda l)^2}\left(\sec\frac{\lambda l}{2} - 1\right)$$

## S.8.5

Under the action of the compressive load $P$ the column will be displaced to the position shown in Fig. S.8.5. As in P.8.4 the bending moment at any point is proportional to the change in curvature. Then, from Eq. (8.1)

$$EI\frac{\mathrm{d}^2 v}{\mathrm{d}z^2} - EI\frac{\mathrm{d}^2 v_0}{\mathrm{d}z^2} = -Pv \tag{i}$$

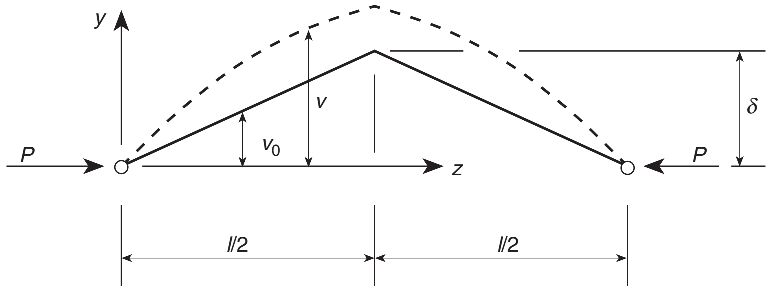


Fig. S.8.5

In this case, since each half of the column is straight before the application of $P$, $\mathrm{d}^2 v_0/\mathrm{d}z^2 = 0$ and Eq. (i) reduces to

$$EI\frac{\mathrm{d}^2 v}{\mathrm{d}z^2} = -Pv \tag{ii}$$

The solution of Eq. (ii) is

$$v = A\cos\mu z + B\sin\mu z \tag{iii}$$

in which $\mu^2 = P/EI$. When $z = 0$, $v = 0$ so that $A = 0$ and Eq. (iii) becomes

$$v = B\sin\mu z \tag{iv}$$

The slope of the column at its mid-point in its unloaded position is $2\delta/l$. This must be the slope of the column at its mid-point in its loaded state since a change of slope over zero distance would require an infinite bending moment. Thus, from Eq. (iv)

$$\frac{\mathrm{d}v}{\mathrm{d}z} = \frac{2\delta}{l} = \mu B\cos\frac{\mu l}{2}$$

so that

$$B = \frac{2\delta}{\mu l\cos(\mu l/2)}$$

and

$$v = \frac{2\delta}{\mu l\cos(\mu l/2)}\sin\mu z \tag{v}$$

The maximum bending moment will occur when $v$ is a maximum, i.e. at the mid-point of the column. Then

$$M(\text{max}) = -Pv_{\text{max}} = -\frac{2P\delta}{\mu l\cos(\mu l/2)}\sin\frac{\mu l}{2}$$

from which

$$M(\text{max}) = -P\frac{2\delta}{l}\sqrt{\frac{EI}{P}}\tan\sqrt{\frac{P}{EI}}\frac{l}{2}$$

## S.8.6

Referring to Fig. S.8.6 the bending moment at any section $z$ is given by

$$M = P(e+v) - \frac{wl}{2}z + w\frac{z^2}{2}$$

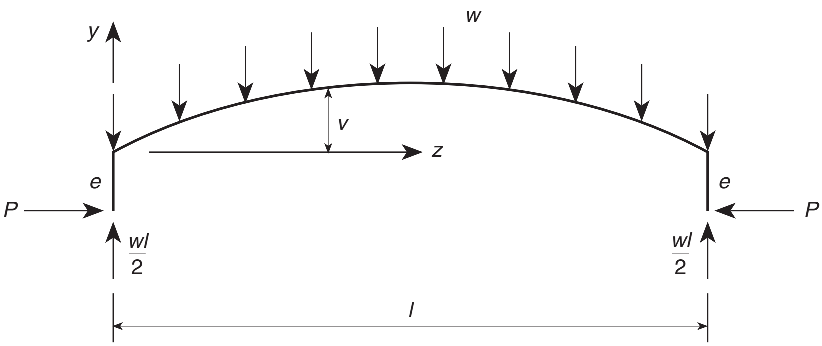


**Fig. S.8.6**

or

$$M = P(e + v) + \frac{w}{2}(z^2 - lz) \tag{i}$$

Substituting for $M$ in Eq. (8.1)

$$EI\frac{\mathrm{d}^2 v}{\mathrm{d}z^2} + Pv = -Pe - \frac{w}{2}(z^2 - lz)$$

or

$$\frac{\mathrm{d}^2 v}{\mathrm{d}z^2} + \mu^2 v = -\mu^2 e - \frac{w\mu^2}{2P}(z^2 - lz) \tag{ii}$$

The solution of Eq. (ii) is

$$v = A\cos\mu z + B\sin\mu z - e + \frac{w}{2P}(lz - z^2) + \frac{w}{\mu^2 P} \tag{iii}$$

When $z = 0$, $v = 0$, hence $A = e - w/\mu^2 P$. When $z = l/2$, $\mathrm{d}v/\mathrm{d}z = 0$ which gives

$$B = A\tan\frac{\mu l}{2} = \left(e - \frac{w}{\mu^2 P}\right)\tan\frac{\mu l}{2}$$

Eq. (iii) then becomes

$$v = \left(e - \frac{w}{\mu^2 P}\right)\left[\frac{\cos\mu(z - l/2)}{\cos\mu l/2} - 1\right] + \frac{w}{2P}(lz - z^2) \tag{iv}$$

The maximum bending moment will occur at mid-span where $z = l/2$ and $v = v_{\max}$. From Eq. (iv)

$$v_{\max} = \left(e - \frac{EIw}{P^2}\right)\left(\sec\frac{\mu l}{2} - 1\right) + \frac{wl^2}{8P}$$

and from Eq. (i)

$$M(\max) = Pe + Pv_{\max} - \frac{wl^2}{8}$$

whence

$$M(\max) = \left(Pe - \frac{w}{\mu^2}\right)\sec\frac{\mu l}{2} + \frac{w}{\mu^2} \qquad \text{(v)}$$

For the maximum bending moment to be as small as possible the bending moment at the ends of the column must be numerically equal to the bending moment at mid-span. Thus

$$Pe + \left(Pe - \frac{w}{\mu^2}\right)\sec\frac{\mu l}{2} + \frac{w}{\mu^2} = 0$$

or

$$Pe\left(1 + \sec\frac{\mu l}{2}\right) = \frac{w}{\mu^2}\left(\sec\frac{\mu l}{2} - 1\right)$$

Then

$$e = \frac{w}{P\mu^2}\left(\frac{1 - \cos\mu l/2}{1 + \cos\mu l/2}\right)$$

i.e.

$$e = \left(\frac{w}{P\mu^2}\right)\tan^2\frac{\mu l}{4} \qquad \text{(vi)}$$

From Eq. (vi) the end moment is

$$Pe = \frac{w}{\mu^2}\tan^2\frac{\mu l}{4} = \frac{wl^2}{16}\left(\frac{\tan\mu l/4}{\mu l/4}\right)\left(\frac{\tan\mu l/4}{\mu l/4}\right)$$

When $P \to 0$, $\tan\mu l/4 \to \mu l/4$ and the end moment becomes $wl^2/16$.

## S.8.7

From Eq. (8.21) the buckling stress, $\sigma_b$, is given by

$$\sigma_b = \frac{\pi^2 E_t}{(l/r)^2} \qquad \text{(i)}$$

The stress–strain relationship is

$$10.5 \times 10^6\varepsilon = \sigma + 21\,000\left(\frac{\sigma}{49\,000}\right)^{16} \qquad \text{(ii)}$$

Hence

$$10.5 \times 10^6\frac{d\varepsilon}{d\sigma} = 1 + \frac{16 \times 21\,000}{(49\,000)^{16}}\sigma^{15}$$

from which

$$E_t = \frac{d\sigma}{d\varepsilon} = \frac{10.5 \times 10^6 \times (49\,000)^{16}}{(49\,000)^{16} + 16 \times 21\,000(\sigma)^{15}}$$

Then, from Eq. (i)

$$\left(\frac{l}{r}\right)^2 = \frac{\pi^2 E_t}{\sigma_b} = \frac{10.36 \times 10^7}{\sigma_b + 336\,000(\sigma_b/49\,000)^{16}} \qquad \text{(iii)}$$

From Eq. (iii) the following $\sigma_b$–$(l/r)$ relationship is found

| | | | | | |
|---|---|---|---|---|---|
| $\sigma_b$ | 4900 | $3 \times 4900$ | $6 \times 4900$ | $9 \times 4900$ | 49 000 |
| $l/r$ | 145.4 | 84.0 | 59.3 | 31.2 | 16.4 |

For the given strut

$$r^2 = \frac{I}{A} = \frac{\pi(D^4 - d^4)/64}{\pi(D^2 - d^2)/4} = \frac{1}{16}(D^2 + d^2)$$

i.e.

$$r^2 = \frac{1}{16}(1.5^2 + 1.34^2) = 0.253\,\text{units}^2$$

Hence

$$r = 0.503\,\text{units}$$

Thus

$$\frac{l}{r} = \frac{20}{0.503} = 39.8$$

Then, from the $\sigma_b$–$(l/r)$ relationship

$$\sigma_b = 40\,500 \text{ force units/units}^2$$

Hence the buckling load is

$$40\,500 \times \frac{\pi}{4}(1.5^2 - 1.34^2)$$

i.e.

$$\text{Buckling load} = 14\,454 \text{ force units}$$

## S.8.8

The deflected shape of each of the members AB and BC is shown in Fig. S.8.8. For the member AB and from Eq. (8.1)

$$EI\frac{d^2 v_1}{dz_1^2} = -M_B$$

so that

$$EI\frac{dv_1}{dz_1} = -M_B z_1 + A$$

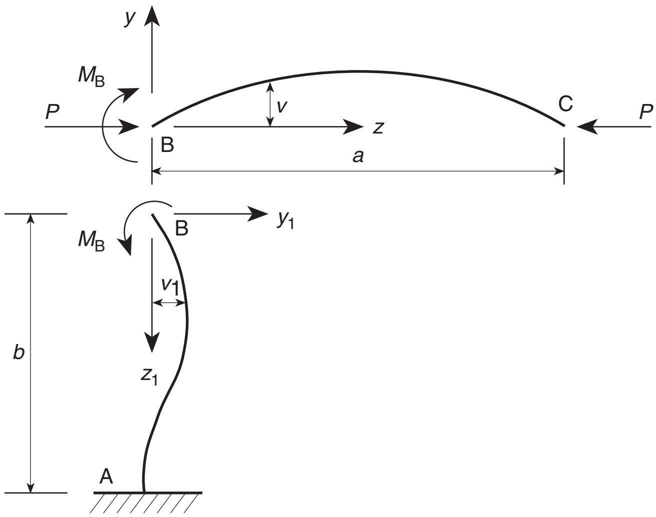


**Fig. S.8.8**

When $z_1 = b$, $dv_1/dz_1 = 0$. Thus $A = M_B b$ and

$$EI\frac{dv_1}{dz_1} = -M_B(z_1 - b) \tag{i}$$

At B, when $z_1 = 0$, Eq. (i) gives

$$\frac{dv_1}{dz_1} = \frac{M_B b}{EI} \tag{ii}$$

In BC Eq. (8.1) gives

$$EI\frac{d^2v}{dz^2} = -Pv + M_B$$

or

$$EI\frac{d^2v}{dz^2} + Pv = M_B \tag{iii}$$

The solution of Eq. (iii) is

$$v = B\cos\lambda z + C\sin\lambda z + M_B/P \tag{iv}$$

When $z = 0$, $v = 0$ so that $B = -M_B/P$.
When $z = a/2$, $dv/dz = 0$ so that

$$C = B\tan\frac{\lambda a}{2} = -\frac{M_B}{P}\tan\frac{\lambda a}{2}$$

Eq. (iv) then becomes

$$v = -\frac{M_B}{P}\left(\cos\lambda z + \tan\frac{\lambda a}{2}\sin\lambda z - 1\right)$$

so that

$$\frac{\mathrm{d}v}{\mathrm{d}z} = -\frac{M_{\mathrm{B}}}{P}\left(-\lambda \sin \lambda z + \lambda \tan \frac{\lambda a}{2} \cos \lambda z\right)$$

At B, when $z = 0$,

$$\frac{\mathrm{d}v}{\mathrm{d}z} = -\frac{M_{\mathrm{B}}}{P}\lambda \tan \frac{\lambda a}{2} \tag{v}$$

Since $\mathrm{d}v_1/\mathrm{d}z_1 = \mathrm{d}v/\mathrm{d}z$ at B then, from Eqs (ii) and (v)

$$\frac{b}{EI} = -\frac{\lambda}{P} \tan \frac{\lambda a}{2}$$

whence

$$\frac{\lambda a}{2} = -\frac{1}{2}\left(\frac{a}{b}\right) \tan \frac{\lambda a}{2}$$

## S.8.9

In an identical manner to S.8.4

$$EI\frac{\mathrm{d}^2 v'}{\mathrm{d}z^2} - EI\frac{\mathrm{d}^2 v}{\mathrm{d}z^2} = -Pv'$$

where $v'$ is the total displacement from the horizontal. Thus

$$\frac{\mathrm{d}^2 v'}{\mathrm{d}z^2} + \frac{P}{EI}v' = \frac{\mathrm{d}^2 v}{\mathrm{d}z^2}$$

or, since

$$\frac{\mathrm{d}^2 v}{\mathrm{d}z^2} = -\frac{\pi^2}{l^2}\delta \sin \frac{\pi}{l}z \quad \text{and} \quad \mu^2 = \frac{P}{EI}$$

$$\frac{\mathrm{d}^2 v'}{\mathrm{d}z^2} + \mu^2 v' = -\frac{\pi^2}{l^2}\delta \sin \frac{\pi z}{l} \tag{i}$$

The solution of Eq. (i) is

$$v' = A \cos \mu z + B \sin \mu z + \frac{\pi^2 \delta}{\pi^2 - \mu^2 l^2} \sin \frac{\pi z}{l} \tag{ii}$$

When $z = 0$ and $l$, $v' = 0$, hence $A = B = 0$ and Eq. (ii) becomes

$$v' = \frac{\pi^2 \delta}{\pi^2 - \mu^2 l^2} \sin \frac{\pi z}{l}$$

The maximum bending moment occurs at the mid-point of the tube so that

$$M(\max) = Pv' = P\frac{\pi^2\delta}{\pi^2 - \mu^2 l^2} = \frac{P\delta}{1 - Pl^2/\pi^2 EI}$$

i.e.

$$M(\max) = \frac{P\delta}{1 - P/P_e} = \frac{P\delta}{1 - \alpha}$$

The total maximum direct stress due to bending and axial load is then

$$\sigma(\max) = \frac{P}{\pi dt} + \left(\frac{P\delta}{1-\alpha}\right)\frac{d/2}{\pi d^3 t/8}$$

Hence

$$\sigma(\max) = \frac{P}{\pi dt}\left(1 + \frac{1}{1-\alpha}\frac{4\delta}{d}\right)$$

## S.8.10

The forces acting on the members AB and BC are shown in Fig. S.8.10

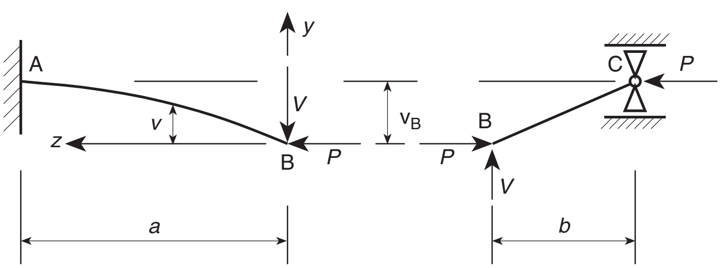


**Fig. S.8.10**

Considering first the moment equilibrium of BC about C

$$Pv_B = Vb$$

from which

$$v_B = \frac{Vb}{P} \tag{i}$$

For the member AB and from Eq. (8.1)

$$EI\frac{d^2v}{dz^2} = -Pv - Vz$$

or

$$\frac{d^2v}{dz^2} + \frac{P}{EI}v = -\frac{Vz}{EI} \tag{ii}$$

The solution of Eq. (ii) is

$$v = A\cos\lambda z + B\sin\lambda z - \frac{Vz}{P} \qquad \text{(iii)}$$

When $z=0$, $v=0$ so that $A=0$. Also when $z=a$, $\mathrm{d}v/\mathrm{d}z=0$, hence

$$0 = \lambda B\cos\lambda a - \frac{V}{P}$$

from which

$$B = \frac{V}{\lambda P\cos\lambda a}$$

and Eq. (iii) becomes

$$v = \frac{V}{P}\left(\frac{\sin\lambda z}{\lambda\cos\lambda a} - z\right)$$

When $z=a$, $v=v_{\mathrm{B}}=Vb/P$ from Eq. (i). Thus

$$\frac{Vb}{P} = \frac{V}{P}\left(\frac{\sin\lambda a}{\lambda\cos\lambda a} - a\right)$$

from which

$$\lambda(a+b) = \tan\lambda a$$

## S.8.11

The bending moment, $M$, at any section of the column is given by

$$M = P_{\mathrm{CR}}v = P_{\mathrm{CR}}k(lz - z^2) \qquad \text{(i)}$$

Also

$$\frac{\mathrm{d}v}{\mathrm{d}z} = k(l-2z) \qquad \text{(ii)}$$

Substituting from Eqs (i) and (ii) in Eq. (8.47)

$$U + V = \frac{P_{\mathrm{CR}}^2k^2}{2E}\left\{\frac{1}{I_1}\int_0^a (lz-z^2)^2\mathrm{d}z + \frac{1}{I_2}\int_a^{l-a}(lz-z^2)^2\mathrm{d}z + \frac{1}{I_1}\int_{l-a}^{l}(lz-z^2)^2\mathrm{d}z\right\}$$
$$- \frac{P_{\mathrm{CR}}k^2}{2}\int_0^l (l-2z)^2\mathrm{d}z$$

i.e.

$$U+V=\frac{P_{\mathrm{CR}}^2k^2}{2E}\left\{\frac{1}{I_1}\left[\frac{l^2z^3}{3}-\frac{lz^4}{2}+\frac{z^5}{5}\right]_0^a+\frac{1}{I_2}\left[\frac{l^2z^3}{3}-\frac{lz^4}{2}+\frac{z^5}{5}\right]_a^{l-a}\right.$$

$$\left.+\frac{1}{I_1}\left[\frac{l^2z^3}{3}-\frac{lz^4}{2}+\frac{l^5}{5}\right]_{l-a}^{l}\right\}-\frac{P_{\mathrm{CR}}k^2}{2}\left[l^2z-2lz^2+\frac{4z^3}{3}\right]_0^l$$

i.e.

$$U+V=\frac{P_{\mathrm{CR}}^2k^2}{2EI_2}\left\{\left(\frac{I_2}{I_1}-1\right)\left[\frac{l^2a^3}{3}-\frac{la^4}{2}+\frac{a^5}{5}-\frac{l^2(l-a)^3}{3}+\frac{l(l-a)^4}{2}-\frac{(l-a)^5}{5}\right]\right.$$

$$\left.+\frac{I_2}{I_1}\frac{l^5}{30}\right\}-\frac{P_{\mathrm{CR}}k^2l^3}{6}$$

From the principle of the stationary value of the total potential energy

$$\frac{\partial(U+V)}{\partial k}=\frac{P_{\mathrm{CR}}^2k}{EI_2}\left\{\left(\frac{I_2}{I_1}-1\right)\left[\frac{l^2a^3}{3}-\frac{la^4}{2}+\frac{a^5}{5}-\frac{l^2(l-a)^3}{3}\right.\right.$$

$$\left.\left.+\frac{l(l-a)^4}{2}-\frac{(l-a)^5}{5}\right]+\frac{I_2}{I_1}\frac{l^5}{30}\right\}-\frac{P_{\mathrm{CR}}kl^3}{3}=0$$

Hence

$$P_{\mathrm{CR}}=\frac{EI_2l^3}{3\left\{\left(\frac{I_2}{I_1}-1\right)\left[\frac{l^2a^3}{3}-\frac{la^4}{2}+\frac{a^5}{5}-\frac{l^2(l-a)^3}{3}+\frac{l(l-a)^4}{2}-\frac{(l-a)^5}{5}\right]+\frac{I_2}{I_1}\frac{l^5}{30}\right\}} \tag{iii}$$

When $I_2=1.6I_1$ and $a=0.2l$, Eq. (iii) becomes

$$P_{\mathrm{CR}}=\frac{14.96EI_1}{l^2} \tag{iv}$$

Without the reinforcement

$$P_{\mathrm{CR}}=\frac{\pi^2EI_1}{l^2} \tag{v}$$

Therefore, from Eqs (iv) and (v) the increase in strength is

$$\frac{EI_1}{l^2}(14.96-\pi^2)$$

Thus the percentage increase in strength is

$$\left[\frac{EI}{l^2}(14.96-\pi^2)\bigg/\frac{l^2}{\pi^2 EI}\right]\times 100 = 52\%$$

Since the radius of gyration of the cross-section of the column remains unchanged

$$I_1 = A_1 r^2 \quad \text{and} \quad I_2 = A_2 r^2$$

Hence

$$\frac{A_2}{A_1} = \frac{I_2}{I_1} = 1.6 \tag{vi}$$

The original weight of the column is $lA_1\rho$ where $\rho$ is the density of the material of the column. Then, the increase in weight $= 0.4lA_1\rho + 0.6lA_2\rho - lA_1\rho = 0.6l\rho(A_2 - A_1)$.

Substituting for $A_2$ from Eq. (vi)

$$\text{Increase in weight} = 0.6l\rho(1.6A_1 - A_1) = 0.36lA_1\rho$$

i.e. an increase of 36%.

## S.8.12

The equation for the deflected centre line of the column is

$$v = \frac{4\delta}{l^2}z^2 \tag{i}$$

in which $\delta$ is the deflection at the ends of the column relative to its centre and the origin for $z$ is at the centre of the column. Also, the second moment of area of its cross-section varies, from the centre to its ends, in accordance with the relationship

$$I = I_1\left(1 - 1.6\frac{z}{l}\right) \tag{ii}$$

At any section of the column the bending moment, $M$, is given by

$$M = P_{\text{CR}}(\delta - v) = P_{\text{CR}}\delta\left(1 - 4\frac{z^2}{l^2}\right) \tag{iii}$$

Also, from Eq. (i)

$$\frac{dv}{dz} = \frac{8\delta}{l^2}z \tag{iv}$$

Substituting in Eq. (8.47) for $M$, $I$ and $dv/dz$

$$U + V = 2\int_0^{l/2}\frac{P_{\text{CR}}^2\delta^2(1-4z^2/l^2)^2}{2EI_1(1-1.6z/l)}dz - \frac{P_{\text{CR}}}{2}2\int_0^{l/2}\frac{64\delta^2}{l^4}z^2 dz$$

or

$$U + V = \frac{P_{CR}^2 \delta^2}{EI_1 l^3} \int_0^{l/2} \frac{(l^2 - 4z^2)^2}{(l - 1.6z)} dz - \frac{64 P_{CR} \delta^2}{l^4} \int_0^{l/2} z^2 dz \qquad \text{(v)}$$

Dividing the numerator by the denominator in the first integral in Eq. (v) gives

$$U + V = \frac{P_{CR}^2 \delta^2}{EI_1 l^3} \left[ \int_0^{l/2} (-10z^3 - 6.25lz^2 + 1.09l^2 z + 0.683 l^3) dz \right.$$
$$\left. + 0.317 l^3 \int_0^{l/2} \frac{dz}{(1 - 1.6z/l)} \right] - \frac{64 P_{CR} \delta^2}{l^4} \left[ \frac{z^3}{3} \right]_0^{l/2}$$

Hence

$$U + V = \frac{P_{CR}^2 \delta^2}{EIl^3} \left[ -10\frac{z^4}{4} - 6.25l\frac{z^3}{3} + 1.09l^2\frac{z^2}{2} + 0.683l^3 z \right.$$
$$\left. - \frac{0.317}{1.6} l^4 \log_e \left( 1 - \frac{1.6z}{l} \right) \right]_0^{l/2} - \frac{8 P_{CR} \delta^2}{3l}$$

i.e.

$$U + V = \frac{0.3803 P_{CR}^2 \delta^2 l}{EI_1} - \frac{8 P_{CR} \delta^2}{3l}$$

From the principle of the stationary value of the total potential energy

$$\frac{\partial(U + V)}{\partial \delta} = \frac{0.7606 P_{CR}^2 \delta l}{EI_1} - \frac{16 P_{CR} \delta}{3l} = 0$$

Hence

$$P_{CR} = \frac{7.01 EI_1}{l^2}$$

For a column of constant thickness and second moment of area $I_2$,

$$P_{CR} = \frac{\pi^2 EI_2}{l^2} \quad \text{(see Eq. (8.5))}$$

For the columns to have the same buckling load

$$\frac{\pi^2 EI_2}{l^2} = \frac{7.01 EI_1}{l^2}$$

so that

$$I_2 = 0.7 I_1$$

Thus, since the radii of gyration are the same

$$A_2 = 0.7 A_1$$

Therefore, the weight of the constant thickness column is equal to $\rho A_2 l = 0.7\rho A_1 l$.
The weight of the tapered column $= \rho \times$ average thickness $\times\, l = \rho \times 0.6 A_1 l$.
Hence the saving in weight $= 0.7\rho A_1 l - 0.6\rho A_1 l = 0.1\rho A_1 l$.
Expressed as a percentage

$$\text{saving in weight} = \frac{0.1\rho A_1 l}{0.7\rho A_1 l} \times 100 = 14.3\%$$

## S.8.13

There are four boundary conditions to be satisfied, namely, $v = 0$ at $z = 0$ and $z = l$, $\mathrm{d}v/\mathrm{d}z = 0$ at $z = 0$ and $\mathrm{d}^2 v/\mathrm{d}z^2$ (i.e. bending moment) $= 0$ at $z = l$. Thus, since only one arbitrary constant may be allowed for, there cannot be more than five terms in the polynomial. Suppose

$$v = a_0 + a_1\left(\frac{z}{l}\right) + a_2\left(\frac{z}{l}\right)^2 + a_3\left(\frac{z}{l}\right)^3 + a_4\left(\frac{z}{l}\right)^4 \qquad \text{(i)}$$

Then, since $v = 0$ at $z = 0$, $a_0 = 0$. Also, since $\mathrm{d}v/\mathrm{d}z = 0$ at $z = 0$, $a_1 = 0$. Hence, Eq. (i) becomes

$$v = a_2\left(\frac{z}{l}\right)^2 + a_3\left(\frac{z}{l}\right)^3 + a_4\left(\frac{z}{l}\right)^4 \qquad \text{(ii)}$$

When $z = l$, $v = 0$, thus

$$0 = a_2 + a_3 + a_4 \qquad \text{(iii)}$$

When $z = l$, $\mathrm{d}^2 v/\mathrm{d}z^2 = 0$, thus

$$0 = a_2 + 3a_3 + 6a_4 \qquad \text{(iv)}$$

Subtracting Eq. (iv) from Eq. (ii)

$$0 = -2a_3 - 5a_4$$

from which $a_3 = -5a_4/2$.

Substituting for $a_3$ in Eq. (iii) gives $a_4 = 2a_2/3$ so that $a_3 = -5a_2/3$. Eq. (ii) then becomes

$$v = a_2\left(\frac{z}{l}\right)^2 - \frac{5a_2}{3}\left(\frac{z}{l}\right)^3 + \frac{2a_2}{3}\left(\frac{z}{l}\right)^4 \qquad \text{(v)}$$

Then

$$\frac{\mathrm{d}v}{\mathrm{d}z} = 2a_2\frac{z}{l} - 5a_2\frac{z^2}{l^3} + \frac{8a_2}{3}\frac{z^3}{l^4} \qquad \text{(vi)}$$

and

$$\frac{\mathrm{d}^2 v}{\mathrm{d}z^2} = 2\frac{a_2}{l} - 10a_2\frac{z}{l^3} + 8a_2\frac{z^2}{l^4} \qquad \text{(vii)}$$

The total strain energy of the column will be the sum of the strain energy due to bending and the strain energy due to the resistance of the elastic foundation. For the latter, consider an element, $\delta z$, of the column. The force on the element when subjected to a small displacement, $v$, is $k\delta z v$. Thus, the strain energy of the element is $\frac{1}{2}kv^2\delta z$ and the strain energy of the column due to the resistance of the elastic foundation is

$$\int_0^l \frac{1}{2}kv^2\mathrm{d}z$$

Substituting for $v$ from Eq. (v)

$$U\text{ (elastic foundation)} = \frac{1}{2}k\frac{a_2^2}{l^4}\int_0^l \left(z^4 - \frac{10z^5}{3l} + \frac{37z^6}{9l^2} - \frac{20z^7}{9l^3} + \frac{4z^8}{9l^4}\right)\mathrm{d}z$$

i.e. $U$ (elastic foundation) $= 0.0017ka_2^2l$.

Now substituting for $\mathrm{d}^2v/\mathrm{d}z^2$ and $\mathrm{d}v/\mathrm{d}z$ in Eq. (8.48) and adding $U$ (elastic foundation) gives

$$U + V = \frac{EI}{2}\int_0^l \frac{4a_2^2}{l^4}\left(1 - \frac{10z}{l} + \frac{33z^2}{l^2} - \frac{40z^3}{l^3} + \frac{16z^4}{l^4}\right)\mathrm{d}z + 0.0017ka_2^2l$$
$$- \frac{P_{\mathrm{CR}}}{2}\int_0^l \frac{a_2^2}{l^4}\left(4z^2 - \frac{20z^3}{l} + \frac{107z^4}{3l^2} - \frac{80z^5}{3l^3} + \frac{64z^6}{9l^4}\right)\mathrm{d}z \qquad \text{(viii)}$$

Eq. (viii) simplifies to

$$U + V = \frac{0.4EI}{l^3}a_2^2 + 0.0017ka_2^2l - \frac{0.019a_2^2P_{\mathrm{CR}}}{l}$$

From the principle of the stationary value of the total potential energy

$$\frac{\partial(U+V)}{\partial a_2} = \frac{0.8EI}{l^3}a_2 + 0.0034ka_2l - \frac{0.038a_2P_{\mathrm{CR}}}{l}$$

whence

$$P_{\mathrm{CR}} = \frac{21.05EI}{l^2} + 0.09kl^2$$

## S.8.14

The purely flexural instability load is given by Eq. (8.7) in which, from Table 8.1 $l_e = 0.5l$ where $l$ is the actual column length. Also it is clear that the least second moment of area of the column cross-section occurs about an axis coincident with the web. Thus

$$I = 2\times\frac{2tb^3}{12} = \frac{tb^3}{3}$$

Then

$$P_{\text{CR}} = \frac{\pi^2 EI}{(0.5l)^2}$$

i.e.

$$P_{\text{CR}} = \frac{4\pi^2 Etb^3}{3l^2} \quad \text{(i)}$$

The purely torsional buckling load is given by the last of Eqs (8.77), i.e.

$$P_{\text{CR}(\theta)} = \frac{A}{I_0}\left(GJ + \frac{\pi^2 E\Gamma}{l^2}\right) \quad \text{(ii)}$$

In Eq. (ii) $A = 5bt$ and

$$I_0 = I_x + I_y = 2 \times 2tb\frac{b^2}{4} + \frac{tb^3}{12} + \frac{tb^3}{3}$$

i.e.

$$I_0 = \frac{17tb^3}{12}$$

Also, from Eq. (18.11)

$$J = \sum \frac{st^3}{3} = \frac{1}{3}(2b8t^3 + bt^3) = \frac{17bt^3}{3}$$

and, referring to S.27.4

$$\Gamma = \frac{tb^5}{12}$$

Then, from Eq. (ii)

$$P_{\text{CR}(\theta)} = \frac{20}{17b}\left(17Gt^3 + \frac{\pi^2 Etb^4}{l^2}\right) \quad \text{(iii)}$$

Now equating Eqs (i) and (iii)

$$\frac{4\pi^2 Etb^3}{3l^2} = \frac{20}{17b}\left(17Gt^3 + \frac{\pi^2 Etb^4}{l^2}\right)$$

from which

$$l^2 = \frac{2\pi^2 Eb^4}{255Gt^2}$$

From Eq. (1.50), $E/G = 2(1+\nu)$. Hence

$$l = \frac{2\pi b^2}{t}\sqrt{\frac{1+\nu}{255}}$$

Eqs (i) and (iii) may be written, respectively, as

$$P_{CR} = \frac{1.33C_1}{l^2}$$

and

$$P_{CR(\theta)} = C_2 + \frac{1.175C_1}{l^2}$$

where $C_1$ and $C_2$ are constants. Thus, if $l$ were less than the value found, the increase in the last term in the expression for $P_{CR(\theta)}$ would be less than the increase in the value of $P_{CR}$, i.e. $P_{CR(\theta)} < P_{CR}$ for a decrease in $l$ and the column would fail in torsion.

## S.8.15

In this case Eqs (8.77) do not apply since the ends of the column are not free to warp. From Eq. (8.70) and since, for the cross-section of the column, $x_s = y_s = 0$,

$$E\Gamma \frac{d^4\theta}{dz^4} + \left(I_0 \frac{P}{A} - GJ\right) \frac{d^2\theta}{dz^2} = 0 \qquad \text{(i)}$$

For buckling, $P = P_{CR}$, the critical load and $P_{CR}/A = \sigma_{CR}$, the critical stress. Eq. (i) may then be written

$$\frac{d^4\theta}{dz^4} + \lambda^2 \frac{d^2\theta}{dz^2} = 0 \qquad \text{(ii)}$$

in which

$$\lambda^2 = \frac{(I_0\sigma_{CR} - GJ)}{E\Gamma} \qquad \text{(iii)}$$

The solution of Eq. (ii) is

$$\theta = A\cos\lambda z + B\sin\lambda z + Dz + F \qquad \text{(iv)}$$

The boundary conditions are:

$$\theta = 0 \text{ at } z = 0 \text{ and } z = 2l$$

$$\frac{d\theta}{dz} = 0 \text{ at } z = 0 \text{ and } z = 2l \quad \text{(see Eq. (18.19))}$$

Then $B = D = 0$, $F = -A$ and Eq. (iv) becomes

$$\theta = A(\cos\lambda z - 1) \qquad \text{(v)}$$

Since $\theta = 0$ when $z = 2l$

$$\cos\lambda 2l = 1$$

or

$$\lambda 2l = 2n\pi$$

Hence, for $n = 1$

$$\lambda^2 = \frac{\pi^2}{l^2}$$

i.e. from Eq. (iii)

$$\frac{I_0\sigma_{\mathrm{CR}} - GJ}{E\Gamma} = \frac{\pi^2}{l^2}$$

so that

$$\sigma_{\mathrm{CR}} = \frac{1}{I_0}\left(GJ + \frac{\pi^2 E\Gamma}{l^2}\right) \qquad \text{(vi)}$$

For the cross-section of Fig. P.8.15

$$J = \sum \frac{st^3}{3} \quad \text{(see Eq. (18.11))}$$

i.e.

$$J = \frac{8bt^3}{3} = \frac{8 \times 25.0 \times 2.5^3}{3} = 1041.7\,\mathrm{mm}^4$$

and

$$I_{xx} = 4bt(b\cos 30°)^2 + 2\frac{(2b)^3 t \sin^2 60°}{12} \quad \text{(see Section 16.4.5)}$$

i.e.

$$I_{xx} = 4b^3t = 4 \times 25.0^3 \times 2.5 = 156\,250.0\,\mathrm{mm}^4$$

Similarly

$$I_{yy} = 4\left(\frac{bt^3}{12} + btb^2\right) + 2\frac{(2b)^3 t \cos^2 60°}{12} = \frac{14b^3t}{3}$$

so that

$$I_{yy} = 14 \times 25.0^3 \times 2.5/3 = 182\,291.7\,\mathrm{mm}^4$$

Then

$$I_0 = I_{xx} + I_{yy} = 338\,541.7\,\mathrm{mm}^4$$

The torsion-bending constant, $\Gamma$, is found by the method described in Section 27.2 and is given by

$$\Gamma = b^5 t = 25.0^5 \times 2.5 = 24.4 \times 10^6\,\mathrm{mm}^4$$

Substituting these values in Eq. (vi) gives

$$\sigma_{\mathrm{CR}} = 282.0\,\mathrm{N/mm}^2$$

## S.8.16

The three possible buckling modes of the column are given by Eqs (8.77) i.e.

$$P_{\text{CR}(xx)} = \frac{\pi^2 EI_{xx}}{L^2} \tag{i}$$

$$P_{\text{CR}(yy)} = \frac{\pi^2 EI_{yy}}{L^2} \tag{ii}$$

$$P_{\text{CR}(\theta)} = \frac{A}{I_0}\left(GJ + \frac{\pi^2 E\Gamma}{L^2}\right) \tag{iii}$$

From Fig. P.8.16 and taking the $x$ axis parallel to the flanges

$$A = (2 \times 20 + 40) \times 1.5 = 120\,\text{mm}^2$$

$$I_{xx} = 2 \times 20 \times 1.5 \times 20^2 + 1.5 \times 40^3/12 = 3.2 \times 10^4\,\text{mm}^4$$

$$I_{yy} = 1.5 \times 40^3/12 = 0.8 \times 10^4\,\text{mm}^4$$

$$I_0 = I_{xx} + I_{yy} = 4.0 \times 10^4\,\text{mm}^4$$

$$J = (20 + 40 + 20) \times 1.5^3/3 = 90.0\,\text{mm}^4 \quad \text{(see Eq. (18.11))}$$

$$\Gamma = \frac{1.5 \times 20^3 \times 40^2}{12}\left(\frac{2 \times 40 + 20}{40 + 2 \times 20}\right)$$

$$= 2.0 \times 10^6\,\text{mm}^6 \quad \text{(see Eq. (ii) of Example 27.1)}$$

Substituting the appropriate values in Eqs (i), (ii) and (iii) gives

$$P_{\text{CR}(xx)} = 22\,107.9\,\text{N}$$

$$P_{\text{CR}(yy)} = 5527.0\,\text{N}$$

$$P_{\text{CR}(\theta)} = 10\,895.2\,\text{N}$$

Thus the column will buckle in bending about the $y$ axis at a load of 5527.0 N.

## S.8.17

The separate modes of buckling are obtained from Eqs (8.77), i.e.

$$P_{\text{CR}(xx)} = P_{\text{CR}(yy)} = \frac{\pi^2 EI}{L^2} (I_{xx} = I_{yy} = I, \text{ say}) \tag{i}$$

and

$$P_{\text{CR}(\theta)} = \frac{A}{I_0}\left(GJ + \frac{\pi^2 E\Gamma}{L^2}\right) \tag{ii}$$

In this case

$$I_{xx} = I_{yy} = \pi r^3 t = \pi \times 40^3 \times 2.0 = 4.02 \times 10^5\,\text{mm}^4$$
$$A = 2\pi r t = 2\pi \times 40 \times 2.0 = 502.7\,\text{mm}^2$$
$$J = 2\pi r t^3/3 = 2\pi \times 40 \times 2.0^3/3 = 670.2\,\text{mm}^4$$

From Eq. (8.68)

$$I_0 = I_{xx} + I_{yy} + A x_s^2 \quad \text{(note that } y_s = 0\text{)}$$

in which $x_s$ is the distance of the shear centre of the section from its vertical diameter; it may be shown that $x_s = 80$ mm (see S.17.3). Then

$$I_0 = 2 \times 4.02 \times 10^5 + 502.7 \times 80^2 = 4.02 \times 10^6\,\text{mm}^4$$

The torsion-bending constant $\Gamma$ is found in a similar manner to that for the section shown in Fig. P.27.3 and is given by

$$\Gamma = \pi r^5 t \left(\frac{2}{3}\pi^2 - 4\right)$$

i.e.

$$\Gamma = \pi \times 40^5 \times 2.0 \left(\frac{2}{3}\pi^2 - 4\right) = 1.66 \times 10^9\,\text{mm}^6$$

(a) $$P_{\text{CR}(xx)} = P_{\text{CR}(yy)} = \frac{\pi^2 \times 70\,000 \times 4.02 \times 10^5}{(3.0 \times 10^3)^2} = 3.09 \times 10^4\,\text{N}$$

(b) $$P_{\text{CR}(\theta)} = \frac{502.7}{4.02 \times 10^6}\left(22\,000 \times 670.2 + \frac{\pi^2 \times 70\,000 \times 1.66 \times 10^9}{(3.0 \times 10^3)^2}\right) = 1.78 \times 10^4\,\text{N}$$

The flexural–torsional buckling load is obtained by expanding Eq. (8.79). Thus

$$(P - P_{\text{CR}(xx)})(P - P_{\text{CR}(\theta)})I_0/A - P^2 x_s^2 = 0$$

from which

$$P^2(1 - A x_s^2/I_0) - P(P_{\text{CR}(xx)} + P_{\text{CR}(\theta)}) + P_{\text{CR}(xx)}P_{\text{CR}(\theta)} = 0 \qquad \text{(iii)}$$

Substituting the appropriate values in Eq. (iii) gives

$$P^2 - 24.39 \times 10^4 P + 27.54 \times 10^8 = 0 \qquad \text{(iv)}$$

The solutions of Eq. (iv) are

$$P = 1.19 \times 10^4\,\text{N} \quad \text{or} \quad 23.21 \times 10^4\,\text{N}$$

Therefore, the least flexural–torsional buckling load is $1.19 \times 10^4$ N.