Then

$$[B]^{\mathrm{T}}[D][B] = \frac{1}{64}\begin{bmatrix} -(2-y) & 0 & -(1-x) \\ 0 & -(1-x) & -(2-y) \\ & \vdots & \\ & \vdots & \\ & \vdots & \\ & \vdots & \\ & \vdots & \\ & \vdots & \end{bmatrix}\begin{bmatrix} -c(2-y) & -d(1-x) & \dots & \dots & \dots \\ -d(2-y) & -c(1-x) & \dots & \dots & \dots \\ -e(1-x) & -e(2-y) & \dots & \dots & \dots \end{bmatrix}$$

Therefore

$$K_{11} = \frac{t}{64}\int_{-2}^{2}\int_{-1}^{1}[c(2-y)^2 + e(1-x)^2]\mathrm{d}x\,\mathrm{d}y$$

which gives $K_{11} = \dfrac{t}{6}(4c+e)$

$$K_{12} = \frac{t}{64}\int_{-2}^{2}\int_{-1}^{1}[d(2-y)(1-x) + e(1-x)(2-y)]\mathrm{d}x\,\mathrm{d}y$$

which gives $K_{12} = \frac{t}{4}(d+e)$.

Solutions to Chapter 7 Problems

S.7.1

Substituting for $((1/\rho_x) + (\nu/\rho_y))$ and $((1/\rho_y) + (\nu/\rho_x))$ from Eqs (7.5) and (7.6), respectively in Eqs (7.3)

$$\sigma_x = \frac{Ez}{1-\nu^2}\frac{M_x}{D} \quad \text{and} \quad \sigma_y = \frac{Ez}{1-\nu^2}\frac{M_y}{D} \tag{i}$$

Hence, since, from Eq. (7.4), $D = Et^3/12(1-\nu^2)$, Eqs (i) become

$$\sigma_x = \frac{12zM_x}{t^3} \quad \sigma_y = \frac{12zM_y}{t^3} \tag{ii}$$

The maximum values of σ_x and σ_y will occur when $z = \pm t/2$. Hence

$$\sigma_x(\max) = \pm\frac{6M_x}{t^2} \quad \sigma_y(\max) = \pm\frac{6M_y}{t^2} \tag{iii}$$

Then

$$\sigma_x(\max) = \pm\frac{6 \times 10 \times 10^3}{10^2} = \pm 600\,\text{N/mm}^2$$

$$\sigma_y(\max) = \pm\frac{6 \times 5 \times 10^3}{10^2} = \pm 300\,\text{N/mm}^2$$

S.7.2

From Eq. (7.11) and since $M_{xy} = 0$

$$M_t = \frac{M_x - M_y}{2}\sin 2\alpha \tag{i}$$

M_t will be a maximum when $2\alpha = \pi/2$, i.e. $\alpha = \pi/4$ (45°). Thus, from Eq. (i)

$$M_t(\max) = \frac{10-5}{2} = 2.5\,\text{Nm/mm}$$

S.7.3

The relationship between M_n and M_x, M_y and M_{xy} in Eq. (7.10) and between M_t and M_x, M_y and M_{xy} in Eq. (7.11) are identical in form to the stress relationships in Eqs (1.8) and (1.9). Therefore, by deduction from Eqs (1.11) and (1.12)

$$M_{\text{I}} = \frac{M_x + M_y}{2} + \frac{1}{2}\sqrt{(M_x - M_y)^2 + 4M_{xy}^2} \tag{i}$$

and

$$M_{\text{II}} = \frac{M_x + M_y}{2} - \frac{1}{2}\sqrt{(M_x - M_y)^2 + 4M_{xy}^2} \tag{ii}$$

Further, Eq. (7.11) gives the inclination of the planes on which the principal moments occur, i.e. when $M_t = 0$. Thus

$$\tan 2\alpha = -\frac{2M_{xy}}{M_x - M_y} \tag{iii}$$

Substituting the values $M_x = 10\,\text{Nm/mm}$, $M_y = 5\,\text{Nm/mm}$ and $M_{xy} = 5\,\text{Nm/mm}$ in Eqs (i), (ii) and (iii) gives

$$M_{\text{I}} = 13.1\,\text{Nm/mm}$$

$$M_{\text{II}} = 1.9\,\text{Nm/mm}$$

and

$$\alpha = -31.7^\circ \quad \text{or} \quad 58.3^\circ$$

The corresponding principal stresses are obtained directly from Eqs (iii) of S.7.1. Hence

$$\sigma_{\mathrm{I}} = \pm \frac{6 \times 13.1 \times 10^3}{10^2} = \pm 786\,\mathrm{N/mm^2}$$

$$\sigma_{\mathrm{II}} = \pm \frac{6 \times 1.9 \times 10^3}{10^2} = \pm 114\,\mathrm{N/mm^2}$$

S.7.4

From the deflection equation

$$\frac{\partial^2 w}{\partial x^2} = -\frac{q_0 a^2}{D\pi^2}\left(1 + A\cosh\frac{\pi y}{a} + B\frac{\pi y}{a}\sinh\frac{\pi y}{a}\right)\sin\frac{\pi x}{a}$$

$$\frac{\partial^2 w}{\partial y^2} = \frac{q_0 a^2}{D\pi^2}\left(A\cosh\frac{\pi y}{a} + 2B\cosh\frac{\pi y}{a} + B\frac{\pi y}{a}\sinh\frac{\pi y}{a}\right)\sin\frac{\pi x}{a}$$

Now $w = 0$ and $M_x = 0$ at $x = 0$ and a. From Eq. (7.7) this is satisfied implicitly.

Also $w = 0$ and $M_y = 0$ at $y = \pm a$ so that, from the deflection equation

$$\mathrm{O} = \frac{q_0 a^4}{D\pi^4}(1 + A\cosh\pi + B\pi\sinh\pi)\sin\frac{\pi x}{a}$$

i.e.

$$1 + A\cosh\pi + B\pi\sinh\pi = 0 \tag{i}$$

Also, from Eq. (7.8)

$$\mathrm{O} = -\frac{q_0 a^2}{D\pi^2}[(A\cosh\pi + 2B\cosh\pi + B\pi\sinh\pi) - 0.3(1 + A\cosh\pi + B\pi\sinh\pi)]\sin\frac{\pi x}{a}$$

or

$$\mathrm{O} = -0.3 + 0.7A\cosh\pi + 2B\cosh\pi + 0.7B\pi\sinh\pi \tag{ii}$$

Solving Eqs (i) and (ii)

$$A = -0.2213 \quad B = 0.0431$$

S.7.5

The deflection is zero at $x = a/2$, $y = a/2$. Then, from the deflection equation

$$\mathrm{O} = \frac{a^4}{4} - \frac{3}{2}a^4(1-\nu) - \frac{3}{4}a^4\nu + A$$

Hence

$$A = \frac{a^4}{4}(5 - 3\nu)$$

The central deflection, i.e. at $x = 0$, $y = 0$ is then

$$= \frac{qa^4}{96(1-\nu)D} \times \frac{1}{4}(5 - 3\nu)$$
$$= \frac{qa^4}{384D}\left(\frac{5 - 3\nu}{1 - \nu}\right)$$

S.7.6

From the equation for deflection

$$\frac{\partial^4 w}{\partial x^4} = w_0 \left(\frac{\pi}{a}\right)^4 \cos\frac{\pi x}{a}\cos\frac{3\pi y}{a}$$
$$\frac{\partial^4 w}{\partial y^4} = w_0 \left(\frac{3\pi}{a}\right)^4 \cos\frac{\pi x}{a}\sin\frac{3\pi y}{a}$$
$$\frac{\partial^4 w}{\partial x^2\,\partial y^2} = w_0 \left(\frac{\pi}{a}\right)^2\left(\frac{3\pi}{a}\right)^2 \cos\frac{\pi x}{a}\cos\frac{3\pi y}{a}$$

Substituting in Eq. (7.20)

$$\frac{q(x,y)}{D} = w_0 \cos\frac{\pi x}{a}\cos\frac{3\pi y}{a}(1 + 2\times 9 + 81)\left(\frac{\pi}{a}\right)^4$$

i.e.

$$q(x,y) = w_0 D 100\frac{\pi^4}{a^4}\cos\frac{\pi x}{a}\cos\frac{3\pi y}{a}$$

From the deflection equation

$$w = 0 \quad \text{at } x = \pm a/2, y = \pm a/2$$

The plate is therefore supported on all four edges.
Also

$$\frac{\partial w}{\partial x} = -w_0\frac{\pi}{a}\sin\frac{\pi x}{a}\cos\frac{3\pi y}{a}$$
$$\frac{\partial w}{\partial y} = -w_0\frac{3\pi}{a}\cos\frac{\pi x}{a}\sin\frac{3\pi y}{a}$$

When

$$x = \pm\frac{a}{2} \quad \frac{\partial w}{\partial x} \neq 0$$
$$y = \pm\frac{a}{2} \quad \frac{\partial w}{\partial y} \neq 0$$

The plate is therefore not clamped on its edges.

Further

$$\frac{\partial^2 w}{\partial x^2} = -w_0\left(\frac{\pi}{a}\right)^2 \cos\frac{\pi x}{a}\cos\frac{3\pi y}{a}$$
$$\frac{\partial^2 w}{\partial y^2} = -w_0\left(\frac{3\pi}{a}\right)^2 \cos\frac{\pi x}{a}\cos\frac{3\pi y}{a}$$

Substituting in Eq. (7.7)

$$M_x = -Dw_0\left(\frac{\pi}{a}\right)^2 \cos\frac{\pi x}{a}\cos\frac{3\pi y}{a}(-1 - 9\nu) \qquad \text{(i)}$$

Similarly, from Eq. (7.8)

$$M_y = w_0 D\left(\frac{\pi}{a}\right)^2 \cos\frac{\pi x}{a}\cos\frac{3\pi y}{a}(9 + \nu) \qquad \text{(ii)}$$

Then, at $x = \pm a/2$, $M_x = 0$ (from Eq. (i)) and at $y = \pm a/2$, $M_y = 0$ (from Eq. (ii)).

The plate is therefore simply supported on all edges.

The corner reactions are given by

$$2D(1 - \nu)\frac{\partial^2 w}{\partial x\,\partial y} \quad \text{(see Eq. (7.14))}$$

Then, since

$$\frac{\partial^2 w}{\partial x\,\partial y} = w_0\frac{\pi}{a}\frac{3\pi}{a}\sin\frac{\pi x}{a}\sin\frac{3\pi y}{a} \qquad \text{at } x = a/2,\ y = a/2$$

$$\text{Corner reactions} = -6w_0 D\left(\frac{\pi}{a}\right)^2(1 - \nu)$$

From Eqs (7.7) and (7.8) and the above, at the centre of the plate

$$M_x = w_0 D\left(\frac{\pi}{a}\right)^2(1 + 9\nu),\ M_y = w_0 D\left(\frac{\pi}{a}\right)^2(9 + \nu).$$

S.7.7

Substituting $q(x, y) = q_0 x/a$ in Eq. (7.29) and noting that the plate is square and of side a

$$a_{mn} = \frac{4}{a^2} \int_0^a \int_0^a q_0 \frac{x}{a} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{a} \mathrm{d}x \, \mathrm{d}y$$

i.e.

$$a_{mn} = \frac{4q_0}{a^3} \int_0^a x \sin \frac{m\pi x}{a} \left[-\frac{a}{n\pi} \cos \frac{n\pi y}{a} \right]_0^a \mathrm{d}x$$

Hence

$$a_{mn} = -\frac{4q_0}{a^2 n\pi} \int_0^a x \sin \frac{m\pi x}{a} (\cos n\pi - 1) \mathrm{d}x$$

The term in brackets is zero when n is even and equal to -2 when n is odd. Thus

$$a_{mn} = \frac{8q_0}{a^2 n\pi} \int_0^a x \sin \frac{m\pi x}{a} \mathrm{d}x \quad (n \text{ odd}) \tag{i}$$

Integrating Eq. (i) by parts

$$a_{mn} = \frac{8q_0}{a^2 n\pi} \left[-x \frac{a}{m\pi} \cos \frac{m\pi x}{a} + \int \frac{a}{m\pi} \cos \frac{m\pi x}{a} \mathrm{d}x \right]_0^a$$

i.e.

$$a_{mn} = \frac{8q_0}{amn\pi^2} \left[-x \cos \frac{m\pi x}{a} + \frac{a}{m\pi} \sin \frac{m\pi x}{a} \right]_0^a$$

The second term in square brackets is zero for all integer values of m. Thus

$$a_{mn} = \frac{8q_0}{amn\pi^2} (-a \cos m\pi)$$

The term in brackets is positive when m is odd and negative when m is even. Thus

$$a_{mn} = \frac{8q_0}{mn\pi^2} (-1)^{m+1}$$

Substituting for a_{mn} in Eq. (7.30) gives the displaced shape of the plate, i.e.

$$w = \frac{1}{\pi^4 D} \sum_{m=1,2,3}^{\infty} \sum_{n=1,3,5}^{\infty} \frac{8q_0(-1)^{m+1}}{mn\pi^2 \left[\left(\frac{m^2}{a^2} \right) + \left(\frac{n^2}{a^2} \right) \right]^2} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{a}$$

or

$$w = \frac{8q_0 a^4}{\pi^6 D} \sum_{m=1,2,3}^{\infty} \sum_{n=1,3,5}^{\infty} \frac{(-1)^{m+1}}{mn(m^2 + n^2)^2} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{a}$$

S.7.8

The boundary conditions which must be satisfied by the equation for the displaced shape of the plate are $w=0$ and $\partial w/\partial n=0$ at all points on the boundary; n is a direction normal to the boundary at any point.

The equation of the ellipse representing the boundary is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \tag{i}$$

Substituting for $x^2/a^2+y^2/b^2$ in the equation for the displaced shape clearly gives $w=0$ for all values of x and y on the boundary of the plate. Also

$$\frac{\partial w}{\partial n}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial n}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial n} \tag{ii}$$

Now

$$w=w_0\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^2$$

so that

$$\frac{\partial w}{\partial x}=-\frac{4w_0x}{a^2}\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right) \tag{iii}$$

and

$$\frac{\partial w}{\partial y}=-\frac{4w_0y}{b^2}\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right) \tag{iv}$$

From Eqs (i), (ii) and (iv) it can be seen that $\partial w/\partial x$ and $\partial w/\partial y$ are zero for all values of x and y on the boundary of the plate. It follows from Eq. (ii) that $\partial w/\partial n=0$ at all points on the boundary of the plate. Thus the equation for the displaced shape satisfies the boundary conditions.

From Eqs (iii) and (iv)

$$\frac{\partial^4 w}{\partial x^4}=\frac{24w_0}{a^4}\qquad \frac{\partial^4 w}{\partial y^4}=\frac{24w_0}{b^4}\qquad \frac{\partial^4 w}{\partial x^2\,\partial y^2}=\frac{8w_0}{a^2b^2}$$

Substituting these values in Eq. (7.20)

$$w_0\left(\frac{24}{a^4}+\frac{16}{a^2b^2}+\frac{24}{b^4}\right)=\frac{p}{D}$$

whence

$$w_0=\frac{p}{8D\left(\dfrac{3}{a^4}+\dfrac{2}{a^2b^2}+\dfrac{3}{b^4}\right)}$$

Now substituting for D from Eq. (7.4)

$$w_0 = \frac{3p(1-\nu^2)}{2Et^3\left(\dfrac{3}{a^4} + \dfrac{2}{a^2b^2} + \dfrac{3}{b^4}\right)} \tag{v}$$

From Eqs (7.3), (7.5) and (7.7)

$$\sigma_x = -\frac{Ez}{1-\nu^2}\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right) \tag{vi}$$

and from Eqs (7.3), (7.6) and (7.8)

$$\sigma_y = -\frac{Ez}{1-\nu^2}\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right) \tag{vii}$$

From Eqs (iii) and (iv)

$$\frac{\partial^2 w}{\partial x^2} = -\frac{4w_0}{a^2}\left(1 - \frac{3x^2}{a^2} - \frac{y^2}{b^2}\right) \qquad \frac{\partial^2 w}{\partial y^2} = -\frac{4w_0}{b^2}\left(1 - \frac{x^2}{a^2} - \frac{3y^2}{b^2}\right)$$

Substituting these expressions in Eq. (vi) and noting that the maximum values of direct stress occur at $z = \pm t/2$

$$\sigma_x(\max) = \pm\frac{Et}{2(1-\nu^2)}\left[-\frac{4w_0}{a^2}\left(1 - \frac{3x^2}{a^2} - \frac{y^2}{b^2}\right) - \frac{4w_0\nu}{b^2}\left(1 - \frac{x^2}{a^2} - \frac{3y^2}{b^2}\right)\right] \tag{viii}$$

At the centre of the plate, $x = y = 0$. Then

$$\sigma_x(\max) = \pm\frac{2Etw_0}{(1-\nu^2)}\left(\frac{1}{a^2} + \frac{\nu}{b^2}\right) \tag{ix}$$

Substituting for w_0 in Eq. (ix) from Eq. (v) gives

$$\sigma_x(\max) = \pm\frac{3pa^2b^2(b^2 + \nu a^2)}{t^2(3b^4 + 2a^2b^2 + 3a^4)} \tag{x}$$

Similarly

$$\sigma_y(\max) = \pm\frac{3pa^2b^2(a^2 + \nu b^2)}{t^2(3b^4 + 2a^2b^2 + 3a^4)} \tag{xi}$$

At the ends of the minor axis, $x = 0$, $y = b$. Thus, from Eq. (viii)

$$\sigma_x(\max) = \pm\frac{2Etw_0}{(1-\nu^2)}\left(\frac{1}{a^2} - \frac{1}{a^2} + \frac{\nu}{b^2} - \frac{3\nu}{b^2}\right)$$

i.e.

$$\sigma_x(\max) = \pm\frac{4Etw_0\nu}{b^2(1-\nu^2)} \tag{xii}$$

Again substituting for w_0 from Eq. (v) in Eq. (xii)

$$\sigma_x(\max) = \pm \frac{6pa^4b^2}{t^2(3b^4 + 2a^2b^2 + 3a^4)}$$

Similarly

$$\sigma_y(\max) = \pm \frac{6pb^4a^2}{t^2(3b^4 + 2a^2b^2 + 3a^4)}$$

S.7.9

The potential energy, V, of the load W is given by

$$V = -Ww$$

i.e.

$$V = -W \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi\xi}{a} \sin \frac{n\pi\eta}{b}$$

Therefore, it may be deduced from Eq. (7.47) that the total potential energy, $U + V$, of the plate is

$$U + V = \frac{D}{2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn}^2 \frac{\pi^4 ab}{4} \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^2 - W \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi\xi}{a} \sin \frac{n\pi\eta}{b}$$

From the principle of the stationary value of the total potential energy

$$\frac{\partial(U + V)}{\partial A_{mn}} = DA_{mn} \frac{\pi^4 ab}{4} \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^2 - W \sin \frac{m\pi\xi}{a} \sin \frac{n\pi\eta}{b} = 0$$

Hence

$$A_{mn} = \frac{4W \sin \dfrac{m\pi\xi}{a} \sin \dfrac{n\pi\eta}{b}}{\pi^4 Dab \left[\left(\dfrac{m^2}{a^2} \right) + \left(\dfrac{n^2}{b^2} \right) \right]^2}$$

so that the deflected shape is obtained.

S.7.10

From Eq. (7.45) the potential energy of the in-plane load, N_x, is

$$-\frac{1}{2} \int_0^a \int_0^b N_x \left(\frac{\partial w}{\partial x} \right)^2 \mathrm{d}x\, \mathrm{d}y$$

The combined potential energy of the in-plane load, N_x, and the load, W, is then, from S.7.9

$$V = -\frac{1}{2}\int_0^a \int_0^b N_x \left(\frac{\partial w}{\partial x}\right)^2 \mathrm{d}x\,\mathrm{d}y - W \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\frac{m\pi\xi}{a} \sin\frac{n\pi\eta}{b}$$

or, since,

$$\frac{\partial w}{\partial x} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\frac{m\pi}{a} \cos\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

$$V = -\frac{1}{2}\int_0^a \int_0^b N_x \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}^2 \frac{m^2\pi^2}{a^2} \cos^2\frac{m\pi x}{a} \sin^2\frac{n\pi y}{b} \mathrm{d}x\,\mathrm{d}y$$
$$- W \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\frac{m\pi\xi}{a} \sin\frac{n\pi\eta}{b}$$

i.e.

$$V = -\frac{ab}{8} N_x \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}^2 \frac{m^2\pi^2}{a^2} - W \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\frac{m\pi\xi}{a} \sin\frac{n\pi\eta}{b}$$

Then, from Eq. (7.47), the total potential energy of the plate is

$$U + V = \frac{D}{2} \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}^2 \frac{\pi^4 ab}{4}\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2 - \frac{ab}{8} N_x \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}^2 \frac{m^2\pi^2}{a^2}$$
$$- W \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\frac{m\pi\xi}{a} \sin\frac{n\pi\eta}{b}$$

Then, from the principle of the stationary value of the total potential energy

$$\frac{\partial(U+V)}{\partial A_{mn}} = DA_{mn}\frac{\pi^4 ab}{4}\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2 - \frac{ab}{4} N_x A_{mn} \frac{m^2\pi^2}{a^2} - W \sin\frac{m\pi\xi}{a} \sin\frac{n\pi\eta}{b} = 0$$

from which

$$A_{mn} = \frac{4W \sin\dfrac{m\pi\xi}{a} \sin\dfrac{n\pi\eta}{b}}{abD\pi^4\left[\left(\dfrac{m^2}{a^2} + \dfrac{n^2}{b^2}\right)^2 - \dfrac{m^2 N_x}{\pi^2 a^2 D}\right]}$$

S.7.11

The guessed form of deflection is

$$w = A_{11}\left(1 - \frac{4x^2}{a^2}\right)\left(1 - \frac{4y^2}{a^2}\right) \tag{i}$$

Clearly when $x = \pm a/2$, $w = 0$ and when $y = \pm a/2$, $w = 0$. Therefore, the equation for the displaced shape satisfies the displacement boundary conditions.

From Eq. (i)

$$\frac{\partial^2 w}{\partial x^2} = -8\frac{A_{11}}{a^2}\left(1 - \frac{4y^2}{a^2}\right) \quad \frac{\partial^2 w}{\partial y^2} = -8\frac{A_{11}}{a^2}\left(1 - \frac{4x^2}{a^2}\right)$$

Substituting in Eq. (7.7)

$$M_x = -\frac{8A_{11}D}{a^2}\left[1 - \frac{4y^2}{a^2} + \nu\left(1 - \frac{4x^2}{a^2}\right)\right]$$

Clearly, when $x = \pm a/2$, $M_x \neq 0$ and when $y = \pm a/2$, $M_x \neq 0$. Similarly for M_y. Thus the assumed displaced shape does not satisfy the condition of zero moment at the simply supported edges.

From Eq. (i)

$$\frac{\partial^2 w}{\partial x\,\partial y} = \frac{64A_{11}xy}{a^4}$$

Substituting for $\partial^2 w/\partial x^2$, $\partial^2 w/\partial y^2$, $\partial^2 w/\partial x\,\partial y$ and w in Eq. (7.46) and simplifying gives

$$U + V = \int_{-a/2}^{a/2}\int_{-a/2}^{a}\left\{\frac{32A_{11}^2 D}{a^4}\left[4 - \frac{16}{a^2}(x^2 + y^2) + \frac{16}{a^4}(x^4 + 2x^2y^2 + y^4) - 1.4 + \frac{5.6}{a^2}(x^2 + y^2) + \frac{67.2x^2y^2}{a^4}\right] - q_0A_{11}\left(1 - \frac{4x^2}{a^2} - \frac{4y^2}{a^2} + \frac{16x^2y^2}{a^4}\right)\right\}\mathrm{d}x\,\mathrm{d}y$$

from which

$$U + V = \frac{62.4A_{11}^2 D}{a^2} - \frac{4q_0A_{11}a^2}{9}$$

From the principle of the stationary value of the total potential energy

$$\frac{\partial(U + V)}{\partial A_{11}} = \frac{124.8A_{11}D}{a^2} - \frac{4q_0a^2}{9} = 0$$

Hence, since $D = Et^3/12(1 - \nu^2)$

$$A_{11} = 0.0389q_0a^4/Et^3$$

S.7.12

From Eq. (7.36) the deflection of the plate from its initial curved position is

$$w_1 = B_{11} \sin \frac{\pi x}{a} \sin \frac{\pi y}{b}$$

in which

$$B_{11} = \frac{A_{11} N_x}{\dfrac{\pi^2 D}{a^2}\left(1 + \dfrac{a^2}{b^2}\right)^2 - N_x}$$

The total deflection, w, of the plate is given by

$$w = w_1 + w_0$$

i.e.

$$w = \left[\frac{A_{11} N_x}{\dfrac{\pi^2 D}{a^2}\left(1 + \dfrac{a^2}{b^2}\right)^2 - N_x} + A_{11} \right] \sin \frac{\pi x}{a} \sin \frac{\pi y}{b}$$

i.e.

$$w = \frac{A_{11}}{1 - \dfrac{N_x a^2}{\pi^2 D} \Big/ \left(1 + \dfrac{a^2}{b^2}\right)^2} \sin \frac{\pi x}{a} \sin \frac{\pi y}{b}$$

Solutions to Chapter 8 Problems

S.8.1

The forces on the bar AB are shown in Fig. S.8.1 where

$$M_{\mathrm{B}} = K\left(\frac{\mathrm{d}v}{\mathrm{d}z}\right)_{\mathrm{B}} \tag{i}$$

and P is the buckling load.

From Eq. (8.1)

$$EI \frac{\mathrm{d}^2 v}{\mathrm{d}z^2} = -Pv \tag{ii}$$

The solution of Eq. (ii) is

$$v = A \cos \mu z + B \sin \mu z \tag{iii}$$