Solutions to Chapter 6 Problems

S.6.1

Referring to Fig. P.6.1 and Fig. 6.3

Member	12	23	34	41	13
Length	L	L	L	L	$\sqrt{2}L$
$\lambda(\cos\theta)$	$1/\sqrt{2}$	$-1/\sqrt{2}$	$-1/\sqrt{2}$	$1/\sqrt{2}$	0
$\mu(\sin\theta)$	$1/\sqrt{2}$	$1/\sqrt{2}$	$-1/\sqrt{2}$	$-1/\sqrt{2}$	1

The stiffness matrix for each member is obtained using Eq. (6.30). Thus

$$[K_{12}] = \frac{AE}{2L}\begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix} \quad [K_{23}] = \frac{AE}{2L}\begin{bmatrix} 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

$$[K_{34}] = \frac{AE}{2L}\begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix} \quad [K_{41}] = \frac{AE}{2L}\begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

$$[K_{13}] = \frac{AE}{\sqrt{2}L}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

The stiffness matrix for the complete framework is now assembled using the method described in Example 6.1. Equation (6.29) then becomes

$$\begin{Bmatrix} F_{x,1} \\ F_{y,1} \\ F_{x,2} \\ F_{y,2} \\ F_{x,3} \\ F_{y,3} \\ F_{x,4} \\ F_{y,4} \end{Bmatrix} = \frac{AE}{2L}\begin{bmatrix} 2 & 0 & -1 & -1 & 0 & 0 & -1 & 1 \\ 0 & 2+\sqrt{2} & -1 & -1 & 0 & -\sqrt{2} & 1 & -1 \\ -1 & -1 & 2 & 0 & -1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 2 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 2 & 0 & -1 & -1 \\ 0 & -\sqrt{2} & 1 & -1 & 0 & 2+\sqrt{2} & -1 & -1 \\ -1 & 1 & 0 & 0 & -1 & -1 & 2 & 0 \\ 1 & -1 & 0 & 0 & -1 & -1 & 0 & 2 \end{bmatrix}\begin{Bmatrix} u_1 = 0 \\ v_1 \\ u_2 = 0 \\ v_2 = 0 \\ u_3 = 0 \\ v_3 \\ u_4 = 0 \\ v_4 = 0 \end{Bmatrix} \quad \text{(i)}$$

In Eq. (i)

$$F_{y,1} = -P \quad F_{x,1} = F_{x,3} = F_{y,3} = 0$$

Then

$$F_{y,1} = -P = \frac{AE}{2L}[(2+\sqrt{2})v_1 - \sqrt{2}v_3] \tag{ii}$$

$$F_{y,3} = 0 = \frac{AE}{2L}[-\sqrt{2}v_1 + (2+\sqrt{2})v_3] \tag{iii}$$

From Eq. (iii)

$$v_1 = (1+\sqrt{2})v_3 \tag{iv}$$

Substituting for v_1 in Eq. (ii) gives

$$v_3 = -\frac{0.293PL}{AE}$$

Hence, from Eq. (iv)

$$v_1 = -\frac{0.707PL}{AE}$$

The forces in the members are obtained using Eq. (6.32), i.e.

$$S_{12} = \frac{AE}{\sqrt{2}L}[1 \quad 1]\begin{Bmatrix} 0-0 \\ 0+\dfrac{0.707PL}{AE}\end{Bmatrix} = \frac{P}{2} = S_{14} \text{ from symmetry}$$

$$S_{13} = \frac{AE}{\sqrt{2}L}[0 \quad 1]\begin{Bmatrix} 0-0 \\ -\dfrac{0.293PL}{AE}+\dfrac{0.707PL}{AE}\end{Bmatrix} = 0.293P$$

$$S_{23} = \frac{AE}{\sqrt{2}L}[-1 \quad 1]\begin{Bmatrix} 0-0 \\ -\dfrac{0.293PL}{AE}-0\end{Bmatrix} = -0.207P = S_{43} \text{ from symmetry}$$

The support reactions are $F_{x,2}, F_{y,2}, F_{x,4}$ and $F_{y,4}$. From Eq. (i)

$$F_{x,2} = \frac{AE}{2L}(-v_1 + v_3) = 0.207P$$

$$F_{y,2} = \frac{AE}{2L}(-v_1 - v_3) = 0.5P$$

$$F_{x,4} = \frac{AE}{2L}(v_1 - v_3) = -0.207P$$

$$F_{y,4} = \frac{AE}{2L}(-v_1 - v_3) = 0.5P$$

S.6.2

Referring to Fig. P.6.2 and Fig. 6.3

Member	12	23	34	31	24
Length	$l/\sqrt{3}$	$l/\sqrt{3}$	l	l	$l/\sqrt{3}$
$\lambda(\cos\theta)$	$\sqrt{3}/2$	0	1/2	−1/2	$\sqrt{3}/2$
$\mu(\sin\theta)$	1/2	1	$-\sqrt{3}/2$	$-\sqrt{3}/2$	−1/2

From Eq. (6.30) the member stiffness matrices are

$$[K_{12}] = \frac{AE}{l}\begin{bmatrix} 3\sqrt{3}/4 & 3/4 & -3\sqrt{3}/4 & -3/4 \\ 3/4 & \sqrt{3}/4 & -3/4 & -\sqrt{3}/4 \\ -3\sqrt{3}/4 & -3/4 & 3\sqrt{3}/4 & 3/4 \\ -3/4 & -\sqrt{3}/4 & 3/4 & \sqrt{3}/4 \end{bmatrix}$$

$$[K_{23}] = \frac{AE}{l}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \sqrt{3} & 0 & -\sqrt{3} \\ 0 & 0 & 0 & 0 \\ 0 & -\sqrt{3} & 0 & \sqrt{3} \end{bmatrix}$$

$$[K_{34}] = \frac{AE}{l}\begin{bmatrix} 1/4 & -\sqrt{3}/4 & -1/4 & \sqrt{3}/4 \\ -\sqrt{3}/4 & 3/4 & \sqrt{3}/4 & -3/4 \\ -1/4 & \sqrt{3}/4 & 1/4 & -\sqrt{3}/4 \\ \sqrt{3}/4 & -3/4 & -\sqrt{3}/4 & 3/4 \end{bmatrix}$$

$$[K_{31}] = \frac{AE}{l}\begin{bmatrix} 1/4 & \sqrt{3}/4 & -1/4 & -\sqrt{3}/4 \\ \sqrt{3}/4 & 3/4 & -\sqrt{3}/4 & -3/4 \\ -1/4 & -\sqrt{3}/4 & 1/4 & \sqrt{3}/4 \\ -\sqrt{3}/4 & -3/4 & \sqrt{3}/4 & 3/4 \end{bmatrix}$$

$$[K_{24}] = \frac{AE}{l}\begin{bmatrix} 3\sqrt{3}/4 & -3/4 & -3\sqrt{3}/4 & 3/4 \\ -3/4 & \sqrt{3}/4 & 3/4 & -\sqrt{3}/4 \\ -3\sqrt{3}/4 & 3/4 & 3\sqrt{3}/4 & -3/4 \\ 3/4 & -\sqrt{3}/4 & -3/4 & \sqrt{3}/4 \end{bmatrix}$$

The stiffness matrix for the complete framework is now assembled using the method described in Example 6.1. Equation (6.29) then becomes

$$\begin{Bmatrix} F_{x,1} \\ F_{y,1} \\ F_{x,2} \\ F_{y,2} \\ F_{x,3} \\ F_{y,3} \\ F_{x,4} \\ F_{y,4} \end{Bmatrix} = \frac{AE}{l}\begin{bmatrix} \frac{1+3\sqrt{3}}{4} & \frac{3+\sqrt{3}}{4} & -\frac{3\sqrt{3}}{4} & -\frac{3}{4} & -\frac{1}{4} & -\frac{\sqrt{3}}{4} & 0 & 0 \\ \frac{3+\sqrt{3}}{4} & \frac{3+\sqrt{3}}{4} & -\frac{3}{4} & -\frac{\sqrt{3}}{4} & -\frac{\sqrt{3}}{4} & -\frac{3}{4} & 0 & 0 \\ -\frac{3\sqrt{3}}{4} & -\frac{3}{4} & \frac{3\sqrt{3}}{2} & 0 & 0 & 0 & -\frac{3\sqrt{3}}{4} & \frac{3}{4} \\ -\frac{3}{4} & -\frac{\sqrt{3}}{4} & 0 & \frac{3\sqrt{3}}{2} & 0 & -\sqrt{3} & \frac{3}{4} & -\frac{\sqrt{3}}{4} \\ -\frac{1}{4} & -\frac{\sqrt{3}}{4} & 0 & 0 & \frac{1}{2} & 0 & -\frac{1}{4} & \frac{\sqrt{3}}{4} \\ -\frac{\sqrt{3}}{4} & -\frac{3}{4} & 0 & -\sqrt{3} & 0 & \frac{3}{2}+\sqrt{3} & \frac{\sqrt{3}}{4} & -\frac{3}{4} \\ 0 & 0 & -\frac{3\sqrt{3}}{4} & \frac{3}{4} & -\frac{1}{4} & \frac{\sqrt{3}}{4} & \frac{1+3\sqrt{3}}{4} & -\frac{3+\sqrt{3}}{4} \\ 0 & 0 & \frac{3}{4} & -\frac{\sqrt{3}}{4} & \frac{\sqrt{3}}{4} & -\frac{3}{4} & -\frac{3+\sqrt{3}}{4} & \frac{3+\sqrt{3}}{4} \end{bmatrix}\begin{Bmatrix} u_1 = 0 \\ v_1 = 0 \\ u_2 = 0 \\ v_2 \\ u_3 = 0 \\ v_3 \\ u_4 = 0 \\ v_4 = 0 \end{Bmatrix} \tag{i}$$

In Eq. (i) $F_{x,2} = F_{y,2} = 0, F_{x,3} = 0, F_{y,3} = -P, F_{x,4} = -H$. Then

$$F_{y,2} = 0 = \frac{AE}{l}\left(\frac{3\sqrt{3}}{2}v_2 - \sqrt{3}v_3\right) \tag{ii}$$

and

$$F_{y,3} = -P = \frac{AE}{l}\left[-\sqrt{3}v_2 + \left(\frac{3}{2} + \sqrt{3}\right)v_3\right] \tag{iii}$$

From Eq. (ii)

$$v_2 = \frac{2}{3}v_3 \tag{iv}$$

Now substituting for v_2 in Eq. (iii)

$$-\frac{Pl}{AE} = -\frac{2\sqrt{3}}{3}v_3 + \frac{3}{2}v_3 + \sqrt{3}v_3$$

Hence

$$v_3 = -\frac{6Pl}{(9 + 2\sqrt{3})AE}$$

and, from Eq. (iv)

$$v_2 = -\frac{4Pl}{(9 + 2\sqrt{3})AE}$$

Also from Eq. (i)

$$F_{x,4} = -H = \frac{AE}{l}\left(\frac{3}{4}v_2 + \frac{\sqrt{3}}{4}v_3\right)$$

Substituting for v_2 and v_3 gives

$$H = 0.449P$$

S.6.3

Referring to Fig. P.6.3 and Fig. 6.3

Member	12	23	34	45	24
Length	l	l	l	l	l
$\lambda(\cos\theta)$	$-1/2$	$1/2$	$-1/2$	$1/2$	1
$\mu(\sin\theta)$	$\sqrt{3}/2$	$\sqrt{3}/2$	$\sqrt{3}/2$	$\sqrt{3}/2$	0

From Eq. (6.30) the member stiffness matrices are

$$[K_{12}] = \frac{AE}{l}\begin{bmatrix} 1/4 & -\sqrt{3}/4 & -1/4 & \sqrt{3}/4 \\ -\sqrt{3}/4 & 3/4 & \sqrt{3}/4 & -3/4 \\ -1/4 & \sqrt{3}/4 & 1/4 & -\sqrt{3}/4 \\ \sqrt{3}/4 & -3/4 & -\sqrt{3}/4 & 3/4 \end{bmatrix}$$

$$[K_{23}] = \frac{AE}{l}\begin{bmatrix} 1/4 & \sqrt{3}/4 & -1/4 & -\sqrt{3}/4 \\ \sqrt{3}/4 & 3/4 & -\sqrt{3}/4 & -3/4 \\ -1/4 & -\sqrt{3}/4 & 1/4 & \sqrt{3}/4 \\ -\sqrt{3}/4 & -3/4 & \sqrt{3}/4 & 3/4 \end{bmatrix}$$

$$[K_{34}] = \frac{AE}{l}\begin{bmatrix} 1/4 & -\sqrt{3}/4 & -1/4 & \sqrt{3}/4 \\ -\sqrt{3}/4 & 3/4 & \sqrt{3}/4 & -3/4 \\ -1/4 & \sqrt{3}/4 & 1/4 & -\sqrt{3}/4 \\ \sqrt{3}/4 & -3/4 & -\sqrt{3}/4 & 3/4 \end{bmatrix}$$

$$[K_{45}] = \frac{AE}{l}\begin{bmatrix} 1/4 & \sqrt{3}/4 & -1/4 & -\sqrt{3}/4 \\ \sqrt{3}/4 & 3/4 & -\sqrt{3}/4 & -3/4 \\ -1/4 & -\sqrt{3}/4 & 1/4 & \sqrt{3}/4 \\ -\sqrt{3}/4 & -3/4 & \sqrt{3}/4 & 3/4 \end{bmatrix}$$

$$[K_{24}] = \frac{AE}{l}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The stiffness matrix for the complete truss is now assembled using the method described in Example 6.1. Equation (6.29) then becomes

$$\begin{Bmatrix} F_{x,1} \\ F_{y,1} \\ F_{x,2} \\ F_{y,2} \\ F_{x,3} \\ F_{y,3} \\ F_{x,4} \\ F_{y,4} \\ F_{x,5} \\ F_{y,5} \end{Bmatrix} = \frac{AE}{4l}\begin{bmatrix} 1 & -\sqrt{3} & -1 & \sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\sqrt{3} & 3 & \sqrt{3} & -3 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & \sqrt{3} & 6 & 0 & -1 & -\sqrt{3} & -4 & 0 & 0 & 0 \\ \sqrt{3} & -3 & 0 & 6 & -\sqrt{3} & -3 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -\sqrt{3} & 2 & 0 & -1 & \sqrt{3} & 0 & 0 \\ 0 & 0 & -\sqrt{3} & -3 & 0 & 6 & \sqrt{3} & -3 & 0 & 0 \\ 0 & 0 & -4 & 0 & -1 & \sqrt{3} & 6 & 0 & -1 & -\sqrt{3} \\ 0 & 0 & 0 & 0 & \sqrt{3} & -3 & 0 & 6 & -\sqrt{3} & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -\sqrt{3} & 1 & \sqrt{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{3} & -3 & \sqrt{3} & 3 \end{bmatrix}\begin{Bmatrix} u_1 = 0 \\ v_1 = 0 \\ u_2 \\ v_2 \\ u_3 = 0 \\ v_3 = 0 \\ u_4 \\ v_4 \\ u_5 = 0 \\ v_5 = 0 \end{Bmatrix} \tag{i}$$

In Eq. (i) $F_{x,2} = F_{y,2} = 0, F_{x,4} = 0, F_{y,4} = -P$. Thus from Eq. (i)

$$F_{x,2} = 0 = \frac{AE}{4l}(6u_2 - 4u_4) \tag{ii}$$

$$F_{y,2} = 0 = \frac{AE}{4l}(6v_2) \tag{iii}$$

$$F_{x,4} = 0 = \frac{AE}{4l}(-4u_2 + 6u_4) \tag{iv}$$

$$F_{y,4} = -P = \frac{AE}{4l}(6v_4) \tag{v}$$

From Eq. (v)

$$v_4 = -\frac{2Pl}{3AE}$$

From Eq. (iii)

$$v_2 = 0$$

and from Eqs (ii) and (iv)

$$u_2 = u_4 = 0$$

Hence, from Eq. (6.32)

$$S_{24} = \frac{AE}{l}[1 \quad 0]\left\{\begin{matrix} 0 - 0 \\ \dfrac{-2Pl}{3AE} - 0 \end{matrix}\right\}$$

which gives

$$S_{24} = 0$$

S.6.4

The uniformly distributed load on the member 26 is equivalent to concentrated loads of $wl/4$ at nodes 2 and 6 together with a concentrated load of $wl/2$ at node 4. Thus, referring to Fig. P.6.4 and Fig. 6.3

Member	12	23	24	46	56	67
Length	l	l	$l/2$	$l/2$	l	l
$\lambda(\cos\theta)$	0	$-1/\sqrt{2}$	1	1	0	$1/\sqrt{2}$
$\mu(\sin\theta)$	1	$1/\sqrt{2}$	0	0	1	$1/\sqrt{2}$

From Eq. (6.47) and using the alternative form of Eq. (6.44)

$$[K_{12}] = \frac{EI}{l^3}\begin{bmatrix} 12 & & & & & \text{SYM} \\ 0 & 0 & & & & \\ 6 & 0 & 4 & & & \\ -12 & 0 & -6 & 12 & & \\ 0 & 0 & 0 & 0 & 0 & \\ 6 & 0 & 2 & 6 & 0 & 0 \end{bmatrix}$$

$$[K_{23}] = \frac{EI}{l^3}\begin{bmatrix} 6 & & & & & \text{SYM} \\ 6 & 6 & & & & \\ 6/\sqrt{2} & 6/\sqrt{2} & 4 & & & \\ 6 & 6 & -6\sqrt{2} & 6 & & \\ -6 & -6 & -6/\sqrt{2} & 6 & 6 & \\ 6/\sqrt{2} & 6/\sqrt{2} & 2 & 6/\sqrt{2} & -6/\sqrt{2} & -4/\sqrt{2} \end{bmatrix}$$

$$[K_{24}] = [K_{46}] = \frac{EI}{l^3}\begin{bmatrix} 0 & & & & & \text{SYM} \\ 0 & 96 & & & & \\ 0 & -24 & 8 & & & \\ 0 & 0 & 0 & 0 & & \\ 0 & -96 & 24 & 0 & 96 & \\ 0 & -24 & 4 & 0 & 24 & 8 \end{bmatrix}$$

$$[K_{56}] = \frac{EI}{l^3}\begin{bmatrix} 12 & & & & & \text{SYM} \\ 0 & 0 & & & & \\ 6 & 0 & 4 & & & \\ -12 & 0 & -6 & 12 & & \\ 0 & 0 & 0 & 0 & 0 & \\ 6 & 0 & 2 & 6 & 0 & 0 \end{bmatrix}$$

$$[K_{67}] = \frac{EI}{l^3}\begin{bmatrix} 6 & & & & & \text{SYM} \\ -6 & 6 & & & & \\ 6/\sqrt{2} & -6/\sqrt{2} & 4 & & & \\ -6 & 6 & -6/\sqrt{2} & 6 & & \\ 6 & -6 & 6/\sqrt{2} & -6 & 6 & \\ 6/\sqrt{2} & -6/\sqrt{2} & 2 & 6/\sqrt{2} & 6/\sqrt{2} & 4/\sqrt{2} \end{bmatrix}$$

The member stiffness matrices are then assembled into a 21×21 symmetrical matrix using the method described in Example 6.1. The known nodal displacements are $u_1 = v_1 = \theta_1 = u_5 = v_5 = \theta_5 = u_2 = u_4 = u_6 = \theta_3 = \theta_7 = 0$ and the support reactions are obtained from $\{F\} = [\text{K}]\{\delta\}$. Having obtained the support reactions the internal shear force and bending moment distributions in each member follow (see Example 6.2).

S.6.5

Referring to Fig. P.6.5, $u_2 = 0$ from symmetry. Consider the members 23 and 29. The forces acting on the member 23 are shown in Fig. S.6.5(a) in which F_{29} is the force applied at 2 in the member 23 due to the axial force in the member 29. Suppose that the node 2 suffers a vertical displacement v_2. The shortening in the member 29 is then $v_2 \cos\theta$ and the corresponding strain is $-(v_2 \cos\theta)/l$. Thus the compressive stress in 29 is $-(Ev_2 \cos\theta)/l$ and the corresponding compressive force is $-(AEv_2 \cos\theta)/l$. Thus

$$F_{29} = -(AEv_2 \cos^2\theta)/l$$

Now $AE = 6\sqrt{2}EI/L^2$. $\theta = 45°$ and $l = \sqrt{2}L$. Hence

$$F_{29} = -\frac{3EI}{L^3}v_2$$

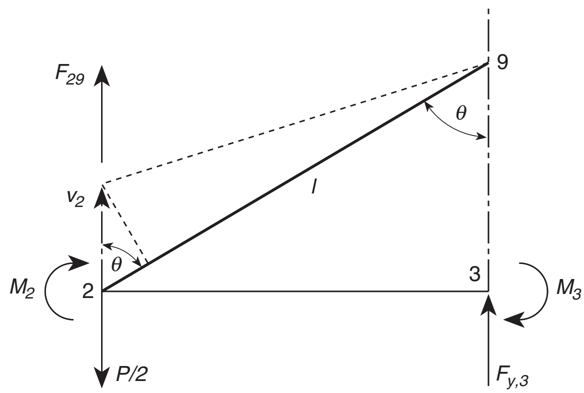

Fig. S.6.5(a)

and

$$F_{y,2} = -\frac{P}{2} - \frac{3EI}{L^3}v_2 \tag{i}$$

Further, from Eq. (3.12)

$$M_3 = GJ\frac{d\theta}{dz} = -2 \times 0.8EI\frac{\theta_3}{0.8L} = -\frac{2EI}{L}\theta_3 \tag{ii}$$

From the alternative form of Eq. (6.44), for the member 23

$$\begin{Bmatrix} F_{y,2} \\ M_2/L \\ F_{y,3} \\ M_3/L \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & -6 & -12 & -6 \\ -6 & 4 & 6 & 2 \\ -12 & 6 & 12 & 6 \\ -6 & 2 & 6 & 4 \end{bmatrix}\begin{Bmatrix} v_2 \\ \theta_2 L = 0 \\ v_3 = 0 \\ \theta_3 L \end{Bmatrix} \tag{iii}$$

Then, from Eqs (i) and (iii)

$$F_{y,2} = -\frac{P}{2} - \frac{3EI}{L^3}v_2 = \frac{12EI}{L^3}v_2 - \frac{6EI}{L^2}\theta_3$$

Hence

$$15v_2 - 6\theta_3 L = -\frac{PL^3}{2EI} \tag{iv}$$

From Eqs (ii) and (iii)

$$\frac{M_3}{L} = -\frac{2EI}{L^2}\theta_3 = -\frac{6EI}{L^3}v_2 + \frac{4EI}{L^2}\theta_3$$

which gives $\theta_3 = v_2/L$.

Substituting for θ_3 in Eq. (iv) gives

$$v_2 = -\frac{PL^3}{18EI}$$

Then

$$\theta_3 = -\frac{PL^2}{18EI}$$

From Eq. (i)

$$F_{y,2} = -\frac{P}{2} + \frac{3EI}{L^3}\frac{PL^3}{18EI} = -\frac{P}{3}$$

and from Eq. (ii)

$$M_3 = \frac{2EI}{L}\frac{PL^2}{18EI} = \frac{PL}{9} = -M_1$$

Now, from Eq. (iii)

$$\frac{M_2}{L} = -\frac{EI}{L^3}6v_2 + \frac{2EI}{L^3}\theta_3 L = \frac{2PL}{9}$$

$$F_{y,3} = -\frac{12EI}{L^3}v_2 + \frac{6EI}{L^3}\theta_3 L = \frac{P}{3}$$

The force in the member 29 is $F_{29}/\cos\theta = \sqrt{2}F_{29}$. Thus

$$S_{29} = S_{28} = \sqrt{2}\frac{3EI}{L^3}\frac{PL^3}{18EI} = \frac{\sqrt{2}P}{6} \quad \text{(tension)}$$

The torques in the members 36 and 37 are given by $M_3/2$, i.e.

$$M_{36} = M_{37} = PL/18$$

The shear force and bending moment diagrams for the member 123 follow and are shown in Figs S.6.5(b) and (c), respectively.

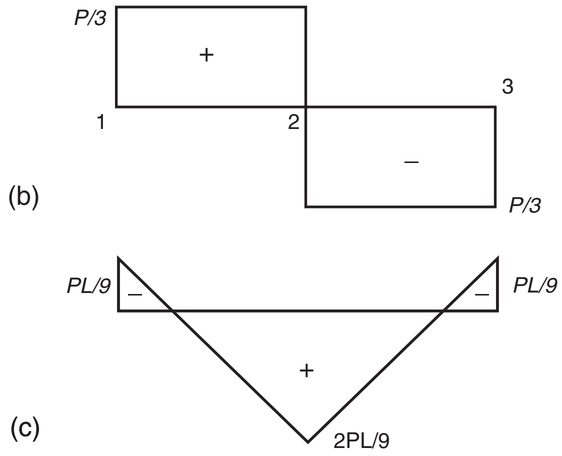

Fig. S.6.5(b) and (c)

S.6.6

The stiffness matrix for each element of the beam is obtained using the given force–displacement relationship, the complete stiffness matrix for the beam is then obtained using the method described in Example 6.1. This gives

$$\begin{Bmatrix} F_{y,1} \\ M_1/L \\ F_{y,2} \\ M_2/L \\ F_{y,3} \\ M_3/L \\ F_{y,4} \\ M_4/L \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 24 & -12 & -24 & -12 & & & & \\ -12 & 8 & 12 & 4 & & & & \\ -24 & 12 & 36 & 6 & -12 & -6 & & \\ -12 & 4 & 6 & 12 & 6 & 2 & & \\ & & -12 & 6 & 36 & -24 & -24 & -12 \\ & & -6 & 2 & -6 & 12 & 12 & 4 \\ & & & & -24 & 12 & 24 & 12 \\ & & & & -12 & 4 & 12 & 8 \end{bmatrix}\begin{Bmatrix} v_1 \\ \theta_1 L \\ v_2 \\ \theta_2 L \\ v_3 \\ \theta_3 L \\ v_4 \\ \theta_4 L \end{Bmatrix} \quad \text{(i)}$$

The ties FB, CH, EB and CG produce vertically upward forces F_2 and F_3 at B and C, respectively. These may be found using the method described in S.6.5. Thus

$$F_2 = -\left(\frac{a_1 E \cos^2 60^\circ}{2L/\sqrt{3}} + \frac{a_2 E \cos^2 45^\circ}{\sqrt{2}L}\right) v_2$$

But $a_1 = 384I/5\sqrt{3}L^2$ and $a_2 = 192I/5\sqrt{2}L^2$ so that

$$F_2 = -\frac{96EI}{5L^3} v_2$$

Similarly

$$F_3 = -\frac{96EI}{5L^3} v_3$$

Then

$$F_{y,2} = -P - \frac{96EI}{5L^3}v_2 \quad \text{and} \quad F_{y,3} = -P - \frac{96EI}{5L^3}v_3$$

In Eq. (i), $v_1 = \theta_1 = v_4 = \theta_4 = 0$ and $M_2 = M_3 = 0$. Also, from symmetry, $v_2 = v_3$, and $\theta_2 = -\theta_3$. Then, from Eq. (i)

$$M_2 = 0 = 6v_2 + 12\theta_2 L + 6v_3 + 2\theta_3 L$$

i.e.

$$12v_2 + 10\theta_2 L = 0$$

which gives

$$\theta_2 = -\frac{6}{5L}v_2$$

Also from Eq. (i)

$$F_{y,2} = -P - \frac{96EI}{5L^3}v_2 = \frac{EI}{L^3}(36v_2 + 6\theta_2 L - 12v_3 - 6\theta_3 L)$$

i.e.

$$-P - \frac{96EI}{5L^3}v_2 = \frac{48EI}{5L^3}v_2$$

whence

$$v_2 = -\frac{5PL^3}{144EI} = v_3$$

and

$$\theta_2 = \frac{PL^2}{24EI} = -\theta_3$$

The reactions at the ends of the beam now follow from the above values and Eq. (i). Thus

$$F_{y,1} = \frac{EI}{L^3}(-24v_2 - 12\theta_2 L) = \frac{P}{3} = F_{y,4}$$

$$M_1 = \frac{EI}{L^2}(12v_2 + 4\theta_2 L) = -\frac{PL}{4} = -M_4$$

Also

$$F_2 = F_3 = \frac{96EI}{5L^3}\frac{5PL^3}{144EI} = \frac{2P}{3}$$

The forces on the beam are then as shown in Fig. S.6.6(a). The shear force and bending moment diagrams for the beam follow and are shown in Figs S.6.6(b) and (c), respectively.

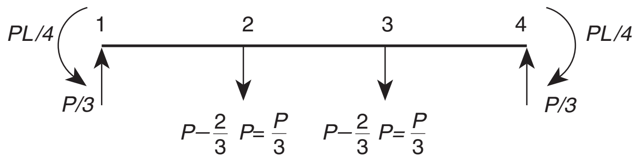

Fig. S.6.6(a)

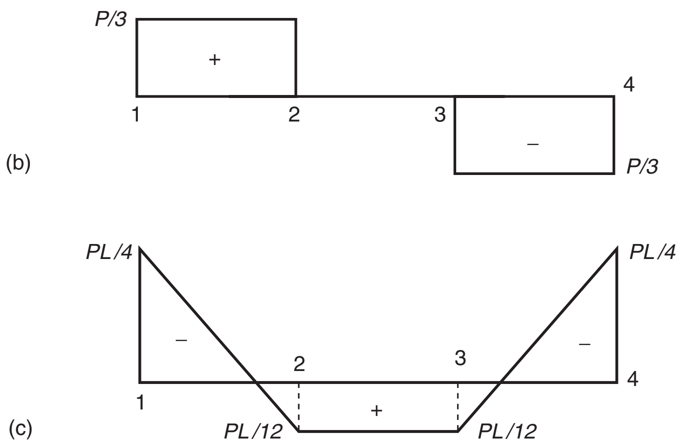

Fig. S.6.6(b) and (c)

The forces in the ties are obtained using Eq. (6.32). Thus

$$S_{\mathrm{BF}} = S_{\mathrm{CH}} = \frac{a_1 E}{2L/\sqrt{3}} \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{Bmatrix} 0 - 0 \\ v_2 - 0 \end{Bmatrix}$$

i.e.

$$S_{\mathrm{BF}} = S_{\mathrm{CH}} = \frac{384EI\sqrt{3}}{5\sqrt{3} \times 2L^3} \frac{1}{2} \frac{5PL^3}{144EI} = \frac{2}{3}P$$

and

$$S_{\mathrm{BE}} = S_{\mathrm{CG}} = \frac{a_2 E}{\sqrt{2}L} \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{Bmatrix} 0 - 0 \\ v_2 - 0 \end{Bmatrix}$$

i.e.

$$S_{\mathrm{BE}} = S_{\mathrm{CG}} = \frac{192EI}{5\sqrt{2} \times \sqrt{2}L^3} \frac{1}{\sqrt{2}} \frac{5PL^3}{144EI} = \frac{\sqrt{2}P}{3}$$

S.6.7

The forces acting on the member 123 are shown in Fig. S.6.7(a). The moment M_2 arises from the torsion of the members 26 and 28 and, from Eq. (3.12), is given by

$$M_2 = -2GJ\frac{\theta_2}{1.6l} = -EI\frac{\theta_2}{l} \qquad \text{(i)}$$

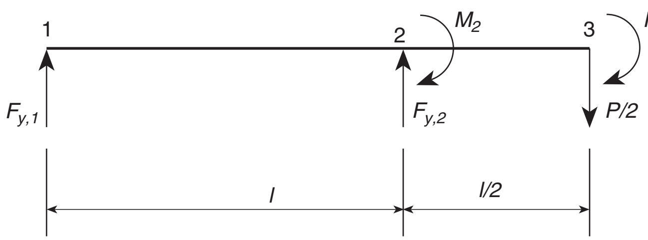

Fig. S.6.7(a)

Now using the alternative form of Eq. (6.44) for the member 12

$$\begin{Bmatrix} F_{y,1} \\ M_1/l \\ F_{y,2} \\ M_2/l \end{Bmatrix} = \frac{EI}{l^3}\begin{bmatrix} 12 & -6 & -12 & -6 \\ -6 & 4 & 6 & 2 \\ -12 & 6 & 12 & 6 \\ -6 & 2 & 6 & 4 \end{bmatrix}\begin{Bmatrix} v_1 \\ \theta_1 L \\ v_2 \\ \theta_2 L \end{Bmatrix} \qquad \text{(ii)}$$

and for the member 23

$$\begin{Bmatrix} F_{y,2} \\ M_2/l \\ F_{y,3} \\ M_3/l \end{Bmatrix} = \frac{EI}{l^3}\begin{bmatrix} 96 & -24 & -96 & -24 \\ -24 & 8 & 24 & 4 \\ -96 & 24 & 96 & 24 \\ -24 & 4 & 24 & 8 \end{bmatrix}\begin{Bmatrix} v_2 \\ \theta_2 L \\ v_3 \\ \theta_3 L \end{Bmatrix} \qquad \text{(iii)}$$

Combining Eqs (ii) and (iii) using the method described in Example 6.1

$$\begin{Bmatrix} F_{y,1} \\ M_1/l \\ F_{y,2} \\ M_2/l \\ F_{y,3} \\ M_3/l \end{Bmatrix} = \frac{EI}{l^3}\begin{bmatrix} 12 & -6 & -12 & -6 & 0 & 0 \\ -6 & 4 & 6 & 2 & 0 & 0 \\ -12 & 6 & 108 & -18 & -96 & -24 \\ -6 & 2 & -18 & 12 & 24 & 4 \\ 0 & 0 & -96 & 24 & 96 & 24 \\ 0 & 0 & -24 & 4 & 24 & 8 \end{bmatrix}\begin{Bmatrix} v_1 \\ \theta_1 l \\ v_2 \\ \theta_2 l \\ v_3 \\ \theta_3 l \end{Bmatrix} \qquad \text{(iv)}$$

In Eq. (iv) $v_1 = v_2 = 0$ and $\theta_3 = 0$. Also $M_1 = 0$ and $F_{y,3} = -P/2$. Then from Eq. (iv)

$$\frac{M_1}{l} = 0 = \frac{EI}{l^3}(4\theta_1 l + 2\theta_2 l)$$

from which

$$\theta_1 = -\frac{\theta_2}{2} \qquad \text{(v)}$$

Also, from Eqs (i) and (iv)

$$\frac{M_2}{l} = -\frac{EI}{l^2}\theta_2 = \frac{EI}{l^3}(2\theta_1 l + 12\theta_2 l + 24v_3)$$

so that

$$13\theta_2 l + 2\theta_1 l + 24v_3 = 0 \qquad \text{(vi)}$$

Finally from Eq. (iv)

$$F_{y,3} = -\frac{P}{2} = \frac{EI}{l^3}(24\theta_2 l + 96v_3)$$

which gives

$$v_3 = -\frac{Pl^3}{192EI} - \frac{\theta_2 l}{4} \qquad \text{(vii)}$$

Substituting in Eq. (vi) for θ_1 from Eq. (v) and v_3 from Eq. (vii) gives

$$\theta_2 = \frac{Pl^2}{48EI}$$

Then, from Eq. (v)

$$\theta_1 = -\frac{Pl^2}{96EI}$$

and from Eq. (vii)

$$v_3 = -\frac{Pl^3}{96EI}$$

Now substituting for θ_1, θ_2 and v_3 in Eq. (iv) gives $F_{y,1} = -P/16$, $F_{y,2} = 9P/16$, $M_2 = -Pl/48$ (from Eq. (i)) and $M_3 = -Pl/6$. Then the bending moment at 2 in 12 is $F_{y,1}l = -Pl/12$ and the bending moment at 2 in 32 is $-(P/2)\,(l/2) + M_3 = -Pl/12$. Also $M_3 = -Pl/6$ so that the bending moment diagram for the member 123 is that shown in Fig. S.6.7(b).

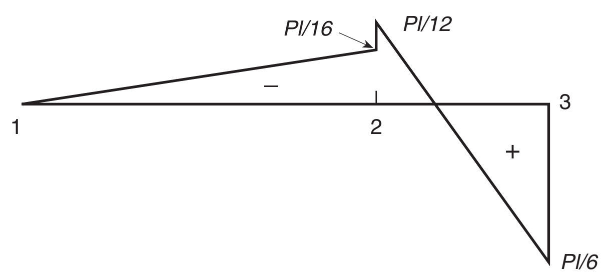

Fig. S.6.7(b)

S.6.8

(a) The element is shown in Fig. S.6.8. The displacement functions for a triangular element are given by Eqs (6.82). Thus

$$\left.\begin{aligned} u_1 &= \alpha_1, \quad v_1 = \alpha_4 \\ u_2 &= \alpha_1 + a\alpha_2, \quad v_2 = \alpha_4 + a\alpha_5 \\ u_3 &= \alpha_1 + a\alpha_3, \quad v_3 = \alpha_4 + a\alpha_6 \end{aligned}\right\} \tag{i}$$

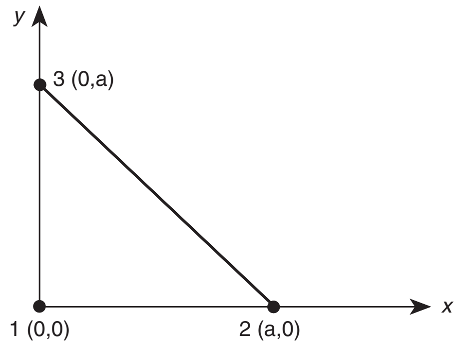

Fig. S.6.8

From Eq. (i)

$$\alpha_1 = u_1 \quad \alpha_2 = (u_2 - u_1)/a \quad \alpha_3 = (u_3 - u_1)/a$$
$$\alpha_4 = v_1 \quad \alpha_5 = (v_2 - v_1)/a \quad \alpha_6 = (v_3 - v_1)/a$$

Hence in matrix form

$$\begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1/a & 0 & 1/a & 0 & 0 & 0 \\ -1/a & 0 & 0 & 0 & 1/a & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1/a & 0 & 1/a & 0 & 0 \\ 0 & -1/a & 0 & 0 & 0 & 1/a \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

which is of the form

$$\{x\} = [A^{-1}]\{\delta^e\}$$

Also, from Eq. (6.89)

$$[C] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

Hence

$$[B] = [C][A^{-1}] = \begin{bmatrix} -1/a & 0 & 1/a & 0 & 0 & 0 \\ 0 & -1/a & 0 & 0 & 0 & 1/a \\ -1/a & -1/a & 0 & 1/a & 1/a & 0 \end{bmatrix}$$

(b) From Eq. (6.94)

$$[K^e] = \begin{bmatrix} -1/a & 0 & -1/a \\ 0 & -1/a & -1/a \\ 1/a & 0 & 0 \\ 0 & 0 & 1/a \\ 0 & 0 & 1/a \\ 0 & 1/a & 0 \end{bmatrix} \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix}$$

$$\times \begin{bmatrix} -1/a & 0 & 1/a & 0 & 0 & 0 \\ 0 & -1/a & 0 & 0 & 0 & 1/a \\ -1/a & -1/a & 0 & 1/a & 1/a & 0 \end{bmatrix} \frac{1}{2}a^2 t$$

which gives

$$[K^e] = \frac{Et}{4(1-\nu^2)} \begin{bmatrix} 3-\nu & 1+\nu & -2 & -(1-\nu) & -(1-\nu) & -2\nu \\ 1+\nu & 3-\nu & -2\nu & -(1-\nu) & -(1-\nu) & -2 \\ -2 & -2\nu & 2 & 0 & 0 & 2\nu \\ -(1-\nu) & -(1-\nu) & 0 & 1-\nu & 1-\nu & 0 \\ -(1-\nu) & -(1-\nu) & 0 & 1-\nu & 1-\nu & 0 \\ -2\nu & -2 & -2\nu & 0 & 0 & 2 \end{bmatrix}$$

Continuity of displacement is only ensured at nodes, not along their edges.

S.6.9

(a) There are six degrees of freedom so that the displacement field must include six coefficients. Thus

$$u = \alpha_1 + \alpha_2 x + \alpha_3 y \qquad \text{(i)}$$

$$v = \alpha_4 + \alpha_5 x + \alpha_6 y \qquad \text{(ii)}$$

(b) From Eqs (i) and (ii) and referring to Fig. S.6.9

$$u_1 = \alpha_1 + \alpha_2 + \alpha_3 \quad v_1 = \alpha_4 + \alpha_5 + \alpha_6$$

$$u_2 = \alpha_1 + 2\alpha_2 + \alpha_3 \quad v_2 = \alpha_4 + 2\alpha_5 + \alpha_6$$

$$u_3 = \alpha_1 + 2\alpha_2 + 2\alpha_3 \quad v_3 = \alpha_4 + 2\alpha_5 + 2\alpha_6$$

Thus

$$\alpha_2 = u_2 - u_1 \quad \alpha_3 = u_3 - u_2 \quad \alpha_1 = 2u_1 - u_3$$

$$\alpha_5 = v_2 - v_1 \quad \alpha_6 = v_3 - v_2 \quad \alpha_4 = 2v_1 - v_3$$

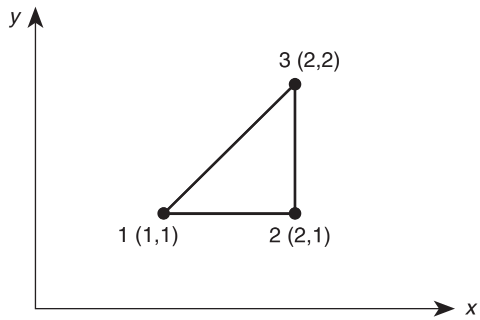

Fig. S.6.9

Therefore

$$\begin{Bmatrix}\alpha_1\\\alpha_2\\\alpha_3\\\alpha_4\\\alpha_5\\\alpha_6\end{Bmatrix} = \begin{bmatrix}2 & 0 & 0 & 0 & -1 & 0\\-1 & 0 & 1 & 0 & 0 & 0\\0 & 0 & -1 & 0 & 1 & 0\\0 & 2 & 0 & 0 & 0 & -1\\0 & -1 & 0 & 1 & 0 & 0\\0 & 2 & 0 & 0 & 0 & -1\end{bmatrix}\begin{Bmatrix}u_1\\v_1\\u_2\\v_2\\u_3\\v_3\end{Bmatrix} \qquad \text{(iii)}$$

which is of the form

$$\{\alpha\} = [A^{-1}]\{\delta^e\}$$

From Eq. (6.89)

$$[C] = \begin{bmatrix}0 & 1 & 0 & 0 & 0 & 0\\0 & 0 & 0 & 0 & 0 & 1\\0 & 0 & 1 & 0 & 1 & 0\end{bmatrix}$$

Hence

$$[B] = [C][A^{-1}] = \begin{bmatrix}-1 & 0 & 1 & 0 & 0 & 0\\0 & 2 & 0 & 0 & 0 & -1\\0 & -1 & -1 & 1 & 1 & 0\end{bmatrix}$$

(c) From Eq. (6.69)

$$\{\sigma\} = [D][B]\{\delta^e\}$$

Thus, for plane stress problems (see Eq. (6.92))

$$[D][B] = \frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu & 0\\\nu & 1 & 0\\0 & 0 & \frac{1}{2}(1-\nu)\end{bmatrix}\begin{bmatrix}-1 & 0 & 1 & 0 & 0 & 0\\0 & 2 & 0 & 0 & 0 & -1\\0 & -1 & -1 & 1 & 1 & 0\end{bmatrix}$$

i.e.

$$[D][B] = \frac{E}{1-\nu^2}\begin{bmatrix} -1 & 2\nu & 1 & 0 & 0 & -\nu \\ -\nu & 2 & \nu & 0 & 0 & -1 \\ 0 & -\frac{1}{2}(1-\nu) & -\frac{1}{2}(1-\nu) & \frac{1}{2}(1-\nu) & \frac{1}{2}(1-\nu) & 0 \end{bmatrix}$$

For plain strain problems (see Eq. (6.93))

$$[D][B] = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1 & \dfrac{\nu}{(1-\nu)} & 0 \\ \dfrac{\nu}{(1-\nu)} & 1 & 0 \\ 0 & 0 & \dfrac{(1-2\nu)}{2(1-\nu)} \end{bmatrix}$$

$$\times\begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & -1 \\ 0 & -1 & -1 & 1 & 1 & 0 \end{bmatrix}$$

$$[D][B] = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}$$

$$\times\begin{bmatrix} -1 & \dfrac{2\nu}{1-\nu} & 1 & 0 & 0 & -\dfrac{\nu}{1-\nu} \\ -\dfrac{\nu}{1-\nu} & 2 & \dfrac{\nu}{1-\nu} & 0 & 0 & -1 \\ 0 & -\dfrac{1-2\nu}{2(1-\nu)} & -\dfrac{1-2\nu}{2(1-\nu)} & \dfrac{1-2\nu}{2(1-\nu)} & \dfrac{1-2\nu}{2(1-\nu)} & 0 \end{bmatrix}$$

S.6.10

(a) The element is shown in Fig. S.6.10. There are eight degrees of freedom so that a displacement field must include eight coefficients. Therefore assume

$$u = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy \tag{i}$$

$$v = \alpha_5 + \alpha_6 x + \alpha_y y + \alpha_8 xy \tag{ii}$$

(b) From Eqs (6.88) and Eqs (i) and (ii)

$$\varepsilon_x = \frac{\partial u}{\partial x} = \alpha_2 + \alpha_4 y$$

$$\varepsilon_y = \frac{\partial v}{\partial y} = \alpha_7 + \alpha_8 x$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \alpha_3 + \alpha_4 x + \alpha_6 + \alpha_8 y$$

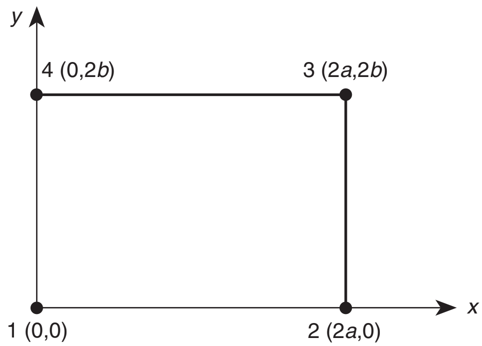

Fig. S.6.10

Thus since $\{\varepsilon\} = [C]\{\alpha\}$

$$[C] = \begin{bmatrix} 0 & 1 & 0 & y & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & x \\ 0 & 0 & 1 & x & 0 & 1 & 0 & y \end{bmatrix} \quad \text{(iii)}$$

(c) From Eq. (iii)

$$[C]^{\mathrm{T}} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ y & 0 & x \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & x \\ 0 & x & y \end{bmatrix}$$

and from Eq. (6.92)

$$[D] = \frac{E}{1 - \nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1 - \nu) \end{bmatrix}$$

Thus

$$\int_{\text{vol}} [C]^{\mathrm{T}}[D][C]\mathrm{d}V = \int_0^{2a} \int_0^{2b} [C]^{\mathrm{T}}[D][C]t \, \mathrm{d}x \, \mathrm{d}y \quad \text{(iv)}$$

Substituting in Eq. (iv) for $[C]^{\mathrm{T}}$, $[D]$ and $[C]$ and multiplying out gives

$$\int_0^{2a}\int_0^{2b}[C]^{\mathrm{T}}[D][C]t\,\mathrm{d}x\,\mathrm{d}y$$

$$= \frac{Et}{1-\nu^2}\int_0^{2a}\int_0^{2b}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & y & 0 & 0 & \nu & \nu x \\ 0 & 0 & \frac{1}{2}(1-\nu) & \frac{x}{2}(1-\nu) & 0 & \frac{1}{2}(1-\nu) & 0 & \frac{y}{2}(1-\nu) \\ 0 & y & \frac{x}{2}(1-\nu) & y^2+\frac{x^2(1-\nu)}{2} & 0 & \frac{x}{2}(1-\nu) & \nu y & \nu xy+\frac{xy}{2}(1-\nu) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) & \frac{x}{2}(1-\nu) & 0 & \frac{1}{2}(1-\nu) & 0 & \frac{y}{2}(1-\nu) \\ 0 & \nu & 0 & \nu y & 0 & 0 & 1 & x \\ 0 & \nu x & \frac{y}{2}(1-\nu) & \nu xy+\frac{xy}{2}(1-\nu) & 0 & \frac{y}{2}(1-\nu) & x & x^2+\frac{y^2}{2}(1-\nu) \end{bmatrix}\mathrm{d}x\,\mathrm{d}y$$

$$= \frac{Et}{1-\nu^2}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4ab & 0 & 4ab^2 & 0 & 0 & 4ab\nu & 4ba^2\nu \\ 0 & 0 & 2ab(1-\nu) & 2a^2b(1-\nu) & 0 & 2ab(1-\nu) & 0 & 2ab^2(1-\nu) \\ 0 & 4ab^2 & 2a^2b(1-\nu) & \frac{8}{3}\{2ab^3+a^3b(1-\nu)\} & 0 & 2a^2b(1-\nu) & 4ab^2\nu & 2a^2b^2(1+\nu) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2ab(1-\nu) & 2a^2b(1-\nu) & 0 & 2ab(1-\nu) & 0 & 2ab^2(1-\nu) \\ 0 & 4ab\nu & 0 & 4ab^2\nu & 0 & 0 & 4ab & 4a^2b \\ 0 & 4a^2b\nu & 2ab^2(1-\nu) & 2a^2b^2(1+\nu) & 0 & 2ab^2(1-\nu) & 4a^2b & \frac{8}{3}\{2a^3b+ab^3(1-\nu)\} \end{bmatrix}$$

S.6.11

From the first of Eqs (6.96)

$$u_1 = \alpha_1 - \alpha_2 - \alpha_3 + \alpha_4 = 0.1/10^3 \quad \text{(i)}$$

$$u_2 = \alpha_1 + \alpha_2 - \alpha_3 - \alpha_4 = 0.3/10^3 \quad \text{(ii)}$$

$$u_3 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 0.6/10^3 \quad \text{(iii)}$$

$$u_4 = \alpha_1 - \alpha_2 + \alpha_3 - \alpha_4 = 0.1/10^3 \quad \text{(iv)}$$

Adding Eqs (i) and (ii)

$$u_1 + u_2 = 2\alpha_1 - 2\alpha_3 = 0.4/10^3$$

i.e.

$$\alpha_1 - \alpha_3 = 0.2/10^3 \qquad \text{(v)}$$

Adding Eqs (iii) and (iv)

$$u_3 + u_4 = 2\alpha_1 + 2\alpha_3 = 0.7/10^3$$

i.e.

$$\alpha_1 + \alpha_3 = 0.35/10^3 \qquad \text{(vi)}$$

Adding Eqs (v) and (vi)

$$\alpha_1 = 0.275/10^3$$

Then from Eq. (v)

$$\alpha_3 = 0.075/10^3$$

Now subtracting Eq. (ii) from Eq. (i)

$$u_1 - u_2 = -2\alpha_2 + 2\alpha_4 = -0.2/10^3$$

i.e.

$$\alpha_2 - \alpha_4 = 0.1/10^3 \qquad \text{(vii)}$$

Subtracting Eq. (iv) from Eq. (iii)

$$u_3 - u_4 = 2\alpha_2 + 2\alpha_4 = 0.5/10^3$$

i.e.

$$\alpha_2 + \alpha_4 = 0.25/10^3 \qquad \text{(viii)}$$

Now adding Eqs (vii) and (viii)

$$2\alpha_2 = 0.35/10^3$$

whence

$$\alpha_2 = 0.175/10^3$$

Then from Eq. (vii)

$$\alpha_4 = 0.075/10^3$$

From the second of Eqs (6.96)

$$v_1 = \alpha_5 - \alpha_6 - \alpha_7 + \alpha_8 = 0.1/10^3 \quad \text{(ix)}$$

$$v_2 = \alpha_5 + \alpha_6 - \alpha_7 - \alpha_8 = 0.3/10^3 \quad \text{(x)}$$

$$v_3 = \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 = 0.7/10^3 \quad \text{(xi)}$$

$$v_4 = \alpha_5 - \alpha_6 + \alpha_7 - \alpha_8 = 0.5/10^3 \quad \text{(xii)}$$

Then, in a similar manner to the above

$$\alpha_5 = 0.4/10^3$$

$$\alpha_7 = 0.2/10^3$$

$$\alpha_6 = 0.1/10^3$$

$$\alpha_8 = 0$$

Eqs (6.96) are now written

$$u_i = (0.275 + 0.175x + 0.075y + 0.075xy) \times 10^{-3}$$

$$v_i = (0.4 + 0.1x + 0.2y) \times 10^{-3}$$

Then, from Eqs (6.88)

$$\varepsilon_x = (0.175 + 0.075y) \times 10^{-3}$$

$$\varepsilon_y = 0.2 \times 10^{-3}$$

$$\gamma_{xy} = (0.075 + 0.075x + 0.1) \times 10^{-3} = (0.175 + 0.075x) \times 10^{-3}$$

At the centre of the element $x = y = 0$. Then

$$\varepsilon_x = 0.175 \times 10^{-3}$$

$$\varepsilon_y = 0.2 \times 10^{-3}$$

$$\gamma_{xy} = 0.175 \times 10^{-3}$$

so that, from Eqs (6.92)

$$\sigma_x = \frac{200\,000}{1 - 0.3^2}(0.175 + 0.3 \times 0.2) \times 10^{-3} = 51.65\,\text{N/mm}^2$$

$$\sigma_y = \frac{200\,000}{1 - 0.3^2}(0.2 + 0.3 \times 0.175) \times 10^{-3} = 55.49\,\text{N/mm}^2$$

$$\tau_{xy} = \frac{200\,000}{2(1 + 0.3)} \times 0.175 \times 10^{-3} = 13.46\,\text{N/mm}^2$$

S.6.12

Suitable displacement functions are:

$$u = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy$$
$$v = \alpha_5 + \alpha_6 x + \alpha_7 y + \alpha_8 xy$$

Then

$$u_1 = \alpha_1 - 2\alpha_2 - \alpha_3 + 2\alpha_4 = 0.001 \quad \text{(i)}$$
$$u_2 = \alpha_1 + 2\alpha_2 - \alpha_3 - 2\alpha_4 = 0.003 \quad \text{(ii)}$$
$$u_3 = \alpha_1 + 2\alpha_2 + \alpha_3 + 2\alpha_4 = -0.003 \quad \text{(iii)}$$
$$u_4 = \alpha_1 - 2\alpha_2 + \alpha_3 - 2\alpha_4 = 0 \quad \text{(iv)}$$

Subtracting Eq. (ii) from Eq. (i)

$$\alpha_2 - \alpha_4 = 0.0005 \quad \text{(v)}$$

Subtracting Eq. (iv) from Eq. (iii)

$$\alpha_2 + \alpha_4 = -0.00075 \quad \text{(vi)}$$

Subtracting Eq. (vi) from Eq. (v)

$$\alpha_4 = -0.000625$$

Then, from either of Eqs (v) or (vi)

$$\alpha_2 = -0.000125$$

Adding Eqs (i) and (ii)

$$\alpha_1 - \alpha_3 = 0.002 \quad \text{(vii)}$$

Adding Eqs (iii) and (iv)

$$\alpha_1 + \alpha_3 = -0.0015 \quad \text{(viii)}$$

Adding Eqs (vii) and (viii)

$$\alpha_1 = 0.00025$$

Then from either of Eqs (vii) or (viii)

$$\alpha_3 = -0.00175$$

Similarly

$$\alpha_5 = -0.001$$
$$\alpha_6 = 0.00025$$
$$\alpha_7 = 0.002$$
$$\alpha_8 = -0.00025$$

Then

$$u_i = 0.00025 - 0.00125x - 0.00175y - 0.000625xy$$
$$v_i = -0.001 + 0.00025x + 0.002y - 0.00025xy$$

From Eqs (1.18) and (1.20)

$$\varepsilon_x = \frac{\partial u}{\partial x} = -0.000125 - 0.000625y$$
$$\varepsilon_y = \frac{\partial v}{\partial y} = 0.002 - 0.00025x$$
$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -0.0015 - 0.000625x - 0.00025y$$

At the centre of the element where $x = y = 0$

$$\varepsilon_x = -0.000125 \quad \varepsilon_y = 0.002 \quad \gamma_{xy} = -0.0015.$$

S.6.13

Assume displacement functions

$$u(x, y) = \alpha_1 + \alpha_2 x + \alpha_3 y$$
$$v(x, y) = \alpha_4 + \alpha_5 x + \alpha_6 y$$

Then

$$u_1 = \alpha_1$$
$$u_2 = \alpha_1 + 4\alpha_2$$
$$u_3 = \alpha_1 + 2\alpha_2 + 2\alpha_3$$

Solving

$$\alpha_2 = \frac{u_2 - u_1}{4} \quad \alpha_3 = \frac{2u_3 - u_1 - u_2}{4}$$

Therefore

$$u = u_1 + \left(\frac{u_2 - u_1}{4}\right) x + \left(\frac{2u_3 - u_1 - u_2}{4}\right) y$$

or

$$u = \left(1 - \frac{x}{4} - \frac{y}{4}\right) u_1 + \left(\frac{x}{4} - \frac{y}{4}\right) u_2 + \frac{y}{2} u_3$$

Similarly

$$v = \left(1 - \frac{x}{4} - \frac{y}{4}\right) v_1 + \left(\frac{x}{4} - \frac{y}{4}\right) v_2 + \frac{y}{2} v_3$$

Then, from Eqs (1.18) and (1.20)

$$\varepsilon_x = \frac{\partial u}{\partial x} = -\frac{u_1}{4} + \frac{u_2}{4}$$
$$\varepsilon_y = \frac{\partial v}{\partial y} = -\frac{v_1}{4} - \frac{v_2}{4} + \frac{v_3}{2}$$
$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -\frac{u_1}{4} - \frac{u_2}{4} - \frac{v_1}{4} + \frac{v_2}{4}$$

Hence

$$[B]\{\delta^e\} = \begin{bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 2 \\ -1 & -1 & -1 & 1 & 2 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

But

$$[D] = \begin{bmatrix} a & b & 0 \\ b & a & 0 \\ 0 & 0 & c \end{bmatrix}$$

so that

$$[D][B] = \frac{1}{4}\begin{bmatrix} -a & -b & a & -b & 0 & 2b \\ -b & -a & b & -a & 0 & 2a \\ -c & -c & -c & c & 2c & 0 \end{bmatrix}$$

and

$$[B]^{\mathrm{T}}[D][B] = \frac{1}{16}\begin{bmatrix} a+c & b+c & -a+c & b-c & -2c & -2b \\ b+c & a+c & -b+c & a-c & -2c & -2a \\ -a+c & -b+c & a+c & -b-c & -2c & 2b \\ b-c & a-c & -b-c & a+c & 2c & -2a \\ -2c & -2c & -2c & 2c & 4c & 0 \\ -2b & -2a & 2b & -2a & 0 & 4a \end{bmatrix}$$

Since $[K^e] = [B]^{\mathrm{T}}[D][B] \times 4 \times 1$

$$[K^e] = \frac{1}{4}\begin{bmatrix} a+c & & & & & \text{SYM} \\ b+c & a+c & & & & \\ -a+c & -b+c & a+c & & & \\ b-c & a-c & -b-c & a+c & & \\ -2c & -2c & -2c & 2c & 4c & \\ -2b & -2a & 2b & -2a & 0 & 4a \end{bmatrix}$$

S.6.14

For $a = 1, b = 2$

$$u = \tfrac{1}{8}[(1-x)(2-y)u_1 + (1+x)(2-y)u_2 + (1+x)(2+y)u_3 + (1-x)(2+y)u_4]$$

Similarly for v

Then

$$\frac{\partial u}{\partial x} = \tfrac{1}{8}[-(2-y)u_1 + (2-y)u_2 + (2+y)u_3 - (2+y)u_4]$$

$$\frac{\partial v}{\partial y} = \tfrac{1}{8}[-(1-x)v_1 - (1+x)v_2 + (1+x)v_3 - (1-x)v_4]$$

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \tfrac{1}{8}[-(1-x)u_1 - (2-y)v_1 - (1+x)u_2 + (2-y)v_2 + (1+x)u_3 + (2+y)v_3 + (1-x)u_4 - (2+y)v_4]$$

In matrix form

$$\begin{bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{bmatrix}$$

$$= \frac{1}{8}\begin{bmatrix} -(2-y) & 0 & (2-y) & 0 & (2+y) & 0 & -(2+y) & 0 \\ 0 & -(1-x) & 0 & -(1+x) & 0 & (1+x) & 0 & (1-x) \\ -(1-x) & -(2-y) & -(1+x) & (2-y) & (1+x) & (2+y) & (1-x) & -(2+y) \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

Also

$$D = \begin{bmatrix} c & d & 0 \\ d & c & 0 \\ 0 & 0 & e \end{bmatrix}$$

Then

$[D][B]$

$$= \frac{1}{8}\begin{bmatrix} -c(2-y) & -d(1-x) & e(2-y) & -d(1+x) & e(2+y) & d(1+x) & -c(2+y) & d(1-x) \\ -d(2-y) & -c(1-x) & d(2-y) & -c(1+x) & d(2+y) & e(1+x) & -d(2+y) & c(1-x) \\ -e(1-x) & -e(2-y) & -e(1+x) & e(2-y) & e(1+x) & e(2+y) & e(1-x) & -e(2+y) \end{bmatrix}$$

Then

$$[B]^{\mathrm{T}}[D][B] = \frac{1}{64}\begin{bmatrix} -(2-y) & 0 & -(1-x) \\ 0 & -(1-x) & -(2-y) \\ & \vdots & \\ & \vdots & \\ & \vdots & \\ & \vdots & \\ & \vdots & \\ & \vdots & \end{bmatrix} \begin{bmatrix} -c(2-y) & -d(1-x) & \ldots & \ldots & \ldots \\ -d(2-y) & -c(1-x) & \ldots & \ldots & \ldots \\ -e(1-x) & -e(2-y) & \ldots & \ldots & \ldots \end{bmatrix}$$

Therefore

$$K_{11} = \frac{t}{64}\int_{-2}^{2}\int_{-1}^{1}[c(2-y)^2 + e(1-x)^2]\mathrm{d}x\,\mathrm{d}y$$

which gives $K_{11} = \dfrac{t}{6}(4c+e)$

$$K_{12} = \frac{t}{64}\int_{-2}^{2}\int_{-1}^{1}[d(2-y)(1-x) + e(1-x)(2-y)]\mathrm{d}x\,\mathrm{d}y$$

which gives $K_{12} = \frac{t}{4}(d+e)$.

Solutions to Chapter 7 Problems

S.7.1

Substituting for $((1/\rho_x) + (\nu/\rho_y))$ and $((1/\rho_y) + (\nu/\rho_x))$ from Eqs (7.5) and (7.6), respectively in Eqs (7.3)

$$\sigma_x = \frac{Ez}{1-\nu^2}\frac{M_x}{D} \quad \text{and} \quad \sigma_y = \frac{Ez}{1-\nu^2}\frac{M_y}{D} \tag{i}$$

Hence, since, from Eq. (7.4), $D = Et^3/12(1-\nu^2)$, Eqs (i) become

$$\sigma_x = \frac{12zM_x}{t^3} \quad \sigma_y = \frac{12zM_y}{t^3} \tag{ii}$$

The maximum values of σ_x and σ_y will occur when $z = \pm t/2$. Hence

$$\sigma_x(\max) = \pm\frac{6M_x}{t^2} \quad \sigma_y(\max) = \pm\frac{6M_y}{t^2} \tag{iii}$$