Then substituting in Eq. (4.20)

$$v_B = \frac{w}{8EI}\left[\int_0^{L/4} 3(Lx^2 - x^3)\mathrm{d}x + \int_{L/4}^{L} (Lx - x^2)(L - x)\mathrm{d}x\right]$$

which gives

$$v_B = \frac{57wL^4}{6144EI}$$

For the deflection at the mid-span point the bending moment at any section due to the actual loading is identical to the expression above. With the unit load applied at C

$$M_1 = \frac{x}{2} \text{ in AC} \quad \text{and} \quad M_1 = \frac{(L - x)}{2} \text{ in CD}$$

Substituting in Eq. (4.20)

$$v_C = \frac{w}{4EI}\left[\int_0^{L/2} (Lx^2 - x^3)\,\mathrm{d}x + \int_{L/2}^{L} (Lx - x^2)(L - x)\mathrm{d}x\right]$$

from which

$$v_C = \frac{5wL^4}{384EI}.$$

Solutions to Chapter 5 Problems

S.5.1

This problem is most readily solved by the application of the unit load method. Therefore, from Eq. (5.20), the vertical deflection of C is given by

$$\Delta_{V,C} = \sum \frac{F_0 F_{1,V} L}{AE} \tag{i}$$

and the horizontal deflection by

$$\Delta_{H,C} = \sum \frac{F_0 F_{1,H} L}{AE} \tag{ii}$$

in which $F_{1,V}$ and $F_{1,H}$ are the forces in a member due to a unit load positioned at C and acting vertically downwards and horizontally to the right, in turn, respectively. Further, the value of L/AE ($= 1/20$ mm/N) for each member is given and may be omitted from the initial calculation. All member forces (see Table S.5.1) are found using the method of joints which is described in textbooks on structural analysis, for example, *Structural and Stress Analysis* by T. H. G. Megson (Elsevier, 2005).

Table S.5.1

Member	F_0(N)	$F_{1,V}$	$F_{1,H}$	$F_0F_{1,V}$	$F_0F_{1,H}$
DC	16.67	1.67	0	27.84	0
BC	−13.33	−1.33	1.0	17.73	−13.33
ED	13.33	1.33	0	17.73	0
DB	−10.0	−1.0	0	10.0	0
AB	−16.67	−1.67	0.8	27.84	−13.34
EB	0	0	0.6	0	0
				$\sum = 101.14$	$\sum = -26.67$

Note that the loads $F_{1,V}$ are obtained most easily by dividing the loads F by a factor of 10. Then, from Eq. (i)

$$\Delta_{V,C} = 101.4 \times \frac{1}{20} = 5.07\,\text{mm}$$

which is positive and therefore in the same direction as the unit vertical load. Also from Eq. (ii)

$$\Delta_{H,C} = -26.67 \times \frac{1}{20} = -1.33\,\text{mm}$$

which is negative and therefore to the left.

The actual deflection, Δ, is then given by

$$\Delta = \sqrt{\Delta_{V,C}^2 + \Delta_{H,C}^2} = 5.24\,\text{mm}$$

which is downwards and at an angle of $\tan^{-1}(1.33/5.07) = 14.7°$ to the left of vertical.

S.5.2

Figure S.5.2 shows a plan view of the plate. Suppose that the point of application of the load is at D, a distance x from each side of the plate. The deflection of D may be found using the unit load method so that, from Eq. (5.20), the vertical deflection of D is given by

$$\Delta_D = \sum \frac{F_0 F_1 L}{AE} \tag{i}$$

Initially, therefore, the forces, F_0, must be calculated. Suppose that the forces in the wires at A, B and C due to the actual load are $F_{0,A}$, $F_{0,B}$ and $F_{0,C}$, respectively. Then resolving vertically

$$F_{0,A} + F_{0,B} + F_{0,C} = 100 \tag{ii}$$

Taking moments about the edges BC, AC and AB in turn gives

$$F_{0,A} \times 4 = 100x \tag{iii}$$

$$F_{0,B} \times 4 \times \sin A = 100x$$

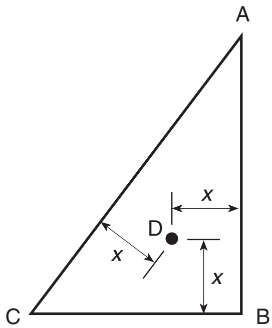

Fig. S.5.2

i.e.

$$F_{0,\mathrm{B}} \times 4 \times 0.6 = 100x \tag{iv}$$

and

$$F_{0,\mathrm{C}} \times 3 = 100x \tag{v}$$

Thus, from Eqs (iii) to (v)

$$4F_{0,\mathrm{A}} = 2.4F_{0,\mathrm{B}} = 3F_{0,\mathrm{C}}$$

so that

$$F_{0,\mathrm{A}} = 0.6F_{0,\mathrm{B}} \quad F_{0,\mathrm{C}} = 0.8F_{0,\mathrm{B}}$$

Substituting in Eq. (ii) gives

$$F_{0,\mathrm{B}} = 41.7\,\mathrm{N}$$

Hence

$$F_{0,\mathrm{A}} = 25.0\,\mathrm{N} \quad \text{and} \quad F_{0,\mathrm{C}} = 33.4\,\mathrm{N}$$

Now apply a unit load at D in the direction of the 100 N load. Then

$$F_{1,\mathrm{A}} = 0.25 \quad F_{1,\mathrm{B}} = 0.417 \quad F_{1,\mathrm{C}} = 0.334$$

Substituting for $F_{0,\mathrm{A}}$, $F_{1,\mathrm{A}}$, etc. in Eq. (i)

$$\Delta_{\mathrm{D}} = \frac{1440}{(\pi/4) \times 1^2 \times 196\,000}(25 \times 0.25 + 41.7 \times 0.417 + 33.4 \times 0.334)$$

i.e.

$$\Delta_{\mathrm{D}} = 0.33\,\mathrm{mm}$$

S.5.3

Suppose that joints 2 and 7 have horizontal and vertical components of displacement u_2, v_2, u_7, and v_7, respectively as shown in Fig. S.5.3. The displaced position of the member 27 is then $2'7'$. The angle α which the member 27 makes with the vertical is then given by

$$\alpha = \tan^{-1} \frac{u_7 - u_2}{3a + v_7 - v_2}$$

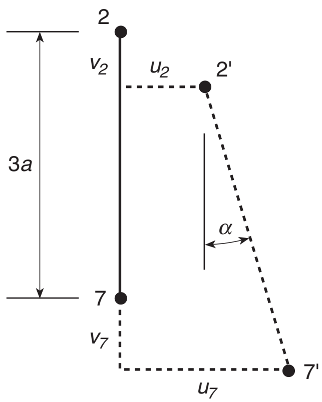

Fig. S.5.3

which, since α is small and v_7 and v_2 are small compared with $3a$, may be written as

$$\alpha = \frac{u_7 - u_2}{3a} \quad \text{(i)}$$

The horizontal components u_2 and u_7 may be found using the unit load method, Eq. (5.20). Thus

$$u_2 = \sum \frac{F_0 F_{1,2} L}{AE} \quad u_7 = \sum \frac{F_0 F_{1,7} L}{AE} \quad \text{(ii)}$$

where $F_{1,2}$ and $F_{1,7}$ are the forces in the members of the framework due to unit loads applied horizontally, in turn, at joints 2 and 7, respectively. The solution is completed in tabular form (Table S.5.3). Substituting the summation terms in Eqs (ii) gives

$$u_2 = -\frac{192Pa}{3AE} \quad u_7 = \frac{570Pa}{9AE}$$

Now substituting for u_2 and u_7 in Eq. (i)

$$\alpha = \frac{382P}{9AE}$$

Table S.5.3

Member	Length	F_0	$F_{1,2}$	$F_{1,7}$	$F_0F_{1,2}L$	$F_0F_{1,7}L$
27	$3a$	$3P$	0	0	0	0
87	$5a$	$5P/3$	0	$5/3$	0	$125Pa/9$
67	$4a$	$-4P/3$	0	$-4/3$	0	$64Pa/9$
21	$4a$	$4P$	$-4/3$	0	$-64Pa/3$	0
23	$5a$	0	$5/3$	0	0	0
26	$5a$	$-5P$	0	0	0	0
38	$3a$	0	0	0	0	0
58	$5a$	0	0	0	0	0
98	$5a$	$5P/3$	0	$5/3$	0	$125Pa/9$
68	$3a$	0	0	0	0	0
16	$3a$	$3P$	0	0	0	0
56	$4a$	$-16P/3$	0	$-4/3$	0	$256Pa/9$
13	$3a$	0	0	0	0	0
43	$5a$	0	$5/3$	0	0	0
93	$\sqrt{34}a$	0	0	0	0	0
03	$5a$	0	0	0	0	0
15	$5a$	$-5P$	0	0	0	0
10	$4a$	$8P$	$-4/3$	0	$-128Pa/3$	0
					$\sum = -192Pa/3$	$\sum = 570Pa/9$

S.5.4

(a) The beam is shown in Fig. S.5.4. The principle of the stationary value of the total complementary energy may be used to determine the deflection at C. From Eq. (5.13)

$$\Delta_C = \int_L d\theta \frac{dM}{dP} \tag{i}$$

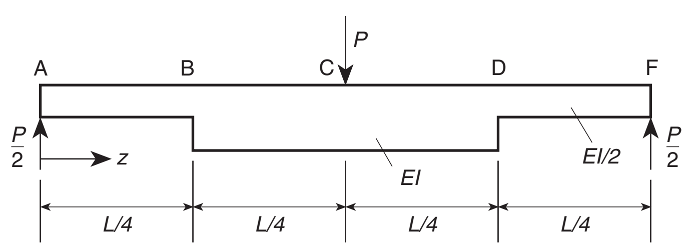

Fig. S.5.4

in which, since the beam is linearly elastic, $d\theta = (M/EI)dz$. Also the beam is symmetrical about its mid-span so that Eq. (i) may be written

$$\Delta_C = 2\int_0^{L/2} \frac{M}{EI}\frac{dM}{dP}dz \tag{ii}$$

In AC

$$M = \frac{P}{2}z$$

so that

$$\frac{\mathrm{d}M}{\mathrm{d}P} = \frac{z}{2}$$

Eq. (ii) then becomes

$$\Delta_{\mathrm{C}} = 2\left[\int_0^{L/4} \frac{Pz^2}{4\left(\dfrac{EI}{2}\right)}\mathrm{d}z + \int_{L/4}^{L/2} \frac{Pz^2}{4EI}\mathrm{d}z\right] \tag{iii}$$

Integrating Eq. (iii) and substituting the limits gives

$$\Delta_{\mathrm{C}} = \frac{3PL^3}{128EI}$$

(b) When the beam is encastré at A and F, fixed end moments M_{A} and M_{F} are induced. From symmetry $M_{\mathrm{A}} = M_{\mathrm{F}}$. The total complementary energy of the beam is, from Eq. (4.18)

$$C = \int_L \int_0^M \mathrm{d}\theta\,\mathrm{d}M - P\Delta_{\mathrm{C}}$$

from which

$$\frac{\partial C}{\partial M_{\mathrm{A}}} = \int_L \mathrm{d}\theta \frac{\partial M}{\partial M_{\mathrm{A}}} = 0 \tag{iv}$$

from the principle of the stationary value. From symmetry the reactions at A and F are each $P/2$. Hence

$$M = \frac{P}{2}z - M_{\mathrm{A}} \quad \text{(assuming } M_{\mathrm{A}} \text{ is a hogging moment)}$$

Then

$$\frac{\partial M}{\partial M_{\mathrm{A}}} = -1$$

Thus, from Eq. (iv)

$$\frac{\partial C}{\partial M_{\mathrm{A}}} = 2\int_0^{L/2} \frac{M}{EI}\frac{\partial M}{\partial M_{\mathrm{A}}}\mathrm{d}z = 0$$

or

$$0 = 2\left[\int_0^{L/4} \frac{1}{(EI/2)}\left(\frac{P}{2}z - M_{\mathrm{A}}\right)(-1)\,\mathrm{d}z + \int_{L/4}^{L/2} \frac{1}{EI}\left(\frac{P}{2}z - M_{\mathrm{A}}\right)(-1)\,\mathrm{d}z\right]$$

from which

$$M_{\mathrm{A}} = \frac{5PL}{48}$$

S.5.5

The unit load method, i.e. the first of Eqs (5.21), may be used to obtain a solution. Thus

$$\delta_{C,H} = \int \frac{M_0 M_1}{EI} dz \qquad \text{(i)}$$

in which the M_1 moments are due to a unit load applied horizontally at C. Then, referring to Fig. S.5.5, in CB

$$M_0 = W(R - R\cos\theta) \quad M_1 = 1 \times z$$

and in BA

$$M_0 = W2R \quad M_1 = 1 \times z$$

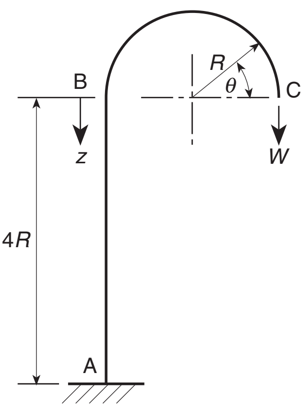

Fig. S.5.5

Hence, substituting these expressions in Eq. (i) and noting that in CB d$s = R\,$dθ and in BA d$s =$ dz

$$\delta_{C,H} = \frac{1}{EI}\left\{\int_0^{\pi} -WR^3(1 - \cos\theta)\sin\theta\, d\theta + \int_0^{4R} 2WRz\, dz\right\}$$

i.e.

$$\delta_{C,H} = \frac{1}{EI}\left\{-WR^3\left[-\cos\theta + \frac{\cos^2\theta}{2}\right]_0^{\pi} + WR[z^2]_0^{4R}\right\}$$

so that

$$\delta_{C,H} = \frac{14WR^3}{EI} \qquad \text{(ii)}$$

The second moment of area of the cross-section of the post is given by

$$I = \frac{\pi}{64}(100^4 - 94^4) = 1.076 \times 10^6 \text{ mm}^4$$

Substituting the value of I and the given values of W and R in Eq. (ii) gives

$$\delta_{\mathrm{C,H}} = 53.3\,\mathrm{mm}$$

S.5.6

Either of the principles of the stationary values of the total complementary energy or the total potential energy may be used to solve this problem.

From Eq. (5.12) the total complementary energy of the system is

$$C = \int_L \int_0^M \mathrm{d}\theta\,\mathrm{d}M - \int_L wv\,\mathrm{d}z \tag{i}$$

in which w is the load intensity at any point in the beam and v the vertical displacement. Equation (i) may be written in the form

$$C = \int_L \int_0^M \frac{M}{EI}\mathrm{d}z\,\mathrm{d}M - \int_L wv\,\mathrm{d}z$$

since, from symmetrical bending theory

$$\delta\theta = \frac{\delta z}{R} = \frac{M}{EI}\delta z$$

Hence

$$C = \int_L \frac{M^2}{2EI}\mathrm{d}z - \int_L wv\,\mathrm{d}z \tag{ii}$$

Alternatively, the total potential energy of the system is the sum of the strain energy due to bending of the beam plus the potential energy V, of the applied load. The strain energy U, due to bending in a beam may be shown to be given by

$$U = \int_L \frac{M^2}{2EI}\mathrm{d}z$$

Hence

$$\mathrm{TPE} = U + V = \int_L \frac{M^2}{2EI}\mathrm{d}z - \int_L wv\,\mathrm{d}z \tag{iii}$$

Eqs (ii) and (iii) are clearly identical.

Now, from symmetrical bending theory

$$\frac{M}{EI} = -\frac{\mathrm{d}^2 v}{\mathrm{d}z^2}$$

Therefore Eq. (ii) (or (iii)) may be rewritten

$$C = \int_0^L \frac{EI}{2}\left(\frac{\mathrm{d}^2 v}{\mathrm{d}z^2}\right)^2 \mathrm{d}z - \int_0^L wv\,\mathrm{d}z \tag{iv}$$

Now

$$v = a_1 \sin\frac{\pi z}{L} + a_2 \sin\frac{2\pi z}{L} \quad w = \frac{2w_0 z}{L}\left(1 - \frac{z}{2L}\right)$$

so that

$$\frac{\mathrm{d}^2 v}{\mathrm{d}z^2} = -a_1 \frac{\pi^2}{L^2}\sin\frac{\pi z}{L} - a_2 \frac{4\pi^2}{L^2}\sin\frac{2\pi z}{L}$$

Substituting in Eq. (iv)

$$C = \frac{EI}{2}\frac{\pi^4}{L^4}\int_0^L \left(a_1 \frac{\pi^2}{L^2}\sin\frac{\pi z}{L} + a_2\frac{4\pi^2}{L^2}\sin\frac{2\pi z}{L}\right)^2 \mathrm{d}z$$
$$-\frac{2w_0}{L}\int_0^L \left(a_1 z \sin\frac{\pi z}{L} + a_2 z\sin\frac{2\pi z}{L} - a_1\frac{z^2}{2L}\sin\frac{\pi z}{L} - a_2\frac{z^2}{2L}\sin\frac{2\pi z}{L}\right)\mathrm{d}z$$

which, on expanding, gives

$$C = \frac{EI\pi^4}{2L^4}\int_0^L \left(a_1^2 \sin^2\frac{\pi z}{L} + 8a_1 a_2 \sin\frac{\pi z}{L}\sin\frac{2\pi z}{L} + 16a_2^2\sin^2\frac{2\pi z}{L}\right)\mathrm{d}z$$
$$-\frac{2w_0}{L}\int_0^L \left(a_1 z \sin\frac{\pi z}{L} + a_2 z\sin\frac{2\pi z}{L} - a_1\frac{z^2}{2L}\sin\frac{\pi z}{L} - a_2\frac{z^2}{2L}\sin\frac{2\pi z}{L}\right)\mathrm{d}z \quad \text{(v)}$$

Eq. (v) may be integrated by a combination of direct integration and integration by parts and gives

$$C = \frac{EI\pi^4}{2L^4}\left(\frac{a_1^2 L}{2} + 8a_2^2 L\right) - a_1 w_0 L\left(\frac{1}{\pi} + \frac{4}{\pi^3}\right) + \frac{a_2 w_0 L}{2\pi} \quad \text{(vi)}$$

From the principle of the stationary value of the total complementary energy

$$\frac{\partial C}{\partial a_1} = 0 \quad \text{and} \quad \frac{\partial C}{\partial a_2} = 0$$

From Eq. (vi)

$$\frac{\partial C}{\partial a_1} = 0 = a_1 \frac{EI\pi^4}{2L^3} - \frac{w_0 L}{\pi^3}(\pi^2 + 4)$$

Hence

$$a_1 = \frac{2w_0 L^4}{EI\pi^7}(\pi^2 + 4)$$

Also

$$\frac{\partial C}{\partial a_2} = 0 = a_2\frac{8EI\pi^4}{L^3} + \frac{w_0 L}{2\pi}$$

whence

$$a_2 = -\frac{w_0 L^4}{16EI\pi^5}$$

The deflected shape of the beam is then

$$v = \frac{w_0 L^4}{EI}\left[\frac{2}{\pi^7}(\pi^2+4)\sin\frac{\pi z}{L} - \frac{1}{16\pi^5}\sin\frac{2\pi z}{L}\right]$$

At mid-span when $z = L/2$

$$v = 0.00918\frac{w_0 L^4}{EI}$$

S.5.7

This problem is solved in a similar manner to P.5.6. Thus Eq. (iv) of S.5.6 is directly applicable, i.e.

$$C = \int_L \frac{EI}{2}\left(\frac{d^2 v}{dz^2}\right)^2 dz - \int_L wv\,dz \tag{i}$$

in which

$$v = \sum_{i=1}^{\infty} a_i \sin\frac{i\pi z}{L} \tag{ii}$$

and w may be expressed as a function of z in the form $w = 4w_0 z(L-z)/L^2$ which satisfies the boundary conditions of $w = 0$ at $z = 0$ and $z = L$ and $w = w_0$ at $z = L/2$.

From Eq. (ii)

$$\frac{d^2 v}{dz^2} = -\sum_{i=1}^{\infty} a_i \frac{i^2\pi^2}{L^2}\sin\frac{i\pi z}{L}$$

Substituting in Eq. (i)

$$C = \frac{EI}{2}\int_0^L \sum_{i=1}^{\infty} a_i^2 \frac{i^4\pi^4}{L^4}\sin^2\frac{i\pi z}{L}dz - \frac{4w_0}{L^2}\int_0^L z(L-z)\sum_{i=1}^{\infty} a_i \sin\frac{i\pi z}{L}dz \tag{iii}$$

Now

$$\int_0^L \sin^2\frac{i\pi z}{L}dz = \int_0^L \frac{1}{2}\left(1-\cos\frac{i2\pi z}{L}\right)dz = \left[\frac{z}{2} - \frac{L}{i2\pi}\sin\frac{i2\pi z}{L}\right]_0^L = \frac{L}{2}$$

$$\int_0^L Lz\sin\frac{i\pi z}{L}dz = L\left[-\frac{zL}{i\pi}\cos\frac{i\pi z}{L} + \int\frac{L}{i\pi}\cos\frac{i\pi z}{L}dz\right]_0^L = -\frac{L^3}{i\pi}\cos i\pi$$

$$\int_0^L z^2\sin\frac{i\pi z}{L}dz = \left[-\frac{z^2 L}{i\pi}\cos\frac{i\pi z}{L} + \int\frac{L}{i\pi}\cos\frac{i\pi z}{L}2z\,dz\right]_0^L$$

$$= -\frac{L^3}{i\pi}\cos i\pi + \frac{2L^3}{i^3\pi^3}(\cos i\pi - 1)$$

Thus Eq. (iii) becomes

$$C = \sum_{i=1}^{\infty} \frac{EIa_i^2 i^4 \pi^4}{4L^3} - \frac{4w_0}{L^2} \sum_{i=1}^{\infty} a_i \left[-\frac{L^3}{i\pi} \cos i\pi + \frac{L^3}{i\pi} \cos i\pi - \frac{2L^3}{i^3\pi^3}(\cos i\pi - 1) \right]$$

or

$$C = \sum_{i=1}^{\infty} \frac{EIa_i^2 i^4 \pi^4}{4L^3} - \frac{4w_0}{L^2} \sum_{i=1}^{\infty} \frac{2a_i L^3}{i^3\pi^3}(1 - \cos i\pi) \qquad \text{(iv)}$$

The value of $(1 - \cos i\pi)$ is zero when i is even and 2 when i is odd. Therefore Eq. (iv) may be written

$$C = \frac{EIa_i^2 i^4 \pi^4}{4L^3} - \frac{16 w_0 a_i L}{i^3 \pi^3} \qquad i \text{ is odd}$$

From the principle of the stationary value of the total complementary energy

$$\frac{\partial C}{\partial a_i} = \frac{EIa_i i^4 \pi^4}{2L^3} - \frac{16 w_0 L}{i^3 \pi^3} = 0$$

Hence

$$a_i = \frac{32 w_0 L^4}{EIi^7 \pi^7}$$

Then

$$v = \sum_{i=1}^{\infty} \frac{32 w_0 L^4}{EIi^7 \pi^7} \sin \frac{i\pi z}{L} \qquad i \text{ is odd}$$

At the mid-span point where $z = L/2$ and using the first term only in the expression for v

$$v_{\text{m.s.}} = \frac{w_0 L^4}{94.4EI}$$

S.5.8

The lengths of the members which are not given are:

$$L_{12} = 9\sqrt{2}a \quad L_{13} = 15a \quad L_{14} = 13a \quad L_{24} = 5a$$

The force in the member 14 due to the temperature change is compressive and equal to $0.7A$. Also the change in length, Δ_{14}, of the member 14 due to a temperature change T is $L_{14}\alpha T = 13a \times 2.4 \times 10^{-6} T$. This must also be equal to the change in length produced by the force in the member corresponding to the temperature rise. Let this force be R.

From the unit load method, Eq. (5.20)

$$\Delta_{14} = \sum \frac{F_0 F_1 L}{AE} \qquad \text{(i)}$$

In this case, since R and the unit load are applied at the same points, in the same direction and no other loads are applied when only the temperature change is being considered, $F_0 = RF_1$. Equation (i) may then be written

$$\Delta_{14} = R \sum \frac{F_1^2 L}{AE} \tag{ii}$$

The method of joints may be used to determine the F_1 forces in the members. Thus

$$F_{14} = 1 \quad F_{13} = \frac{-35}{13} \quad F_{12} = \frac{16\sqrt{2}}{13} \quad F_{24} = \frac{-20}{13} \quad F_{23} = \frac{28}{13}$$

Eq. (ii) then becomes

$$\Delta_{14} = R\left[\frac{1^2 \times 13a}{AE} + \frac{35^2 \times 15a}{13^2 AE} + \frac{(16\sqrt{2})^2 \times 9\sqrt{2}a}{13^2\sqrt{2}AE} + \frac{20^2 \times 5a}{13^2 AE} + \frac{28^2 \times 3a}{13^2 AE}\right]$$

or

$$\Delta_{14} = \frac{Ra}{13^2 AE}(13^3 + 35^2 \times 15 + 16^2 \times 18 + 20^2 \times 5 + 28^2 \times 3)$$

i.e.

$$\Delta_{14} = \frac{29\,532aR}{13^2 AE}$$

Then

$$13a \times 24 \times 10^{-6} T = \frac{29\,532a(0.7A)}{13^2 AE}$$

so that

$$T = 5.6^\circ$$

S.5.9

Referring to Figs P.5.9(a), (b) and S.5.9 it can be seen that the members 12, 24 and 23 remain unloaded until P has moved through a horizontal distance 0.25 $\cos\alpha$, i.e. a distance of $0.25 \times 600/750 = 0.2$ mm. Therefore, until P has moved through a horizontal distance of 0.2 mm P is equilibrated solely by the forces in the members 13, 34 and 41 which therefore form a triangular framework. The method of solution is to find the value of P which causes a horizontal displacement of 0.2 mm of joint 1 in this framework.

Using the unit load method, i.e. Eq. (5.20) and solving in tabular form (see Table S.5.9(a)).

Then

$$0.2 = \frac{1425.0P}{300 \times 70\,000}$$

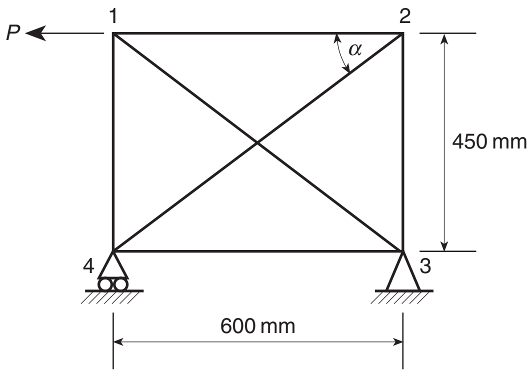

Fig. S.5.9

Table S.5.9(a)

Member	Length (mm)	F_0	F_1	F_0F_1L
13	750	$1.25P$	1.25	$1171.9P$
14	450	$-0.75P$	-0.75	$253.1P$
43	600	0	0	0
				$\sum = 1425.0P$

from which

$$P = 2947\,\mathrm{N}$$

The corresponding forces in the members 13, 14 and 43 are then

$$F_{13} = 3683.8\,\mathrm{N} \quad F_{14} = -2210.3\,\mathrm{N} \quad F_{43} = 0$$

When $P = 10\,000\,\mathrm{N}$ additional forces will be generated in these members corresponding to a load of $P' = 10\,000 - 2947 = 7053\,\mathrm{N}$. Also P' will now produce forces in the remaining members 12, 24 and 23 of the frame. The solution is now completed in a similar manner to that for the frame shown in Fig. 5.8 using Eq. (5.16). Suppose that R is the force in the member 24; the solution is continued in Table S.5.9(b). From Eq. (5.16)

$$2592R + 1140P' = 0$$

Table S.5.9(b)

Member	Length (mm)	F	$\partial F/\partial R$	$FL(\partial F/\partial R)$
12	600	$-0.8R$	-0.8	$384R$
23	450	$-0.6R$	-0.6	$162R$
34	600	$-0.8R$	-0.8	$384R$
41	450	$-(0.6R + 0.75P')$	-0.6	$162R + 202.5P'$
13	750	$R + 1.25P'$	1.0	$750R + 937.5P'$
24	750	R	1.0	$750R$
				$\sum = 2592R + 1140P'$

so that

$$R = -\frac{1140 \times 7053}{2592}$$

i.e.

$$R = -3102\,\mathrm{N}$$

Then

$$\begin{aligned}
F_{12} &= -0.8 \times (-3102) = 2481.6\,\mathrm{N}\ \text{(tension)} \\
F_{23} &= -0.6 \times (-3102) = 1861.2\,\mathrm{N}\ \text{(tension)} \\
F_{34} &= -0.8 \times (-3102) = 2481.6\,\mathrm{N}\ \text{(tension)} \\
F_{41} &= -0.6 \times (-3102) - 0.75 \times 7053 - 2210.3 = -5638.9\,\mathrm{N}\ \text{(compression)} \\
F_{13} &= -3102 + 1.25 \times 7053 + 3683.8 = 9398.1\,\mathrm{N}\ \text{(tension)} \\
F_{24} &= -3102.0\,\mathrm{N}\ \text{(compression)}
\end{aligned}$$

S.5.10

Referring to Fig. S.5.10(a) the vertical reactions at A and D are found from statical equilibrium. Then, taking moments about D

$$R_{\mathrm{A}} \tfrac{2}{3} l + \tfrac{1}{2} l w \tfrac{2}{3} l = 0$$

i.e.

$$R_{\mathrm{A}} = -\frac{wl}{2} \quad \text{(downwards)}$$

Hence

$$R_{\mathrm{D}} = \frac{wl}{2} \quad \text{(upwards)}$$

Also for horizontal equilibrium

$$H_{\mathrm{A}} + \frac{wl}{2} = H_{\mathrm{D}} \tag{i}$$

The total complementary energy of the frame is, from Eq. (5.12)

$$C = \int_L \int_0^M \mathrm{d}\theta\, \mathrm{d}M - H_{\mathrm{A}} \Delta_{\mathrm{A,H}} - R_{\mathrm{A}} \Delta_{\mathrm{A,V}} - H_{\mathrm{D}} \Delta_{\mathrm{D,H}} - R_{\mathrm{D}} \Delta_{\mathrm{D,V}} + \int_0^l w' \Delta\, \mathrm{d}z \tag{ii}$$

in which $\Delta_{\mathrm{A,H}}$, $\Delta_{\mathrm{A,V}}$, $\Delta_{\mathrm{D,H}}$ and $\Delta_{\mathrm{D,V}}$ are the horizontal and vertical components of the displacements at A and D, respectively and Δ is the horizontal displacement of the member AB at any distance z from A. From the principle of the stationary value

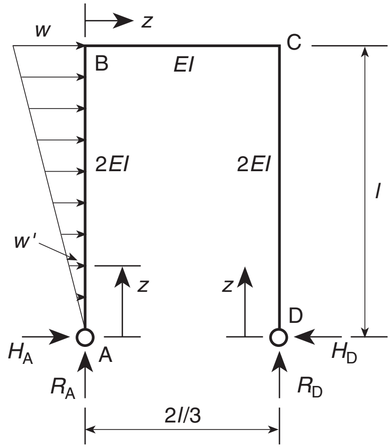

Fig. S.5.10(a)

of the total complementary energy of the frame and selecting $\Delta_{A,H}$ as the required displacement

$$\frac{\partial C}{\partial H_A} = \int_L d\theta \frac{\partial M}{\partial H_A} - \Delta_{A,H} = 0 \tag{iii}$$

In this case $\Delta_{A,H} = 0$ so that Eq. (iii) becomes

$$\int_L d\theta \frac{\partial M}{\partial H_A} = 0$$

or, since $d\theta = (M/EI)dz$

$$\int_L \frac{M}{EI} \frac{\partial M}{\partial H_A} dz = 0 \tag{iv}$$

In AB

$$M = -H_A z - \frac{wz^3}{6l} \qquad \frac{\partial M}{\partial H_A} = -z$$

In BC

$$M = R_A z - H_A l - \frac{wl^2}{6} \qquad \frac{\partial M}{\partial H_A} = -l$$

In DC

$$M = -H_D z = -\left(H_A + \frac{wl}{2}\right) z \quad \text{from Eq. (i),} \quad \frac{\partial M}{\partial H_A} = -z$$

Substituting these expressions in Eq. (iv) gives

$$\int_0^l \frac{1}{2EI}\left(-H_A z - \frac{wz^3}{6l}\right)(-z)\mathrm{d}z + \int_0^{2l/3} \frac{1}{EI}\left(-\frac{wl}{2}z - H_A l - \frac{wl^2}{6}\right)(-l)\mathrm{d}z$$
$$+ \int_0^l \frac{1}{2EI}\left(-H_A - \frac{wl}{2}\right)z(-z)\mathrm{d}z = 0$$

or

$$\frac{1}{2}\int_0^l \left(H_A z^2 + \frac{wz^4}{6l}\right)\mathrm{d}z + \int_0^{2l/3}\left(\frac{wl^2}{2}z + H_A l^2 + \frac{wl^3}{6}\right)\mathrm{d}z$$
$$+ \frac{1}{2}\int_0^l\left(H_A z^2 + \frac{wlz^2}{2}\right)\mathrm{d}z = 0$$

from which

$$2H_A l^3 + \frac{29}{45}wl^4 = 0$$

or

$$H_A = -29wl/90$$

Hence, from Eq. (i)

$$H_D = 8wl/45$$

Thus

$$M_{AB} = -H_A z - \frac{wz^3}{6l} = \frac{29wl}{90}z - \frac{w}{6l}z^3$$

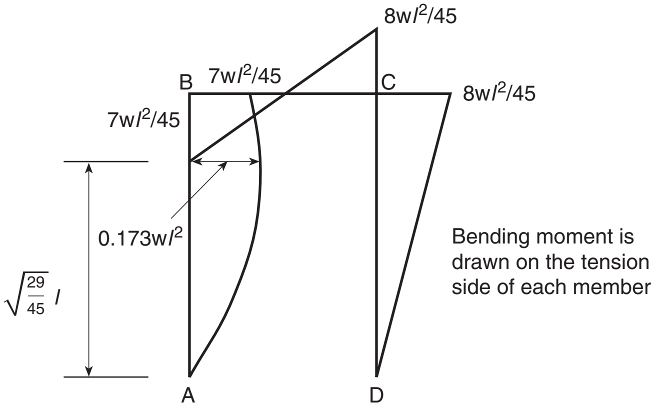

Fig. S.5.10(b)

When $z = 0$, $M_{\text{AB}} = 0$ and when $z = l$, $M_{\text{AB}} = 7wl^2/45$. Also, $\text{d}M_{\text{AB}}/\text{d}z = 0$ for a turning value, i.e.

$$\frac{\text{d}M_{\text{AB}}}{\text{d}z} = \frac{29wl}{90} - \frac{3wz^2}{6l} = 0$$

from which $z = \sqrt{29/45}l$. Hence $M_{\text{AB}}(\text{max}) = 0.173wl^2$.

The bending moment distributions in BC and CD are linear and $M_{\text{B}} = 7wl^2/45$, $M_{\text{D}} = 0$ and $M_{\text{C}} = H_{\text{D}}l = 8wl^2/45$.

The complete bending moment diagram for the frame is shown in Fig. S.5.10(b).

S.5.11

The bracket is shown in Fig. S.5.11 in which R_{C} is the vertical reaction at C and M_{C} is the moment reaction at C in the vertical plane containing AC.

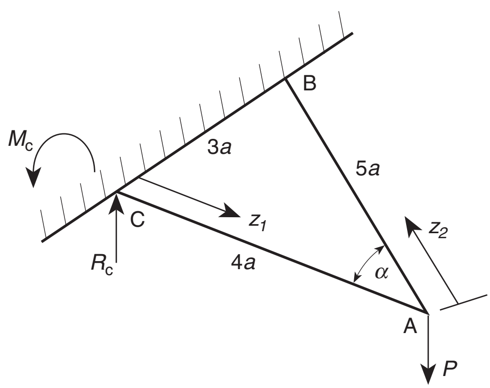

Fig. S.5.11

From Eq. (5.12) the total complementary energy of the bracket is given by

$$C = \int_L \int_0^M \text{d}\theta\,\text{d}M + \int_L \int_0^T \text{d}\phi\,\text{d}T - M_{\text{C}}\theta_{\text{C}} - R_{\text{C}}\Delta_{\text{C}} - P\Delta_{\text{A}}$$

in which T is the torque in AB producing an angle of twist, ϕ, at any section and the remaining symbols have their usual meaning. Then, from the principle of the stationary value of the total complementary energy and since $\theta_{\text{C}} = \Delta_{\text{C}} = 0$

$$\frac{\partial C}{\partial R_{\text{C}}} = \int_L \frac{M}{EI}\frac{\partial M}{\partial R_{\text{C}}}\text{d}z + \int_L \frac{T}{GJ}\frac{\partial T}{\partial R_{\text{C}}}\text{d}z = 0 \qquad \text{(i)}$$

and

$$\frac{\partial C}{\partial M_{\text{C}}} = \int_L \frac{M}{EI}\frac{\partial M}{\partial M_{\text{C}}}\text{d}z + \int_L \frac{T}{GJ}\frac{\partial T}{\partial M_{\text{C}}}\text{d}z = 0 \qquad \text{(ii)}$$

From Fig. S.5.11

$$M_{\text{AC}} = R_{\text{C}}z_1 - M_{\text{C}} \quad T_{\text{AC}} = 0$$

so that

$$\frac{\partial M_{\mathrm{AC}}}{\partial R_{\mathrm{C}}} = z_1 \quad \frac{\partial M_{\mathrm{AC}}}{\partial M_{\mathrm{C}}} = -1 \quad \frac{\partial T_{\mathrm{AC}}}{\partial R_{\mathrm{C}}} = \frac{\partial T_{\mathrm{AC}}}{\partial M_{\mathrm{C}}} = 0$$

Also

$$M_{\mathrm{AB}} = -Pz_2 + R_{\mathrm{C}}(z_2 - 4a\cos\alpha) + M_{\mathrm{C}}\,\cos\alpha$$

i.e.

$$M_{\mathrm{AB}} = -Pz_2 + R_{\mathrm{C}}\left(z_2 - \frac{16a}{5}\right) + \frac{4}{5}M_{\mathrm{C}}$$

Hence

$$\frac{\partial M_{\mathrm{AB}}}{\partial R_{\mathrm{C}}} = z_2 - \frac{16a}{5} \quad \frac{\partial M_{\mathrm{AB}}}{\partial M_{\mathrm{C}}} = \frac{4}{5}$$

Finally

$$T_{\mathrm{AB}} = R_{\mathrm{C}}4a\sin\alpha - M_{\mathrm{C}}\sin\alpha$$

i.e.

$$T_{\mathrm{AB}} = \frac{12a}{5}R_{\mathrm{C}} - \frac{3}{5}M_{\mathrm{C}}$$

so that

$$\frac{\partial T_{\mathrm{AB}}}{\partial R_{\mathrm{C}}} = \frac{12a}{5} \quad \frac{\partial T_{\mathrm{AB}}}{\partial M_{\mathrm{C}}} = -\frac{3}{5}$$

Substituting these expressions in Eq. (i)

$$\int_0^{4a} \frac{1}{EI}(R_{\mathrm{C}}z_1 - M_{\mathrm{C}})z_1\,\mathrm{d}z_1 + \int_0^{5a} \frac{1}{1.5EI}\left[-Pz_2 + R_{\mathrm{C}}\left(z_2 - \frac{16a}{5}\right) + \frac{4}{5}M_{\mathrm{C}}\right] \times \left(z_2 - \frac{16a}{5}\right)\mathrm{d}z_2 + \int_0^{5a} \frac{1}{3GI}\left(\frac{12a}{5}R_{\mathrm{C}} - \frac{3}{5}M_{\mathrm{C}}\right)\frac{12a}{5}\mathrm{d}z_2 = 0 \qquad \text{(iii)}$$

Note that for the circular section tube AC the torsion constant J (i.e. the polar second moment of area) $= 2 \times 1.5I$ from the theorem of perpendicular axes.

Integrating Eq. (iii), substituting the limits and noting that $G/E = 0.38$ gives

$$55.17\,R_{\mathrm{C}}a - 16.18\,M_{\mathrm{C}} - 1.11\,Pa = 0 \qquad \text{(iv)}$$

Now substituting in Eq. (ii) for M_{AC}, $\partial M_{\mathrm{AC}}/\partial M_{\mathrm{C}}$, etc.

$$\int_0^{4a} \frac{1}{EI}(R_{\mathrm{C}}z_1 - M_{\mathrm{C}})(-1)\mathrm{d}z_1 + \int_0^{5a} \frac{1}{1.5EI}\left[-Pz_2 + R_{\mathrm{C}}\left(z_2 - \frac{16a}{5}\right) + \frac{4}{5}M_{\mathrm{C}}\right]\frac{4}{5}\mathrm{d}z_2 + \int_0^{5a} \frac{1}{3GI}\left(\frac{12a}{5}R_{\mathrm{C}} - \frac{3}{5}M_{\mathrm{C}}\right)\left(-\frac{3}{5}\right)\mathrm{d}z_2 = 0 \qquad \text{(v)}$$

from which

$$16.58\,R_{\mathrm{C}}a - 7.71\,M_{\mathrm{C}} + 6.67\,Pa = 0 \qquad \text{(vi)}$$

Solving the simultaneous Eqs (iv) and (vi) gives

$$R_{\mathrm{C}} = 0.72\,P$$

S.5.12

Suppose that R is the tensile force in the member 23, i.e. $R = xP_0$. Then, from Eq. (5.15)

$$\sum \lambda_i \frac{\partial F_i}{\partial R} = 0 \qquad \text{(i)}$$

in which, for members 12, 23 and 34

$$\lambda_i = \varepsilon L_i = \frac{\tau_i L_i}{E}\left[1 + \left(\frac{\tau_i}{\tau_0}\right)^n\right] \qquad \text{(ii)}$$

But $\tau_i = F_i/A_i$ so that Eq. (ii) may be written

$$\lambda_i = \frac{F_i L_i}{A_i E_i}\left[1 + \left(\frac{F_i}{A_i \tau_0}\right)^n\right] \qquad \text{(iii)}$$

For members 15, 25, 35 and 45 which are linearly elastic

$$\lambda_i = \frac{F_i L_i}{A_i E} \qquad \text{(iv)}$$

The solution is continued in Table S.5.12. Summing the final column in Table S.5.12 gives

$$\frac{4RL}{\sqrt{3}AE}[1 + (\alpha x)^n] + \frac{2\sqrt{3}RL}{AE}[1 + (\alpha x)^n] + \frac{8L}{\sqrt{3}AE}\left(P_0 + \frac{2R}{\sqrt{3}}\right) + \frac{16RL}{\sqrt{3}AE} = 0 \quad \text{(v)}$$

from Eq. (i)

Noting that $R = xP_0$, Eq. (v) simplifies to

$$4x[1 + (\alpha x)^n] + 6x[1 + (\alpha x)^n] + 8 + \frac{16x}{\sqrt{3}} + 16x = 0$$

or

$$10x(\alpha x)^n + x\left(10 + \frac{16}{\sqrt{3}} + 16\right) + 8 = 0$$

from which

$$\alpha^n x^{n+1} + 3.5x + 0.80 = 0$$

Table S.5.12

Member	L_i	A_i	F_i	$\partial F_i/\partial R$	λ_i	$\lambda_i\,\partial F_i/\partial R$
12	$2L$	$A/\sqrt{3}$	$R/\sqrt{3}$	$1/\sqrt{3}$	$\dfrac{2RL}{AE}\left[1+\left(\dfrac{R}{A\tau_0}\right)^n\right]$	$\dfrac{2RL}{\sqrt{3}AE}[1+(\alpha x)^n]$
23	$2L/\sqrt{3}$	A	R	1	$\dfrac{2\sqrt{3}RL}{AE}\left[1+\left(\dfrac{R}{A\tau_0}\right)^n\right]$	$\dfrac{2\sqrt{3}RL}{AE}[1+(\alpha x)^n]$
34	$2L$	$A/\sqrt{3}$	$R/\sqrt{3}$	$1/\sqrt{3}$	$\dfrac{2RL}{AE}\left[1+\left(\dfrac{R}{A\tau_0}\right)^n\right]$	$\dfrac{2RL}{\sqrt{3}AE}[1+(\alpha x)^n]$
15	$2L$	A	$-P_0-2R/\sqrt{3}$	$-2/\sqrt{3}$	$-\dfrac{(P_0+2R/\sqrt{3})2L}{AE}$	$\dfrac{4L}{\sqrt{3}AE}(P_0+2R/\sqrt{3})$
25	$2L$	$A/\sqrt{3}$	$-2R/\sqrt{3}$	$-2/\sqrt{3}$	$-\dfrac{4RL}{AE}$	$\dfrac{8RL}{\sqrt{3}AE}$
35	$2L$	$A/\sqrt{3}$	$-2R/\sqrt{3}$	$-2/\sqrt{3}$	$-\dfrac{4RL}{AE}$	$\dfrac{8RL}{\sqrt{3}AE}$
45	$2L$	A	$-P_0-2R/\sqrt{3}$	$-2/\sqrt{3}$	$-\dfrac{2L}{AE}(P_0+2R/\sqrt{3})$	$\dfrac{4L}{\sqrt{3}AE}(P_0+2R/\sqrt{3})$

S.5.13

Suppose that the vertical reaction between the two beams at C is P. Then the force system acting on the beam AB is as shown in Fig. S.5.13. Taking moments about B

$$R_{\text{A}} \times 9.15 + P \times 6.1 - 100 \times 3.05 = 0$$

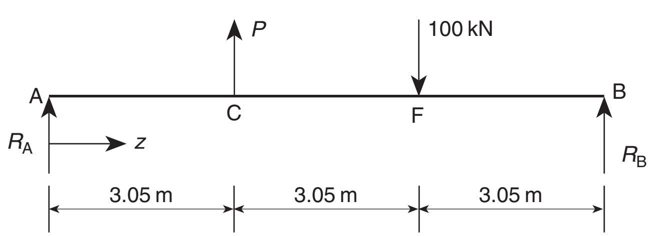

Fig. S.5.13

so that

$$R_{\text{A}} = 33.3 - 0.67P$$

The total complementary energy of the beam is, from Eq. (5.12)

$$C = \int_L \int_0^M \mathrm{d}\theta\,\mathrm{d}M - P\Delta_{\text{C}} - 100\Delta_{\text{F}} = 0$$

where Δ_{C} and Δ_{F} are the vertical displacements at C and F, respectively. Then, from the principle of the stationary value of the total complementary energy of the beam

$$\frac{\partial C}{\partial P} = \int_L \mathrm{d}\theta \frac{\partial M}{\partial P} - \Delta_{\mathrm{C}} = 0$$

whence, as in previous cases

$$\Delta_{\mathrm{C}} = \int_L \frac{M}{EI} \frac{\partial M}{\partial P} \mathrm{d}z \qquad \text{(i)}$$

In AC

$$M_{\mathrm{AC}} = R_{\mathrm{A}} z = (33.3 - 0.67P)z$$

so that

$$\frac{\partial M_{\mathrm{AC}}}{\partial P} = -0.67z$$

In CF

$$M_{\mathrm{CF}} = R_{\mathrm{A}} z + P(z - 3.05) = 33.3z + P(0.33z - 3.05)$$

from which

$$\frac{\partial M_{\mathrm{CF}}}{\partial P} = 0.33z - 3.05$$

In FB

$$M_{\mathrm{FB}} = R_{\mathrm{A}} z + P(z - 3.05) - 100(z - 6.1) = -66.7z + 610 + P(0.33z - 3.04)$$

which gives

$$\frac{\partial M_{\mathrm{FB}}}{\partial P} = 0.33z - 3.04$$

Substituting these expressions in Eq. (i)

$$\begin{aligned} EI\Delta_{\mathrm{C}} = & \int_0^{3.05} (33.3 - 0.67P)z(-0.67z)\mathrm{d}z \\ & + \int_{3.05}^{6.1} [33.3z + P(0.33z - 3.05)](0.33z - 3.05)\mathrm{d}z \\ & + \int_{6.1}^{9.15} [-66.7z + 610 + P(0.33z - 3.05)](0.33z - 3.05)\mathrm{d}z \end{aligned}$$

which simplifies to

$$
\begin{aligned}
EI\Delta_C = &\int_0^{3.05} (-22.2z^2 + 0.44Pz^2)\mathrm{d}z \\
&+ \int_{3.05}^{6.1} (10.99z^2 + 0.11Pz^2 - 2.02Pz + 9.3P - 101.6z)\mathrm{d}z \\
&+ \int_{6.1}^{9.15} (-22.01z^2 + 404.7z + 0.11Pz^2 - 2.02Pz + 9.3P - 1860.5)\mathrm{d}z
\end{aligned}
$$

Integrating this equation and substituting the limits gives

$$EI\Delta_C = 12.78P - 1117.8 \tag{ii}$$

From compatibility of displacement, the displacement at C in the beam AB is equal to the displacement at C in the beam ED. The displacement at the mid-span point in a fixed beam of span L which carries a central load P is $PL^3/192EI$. Hence, equating this value to Δ_C in Eq. (ii) and noting that Δ_C in Eq. (ii) is positive in the direction of P

$$-(12.78P - 1117.8) = P \times \frac{6.1^3}{192}$$

which gives

$$P = 80.1\,\mathrm{kN}$$

Thus

$$\Delta_C = \frac{80.1 \times 10^3 \times 6.1^3 \times 10^9}{192 \times 200\,000 \times 83.5 \times 10^6}$$

i.e.

$$\Delta_C = 5.6\,\mathrm{mm}$$

Note: The use of complementary energy in this problem produces a rather lengthy solution. A quicker approach to finding the displacement Δ_C in terms of P for the beam AB would be to use Macauley's method (see, e.g. *Structural and Stress Analysis* by T. H. G. Megson (Elsevier, 2005)).

S.5.14

The internal force system in the framework and beam is statically determinate so that the unit load method may be used directly to determine the vertical displacement of D. Hence, from the first of Eqs (5.21) and Eq. (5.20)

$$\Delta_{D,V} = \int_L \frac{M_0 M_1}{EI}\mathrm{d}z + \sum_{i=1}^{k} \frac{F_{i,0}F_{i,1}L_i}{A_i E_i} \tag{i}$$

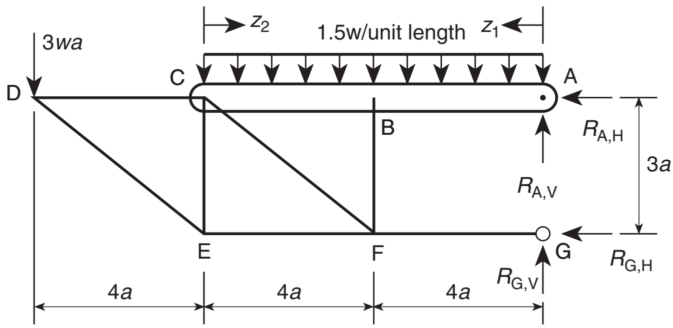

Fig. S.5.14

Referring to Fig. S.5.14 and taking moments about A

$$R_{\mathrm{G,H}}3a - 1.5w\frac{(8a)^2}{2} - 3wa12a = 0$$

from which

$$R_{\mathrm{G,H}} = 28wa$$

Hence

$$R_{\mathrm{A,H}} = -28wa$$

From the vertical equilibrium of the support G, $R_{\mathrm{G,V}} = 0$, so that, resolving vertically

$$R_{\mathrm{A,V}} - 1.5w8a - 3wa = 0$$

i.e.

$$R_{\mathrm{A,V}} = 15wa$$

With a unit vertical load at D

$$R_{\mathrm{G,H}} = 4 \quad R_{\mathrm{A,H}} = -4 \quad R_{\mathrm{A,V}} = 1 \quad R_{\mathrm{G,V}} = 0$$

For the beam ABC, in AB

$$M_0 = R_{\mathrm{A,V}}z_1 - \frac{1.5wz_1^2}{2} = 15waz_1 - 0.75wz_1^2 \quad M_1 = 1 \times z_1$$

and in BC

$$M_0 = 15waz_2 - 0.75wz_2^2 \quad M_1 = 1 \times z_2$$

Hence

$$\int_L \frac{M_0M_1}{EI}\mathrm{d}z = \frac{16}{Aa^2E}\left[\int_0^{4a}(15waz_1^2 - 0.75wz_1^3)\mathrm{d}z_1 + \int_0^{4a}(15waz_2^2 - 0.75wz_2^3)\mathrm{d}z_2\right]$$

Suppose $z_1 = z_2 = z$ say, then

$$\int_L \frac{M_0 M_1}{EI} \mathrm{d}z = \frac{16}{Aa^2 E} 2 \int_0^{4a} (15waz^2 - 0.75wz^3) \mathrm{d}z = \frac{32w}{Aa^2 E} \left[5az^3 - \frac{0.75}{4} z^4 \right]_0^{4a}$$

i.e.

$$\int_L \frac{M_0 M_1}{EI} \mathrm{d}z = \frac{8704wa^2}{AE}$$

The solution is continued in Table S.5.14.

Table S.5.14

Member	L	A	F_0	F_1	$F_0 F_1 L/A$
AB	$4a$	$4A$	$28wa$	4	$112wa^2/A$
BC	$4a$	$4A$	$28wa$	4	$112wa^2/A$
CD	$4a$	A	$4wa$	$4/3$	$64wa^2/3A$
DE	$5a$	A	$-5wa$	$-5/3$	$125wa^2/3A$
EF	$4a$	A	$-4wa$	$-4/3$	$64wa^2/3A$
FG	$4a$	A	$-28wa$	-4	$448wa^2/A$
CE	$3a$	A	$3wa$	1	$9wa^2/A$
CF	$5a$	A	$-30wa$	$-10/3$	$500wa^2/A$
BF	$3a$	A	$18wa$	2	$108wa^2/A$
					$\sum = 4120wa^2/3A$

Thus

$$\Delta_\mathrm{D} = \frac{8704wa^2}{AE} + \frac{4120wa^2}{3AE}$$

i.e.

$$\Delta_\mathrm{D} = \frac{30\,232wa^2}{3AE}$$

S.5.15

The internal force systems at C and D in the ring frame are shown in Fig. S.5.15. The total complementary energy of the half-frame is, from Eq. (5.12)

$$C = \int_L \int_0^M \mathrm{d}\theta \, \mathrm{d}M - F\Delta_\mathrm{B}$$

in which Δ_B is the horizontal displacement of the joint B. Note that, from symmetry, the translational and rotational displacements at C and D are zero. Hence, from the principle of the stationary value of the total complementary energy and choosing the horizontal displacement at C (=0) as the unknown

$$\frac{\partial C}{\partial N_\mathrm{C}} = \int_L \frac{M}{EI} \frac{\partial M}{\partial N_\mathrm{C}} \mathrm{d}z = 0 \qquad \text{(i)}$$

In CB

$$M_{\mathrm{CB}} = M_{\mathrm{C}} - N_{\mathrm{C}}(r - r\cos\theta_1) \tag{ii}$$

At B, $M_{\mathrm{CB}} = 0$. Thus

$$M_{\mathrm{C}} = N_{\mathrm{C}}(r + r\sin 30^\circ) = 1.5N_{\mathrm{C}}r \tag{iii}$$

Eq. (ii) then becomes

$$M_{\mathrm{CB}} = N_{\mathrm{C}}r(0.5 + \cos\theta_1) \tag{iv}$$

Then

$$\frac{\partial M_{\mathrm{CB}}}{\partial N_{\mathrm{C}}} = r(0.5 + \cos\theta_1) \tag{v}$$

In DB

$$M_{\mathrm{DB}} = M_{\mathrm{D}} - N_{\mathrm{D}}(r - r\cos\theta_2) \tag{vi}$$

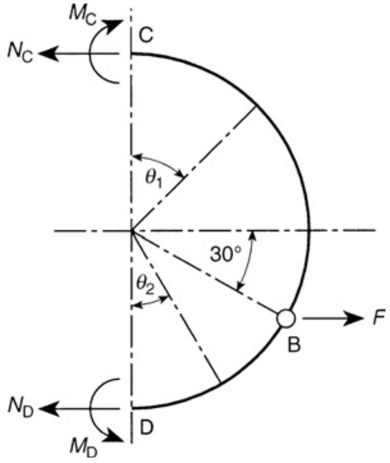

Fig. S.5.15

Again the internal moment at B is zero so that

$$M_{\mathrm{D}} = N_{\mathrm{D}}(r - r\sin 30^\circ) = 0.5N_{\mathrm{D}}r \tag{vii}$$

Hence

$$M_{\mathrm{DB}} = N_{\mathrm{D}}r(\cos\theta_2 - 0.5) \tag{viii}$$

Also, from horizontal equilibrium

$$N_{\mathrm{D}} + N_{\mathrm{C}} = F$$

so that

$$N_{\mathrm{D}} = F - N_{\mathrm{C}}$$

and Eq. (viii) may be written

$$M_{\mathrm{DB}} = (F - N_{\mathrm{C}})r(\cos\theta_2 - 0.5) \tag{ix}$$

whence

$$\frac{\partial M_{\mathrm{DB}}}{\partial N_{\mathrm{C}}} = -r(\cos\theta_2 - 0.5) \tag{x}$$

Substituting from Eqs (iv), (v), (ix) and (x) in Eq. (i)

$$\int_0^{120^\circ} \frac{1}{EI} N_{\mathrm{C}} r^3 (0.5 + \cos\theta_1)^2 \mathrm{d}\theta_1 - \int_0^{60^\circ} \frac{(F - N_{\mathrm{C}})}{xEI} r^3 (\cos\theta_2 - 0.5)^2 \mathrm{d}\theta_2 = 0$$

i.e.

$$N_{\mathrm{C}} \int_0^{120^\circ} (0.25 + \cos\theta_1 + \cos^2\theta_1)\mathrm{d}\theta_1 - \frac{(F - N_{\mathrm{C}})}{x} \int_0^{60^\circ} (\cos^2\theta_2 - \cos\theta_2 + 0.25)\mathrm{d}\theta_2 = 0$$

which, when expanded becomes

$$N_{\mathrm{C}} \int_0^{120^\circ} \left(0.75 + \cos\theta_1 + \frac{\cos 2\theta_1}{2}\right) \mathrm{d}\theta_1 - \frac{(F - N_{\mathrm{C}})}{x} \times \int_0^{60^\circ} \left(\frac{\cos 2\theta_2}{2} - \cos\theta_2 + 0.75\right) \mathrm{d}\theta_2 = 0$$

Hence

$$N_{\mathrm{C}} \left[0.75\theta_1 + \sin\theta_1 + \frac{\sin 2\theta_1}{4}\right]_0^{120^\circ} - \frac{(F - N_{\mathrm{C}})}{x} \left[\frac{\sin 2\theta_2}{4} - \sin\theta_2 + 0.75\theta_2\right]_0^{60^\circ} = 0$$

from which

$$2.22 N_{\mathrm{C}} - 0.136 \frac{(F - N_{\mathrm{C}})}{x} = 0 \tag{xi}$$

The maximum bending moment in ADB is equal to half the maximum bending moment in ACB. Thus

$$M_{\mathrm{D}} = \tfrac{1}{2} M_{\mathrm{C}}$$

Then, from Eqs (vii) and (iii)

$$0.5 N_{\mathrm{D}} r = 0.75 N_{\mathrm{C}} r$$

so that

$$0.5(F - N_{\mathrm{C}}) = 0.75 N_{\mathrm{C}}$$

i.e.

$$F - N_{\mathrm{C}} = 1.5 N_{\mathrm{C}}$$

Substituting for $F - N_C$ in Eq. (xi)

$$2.22N_C - 0.136 \times \frac{1.5N_C}{x} = 0$$

whence

$$x = 0.092$$

S.5.16

From symmetry the shear force in the tank wall at the lowest point is zero. Let the normal force and bending moment at this point be N_O and M_O, respectively as shown in Fig. S.5.16.

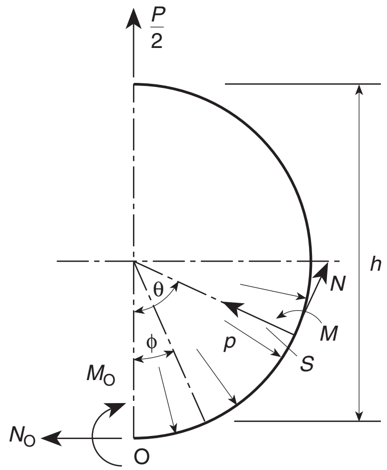

Fig. S.5.16

The total complementary energy of the half-tank is, from Eq. (5.12)

$$C = \int_L \int_0^M \mathrm{d}\theta \, \mathrm{d}M - \frac{P}{2}\Delta_P$$

where Δ_P is the vertical displacement at the point of application of P. Since the rotation and translation at O are zero from symmetry then, from the principle of the stationary value of the total complementary energy

$$\frac{\partial C}{\partial M_O} = \int_L \frac{M}{EI} \frac{\partial M}{\partial M_O} \mathrm{d}z = 0 \tag{i}$$

and

$$\frac{\partial C}{\partial N_O} = \int_L \frac{M}{EI} \frac{\partial M}{\partial N_O} \mathrm{d}z = 0 \tag{ii}$$

At any point in the tank wall

$$M = M_{\mathrm{O}} + N_{\mathrm{O}}(r - r\cos\theta) - \int_0^\theta pr^2 \sin(\theta - \phi)\mathrm{d}\phi \tag{iii}$$

For unit length of tank

$$p = \pi r^2 \rho$$

where ρ is the density of the fuel.

At the position θ,

$$p = \rho h = \rho(r + r\cos\phi)$$

Hence

$$p = \frac{P}{\pi r}(1 + \cos\phi)$$

and the last term in Eq. (iii) becomes

$$\int_0^\theta \frac{Pr}{\pi}(1 + \cos\phi)\sin(\theta - \phi)\mathrm{d}\phi = \frac{Pr}{\pi}\int_0^\theta (1 + \cos\phi)(\sin\theta\cos\phi - \cos\theta\sin\phi)\mathrm{d}\phi$$

Expanding the expression on the right-hand side gives

$$\frac{Pr}{\pi}\int_0^\theta (\sin\theta\cos\phi - \cos\theta\sin\phi + \sin\theta\cos^2\phi - \cos\theta\sin\phi\cos\phi)\mathrm{d}\phi$$
$$= \frac{Pr}{\pi}\left(1 + \frac{\theta}{2}\sin\theta - \cos\theta\right)$$

Hence Eq. (iii) becomes

$$M = M_{\mathrm{O}} + N_{\mathrm{O}}r(1 - \cos\theta) - \frac{Pr}{\pi}\left(1 + \frac{\theta}{2}\sin\theta - \cos\theta\right) \tag{iv}$$

so that

$$\frac{\partial M}{\partial M_{\mathrm{O}}} = 1 \quad \text{and} \quad \frac{\partial M}{\partial N_{\mathrm{O}}} = r(1 - \cos\theta)$$

Substituting for M and $\partial M/\partial M_{\mathrm{O}}$ in Eq. (i) and noting that $EI = \text{constant}$,

$$\int_0^\pi \left[M_{\mathrm{O}} + N_{\mathrm{O}}r(1 - \cos\theta) - \frac{Pr}{\pi}\left(1 + \frac{\theta}{2}\sin\theta - \cos\theta\right)\right]\mathrm{d}\theta = 0 \tag{v}$$

from which

$$M_{\mathrm{O}} + N_{\mathrm{O}}r - \frac{3Pr}{2\pi} = 0 \tag{vi}$$

Now substituting for M and $\partial M/\partial N_{\mathrm{O}}$ in Eq. (ii)

$$\int_0^\pi \left[M_{\mathrm{O}} + N_{\mathrm{O}}r(1 - \cos\theta) - \frac{Pr}{\pi}\left(1 + \frac{\theta}{2}\sin\theta - \cos\theta\right)\right] r(1 - \cos\theta)\mathrm{d}\theta = 0$$

The first part of this integral is identical to that in Eq. (v) and is therefore zero. The remaining integral is then

$$\int_0^{\pi} \left[M_O + N_O r(1 - \cos\theta) - \frac{Pr}{\pi}\left(1 + \frac{\theta}{2}\sin\theta - \cos\theta\right)\right]\cos\theta\,\mathrm{d}\theta = 0$$

which gives

$$\frac{N_O}{2} - \frac{5}{8}\frac{Pr}{\pi} = 0$$

Hence

$$N_O = 0.398P$$

and from Eq. (vi)

$$M_O = 0.080Pr$$

Substituting these values in Eq. (iv)

$$M = Pr(0.160 - 0.080\cos\theta - 0.159\theta\sin\theta)$$

S.5.17

The internal force systems at A and B are shown in Fig. S.5.17; from symmetry the shear forces at these points are zero as are the translations and rotations. It follows that the total complementary energy of the half-frame is, from Eq. (5.12)

$$C = \int_L \int_0^M \mathrm{d}\theta\,\mathrm{d}M$$

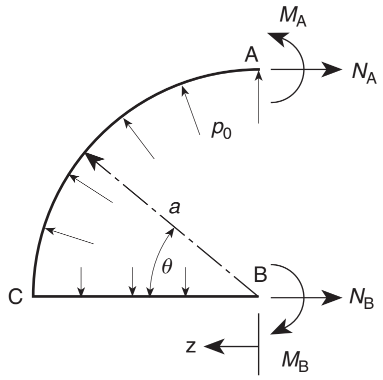

Fig. S.5.17

From the principle of the stationary value of the total complementary energy

$$\frac{\partial C}{\partial M_B} = \int_L \frac{M}{EI}\frac{\partial M}{\partial M_B}\mathrm{d}z = 0 \tag{i}$$

and

$$\frac{\partial C}{\partial N_B} = \int_L \frac{M}{EI} \frac{\partial M}{\partial N_B} dz = 0 \tag{ii}$$

In BC

$$M = M_B + \frac{p_0 z^2}{2} \tag{iii}$$

so that

$$\frac{\partial M}{\partial M_B} = 1 \quad \frac{\partial M}{\partial N_B} = 0$$

In CA

$$M = M_B - N_B a \sin\theta + p_0 a \left(a\cos\theta - \frac{a}{2}\right) + p_0 \frac{(a\sin\theta)^2}{2} + \frac{p_0}{2}(a - a\cos\theta)^2$$

which simplifies to

$$M = M_B - N_B a \sin\theta + \frac{p_0 a^2}{2} \tag{iv}$$

Hence

$$\frac{\partial M}{\partial M_B} = 1 \quad \frac{\partial M}{\partial N_B} = -a\sin\theta$$

Substituting for M and $\partial M/\partial M_B$ in Eq. (i)

$$\int_0^a \frac{1}{2EI}\left(M_B + \frac{p_0 z^2}{2}\right) dz + \int_0^{\pi/2} \frac{1}{EI}\left(M_B - N_B a\sin\theta + \frac{p_0 a^2}{2}\right) a\, d\theta = 0$$

i.e.

$$\frac{1}{2}\left[M_B z + \frac{p_0 z^3}{6}\right]_0^a + a\left[M_B\theta + N_B a\cos\theta + \frac{p_0 a^2}{2}\right]_0^{\pi/2} = 0$$

which simplifies to

$$2.071 M_B - N_B a + 0.869 p_0 a^2 = 0$$

Thus

$$M_B - 0.483 N_B a + 0.420 p_0 a^2 = 0 \tag{v}$$

Now substituting for M and $\partial M/\partial N_B$ in Eq. (ii)

$$\int_0^{\pi/2} \frac{1}{EI}\left(M_B - N_B a\sin\theta + \frac{p_0 a^2}{2}\right)(-a\sin\theta) a\, d\theta = 0$$

or

$$\int_0^{\pi/2} \left(M_B \sin\theta - N_B a \sin^2\theta + \frac{p_0 a^2}{2}\sin\theta\right) d\theta = 0$$

which gives

$$M_B - 0.785N_B a + 0.5p_0a^2 = 0 \tag{vi}$$

Subtracting Eq. (vi) from Eq. (v)

$$0.302N_B a - 0.08p_0a^2 = 0$$

so that

$$N_B = 0.265p_0a$$

Substituting for N_B in Eq. (v) gives

$$M_B = -0.292p_0a^2$$

Therefore, from Eq. (iii)

$$M_C = M_B + \frac{p_0a^2}{2} = -0.292p_0a^2 + \frac{p_0a^2}{2}$$

i.e.

$$M_C = 0.208p_0a^2$$

and from Eq. (iv)

$$M_A = -0.292p_0a^2 - 0.265p_0a^2 + \frac{p_0a^2}{2}$$

i.e.

$$M_A = -0.057p_0a^2$$

Also, from Eq. (iii)

$$M_{BC} = -0.292p_0a^2 + \frac{p_0}{2}z^2 \tag{vii}$$

At a point of contraflexure $M_{BC} = 0$. Thus, from Eq. (vii), a point of contraflexure occurs in BC when $z^2 = 0.584a^2$, i.e. when $z = 0.764a$. Also, from Eq. (iv), $M_{CA} = 0$ when $\sin\theta = 0.208/0.265 = 0.785$, i.e. when $\theta = 51.7°$.

S.5.18

Consider the half-frame shown in Fig. S.5.18(a). On the plane of antisymmetry through the points 7, 8 and 9 only shear forces S_7, S_8 and S_9 are present. Thus from the horizontal equilibrium of the frame

$$S_7 + S_8 + S_9 - 6aq = 0 \tag{i}$$

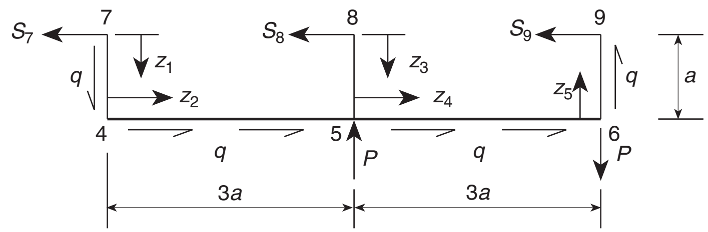

Fig. S.5.18(a)

Also, from the overall equilibrium of the complete frame and taking moments about the corner 6

$$2aq6a + 6aq2a - 2P3a = 0$$

which gives

$$q = P/4a$$

The total complementary energy of the half-frame is, from Eq. (5.12)

$$C = \int_L \int_0^M \mathrm{d}\theta \, \mathrm{d}M - P\Delta_5 - P\Delta_6 = 0$$

Noting that the horizontal displacements at 7, 8 and 9 are zero from antisymmetry, then

$$\frac{\partial C}{\partial S_7} = \int_L \frac{M}{EI} \frac{\partial M}{\partial S_7} \mathrm{d}z = 0 \tag{ii}$$

and

$$\frac{\partial C}{\partial S_8} = \int_L \frac{M}{EI} \frac{\partial M}{\partial S_8} \mathrm{d}z = 0 \tag{iii}$$

In 74

$$M = S_7 z_1 \quad \text{and} \quad \partial M/\partial S_7 = z_1 \quad \partial M/\partial S_8 = 0$$

In 45

$$M = S_7 a + qaz_2 \quad \text{and} \quad \partial M/\partial S_7 = a \quad \partial M/\partial S_8 = 0$$

In 85

$$M = S_8 z_3 \quad \text{and} \quad \partial M/\partial S_7 = 0 \quad \partial M/\partial S_8 = z_3$$

In 56

$$M = S_7 a + S_8 a + qa(3a + z_4) - Pz_4 \quad \text{and} \quad \partial M/\partial S_7 = a \quad \partial M/\partial S_8 = a$$

In 69

$$M = S_7(a - z_5) + S_8(a - z_5) + 6a^2q - 3Pa + 6aqz_5$$

and

$$\partial M/\partial S_7 = (a - z_5) \quad \partial M/\partial S_8 = (a - z_5)$$

Substituting the relevant expressions in Eq. (ii) gives

$$\int_0^a S_7 z_1^2\,\mathrm{d}z_1 + \int_0^{3a} (S_7a^2 + qa^2z_2)\mathrm{d}z_2 + \int_0^{3a} [S_7a + S_8a + qa(3a + z_4) - Pz_4]a\,\mathrm{d}z_4$$
$$+ \int_0^a [S_7(a - z_5) + S_8(a - z_5) + 6a^2q - 3Pa + 6aqz_5](a - z_5)\mathrm{d}z_5 = 0 \qquad \text{(iv)}$$

from which

$$20S_7 + 10S_8 + 66aq - 18P = 0 \qquad \text{(v)}$$

Now substituting for M and $\partial M/\partial S_8$ in Eq. (iii)

$$\int_0^a S_8 z_3^2\,\mathrm{d}z_3 + \int_0^{3a} [S_7a + S_8a + qa(3a + z_4) - Pz_4]a\,\mathrm{d}z_4$$
$$+ \int_0^a [S_7(a - z_5) + S_8(a - z_5) + 6a^2q - 3Pa + 6aqz_5](a - z_5)\mathrm{d}z_5 = 0 \qquad \text{(vi)}$$

The last two integrals in Eq. (vi) are identical to the last two integrals in Eq. (iv). Thus, Eq. (vi) becomes

$$10S_7 + 11S_8 + 52.5aq - 18P = 0 \qquad \text{(vii)}$$

The simultaneous solution of Eqs (v) and (vii) gives

$$S_8 = -\frac{39}{12}aq + \frac{3}{2}P$$

whence, since $q = P/4a$

$$S_8 = 0.69P$$

Substituting for S_8 in either of Eqs (v) or (vii) gives

$$S_7 = -0.27P$$

Then, from Eq. (i)

$$S_9 = 1.08P$$

The bending moment diagram is shown in Fig. S.5.18(b) in which the bending moments are drawn on the tension side of each member.

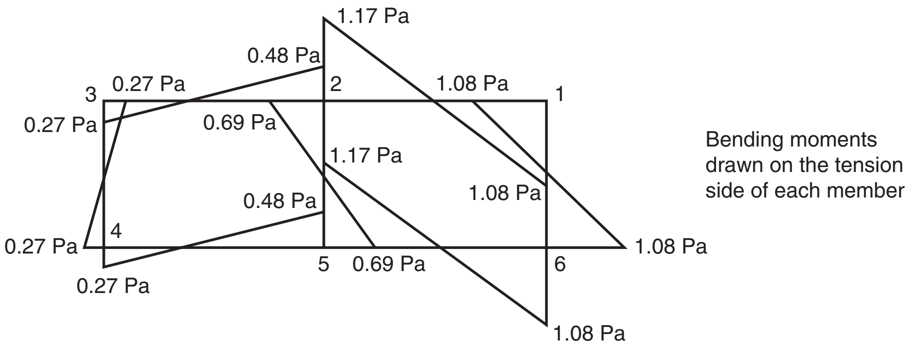

Fig. S.5.18(b)

S.5.19

From the overall equilibrium of the complete frame

$$\int_0^{2\pi r} qr \, \mathrm{d}s = T$$

which gives

$$2\pi r^2 q = T$$

i.e.

$$q = \frac{T}{2\pi r^2} \qquad \text{(i)}$$

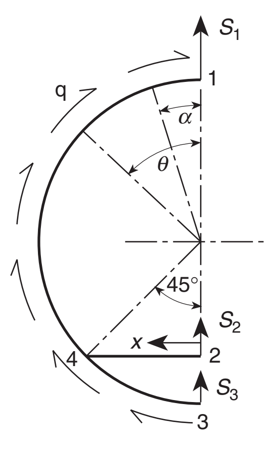

Fig. S.5.19

Considering the half frame shown in Fig. S.5.19 there are only internal shear forces on the vertical plane of antisymmetry. From the vertical equilibrium of the half-frame

$$S_1 + S_2 + S_3 + \int_0^{\pi} q \sin \alpha r \, \mathrm{d}\alpha = 0$$

Substituting for q from Eq. (i) and integrating

$$S_1 + S_2 + S_3 + \frac{T}{2\pi r}[-\cos \alpha]_0^{\pi} = 0$$

which gives

$$S_1 + S_2 + S_3 = -\frac{T}{\pi r} \tag{ii}$$

The vertical displacements at the points 1, 2 and 3 are zero from antisymmetry so that, from Eq. (5.12), the total complementary energy of the half-frame is given by

$$C = \int_L \int_0^M \mathrm{d}\theta \, \mathrm{d}M$$

Then, from the principle of the stationary value of the total complementary energy

$$\frac{\partial C}{\partial S_1} = \int_L \frac{M}{EI} \frac{\partial M}{\partial S_1} \mathrm{d}z \tag{iii}$$

and

$$\frac{\partial C}{\partial S_2} = \int_L \frac{M}{EI} \frac{\partial M}{\partial S_2} \mathrm{d}z \tag{iv}$$

In the wall 14

$$M = S_1 r \sin \theta - \int_0^{\theta} q[r - r \cos(\theta - \alpha)] r \, \mathrm{d}\alpha$$

i.e.

$$M = S_1 r \sin \theta - \frac{T}{2\pi}[\alpha - \sin(\alpha - \theta)]_0^{\theta}$$

which gives

$$M = S_1 r \sin \theta - \frac{T}{2\pi}(\theta - \sin \theta) \tag{v}$$

whence

$$\frac{\partial M}{\partial S_1} = r \sin \theta \qquad \frac{\partial M}{\partial S_2} = 0$$

In the wall 24

$$M = S_2 x \tag{vi}$$

and

$$\frac{\partial M}{\partial S_1} = 0 \quad \frac{\partial M}{\partial S_2} = x$$

In the wall 43

$$M = S_1 r \sin\theta - \frac{T}{2\pi}(\theta - \sin\theta) + S_2 r \sin\theta \tag{vii}$$

and

$$\frac{\partial M}{\partial S_1} = r\sin\theta \quad \frac{\partial M}{\partial S_2} = r\sin\theta$$

Substituting for M and $\partial M/\partial S_1$ in Eq. (iii)

$$\int_0^{3\pi/4}\left[S_1 r\sin\theta - \frac{T}{2\pi}(\theta - \sin\theta)\right] r\sin\theta r\,\mathrm{d}\theta \\ + \int_{3\pi/4}^{\pi}\left[S_1 r\sin\theta - \frac{T}{2\pi}(\theta - \sin\theta) + S_2 r\sin\theta\right] r\sin\theta\, r\,\mathrm{d}\theta = 0$$

which simplifies to

$$\int_0^{\pi}\left[S_1 r\sin\theta - \frac{T}{2\pi}(\theta - \sin\theta)\right] r^2\sin\theta\,\mathrm{d}\theta + \int_{3\pi/4}^{\pi} S_2 r^3\sin\theta\,\mathrm{d}\theta = 0$$

Integrating and simplifying gives

$$S_1 r - 0.16T + 0.09S_2 r = 0 \tag{viii}$$

Now substituting for M and $\partial M/\partial S_2$ in Eq. (iv)

$$\int_{3\pi/4}^{\pi}\left[S_1 r\sin\theta - \frac{T}{2\pi}(\theta - \sin\theta) + S_2 r\sin\theta\right] r\sin\theta\, r\,\mathrm{d}\theta + \int_0^{r/\sqrt{2}} S_2 x^2\,\mathrm{d}x = 0$$

Integrating and simplifying gives

$$S_1 r - 0.69T + 1.83S_2 r = 0 \tag{ix}$$

Subtracting Eq. (ix) from Eq. (viii)

$$0.53T - 1.74S_2 r = 0$$

whence

$$S_2 = \frac{0.30T}{r}$$

From Eq. (viii)

$$S_1 = \frac{0.13T}{r}$$

and from Eq. (ii)

$$S_3 = \frac{-0.75T}{r}$$

Hence, from Eqs (v) to (vii)

$$M_{14} = T(0.29\sin\theta - 0.16\theta)$$

$$M_{24} = \frac{0.30Tx}{r}$$

$$M_{43} = T(0.59\sin\theta - 0.16\theta)$$

S.5.20

Initially the vertical reaction at C, R_C, must be found. From Eq. (5.12) the total complementary energy of the member is given by

$$C = \int_L \int_0^M \mathrm{d}\theta\,\mathrm{d}M - R_C\Delta_C - F\Delta_B$$

From the principle of the stationary value of the total complementary energy and since $\Delta_C = 0$

$$\frac{\partial C}{\partial R_C} = \int_L \frac{M}{EI}\frac{\partial M}{\partial R_C}\mathrm{d}s = 0 \tag{i}$$

Referring to Fig. S.5.20

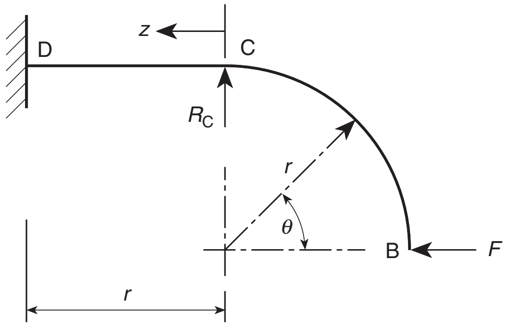

Fig. S.5.20

In BC

$$M = Fr\sin\theta \quad \text{and} \quad \frac{\partial M}{\partial R_C} = 0$$

In CD

$$M = Fr - R_C z \quad \text{and} \quad \frac{\partial M}{\partial R_C} = -z$$

Substituting these expressions in Eq. (i) gives

$$\int_0^r (Fr - R_C z)(-z)\mathrm{d}z = 0$$

from which

$$R_C = 1.5F$$

Note that Eq. (i) does not include the effects of shear and axial force. If these had been included the value of R_C would be $1.4F$; the above is therefore a reasonable approximation. Also, from Eq. (1.50), $G = 3E/8$.

The unit load method may now be used to complete the solution. Thus, from the first of Eqs (5.21), Eq. (5.20) and Eq. (20.18)

$$\delta_{B,H} = \int_L \frac{M_0 M_1}{EI}\mathrm{d}s + \int_L \frac{F_0 F_1}{AE}\mathrm{d}s + \int_L \frac{S_0 S_1}{GA'}\mathrm{d}s \qquad \text{(ii)}$$

In BC

$$M_0 = Fr\sin\theta \quad M_1 = r\sin\theta$$
$$F_0 = F\sin\theta \quad F_1 = \sin\theta$$
$$S_0 = F\cos\theta \quad S_1 = \cos\theta$$

In CD

$$M_0 = F(r - 1.5z) \quad M_1 = (r - 1.5z)$$
$$F_0 = F \qquad F_1 = 1$$
$$S_0 = 1.5F \qquad S_1 = 1.5$$

Substituting these expressions in Eq. (ii) gives

$$\delta_{B,H} = \int_0^{\pi/2} \frac{Fr^3\sin^2\theta}{EI}\mathrm{d}\theta + \int_0^{\pi/2} \frac{Fr\sin^2\theta}{AE}\mathrm{d}\theta + \int_0^{\pi/2} \frac{Fr\cos^2\theta}{GA'}\mathrm{d}\theta$$
$$+ \int_0^r \frac{F}{EI}(r - 1.5z)^2\mathrm{d}z + \int_0^r \frac{F}{AE}\mathrm{d}z + \int_0^r \frac{2.25F}{GA'}\mathrm{d}z$$

or

$$\delta_{B.H} = \frac{400Fr}{AE}\int_0^{\pi/2} \frac{1}{2}(1 - \cos 2\theta)\mathrm{d}\theta + \frac{Fr}{AE}\int_0^{\pi/2} \frac{1}{2}(1 - \cos 2\theta)\mathrm{d}\theta$$
$$+ \frac{32Fr}{3AE}\int_0^{\pi/2} \frac{1}{2}(1 + \cos 2\theta)\mathrm{d}\theta + \frac{400F}{Ar^2E}\int_0^r (r^2 - 3rz + 2.25z^2)\mathrm{d}z$$
$$+ \frac{F}{AE}\int_0^r \mathrm{d}z + \frac{24F}{AE}\int_0^r \mathrm{d}z$$

from which

$$\delta_{B,H} = \frac{448.3Fr}{AE}$$

S.5.21

From Clerk–Maxwell's reciprocal theorem the deflection at A due to W at B is equal to the deflection at B due to W at A, i.e. δ_2.

What is now required is the deflection at B due to W at B.

Since the deflection at A with W at A and the spring removed is δ_3, the load in the spring at A with W at B is $(\delta_2/\delta_3)W$ which must equal the load in the spring at B with W at B. Thus, the resultant load at B with W at B is

$$W - \left(\frac{\delta_2}{\delta_3}\right) W = W\left(1 - \frac{\delta_2}{\delta_3}\right) \qquad \text{(i)}$$

Now the load W at A with the spring in place produces a deflection of δ_1 at A. Thus, the resultant load at A is $(\delta_1/\delta_3)W$ so that, if the load in the spring at A with W at A is F, then $W - F = (\delta_1/\delta_3)W$, i.e.

$$F = W\left(1 - \frac{\delta_1}{\delta_3}\right) \qquad \text{(ii)}$$

This then is the load at B with W at A and it produces a deflection δ_2. Therefore, from Eqs (i) and (ii) the deflection at B due to W at B is

$$\frac{W\left(1 - \dfrac{\delta_2}{\delta_3}\right)}{W\left(1 - \dfrac{\delta_1}{\delta_3}\right)}\delta_2$$

Thus the extension of the spring with W at B is

$$\frac{\left(1 - \dfrac{\delta_2}{\delta_3}\right)}{\left(1 - \dfrac{\delta_1}{\delta_3}\right)}\delta_2 - \delta_2$$

i.e.

$$\delta_2\left(\frac{\delta_1 - \delta_2}{\delta_3 - \delta_1}\right)$$

S.5.22

Referring to Fig. S.5.22

$R_A = R_B = 1000$ N from symmetry.

The slope of the beam at A and B may be obtained from the second of Eqs (16.32), i.e.

$$v'' = -\frac{M}{EI}$$

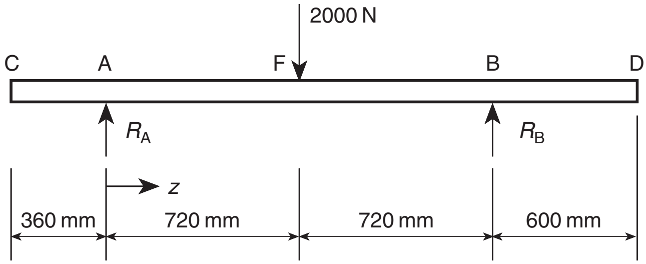

Fig. S.5.22

where, for the half-span AF, $M = R_A z = 1000z$. Thus

$$v'' = -\frac{1000}{EI}z$$

and

$$v' = -\frac{500}{EI}z^2 + C_1$$

When $z = 720$ mm, $v' = 0$ from symmetry and hence $C_1 = 2.59 \times 10^8/EI$. Hence

$$v' = \frac{1}{EI}(-500z^2 + 2.59 \times 10^8)$$

Thus v' (at A) = 0.011 rads = v' (at B). The deflection at C is then = 360 × 0.011 = 3.96 mm and the deflection at D = 600 × 0.011 = 6.6 mm.

From the reciprocal theorem the deflection at F due to a load of 3000 N at C = 3.96 × 3000/2000 = 5.94 mm and the deflection at F due to a load of 3000 N at D = 6.6 × 3000/2000 = 9.9 mm. Therefore the total deflection at F due to loads of 3000 N acting simultaneously at C and D is 5.94 + 9.9 = 15.84 mm.

S.5.23

Since the frame is symmetrical about a vertical plane through its centre only half need be considered. Also, due to symmetry the frame will act as though fixed at C (Fig. S.5.23).

If the frame were unsupported at B the horizontal displacement at B, $\Delta_{B,T}$, due to the temperature rise may be obtained using Eq. (5.32) in which, due to a unit load acting horizontally at B, $M_1 = 1 \times (r \sin 30° + r \sin \theta)$. Hence

$$\Delta_{B,T} = \int_{-\pi/6}^{\pi/2} (0.5r + r\sin\theta)\frac{2\alpha T}{d} r\, d\theta$$

i.e.

$$\Delta_{B,T} = \frac{2\alpha T r^2}{d}[0.5\theta - \cos\theta]_{-\pi/6}^{\pi/2}$$

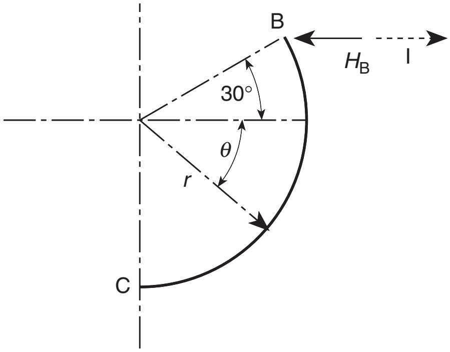

Fig. S.5.23

which gives

$$\Delta_{B,T} = \frac{3.83\alpha T r^2}{d} \quad \text{(to the right)} \tag{i}$$

Suppose that in the actual frame the horizontal reaction at B is H_B. Since B is not displaced, the 'displacement' $\Delta_{B,H}$ produced by H_B must be equal and opposite to $\Delta_{B,T}$ in Eq. (i). Then, from the first of Eqs (5.21) and noting that $M_0 = -H_B(0.5r + r\sin\theta)$

$$\Delta_{B,H} = -\frac{1}{EI}\int_{-\pi/6}^{\pi/2} H_B(0.5r + r\sin\theta)^2 r\,d\theta$$

i.e.

$$\Delta_{B,H} = -\frac{H_B r^3}{EI}\int_{-\pi/6}^{\pi/2} (0.25 + \sin\theta + \sin^2\theta)d\theta$$

Hence

$$\Delta_{B,H} = -\frac{H_B r^3}{EI}\left[0.75\theta - \cos\theta - \frac{\sin 2\theta}{4}\right]_{-\pi/6}^{\pi/2}$$

so that

$$\Delta_{B,H} = -\frac{2.22 H_B r^3}{EI} \quad \text{(to the left)} \tag{ii}$$

Then, since

$$\Delta_{B,H} + \Delta_{B,T} = 0$$

$$-\frac{2.22 H_B r^3}{EI} + \frac{3.83\alpha T r^2}{d} = 0$$

from which

$$H_B = \frac{1.73 EIT\alpha}{d} \tag{iii}$$

The maximum bending moment in the frame will occur at C and is given by

$$M(\max) = H_B \times 1.5r$$

Then, from symmetrical bending theory the direct stress through the depth of the frame section is given by

$$\sigma = \frac{My}{I} \quad \text{(see Eqs (16.21))}$$

and

$$\sigma_{\max} = \frac{M(\max)y(\max)}{I}$$

i.e.

$$\sigma_{\max} = \frac{H_B \times 1.5r \times 0.5d}{I}$$

or, substituting for H_B from Eq. (iii)

$$\sigma_{\max} = 1.30ET\alpha$$

S.5.24

The solution is similar to that for P.5.23 in that the horizontal displacement of B due to the temperature gradient is equal and opposite in direction to the 'displacement' produced by the horizontal reaction at B, H_B. Again only half the frame need be considered from symmetry.

Referring to Fig. S.5.24

$$M_1 = r\cos\psi \text{ in BC and Cd}$$

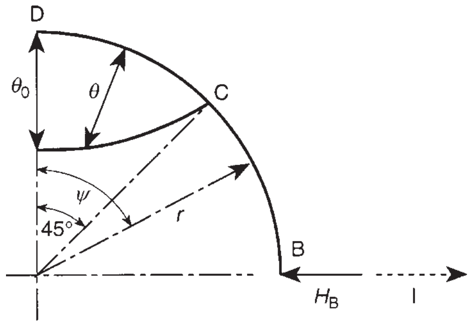

Fig. S.5.24

Then, from Eq. (5.32)

$$\Delta_{B,T} = \int_0^{\pi/4} (r\cos\psi)\alpha\frac{\theta_0\cos 2\psi}{h} r\,d\psi + \int_{\pi/4}^{\pi/2} (r\cos\psi)\alpha\left(\frac{0}{h}\right) r\,d\psi$$

i.e.

$$\Delta_{B,T} = \frac{r^2 \alpha \theta_0}{h} \int_0^{\pi/4} \cos\psi \cos 2\psi \, d\psi$$

or

$$\Delta_{B,T} = \frac{r^2 \alpha \theta_0}{h} \int_0^{\pi/4} (\cos\psi - 2\sin^2\psi \cos\psi) d\psi$$

Hence

$$\Delta_{B,T} = \frac{r^2 \alpha \theta_0}{h} \left[\sin\psi - \frac{2}{3}\sin^3\psi \right]_0^{\pi/4}$$

which gives

$$\Delta_{B,T} = \frac{0.47 r^2 \alpha \theta_0}{h} \quad \text{(to the right)} \tag{i}$$

From the first of Eqs (5.21) in which $M_0 = -H_B r \cos\psi$

$$\Delta_{B,H} = \int_0^{\pi/2} -\frac{H_B r \cos\psi \, r \cos\psi}{EI} r \, d\psi$$

i.e.

$$\Delta_{B,H} = -\frac{H_B r^3}{EI} \int_0^{\pi/2} \cos^2\psi \, d\psi$$

or

$$\Delta_{B,H} = -\frac{H_B r^3}{EI} \int_0^{\pi/2} \frac{1}{2}(1 + \cos 2\psi) d\psi$$

whence

$$\Delta_{B,H} = -\frac{0.79 H_B r^3}{EI} \quad \text{(to the left)} \tag{ii}$$

Then, since $\Delta_{B,H} + \Delta_{B,T} = 0$, from Eqs (i) and (ii)

$$-\frac{0.79 H_B r^3}{EI} + \frac{0.47 r^2 \alpha \theta_0}{h} = 0$$

from which

$$H_B = \frac{0.59 EI \alpha \theta_0}{rh}$$

Then

$$M = H_B r \cos\psi$$

so that

$$M = \frac{0.59 EI \alpha \theta_0 \cos\psi}{h}$$