Solutions to Chapter 3 Problems

S.3.1

Initially the stress function, ϕ, must be expressed in terms of Cartesian coordinates. Thus, from the equation of a circle of radius, a, and having the origin of its axes at its centre.

$$\phi = k(x^2 + y^2 - a^2) \qquad \text{(i)}$$

From Eqs (3.4) and (3.11)

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = F = -2G\frac{\mathrm{d}\theta}{\mathrm{d}z} \qquad \text{(ii)}$$

Differentiating Eq. (i) and substituting in Eq. (ii)

$$4k = -2G\frac{\mathrm{d}\theta}{\mathrm{d}z}$$

or

$$k = -\frac{1}{2}G\frac{\mathrm{d}\theta}{\mathrm{d}z} \qquad \text{(iii)}$$

From Eq. (3.8)

$$T = 2\iint \phi \, \mathrm{d}x \, \mathrm{d}y$$

i.e.

$$T = -G\frac{\mathrm{d}\theta}{\mathrm{d}z}\left[\iint_A x^2 \, \mathrm{d}x \, \mathrm{d}y + \iint_A y^2 \, \mathrm{d}x \, \mathrm{d}y - a^2 \iint_A \mathrm{d}x \, \mathrm{d}y\right] \qquad \text{(iv)}$$

where $\iint_A x^2 \, \mathrm{d}x \, \mathrm{d}y = I_y$, the second moment of area of the cross-section about the y axis; $\iint_A y^2 \, \mathrm{d}x \, \mathrm{d}y = I_x$, the second moment of area of the cross-section about the x axis and $\iint_A \mathrm{d}x \, \mathrm{d}y = A$, the area of the cross-section. Thus, since $I_y = \pi a^4/4$, $I_x = \pi a^4/4$ and $A = \pi a^2$ Eq. (iv) becomes

$$T = G\frac{\mathrm{d}\theta}{\mathrm{d}z}\frac{\pi a^4}{2}$$

or

$$\frac{\mathrm{d}\theta}{\mathrm{d}z} = \frac{2T}{G\pi a^4} = \frac{T}{GI_p} \qquad \text{(v)}$$

From Eqs (3.2) and (v)

$$\tau_{zy} = -\frac{\partial \phi}{\partial x} = -2kx = G\frac{\mathrm{d}\theta}{\mathrm{d}z}x = \frac{Tx}{I_p} \qquad \text{(vi)}$$

and

$$\tau_{zx} = \frac{\partial \phi}{\partial y} = 2ky = -G\frac{\mathrm{d}\theta}{\mathrm{d}z}y = -\frac{Ty}{I_p} \tag{vii}$$

Substituting for τ_{zy} and τ_{zx} from Eqs (vi) and (vii) in the second of Eqs (3.15)

$$\tau_{zs} = \frac{T}{I_p}(xl + ym) \tag{viii}$$

in which, from Eqs (3.6)

$$l = \frac{\mathrm{d}y}{\mathrm{d}s} \quad m = -\frac{\mathrm{d}x}{\mathrm{d}s}$$

Suppose that the bar of Fig. 3.2 is circular in cross-section and that the radius makes an angle α with the x axis. Then.

$$m = \sin\alpha \quad \text{and} \quad l = \cos\alpha$$

Also, at any radius, r

$$y = r\sin\alpha \quad x = r\cos\alpha$$

Substituting for x, l, y and m in Eq. (viii) gives

$$\tau_{zs} = \frac{Tr}{I_p}(=\tau)$$

Now substituting for τ_{zx}, τ_{zy} and $\mathrm{d}\theta/\mathrm{d}z$ from Eqs (vii), (vi) and (v) in Eqs (3.10)

$$\frac{\partial w}{\partial x} = -\frac{Ty}{GI_p} + \frac{Ty}{GI_p} = 0 \tag{ix}$$

$$\frac{\partial w}{\partial y} = \frac{Tx}{GI_p} - \frac{Tx}{GI_p} = 0 \tag{x}$$

The possible solutions of Eqs (ix) and (x) are $w = 0$ and $w = \text{constant}$. The latter solution implies a displacement of the whole bar along the z axis which, under the given loading, cannot occur. Therefore, the first solution applies, i.e. the warping is zero at all points in the cross-section.

The stress function, ϕ, defined in Eq. (i) is constant at any radius, r, in the cross-section of the bar so that there are no shear stresses acting across such a boundary. Thus, the material contained within this boundary could be removed without affecting the stress distribution in the outer portion. Therefore, the stress function could be used for a hollow bar of circular cross-section.

S.3.2

In S.3.1 it has been shown that the warping of the cross-section of the bar is everywhere zero. Then, from Eq. (3.17) and since $\mathrm{d}\theta/\mathrm{d}z \neq 0$

$$\psi(x, y) = 0 \tag{i}$$

This warping function satisfies Eq. (3.20). Also Eq. (3.21) reduces to

$$xm - yl = 0 \tag{ii}$$

On the boundary of the bar $x = al$, $y = am$ so that Eq. (ii), i.e. Eq. (3.21), is satisfied.

Since $\psi = 0$, Eq. (3.23) for the torsion constant reduces to

$$J = \iint_A x^2\,\mathrm{d}x\,\mathrm{d}y + \iint_A y^2\,\mathrm{d}x\,\mathrm{d}y = I_p$$

Therefore, from Eq. (3.12)

$$T = GI_p\frac{\mathrm{d}\theta}{\mathrm{d}z}$$

as in S.3.1.

From Eqs (3.19)

$$\tau_{zx} = G\frac{\mathrm{d}\theta}{\mathrm{d}z}(-y) = -\frac{Ty}{I_p}$$

and

$$\tau_{zy} = G\frac{\mathrm{d}\theta}{\mathrm{d}z}(x) = \frac{Tx}{I_p}$$

which are identical to Eqs (vii) and (vi) in S.3.1. Hence

$$\tau_{zs} = \tau = \frac{Tr}{I_p}$$

as in S.3.1.

S.3.3

Since $\psi = kxy$, Eq. (3.20) is satisfied.

Substituting for ψ in Eq. (3.21)

$$(kx + x)m + (ky - y)l = 0$$

or, from Eqs (3.6)

$$-x(k+1)\frac{\mathrm{d}x}{\mathrm{d}s} + y(k-1)\frac{\mathrm{d}y}{\mathrm{d}s} = 0$$

or

$$\frac{d}{ds}\left[-\frac{x^2}{2}(k+1)+\frac{y^2}{2}(k-1)\right]=0$$

so that

$$-\frac{x^2}{2}(k+1)+\frac{y^2}{2}(k-1)=\text{constant on the boundary of the bar}$$

Rearranging

$$x^2+\left(\frac{1-k}{1+k}\right)y^2=\text{constant} \tag{i}$$

Also, the equation of the elliptical boundary of the bar is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

or

$$x^2+\frac{a^2}{b^2}y^2=a^2 \tag{ii}$$

Comparing Eqs (i) and (ii)

$$\frac{a^2}{b^2}=\left(\frac{1-k}{1+k}\right)$$

from which

$$k=\frac{b^2-a^2}{a^2+b^2} \tag{iii}$$

and

$$\psi=\frac{b^2-a^2}{a^2+b^2}xy \tag{iv}$$

Substituting for ψ in Eq. (3.23) gives the torsion constant, J, i.e.

$$J=\iint_A\left[\left(\frac{b^2-a^2}{a^2+b^2}+1\right)x^2-\left(\frac{b^2-a^2}{a^2+b^2}-1\right)y^2\right]dx\,dy \tag{v}$$

Now $\iint_A x^2\,dx\,dy=I_y=\pi a^3b/4$ for an elliptical cross-section. Similarly $\iint_A y^2\,dx\,dy=I_x=\pi ab^3/4$. Equation (v) therefore simplifies to

$$J=\frac{\pi a^3b^3}{a^2+b^2} \tag{vi}$$

which are identical to Eq. (v) of Example 3.1.

From Eq. (3.22) the rate of twist is

$$\frac{d\theta}{dz}=\frac{T(a^2+b^2)}{G\pi a^3b^3} \tag{vii}$$

The shear stresses are obtained from Eqs (3.19), i.e.

$$\tau_{zx} = \frac{GT(a^2 + b^2)}{G\pi a^3 b^3}\left[\left(\frac{b^2 - a^2}{a^2 + b^2}\right) y - y\right]$$

so that

$$\tau_{zx} = -\frac{2Ty}{\pi a b^3}$$

and

$$\tau_{zy} = \frac{GT(a^2 + b^2)}{G\pi a^3 b^3}\left[\left(\frac{b^2 - a^2}{a^2 + b^2}\right) x + x\right]$$

i.e.

$$\tau_{zy} = \frac{2Tx}{\pi a^3 b}$$

which are identical to Eq. (vi) of Example 3.1.

From Eq. (3.17)

$$w = \frac{T(a^2 + b^2)}{G\pi a^3 b^3}\left(\frac{b^2 - a^2}{a^2 + b^2}\right) xy$$

i.e.

$$w = \frac{T(b^2 - a^2)}{G\pi a^3 b^3} xy \quad \text{(compare with Eq. (viii) of Example 3.1)}$$

S.3.4

The stress function is

$$\phi = -G\frac{\mathrm{d}\theta}{\mathrm{d}z}\left[\frac{1}{2}(x^2 + y^2) - \frac{1}{2a}(x^3 - 3xy^2) - \frac{2}{27}a^2\right] \tag{i}$$

Differentiating Eq. (i) twice with respect to x and y in turn gives

$$\frac{\partial^2 \phi}{\partial x^2} = -G\frac{\mathrm{d}\theta}{\mathrm{d}z}\left(1 - \frac{3x}{a}\right)$$

$$\frac{\partial^2 \phi}{\partial y^2} = -G\frac{\mathrm{d}\theta}{\mathrm{d}z}\left(1 + \frac{3x}{a}\right)$$

Therefore

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -2G\frac{\mathrm{d}\theta}{\mathrm{d}z} = \text{constant}$$

and Eq. (3.4) is satisfied.

Further

$$\text{on AB,} \quad x = \frac{-a}{3} \quad y = y$$

$$\text{on BC,} \quad y = \frac{-x}{\sqrt{3}} + \frac{2a}{3\sqrt{3}}$$

$$\text{on AC,} \quad y = \frac{x}{\sqrt{3}} - \frac{2a}{3\sqrt{3}}$$

Substituting these expressions in turn in Eq. (i) gives

$$\phi_{\text{AB}} = \phi_{\text{BC}} = \phi_{\text{AC}} = 0$$

so that Eq. (i) satisfies the condition $\phi = 0$ on the boundary of the triangle.

From Eqs (3.2) and (i)

$$\tau_{zy} = -\frac{\partial \phi}{\partial x} = G\frac{\mathrm{d}\theta}{\mathrm{d}z}\left(x - \frac{3x^2}{2a} + \frac{3y^2}{2a}\right) \tag{ii}$$

and

$$\tau_{zx} = \frac{\partial \phi}{\partial y} = -G\frac{\mathrm{d}\theta}{\mathrm{d}z}\left(y + \frac{3xy}{a}\right) \tag{iii}$$

At each corner of the triangular section $\tau_{zy} = \tau_{zx} = 0$. Also, from antisymmetry, the distribution of shear stress will be the same along each side. For AB, where $x = -a/3$ and $y = y$, Eqs (ii) and (iii) become

$$\tau_{zy} = G\frac{\mathrm{d}\theta}{\mathrm{d}z}\left(-\frac{a}{2} + \frac{3y^2}{2a}\right) \tag{iv}$$

and

$$\tau_{zx} = 0 \tag{v}$$

From Eq. (iv) the maximum value of τ_{zy} occurs at $y = 0$ and is

$$\tau_{zy}(\max) = -\frac{Ga}{2}\frac{\mathrm{d}\theta}{\mathrm{d}z} \tag{vi}$$

The distribution of shear stress along the x axis is obtained from Eqs (ii) and (iii) in which $x = x$, $y = 0$, i.e.

$$\tau_{zy} = G\frac{\mathrm{d}\theta}{\mathrm{d}z}\left(x - \frac{3x^2}{2a}\right) \tag{vii}$$

$$\tau_{zx} = 0$$

From Eq. (vii) τ_{zy} has a mathematical maximum at $x = +a/3$ which gives

$$\tau_{zy} = \frac{Ga}{6}\frac{\mathrm{d}\theta}{\mathrm{d}z} \tag{viii}$$

which is less than the value given by Eq. (vi). Thus the maximum value of shear stress in the section is $(-Ga/2)\mathrm{d}\theta/\mathrm{d}z$.

The rate of twist may be found by substituting for ϕ from Eq. (i) in (3.8).

Thus

$$T = -2G\frac{\mathrm{d}\theta}{\mathrm{d}z}\iint\left[\frac{1}{2}(x^2+y^2)-\frac{1}{2a}(x^3-3xy^2)-\frac{2}{27}a^2\right]\mathrm{d}x\,\mathrm{d}y \qquad \text{(ix)}$$

The equation of the side AC of the triangle is $y=(x-2a/3)/\sqrt{3}$ and that of BC, $y=-(x-2a/3)/\sqrt{3}$. Equation (ix) then becomes

$$T = -2G\frac{\mathrm{d}\theta}{\mathrm{d}z}\int_{-a/3}^{2a/3}\int_{(x-2a/3)/\sqrt{3}}^{-(x-2a/3)/\sqrt{3}}\left[\frac{1}{2}(x^2+y^2)-\frac{1}{2a}(x^3-3xy^2)-\frac{2}{27}a^2\right]\mathrm{d}x\,\mathrm{d}y$$

which gives

$$T = \frac{Ga^4}{15\sqrt{3}}\frac{\mathrm{d}\theta}{\mathrm{d}z}$$

so that

$$\frac{\mathrm{d}\theta}{\mathrm{d}z} = \frac{15\sqrt{3}T}{Ga^4} \qquad \text{(x)}$$

From the first of Eqs (3.10)

$$\frac{\partial w}{\partial x} = \frac{\tau_{zx}}{G}+\frac{\mathrm{d}\theta}{\mathrm{d}z}y$$

Substituting for τ_{zx} from Eq. (iii)

$$\frac{\partial w}{\partial x} = -\frac{\mathrm{d}\theta}{\mathrm{d}z}\left(y+\frac{3xy}{a}-y\right)$$

i.e.

$$\frac{\partial w}{\partial x} = -\frac{3xy}{a}\frac{\mathrm{d}\theta}{\mathrm{d}z}$$

whence

$$w = -\frac{3x^2y}{2a}\frac{\mathrm{d}\theta}{\mathrm{d}z}+f(y) \qquad \text{(xi)}$$

Similarly from the second of Eqs (3.10)

$$w = -\frac{3x^2y}{2a}\frac{\mathrm{d}\theta}{\mathrm{d}z}+\frac{y^3}{2a}\frac{\mathrm{d}\theta}{\mathrm{d}z}+f(x) \qquad \text{(xii)}$$

Comparing Eqs (xi) and (xii)

$$f(x)=0 \quad \text{and} \quad f(y)=\frac{y^3}{2a}\frac{\mathrm{d}\theta}{\mathrm{d}z}$$

Hence

$$w = \frac{1}{2a}\frac{d\theta}{dz}(y^3 - 3x^2y).$$

S.3.5

The torsion constant, J, for the complete cross-section is found by summing the torsion constants of the narrow rectangular strips which form the section. Then, from Eq. (3.29)

$$J = 2\frac{at^3}{3} + \frac{bt^3}{3} = \frac{(2a+b)t^3}{3}$$

Therefore, from the general torsion equation (3.12)

$$\frac{d\theta}{dz} = \frac{3T}{G(2a+b)t^3} \quad \text{(i)}$$

The maximum shear stress follows from Eqs (3.28) and (i), hence

$$\tau_{max} = \pm Gt\frac{d\theta}{dz} = \pm\frac{3T}{(2a+b)t^2}.$$

Solutions to Chapter 4 Problems

S.4.1

Give the beam at D a virtual displacement δ_D as shown in Fig. S.4.1. The virtual displacements of C and B are then, respectively, $3\delta_D/4$ and $\delta_D/2$.

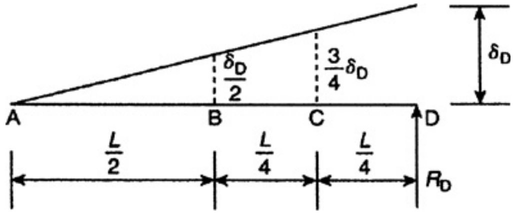

Fig. S.4.1

The equation of virtual work is then

$$R_D\delta_D - \frac{2W\delta_D}{2} - \frac{W3\delta_D}{4} = 0$$

from which

$$R_D = 1.75W$$

It follows that

$$R_A = 1.25W.$$