i.e.

$$1.29 + 8.14 = \sqrt{7^2 + 4\tau_{xy}^2}$$

from which $\tau_{xy} = 3.17\,\text{N/mm}^2$.

The shear force at **P** is equal to $Q$ so that the shear stress at **P** is given by

$$\tau_{xy} = 3.17 = \frac{3Q}{2 \times 150 \times 300}$$

from which

$$Q = 95\,100\,\text{N} = 95.1\,\text{kN}.$$

# Solutions to Chapter 2 Problems

## S.2.1

The stress system applied to the plate is shown in Fig. S.2.1. The origin, O, of the axes may be chosen at any point in the plate; let **P** be the point whose coordinates are (2, 3).

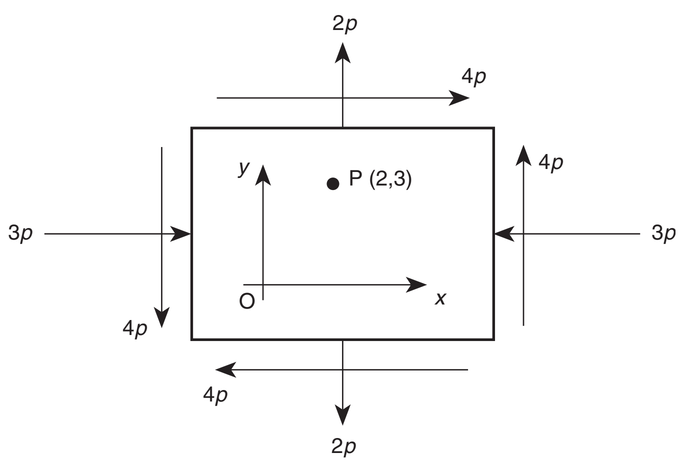


**Fig. S.2.1**

From Eqs (1.42) in which $\sigma_z = 0$

$$\varepsilon_x = -\frac{3p}{E} - \nu\frac{2p}{E} = -\frac{3.5p}{E} \quad \text{(i)}$$

$$\varepsilon_y = \frac{2p}{E} + \nu\frac{3p}{E} = \frac{2.75p}{E} \quad \text{(ii)}$$

Hence, from Eqs (1.27)

$$\frac{\partial u}{\partial x} = -\frac{3.5p}{E} \quad \text{so that} \quad u = -\frac{3.5p}{E}x + f_1(y) \quad \text{(iii)}$$

where $f_1(y)$ is a function of $y$. Also

$$\frac{\partial v}{\partial y} = \frac{2.75p}{E} \quad \text{so that} \quad v = -\frac{2.75p}{E}y + f_2(x) \tag{iv}$$

in which $f_2(x)$ is a function of $x$.

From the last of Eqs (1.52) and Eq. (1.28)

$$\gamma_{xy} = \frac{4p}{G} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = \frac{\partial f_2(x)}{\partial x} + \frac{\partial f_1(y)}{\partial y} \quad \text{(from Eqs (iv) and (iii))}$$

Suppose

$$\frac{\partial f_1(y)}{\partial y} = A$$

then

$$f_1(y) = Ay + B \tag{v}$$

in which $A$ and $B$ are constants.

Similarly, suppose

$$\frac{\partial f_2(x)}{\partial x} = C$$

then

$$f_2(x) = Cx + D \tag{vi}$$

in which $C$ and $D$ are constants.

Substituting for $f_1(y)$ and $f_2(x)$ in Eqs (iii) and (iv) gives

$$u = -\frac{3.5p}{E}x + Ay + B \tag{vii}$$

and

$$v = \frac{2.75p}{E}y + Cx + D \tag{viii}$$

Since the origin of the axes is fixed in space it follows that when $x = y = 0$, $u = v = 0$. Hence, from Eqs (vii) and (viii), $B = D = 0$. Further, the direction of O$x$ is fixed in space so that, when $y = 0$, $\partial v/\partial x = 0$. Therefore, from Eq. (viii), $C = 0$. Thus, from Eqs (1.28) and (vii), when $x = 0$.

$$\frac{\partial u}{\partial y} = \frac{4p}{G} = A$$

Eqs (vii) and (viii) now become

$$u = -\frac{3.5p}{E}x + \frac{4p}{G}y \tag{ix}$$

$$v = \frac{2.75p}{E} y \tag{x}$$

From Eq. (1.50), $G = E/2(1 + \nu) = E/2.5$ and Eq. (ix) becomes

$$u = \frac{p}{E}(-3.5x + 10y) \tag{xi}$$

At the point (2, 3)

$$u = \frac{23p}{E} \quad \text{(from Eq. (xi))}$$

and

$$v = \frac{8.25p}{E} \quad \text{(from Eq. (x))}$$

The point **P** therefore moves at an angle $\alpha$ to the $x$ axis given by

$$\alpha = \tan^{-1} \frac{8.25}{23} = 19.73°$$

## S.2.2

An Airy stress function, $\phi$, is defined by the equations (Eqs (2.8)):

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2} \quad \sigma_y = \frac{\partial^2 \phi}{\partial x^2} \quad \tau_{xy} = -\frac{\partial^2 \phi}{\partial x\, \partial y}$$

and has a final form which is determined by the boundary conditions relating to a particular problem.

Since

$$\phi = Ay^3 + By^3x + Cyx \tag{i}$$

$$\frac{\partial^4 \phi}{\partial x^4} = 0 \quad \frac{\partial^4 \phi}{\partial y^4} = 0 \quad \frac{\partial^4 \phi}{\partial x^2 \partial y^2} = 0$$

and the biharmonic equation (2.9) is satisfied. Further

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2} = 6Ay + 6Byx \tag{ii}$$

$$\sigma_y = \frac{\partial^2 \phi}{\partial x^2} = 0 \tag{iii}$$

$$\tau_{xy} = -\frac{\partial^2 \phi}{\partial x\, \partial y} = -3By^2 - C \tag{iv}$$

The distribution of shear stress in a rectangular section beam is parabolic and is zero at the upper and lower surfaces. Hence, when $y = \pm d/2$, $\tau_{xy} = 0$. Thus, from Eq. (iv)

$$B = \frac{-4C}{3d^2} \tag{v}$$

The resultant shear force at any section of the beam is $-P$. Therefore

$$\int_{-d/2}^{d/2} \tau_{xy} t \, \mathrm{d}y = -P$$

Substituting for $\tau_{xy}$ from Eq. (iv)

$$\int_{-d/2}^{d/2} (-3By^2 - C)t \, \mathrm{d}y = -P$$

which gives

$$2t\left(\frac{Bd^3}{8} + \frac{Cd}{2}\right) = P$$

Substituting for $B$ from Eq. (v) gives

$$C = \frac{3P}{2td} \qquad \text{(vi)}$$

It now follows from Eqs (v) and (vi) that

$$B = \frac{-2P}{td^3} \qquad \text{(vii)}$$

At the free end of the beam where $x = l$ the bending moment is zero and thus $\sigma_x = 0$ for any value of $y$. Therefore, from Eq. (ii)

$$6A + 6Bl = 0$$

whence

$$A = \frac{2Pl}{td^3} \qquad \text{(viii)}$$

Then, from Eq. (ii)

$$\sigma_x = \frac{12Pl}{td^3}y - \frac{12P}{td^3}xy$$

or

$$\sigma_x = \frac{12P(l - x)}{td^3}y \qquad \text{(ix)}$$

Equation (ix) is the direct stress distribution at any section of the beam given by simple bending theory, i.e.

$$\sigma_x = \frac{My}{I}$$

where $M = P(l - x)$ and $I = td^3/12$.

The shear stress distribution given by Eq. (iv) is

$$\tau_{xy} = \frac{6P}{td^3}y^2 - \frac{3P}{2td}$$

or

$$\tau_{xy} = \frac{6P}{td^3}\left(y^2 - \frac{d^2}{4}\right) \tag{x}$$

Equation (x) is identical to that derived from simple bending theory and may be found in standard texts on stress analysis, strength of materials, etc.

## S.2.3

The stress function is

$$\phi = \frac{w}{20h^3}(15h^2x^2y - 5x^2y^3 - 2h^2y^3 + y^5)$$

Then

$$\frac{\partial^2\phi}{\partial x^2} = \frac{w}{20h^3}(30h^2y - 10y^3) = \sigma_y$$

$$\frac{\partial^2\phi}{\partial y^2} = \frac{w}{20h^3}(-30x^2y - 12h^2y + 20y^3) = \sigma_x$$

$$\frac{\partial^2\phi}{\partial x\,\partial y} = \frac{w}{20h^3}(30h^2x - 30xy^2) = -\tau_{xy}$$

$$\frac{\partial^4\phi}{\partial x^4} = 0$$

$$\frac{\partial^4\phi}{\partial y^4} = \frac{w}{20h^3}(120y)$$

$$\frac{\partial^4\phi}{\partial x^2\,\partial y^2} = \frac{w}{20h^3}(-60y)$$

Substituting in Eq. (2.9)

$$\nabla^4\phi = 0$$

so that the stress function satisfies the biharmonic equation.

The boundary conditions are as follows:

- At $y = h$, $\sigma_y = w$ and $\tau_{xy} = 0$ which are satisfied.
- At $y = -h$, $\sigma_y = -w$ and $\tau_{xy} = 0$ which are satisfied.
- At $x = 0$, $\sigma_x = w/20h^3\ (-12h^2y + 20y^3) \neq 0$.

Also

$$\begin{aligned}\int_{-h}^{h}\sigma_x\,\mathrm{d}y &= \frac{w}{20h^3}\int_{-h}^{h}(-12h^2y + 20y^3)\mathrm{d}y\\ &= \frac{w}{20h^3}[-6h^2y^2 + 5y^4]_{-h}^{h}\\ &= 0\end{aligned}$$

i.e. no resultant force.

Finally

$$\int_{-h}^{h} \sigma_x y\,\mathrm{d}y = \frac{w}{20h^3}\int_{-h}^{h}(-12h^2y^2 + 20y^4)\mathrm{d}y$$
$$= \frac{w}{20h^3}[-4h^2y^3 + 4y^5]_{-h}^{h}$$
$$= 0$$

i.e. no resultant moment.

## S.2.4

The Airy stress function is

$$\phi = \frac{p}{120d^3}[5(x^3 - l^2x)(y+d)^2(y-2d) - 3yx(y^2 - d^2)^2]$$

Then

$$\frac{\partial^4\phi}{\partial x^4} = 0 \quad \frac{\partial^4\phi}{\partial y^4} = -\frac{3pxy}{d^3} \quad \frac{\partial^4\phi}{\partial x^2\partial y^2} = \frac{3pxy}{2d^3}$$

Substituting these values in Eq. (2.9) gives

$$0 + 2\times\frac{3pxy}{2d^3} - \frac{3pxy}{d^3} = 0$$

Therefore, the biharmonic equation (2.9) is satisfied.

The direct stress, $\sigma_x$, is given by (see Eqs (2.8))

$$\sigma_x = \frac{\partial^2\phi}{\partial y^2} = \frac{px}{20d^3}[5y(x^2 - l^2) - 10y^3 + 6d^2y]$$

When $x = 0$, $\sigma_x = 0$ for all values of $y$. When $x = l$

$$\sigma_x = \frac{pl}{20d^3}(-10y^3 + 6d^2y)$$

and the total end load $= \int_{-d}^{d}\sigma_x 1\,\mathrm{d}y$

$$= \frac{pl}{20d^3}\int_{-d}^{d}(-10y^3 + 6d^2y)\mathrm{d}y = 0$$

Thus the stress function satisfies the boundary conditions for axial load in the $x$ direction.

Also, the direct stress, $\sigma_y$, is given by (see Eqs (2.8))

$$\sigma_y = \frac{\partial^2\phi}{\partial x^2} = \frac{px}{4d^3}(y^3 - 3yd^2 - 2d^3)$$

When $x = 0$, $\sigma_y = 0$ for all values of $y$. Also at any section $x$ where $y = -d$

$$\sigma_y = \frac{px}{4d^3}(-d^3 + 3d^3 - 2d^3) = 0$$

and when $y = +d$

$$\sigma_y = \frac{px}{4d^3}(d^3 - 3d^3 - 2d^3) = -px$$

Thus, the stress function satisfies the boundary conditions for load in the $y$ direction.

The shear stress, $\tau_{xy}$, is given by (see Eqs (2.8))

$$\tau_{xy} = -\frac{\partial^2 \phi}{\partial x\, \partial y} = -\frac{p}{40d^3}[5(3x^2 - l^2)(y^2 - d^2) - 5y^4 + 6y^2d^2 - d^4]$$

When $x = 0$

$$\tau_{xy} = -\frac{p}{40d^3}[-5l^2(y^2 - d^2) - 5y^4 + 6y^2d^2 - d^4]$$

so that, when $y = \pm d$, $\tau_{xy} = 0$. The resultant shear force on the plane $x = 0$ is given by

$$\int_{-d}^{d} \tau_{xy} 1\, \mathrm{d}y = -\frac{p}{40d^3}\int_{-d}^{d}[-5l^2(y^2 - d^2) - 5y^4 + 6y^2d^2 - d^4]\mathrm{d}y = -\frac{pl^2}{6}$$

From Fig. P.2.4 and taking moments about the plane $x = l$,

$$\tau_{xy}(x = 0)12dl = \frac{1}{2}lpl\frac{2}{3}l$$

i.e.

$$\tau_{xy}(x = 0) = \frac{pl^2}{6d}$$

and the shear force is $pl^2/6$.

Thus, although the resultant of the Airy stress function shear stress has the same magnitude as the equilibrating shear force it varies through the depth of the beam whereas the applied equilibrating shear stress is constant. A similar situation arises on the plane $x = l$.

## S.2.5

The stress function is

$$\phi = \frac{w}{40bc^3}(-10c^3x^2 - 15c^2x^2y + 2c^2y^3 + 5x^2y^3 - y^5)$$

Then

$$\frac{\partial^2\phi}{\partial y^2} = \frac{w}{40bc^3}(12c^2y + 30x^2y - 20y^3) = \sigma_x$$

$$\frac{\partial^2\phi}{\partial x^2} = \frac{w}{40bc^3}(-20c^3 - 30c^2y + 10y^3) = \sigma_y$$

$$\frac{\partial^2\phi}{\partial x\,\partial y} = \frac{w}{40bc^3}(-30c^2x + 30xy^2) = -\tau_{xy}$$

$$\frac{\partial^4\phi}{\partial x^4} = 0$$

$$\frac{\partial^4\phi}{\partial y^4} = \frac{w}{40bc^3}(-120y)$$

$$\frac{\partial^4\phi}{\partial x^2\,\partial y^2} = \frac{w}{40bc^3}(60y)$$

Substituting in Eq. (2.9)

$$\nabla^4\phi = 0$$

so that the stress function satisfies the biharmonic equation.

On the boundary, $y = +c$

$$\sigma_y = -\frac{w}{b} \quad \tau_{xy} = 0$$

At $y = -c$

$$\sigma_y = 0 \quad \tau_{xy} = 0$$

At $x = 0$

$$\sigma_x = \frac{w}{40bc^3}(12c^2y - 20y^3)$$

Then

$$\int_{-c}^{c} \sigma_x \,\mathrm{d}y = \frac{w}{40bc^3}\int_{-c}^{c}(12c^2y - 20y^3)\mathrm{d}y = \frac{w}{40bc^3}[6c^2y^2 - 5y^4]_{-c}^{c} = 0$$

i.e. the direct stress distribution at the end of the cantilever is self-equilibrating.

The axial force at any section is

$$\int_{-c}^{c} \sigma_x \,\mathrm{d}y = \frac{w}{40bc^3}\int_{-c}^{c}(12c^2y + 30x^2y - 20y^3)\mathrm{d}y = \frac{w}{40bc^3}[6c^2y^2 + 15x^2y^2 - 5y^4]_{-c}^{c} = 0$$

i.e. no axial force at any section of the beam.

The bending moment at $x = 0$ is

$$\int_{-c}^{c} \sigma_x y \,\mathrm{d}y = \frac{w}{40bc^3} \int_{-c}^{c} (12c^2y^2 - 20y^4)\mathrm{d}y$$
$$= \frac{w}{40bc^3}[4c^2y^3 - 4y^5]_{-c}^{c} = 0$$

i.e. the beam is a cantilever beam under a uniformly distributed load of $w$/unit area with a self-equilibrating stress application at $x = 0$.

## S.2.6

From physics, the strain due to a temperature rise $T$ in a bar of original length $L_0$ and final length $L$ is given by

$$\varepsilon = \frac{L - L_0}{L_0} = \frac{L_0(1 + \alpha T) - L_0}{L_0} = \alpha T$$

Thus for the isotropic sheet, Eqs (1.52) become

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y) + \alpha T$$
$$\varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x) + \alpha T$$

Also, from the last of Eqs (1.52) and (1.50)

$$\gamma_{xy} = \frac{2(1 + \nu)}{E}\tau_{xy}$$

Substituting in Eq. (1.21)

$$\frac{2(1 + \nu)}{E}\frac{\partial^2 \tau_{xy}}{\partial x\,\partial y} = \frac{1}{E}\left(\frac{\partial^2 \sigma_y}{\partial x^2} - \nu\frac{\partial^2 \sigma_x}{\partial x^2}\right) + \alpha\frac{\partial^2 T}{\partial x^2} + \frac{1}{E}\left(\frac{\partial^2 \sigma_x}{\partial y^2} - \nu\frac{\partial^2 \sigma_y}{\partial y^2}\right) + \alpha\frac{\partial^2 T}{\partial y^2}$$

or

$$2(1 + \nu)\frac{\partial^2 \tau_{xy}}{\partial x\,\partial y} = \frac{\partial^2 \sigma_y}{\partial x^2} + \frac{\partial^2 \sigma_x}{\partial y^2} - \nu\frac{\partial^2 \sigma_x}{\partial x^2} - \nu\frac{\partial^2 \sigma_y}{\partial y^2} + E\alpha\nabla^2 T \qquad \text{(i)}$$

From Eqs (1.6) and assuming body forces $X = Y = 0$

$$\frac{\partial^2 \tau_{xy}}{\partial y\,\partial x} = -\frac{\partial^2 \sigma_x}{\partial x^2} \qquad \frac{\partial^2 \tau_{xy}}{\partial x\,\partial y} = -\frac{\partial^2 \sigma_y}{\partial y^2}$$

Hence

$$2\frac{\partial^2 \tau_{xy}}{\partial x\,\partial y} = -\frac{\partial^2 \sigma_x}{\partial x^2} - \frac{\partial^2 \sigma_y}{\partial y^2}$$

and

$$2\nu\frac{\partial^2\tau_{xy}}{\partial x\,\partial y} = -\nu\frac{\partial^2\sigma_x}{\partial x^2} - \nu\frac{\partial^2\sigma_y}{\partial y^2}$$

Substituting in Eq. (i)

$$-\frac{\partial^2\sigma_x}{\partial x^2} - \frac{\partial^2\sigma_y}{\partial y^2} = \frac{\partial^2\sigma_y}{\partial x^2} + \frac{\partial^2\sigma_x}{\partial y^2} + E\alpha\nabla^2 T$$

Thus

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma_x + \sigma_y) + E\alpha\nabla^2 T = 0$$

and since

$$\sigma_x = \frac{\partial^2\phi}{\partial y^2} \quad \sigma_y = \frac{\partial^2\phi}{\partial x^2} \quad \text{(see Eqs (2.8))}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial x^2}\right) + E\alpha\nabla^2 T = 0$$

or

$$\nabla^2(\nabla^2\phi + E\alpha T) = 0$$

## S.2.7

The stress function is

$$\phi = \frac{3Qxy}{4a} - \frac{Qxy^3}{4a^3}$$

Then

$$\frac{\partial^2\phi}{\partial x^2} = 0 = \sigma_y$$

$$\frac{\partial^2\phi}{\partial y^2} = -\frac{3Qxy}{2a^3} = \sigma_x$$

$$\frac{\partial^2\phi}{\partial x\,\partial y} = \frac{3Q}{4a} - \frac{3Qy^2}{4a^3} = -\tau_{xy}$$

Also

$$\frac{\partial^4\phi}{\partial x^4} = 0 \quad \frac{\partial^4\phi}{\partial y^4} = 0 \quad \frac{\partial^4\phi}{\partial x^2\,\partial y^2} = 0$$

so that Eq. (2.9), the biharmonic equation, is satisfied.

When $x=a$, $\sigma_x=-3Qy/2a^2$, i.e. linear.

Then, when

$$\begin{aligned} y=0 \quad & \sigma_x=0 \\ y=-a \quad & \sigma_x=\frac{3Q}{2a} \\ y=+a \quad & \sigma_x=\frac{-3Q}{2a} \end{aligned}$$

Also, when $x=-a$, $\sigma_x=3Qy/2a^2$, i.e. linear and when

$$\begin{aligned} y=0 \quad & \sigma_x=0 \\ y=-a \quad & \sigma_x=\frac{-3Q}{2a} \\ y=+a \quad & \sigma_x=\frac{3Q}{2a} \end{aligned}$$

The shear stress is given by (see above)

$$\tau_{xy}=-\frac{3Q}{4a}\left(1-\frac{y^2}{a^2}\right),\ \text{i.e. parabolic}$$

so that, when $y=\pm a$, $\tau_{xy}=0$ and when $y=0$, $\tau_{xy}=-3Q/4a$.

The resultant shear force at $x=\pm a$ is

$$=\int_{-a}^{a}-\frac{3Q}{4a}\left(1-\frac{y^2}{a^2}\right)\mathrm{d}y$$

i.e.

$$SF=Q.$$

The resultant bending moment at $x=\pm a$ is

$$\begin{aligned} &=\int_{-a}^{a}\sigma_x y\,\mathrm{d}y \\ &=\int_{-a}^{a}\frac{3Qay^2}{2a^3}\mathrm{d}y \end{aligned}$$

i.e.

$$BM=-Qa.$$