Symmetric responses to turbulence

In the previous chapter, we've created a model for turbulence. Now, we're going to apply it to an aircraft. How will an aircraft in turbulent air behave? In this chapter we'll examine the symmetric response. (The asymmetric response is left for the next chapter.) We start by finding a way to include turbulence into our equations of motion. After that, we will turn these equations of motion into a state space form.

By the way, in this chapter, we will not use the formal notation for stochastic variables $\bar{u}$ anymore, since otherwise some of the formulas will be a bit unreadable. You'll just have to remember yourself which variables are stochastic.

1 The effects of turbulence on the aircraft

1.1 Definitions

Let's consider an aircraft in a steady symmetric flight. Its velocity with respect to the ground is denoted by $\mathbf{V}$. Its velocity with respect to the air is $\mathbf{V_a}$. Finally, the velocity of air with respect to the ground is $\mathbf{V_g}$ (the **gust velocity**). So, we have

$$\mathbf{V} = \mathbf{V_a} + \mathbf{V_g} = \mathbf{V_a} + \mathbf{V_{g,av}} - \mathbf{u}. \tag{1.1}$$

In this chapter, we're considering symmetric aircraft motions. So, we will be interested in the components u_g and w_g of $\mathbf{u}$, but not in v_g. (We leave that for the next chapter.) Also, for simplicity we assume that there is no wind. So, $\mathbf{V_{g,av}} = \mathbf{0}$.

As you probably know, the **pitch angle** θ is defined as the angle between the aircraft's X axis and the horizontal plane. The **flight path angle** γ is the angle between the velocity vector $\mathbf{V}$ and the horizontal plane. The **angle of attack** α is now defined as $\alpha = \theta - \gamma$. That is, it's the angle between the aircraft's X axis and the velocity vector $\mathbf{V}$.

Usually, α is the angle between the aircraft's X axis and the airflow. But when there are gusts, this is not the case anymore. So, assuming that u_g and w_g are small relative to the velocity V, we define the **gust angle of attack** $\alpha_g = w_g/V$. The **total angle of attack** is now defined as $\alpha_{tot} = \alpha + \alpha_g$. Finally, we define the **non-dimensional gust velocity** $\hat{u}_g = u_g/V$. This implies that $V_a = V(1 + \hat{u}_g)$.

1.2 Equations for forces and moments

If we want to consider the aircraft response, we need to look at the forces and moments that act on the aircraft. The symmetric forces and moments acting on the aircraft due to gusts are denoted by X_g, Z_g and M_g. These forces/moments are turned into non-dimensional coefficients using

$$C_{X_g} = \frac{X_g}{\frac{1}{2}\rho V^2 S}, \qquad C_{Z_g} = \frac{Z_g}{\frac{1}{2}\rho V^2 S} \qquad \text{and} \qquad C_{m_g} = \frac{M_g}{\frac{1}{2}\rho V^2 S}. \tag{1.2}$$

To find an expression for C_{X_g}, we can use linearization. This gives us

$$C_{X_g} = \frac{1}{\frac{1}{2}\rho V^2 S}\frac{\partial X_g}{\partial \hat{u}_g}\hat{u}_g + \frac{1}{\frac{1}{2}\rho V^2 S}\frac{\partial X_g}{\partial \frac{\dot{\hat{u}}_g \bar{c}}{V}}\frac{\dot{\hat{u}}_g \bar{c}}{V} + \frac{1}{\frac{1}{2}\rho V^2 S}\frac{\partial X_g}{\partial \alpha_g}\alpha_g + \frac{1}{\frac{1}{2}\rho V^2 S}\frac{\partial X_g}{\partial \frac{\dot{\alpha}_g \bar{c}}{V}}\frac{\dot{\alpha}_g \bar{c}}{V}. \tag{1.3}$$

Using some coefficients, we can make the above equation a lot shorter. We then get

$$C_{X_g} = C_{X_{u_g}}\hat{u}_g + C_{X_{\dot{u}_g}}\frac{\dot{\hat{u}}_g \bar{c}}{V} + C_{X_{\alpha_g}}\alpha_g + C_{X_{\dot{\alpha}_g}}\frac{\dot{\alpha}_g \bar{c}}{V}. \tag{1.4}$$

Here, $\bar{c}$ is the mean chord length of the aircraft. The coefficients C_{Z_g} and C_{m_g} can be written in a similar way. The partial derivatives $C_{X_{u_g}}$, $C_{X_{\dot{u}_g}}$ and such are called **gust derivatives**.

1.3 Implementing the turbulence model

Let's try to implement the turbulence model which was derived in the previous chapter. We often assume that the gust field only varies in the X direction. So, it does not vary in the Y and Z direction. We thus write

$$\hat{u}_g = \hat{u}_{g_{max}} e^{j\Omega x} = \hat{u}_{g_{max}} e^{j\frac{\omega x}{V}} \qquad \text{and} \qquad \alpha_g = \alpha_{g_{max}} e^{j\Omega x} = \alpha_{g_{max}} e^{j\frac{\omega x}{V}}. \tag{1.5}$$

The wavelength in this field is still given by $\lambda = 2\pi/\Omega = 2\pi V/\omega$. However, we often don't work with ω and x, but with the **non-dimensional distance** s_c and the **reduced frequency** k_c, defined as

$$s_c = \frac{x}{\bar{c}} = \frac{Vt}{\bar{c}} \qquad \text{and} \qquad k_c = \Omega\bar{c} = \frac{\omega\bar{c}}{V}. \tag{1.6}$$

Note that now $\Omega x = k_c s_c$. If we use this fact and combine it with equations (1.4) and (1.5), we find that

$$C_{X_g} = \left(C_{X_{u_g}} + C_{X_{\dot{u}_g}} jk_c\right)\hat{u}_g + \left(C_{X_{\alpha_g}} + C_{X_{\dot{\alpha}_g}} jk_c\right)\alpha_g. \tag{1.7}$$

Once more, a similar expression can be derived for C_{Z_g} and C_{m_g}.

2 Finding coefficients and state space representations

2.1 Finding the gust derivatives

It would be nice if we could find expressions for the gust derivatives. First, we will examine the **steady gust derivatives** like $C_{X_{u_g}}$, $C_{Z_{u_g}}$ and $C_{M_{u_g}}$. These coefficients simply represent the forces/moment acting on the aircraft when the velocity changes. However, these coefficients are already known from normal flight dynamics. In fact, we have

$$C_{X_{u_g}} = C_{X_u} \qquad C_{Z_{u_g}} = C_{Z_u} \qquad \text{and} \qquad C_{M_{u_g}} = C_{M_u}. \tag{2.1}$$

Now let's examine the **unsteady gust derivatives** $C_{X_{\dot{u}_g}}$, $C_{Z_{\dot{u}_g}}$ and $C_{M_{\dot{u}_g}}$. It can be shown that the term $C_{X_{\dot{u}_g}} jk_c$ (and also the term $C_{X_{\dot{\alpha}_g}} jk_c$) is very small. Next to this, it is also very hard to derive a relation for it. So, it is usually neglected.

Deriving expressions for $C_{Z_{\dot{u}_g}}$ and $C_{M_{\dot{u}_g}}$ is quite difficult as well. But it can be done for an aircraft with a normal wing-fuselage-horizontal tailplane configuration. In fact, there are two methods for it. Both methods use the fact that the turbulence first hits the main wing. A time $\tau = (x_h - x_w)/V$ later, it hits the horizontal tailplane. (This is called the **gust penetration effect**.) Here, x_w and x_h are the x-positions of the aerodynamic centers of the wing and the horizontal tailplane, respectively. Also, we denote the x-position of the aircraft center of gravity by x_{cg}.

In the first method, we look at dynamic pressures. When a gust hits the wing, the dynamic pressure at the wing changes. The same holds for the horizontal tailplane, but this happens a time τ later. By using this data, we can derive that

$$C_{Z_{\dot{u}_g}} = 2\left(C_{Z_w}\frac{x_{cg} - x_w}{\bar{c}} + C_{Z_h}\frac{x_{cg} - x_h}{\bar{c}}\right) = 2C_{m_{ac}}, \tag{2.2}$$

$$C_{m_{\dot{u}_g}} = 2\left(C_{m_w}\frac{x_{cg} - x_w}{\bar{c}} + C_{m_h}\frac{x_{cg} - x_h}{\bar{c}}\right) = -2C_{m_h}\frac{l_h}{\bar{c}}. \tag{2.3}$$

In the second method, we don't consider the change in dynamic pressure. Instead, we look at how much the wing changes the flow velocity. The gust hits the wing first. When this happens, the velocity of the gust is changed by an amount Δu. A time τ later, a gust with a velocity $\left(1 - \frac{\partial \Delta u}{\partial \hat{u}}\right)\hat{u}_g$ arrives at the horizontal tailplane. Based on this data, we can't only derive relations for the unsteady gust derivatives, but also for the steady gust derivatives. In fact, we will find that

$$C_{X_{u_g}} = C_{X_u} = C_{X_{w_u}} + C_{X_{h_u}} \left(\frac{V_h}{V}\right)^2 \frac{S_h}{S} \left(1 - \frac{\partial \Delta u}{\partial \hat{u}}\right), \tag{2.4}$$

$$C_{X_{\dot{u}_g}} = -C_{X_{h_u}} \left(\frac{V_h}{V}\right)^2 \frac{S_h l_h}{S\bar{c}} \left(1 - \frac{\partial \Delta u}{\partial \hat{u}}\right), \tag{2.5}$$

$$C_{Z_{u_g}} = C_{Z_u} = C_{Z_{w_u}} + C_{Z_{h_u}} \left(\frac{V_h}{V}\right)^2 \frac{S_h}{S} \left(1 - \frac{\partial \Delta u}{\partial \hat{u}}\right), \tag{2.6}$$

$$C_{Z_{\dot{u}_g}} = -C_{Z_{h_u}} \left(\frac{V_h}{V}\right)^2 \frac{S_h l_h}{S\bar{c}} \left(1 - \frac{\partial \Delta u}{\partial \hat{u}}\right), \tag{2.7}$$

$$C_{m_{u_g}} = C_{m_u} = C_{m_{w_u}} + C_{Z_{h_u}} \left(\frac{V_h}{V}\right)^2 \frac{S_h l_h}{S\bar{c}} \left(1 - \frac{\partial \Delta u}{\partial \hat{u}}\right), \tag{2.8}$$

$$C_{m_{\dot{u}_g}} = -C_{Z_{h_u}} \left(\frac{V_h}{V}\right)^2 \frac{S_h l_h^2}{S\bar{c}^2} \left(1 - \frac{\partial \Delta u}{\partial \hat{u}}\right). \tag{2.9}$$

Now let's try to find the coefficients for α_g. This time, we only use one method, which is similar to the second method which we just saw. The gust α_g that hits the wing causes a change in downwash ϵ. By using this knowledge, we can derive for the steady gust derivatives that

$$C_{X_{\alpha_g}} = C_{X_\alpha} \qquad C_{Z_{\alpha_g}} = C_{Z_\alpha} \qquad \text{and} \qquad C_{M_{\alpha_g}} = C_{M_\alpha}. \tag{2.10}$$

(Although usually, it is simply assumed that $C_{X_{\alpha_g}} = 0$.) For the unsteady gust derivates, we have

$$C_{X_{\dot{\alpha}_g}} = C_{X_{\dot{\alpha}}} - C_{X_q} \qquad C_{Z_{\dot{\alpha}_g}} = C_{Z_{\dot{\alpha}}} - C_{Z_q} \qquad \text{and} \qquad C_{M_{\dot{\alpha}_g}} = C_{M_{\dot{\alpha}}} - C_{m_q}. \tag{2.11}$$

2.2 The symmetric equations of motion for aircraft turbulence

Now that we have values for the gust derivatives, we can derive the equations of motion. If we take the equations of motion, known from flight dynamics, and add the gusts in the input vector, we find that

$$\begin{bmatrix} C_{X_u} - 2\mu_c D_c & C_{X_\alpha} & C_{Z_0} & 0 \\ C_{Z_u} & C_{Z_\alpha} + (C_{Z_{\dot{\alpha}}} - 2\mu_c) D_c & -C_{X_0} & 2\mu_c + C_{Z_q} \\ 0 & 0 & -D_c & 1 \\ C_{m_u} & C_{m_\alpha} + C_{m_{\dot{\alpha}}} D_c & 0 & C_{m_q} - 2\mu_c K_Y^2 D_c \end{bmatrix} \begin{bmatrix} \hat{u} \\ \alpha \\ \theta \\ \frac{q\bar{c}}{V} \end{bmatrix} =$$

$$- \begin{bmatrix} C_{X_{\delta_e}} & C_{X_{u_g}} & 0 & C_{X_{\alpha_g}} & 0 \\ C_{Z_{\delta_e}} & C_{Z_{u_g}} & C_{Z_{\dot{u}_g}} & C_{Z_{\alpha_g}} & C_{Z_{\dot{\alpha}_g}} \\ 0 & 0 & 0 & 0 & 0 \\ C_{m_{\delta_e}} & C_{m_{u_g}} & C_{m_{\dot{u}_g}} & C_{m_{\alpha_g}} & C_{m_{\dot{\alpha}_g}} \end{bmatrix} \begin{bmatrix} \delta_e \\ \hat{u}_g \\ D_c \hat{u}_g \\ \alpha_g \\ D_c \alpha_g \end{bmatrix}. \tag{2.12}$$

The above equations can be transformed to a state space form. (To do this, you have to use the definition of the derivative operator $D_c = \frac{\bar{c}}{V}\frac{d}{dt}$.) And, if necessary, this state space form can also be combined with the (normalized) state space form of the forming filter for u_g and w_g. However, this is quite complicated, so we won't go into depth on that here.

2.3 Eigenmotions of the aircraft

Let's suppose that the pilot provides no input to the aircraft. So, $\delta_e = 0$. If the aircraft is in a gust, how does it behave? That's what we'll investigate now.

First, we'll examine the short period motion. During this motion, it is assumed that the velocity and the flight path angle don't change. Also, the forces in X direction are zero. Let's apply these assumptions to the state space matrix above. We then remain with

$$\begin{bmatrix} C_{Z_\alpha} + (C_{Z_{\dot\alpha}} - 2\mu_c)D_c & 2\mu_c + C_{Z_q} \\ C_{m_\alpha} + C_{m_{\dot\alpha}} D_c & C_{m_q} - 2\mu_c K_Y^2 D_c \end{bmatrix} \begin{bmatrix} \alpha \\ \frac{q\bar{c}}{V} \end{bmatrix} = - \begin{bmatrix} C_{Z_{\alpha_g}} & C_{Z_{\dot\alpha_g}} \\ C_{m_{\alpha_g}} & C_{m_{\dot\alpha_g}} \end{bmatrix} \begin{bmatrix} \alpha_g \\ D_c \alpha_g \end{bmatrix}. \tag{2.13}$$

With this state space representation, the short period motion of an aircraft in turbulence can be modeled. A similar trick can be performed for the phugoid motion. This time, we assume that the angle of attack α remains constant. Furthermore, we neglect C_{Z_q} and C_{X_0} and we assume that the moment acting on the aircraft is approximately zero. We now remain with

$$\begin{bmatrix} C_{X_u} - 2\mu_c D_c & C_{Z_0} & 0 \\ C_{Z_u} & 0 & 2\mu_c \\ 0 & -D_c & 1 \end{bmatrix} \begin{bmatrix} \hat{u} \\ \theta \\ \frac{q\bar{c}}{V} \end{bmatrix} = - \begin{bmatrix} C_{X_{u_g}} & 0 \\ C_{Z_{u_g}} & C_{Z_{\dot{u}_g}} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{u}_g \\ D_c \hat{u}_g \end{bmatrix}. \tag{2.14}$$

This state space representation can be used to model the phugoid motion of an aircraft in turbulence.