Spectral analysis of continuous processes

In the previous chapter, we have examined systems in the time-domain. In both this chapter and the next one, we're going to look at the frequency domain. To do that, we first examine Fourier series and the Fourier transform. With this theory, we can then examine the properties of systems in the frequency domain. This chapter concerns the continuous-time case, while the next chapter deals with discrete time.

1 Fourier series

1.1 Continuous-time Fourier series

Let's examine a periodic function $x(t)$. (**Periodic** means that there is some T such that $x(t) = x(t+T)$ for all t.) We can approximate $x(t)$ by summing up several **basis functions**. In the **continuous-time Fourier series** (CTFS) approximation, we use sines and cosines as basis functions. So, we approximate $x(t)$ like

$$\tilde{x}(t) = \sum_{k=0}^{N-1} \left(a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t)\right) = a_0 + \sum_{k=1}^{N-1} \left(a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t)\right). \quad (1.1)$$

The **fundamental frequency** is generally chosen to be $\omega_0 = 2\pi/T$. (The frequency $k\omega_0$ is now called the k**th harmonic**.) In this case, all basis functions are **orthogonal** on the interval $[t_0, t_0+T]$. This means that, if k and l are positive integers, we have

$$\int_{t_0}^{t_0+T} \sin(k\omega_0 t)\cos(l\omega_0 t)\, dt = 0, \quad (1.2)$$

$$\int_{t_0}^{t_0+T} \sin(k\omega_0 t)\sin(l\omega_0 t)\, dt = \begin{cases} 0 & \text{if } k \neq l \\ \frac{T}{2} & \text{if } k = l, \end{cases} \quad (1.3)$$

$$\int_{t_0}^{t_0+T} \cos(k\omega_0 t)\cos(l\omega_0 t)\, dt = \begin{cases} 0 & \text{if } k \neq l \\ \frac{T}{2} & \text{if } k = l. \end{cases} \quad (1.4)$$

We can use the above equations to find the coefficients a_0, a_k and b_k. We will then find that

$$a_0 = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)\, dt, \quad (1.5)$$

$$a_k = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)\cos(k\omega_0 t)\, dt, \quad (1.6)$$

$$b_k = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)\sin(k\omega_0 t)\, dt. \quad (1.7)$$

It is interesting to note that a_0 is, in fact, the average of the signal $x(t)$.

1.2 Continuous-time Fourier series in complex form

Using complex numbers, we can write the equations of the previous paragraph in a much easier form. Let's denote $j = \sqrt{-1}$ as the complex number. As you know, we can write $e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$, so

$$\cos(\omega t) = \frac{1}{2}\left(e^{j\omega t} + e^{-j\omega t}\right) \quad \text{and} \quad \sin(\omega t) = \frac{1}{2j}\left(e^{j\omega t} - e^{-j\omega t}\right). \quad (1.8)$$

We can now rewrite equation (1.1) to

$$\tilde{x}(t) = \sum_{k=-(N-1)}^{N-1} c_k e^{jk\omega_0 t}, \qquad \text{with} \qquad c_k = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)e^{-jk\omega_0 t}\,dt. \tag{1.9}$$

Note that the coefficients c_k are complex numbers as well. However, they are set such that the approximation $\tilde{x}(t)$ is a real-valued function. In the above equation, the left relation is called the **synthesis equation**. This is because it constructs/synthesizes the approximation $\tilde{x}(t)$ from basis functions. The right relation is called the **analysis equation**. This is because it analyses how to approximate $x(t)$ using basis functions.

We can also find the relationships between the coefficients a_k, b_k and c_k. These are

$$c_0 = a_0, \qquad c_k = \frac{1}{2}(a_k - jb_k), \qquad c_{-k} = \frac{1}{2}(a_k + jb_k), \tag{1.10}$$

$$a_0 = c_0, \qquad a_k = 2\text{Re}\{c_k\} = 2\text{Re}\{c_{-k}\}, \qquad b_k = -2\text{Im}\{c_k\} = 2\text{Im}\{c_{-k}\}. \tag{1.11}$$

The latter relations follow from the fact that c_k and c_{-k} are complex conjugates. We denote this by $c_k = c_{-k}^*$.

1.3 Properties of the Fourier series

The Fourier transform has several properties. We'll mention a couple of them. First of all, let's examine N. When N increases, then the approximation $\tilde{x}(t)$ of $x(t)$ becomes better. And, if $N \to \infty$, then $\tilde{x}(t) \to x(t)$.

When we find the Fourier transform of an even function, then we will only get cosine terms. So, $b_k = 0$ for all k. (An **even function** $x(t)$ satisfies $x(t) = x(-t)$.) Similarly, when we find the Fourier transform of an odd function, we only get sine terms. So, $a_k = 0$ for all k. (An **odd function** $x(t)$ satisfies $x(t) = -x(-t)$.)

Let's look at the average of the squared signal $x(t)^2$. It can be shown that this equals the sum of the squared Fourier series coefficients. So,

$$\frac{1}{T}\int_{t_0}^{t_0+T} x(t)^2\,dt = a_0^2 + \sum_{k=1}^{\infty}\frac{1}{2}\left(a_k^2 + b_k^2\right) = \sum_{k=-\infty}^{\infty}|c_k|^2. \tag{1.12}$$

This relation is called **Parseval's theorem for the Fourier series expansion**.

Finally, we can consider the Fourier series expansion of the nth derivative of $x(t)$. We then find that it equals

$$\frac{d^n x(t)}{dt^n} = \sum_{k=-\infty}^{\infty}(jk\omega_0)^n c_k e^{jk\omega_0 t}. \tag{1.13}$$

2 The continuous-time Fourier transform

2.1 The Fourier transform equations

Previously, we have derived the Fourier series of periodic functions. However, now we examine an aperiodic function. This function can, in fact, be seen as a periodic function with period $T = \infty$. So we can approximate it using a Fourier series. If we take $N = \infty$ and $t_0 = -\frac{1}{2}T$, then we get

$$\tilde{x}(t) = \lim_{T\to\infty}\left(\sum_{k=-\infty}^{+\infty}\frac{\omega_0}{2\pi}\left(\int_{-\frac{1}{2}T}^{+\frac{1}{2}T} x(t)e^{-jk\omega_0 t}\,dt\right)e^{jk\omega_0 t}\right). \tag{2.1}$$

However, if $T \to \infty$, then $\omega_0 = 2\pi/T$ becomes infinitesimally small. So, we rewrite it as $d\omega$. This then turns the sum into an integral. We should then also denote $k\omega_0$ simply as ω. This turns the above equation into

$$\tilde{x}(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} x(t) e^{-j\omega t} \, dt \right) e^{j\omega t} \, d\omega. \tag{2.2}$$

The inner integral from the above equation is called the **Fourier integral**. In fact, it is the **Fourier transform** $X(\omega)$ of the signal $x(t)$. This Fourier transform is denoted as

$$X(\omega) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} \, dt. \tag{2.3}$$

The outer integral is the **inverse Fourier transform**. It is written as

$$x(t) = \mathcal{F}^{-1}\{X(\omega)\} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\omega) e^{j\omega t} \, d\omega. \tag{2.4}$$

It must be noted that, in literature, there is no general consensus on where to place the $1/2\pi$ term in the above two equations. Some people put it in the first, other people put it in the second, and other people put the term $\sqrt{1/2\pi}$ in both terms. Also, when you use the frequency in Hertz instead of rad/s, the whole term vanishes altogether. In this summary, however, we'll simply use the notation of the above two equations.

2.2 Fourier transforms of basic functions

It can be worthwhile to remember the Fourier transform of several basic functions. When remembering them, it is convenient to keep in mind that $X(\omega)$ is an indication of how 'strong' the frequency ω is present in the signal. This may make it easier to remember. Now we'll list a couple of basic transforms.

- $\mathcal{F}\{1\} = 2\pi\delta(\omega)$, where $\delta(\omega)$ is again the Dirac delta function. So basically, only the frequency $\omega = 0$ is present in the signal $x(t) = 1$.
- $\mathcal{F}\{\cos(\omega_0 t)\} = \pi\left(\delta(\omega + \omega_0) + \delta(\omega - \omega_0)\right)$. So, the frequencies $\omega = \omega_0$ and $\omega = -\omega_0$ are present in the signal $x(t) = \cos(\omega_0 t)$.
- Let's define the **block function** $b(t)$ with width T and the sinc function according to

$$b(t) = \begin{cases} 1 & \text{if } |t| < T/2, \\ 1/2 & \text{if } |t| = T/2, \\ 0 & \text{if } |t| > T/2, \end{cases} \qquad \text{and} \qquad \text{sinc}(x) = \frac{\sin(x)}{x}. \tag{2.5}$$

  Now, we have $B(\omega) = \mathcal{F}\{b(t)\} = T\text{sinc}\left(\omega\frac{T}{2}\right)$. It is interesting to note that, if $T \to \infty$, then $b(t) = 1$ for all t. Thus, $B(\omega) \to 2\pi\delta(\omega)$.
- Let's consider the block function $b(\omega)$ with width W in the frequency domain. Now let's take the inverse Fourier transform. We then get $\mathcal{F}^{-1}\{b(\omega)\} = \frac{W}{2\pi}\text{sinc}\left(\frac{W}{2}t\right)$.

By the way, the sinc function is quite an important function. This function has a big peak of $\text{sinc}(x) = 1$ at $x = 0$. For the rest, it is zero if $\omega = 2\pi k/T$, with k a nonzero integer. Also, the sinc function is an even function. So, $\text{sinc}(x) = \text{sinc}(-x)$.

It is interesting to note that transforming a block function gives a sinc-function, while transforming the sinc-function gives a block-function. This is due to the **duality property** of the Fourier transform. This property states that

$$\text{if} \quad \mathcal{F}\{x(t)\} = X(\omega) \quad \text{or, equivalently,} \quad x(t) = \mathcal{F}^{-1}\{X(\omega)\} \quad \text{then} \quad \mathcal{F}\{X(t)\} = 2\pi x(-\omega). \tag{2.6}$$

2.3 Making a function periodic

Let's suppose that we have some function $x(t)$. We can make this function periodic by 'copying' it and moving it by integer multiples of T_0. This gives us the periodic function $x_p(t)$, being

$$x_p(t) = \sum_{n=-\infty}^{+\infty} x(t + nT_0). \tag{2.7}$$

Because the above function is periodic with period T_0, we can find the Fourier series. But now it can be shown that the coefficients c_k of this series actually equal $X(k\omega_0)/T_0$, where $X(\omega) = \mathcal{F}\{x(t)\}$ and $\omega_0 = 2\pi/T_0$. So, the coefficients c_k can simply be derived from $X(\omega)$.

Alternatively, we can also find the continuous-time Fourier transform of $x_p(t)$. This then becomes

$$X_p(\omega) = \sum_{n=-\infty}^{+\infty} \frac{X(\omega)}{T_0} \delta(\omega - n\omega_0). \tag{2.8}$$

Note that this is a discrete function: it only has values at certain points.

Let's see how this trick works for the block function $b(t)$. First, we define the **periodic block function** $b_p(t)$ as

$$b_p(t) = \begin{cases} 1 & \text{if } |t| < T/2 + nT_0, \\ 1/2 & \text{if } |t| = T/2 + nT_0, \\ 0 & \text{if } |t| > T/2 + nT_0. \end{cases} \tag{2.9}$$

We can now find the Fourier series coefficients c_k. They will turn out to be equal to $\frac{T}{T_0}\text{sinc}\left(k\omega\frac{T}{2}\right)$, which is exactly what the above trick predicts them to be.

3 Spectral analysis applied to systems

3.1 Spectral analysis

Let's examine a stochastic process $\bar{x}(t)$. If we try to analyze it in the frequency domain, we run into a problem. The resulting Fourier transform will be different for every realization $x(t)$. However, usually we aren't interested in $x(t)$. Instead, we are interested in the **energy** of the process $\bar{x}(t)$. This energy is generally proportional to $\bar{x}(t)^2$ or, when two processes are involved, to $\bar{x}(t)\bar{y}(t+\tau)$. (Here, we do assume that $\bar{x}(t)$ and $\bar{y}(t)$ have zero mean. If not, they can be **normalized** by subtracting the mean from the process.)

We know that the product $\bar{x}(t)\bar{y}(t+\tau)$ is related to $C_{\bar{x}\bar{y}}(\tau)$. So, let's examine this parameter. We assume that both $\bar{x}(t)$ and $\bar{y}(t)$ are ergodic processes. $x(t)$ and $y(t)$ are realizations of these processes, with corresponding Fourier transforms $X(\omega)$ and $Y(\omega)$. It can now be shown that $C_{\bar{x}\bar{y}}(\tau)$ equals

$$C_{\bar{x}\bar{y}}(\tau) = \lim_{T\to\infty} \frac{1}{2T}\frac{1}{2\pi}\int_{-\infty}^{+\infty} Y(\omega)X(-\omega)e^{j\omega\tau}\, d\omega. \tag{3.1}$$

We can define the **power spectral density function** (PSD function) $S_{\bar{x}\bar{y}}(\omega)$ as

$$S_{\bar{x}\bar{y}}(\omega) = \lim_{T\to\infty} \frac{1}{2T} Y(\omega)X(-\omega). \tag{3.2}$$

The relation between the covariance function $C_{\bar{x}\bar{y}}(\tau)$ and the power spectral density function $S_{\bar{x}\bar{y}}(\omega)$ is then very easy. It is given by

$$S_{\bar{x}\bar{y}}(\omega) = \mathcal{F}\{C_{\bar{x}\bar{y}}(\tau)\} = \int_{-\infty}^{+\infty} C_{\bar{x}\bar{y}}(\tau)e^{-j\omega\tau}\,d\tau, \tag{3.3}$$

$$C_{\bar{x}\bar{y}}(\tau) = \mathcal{F}^{-1}\{S_{\bar{x}\bar{y}}(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{\bar{x}\bar{y}}(\omega)e^{j\omega\tau}\,d\omega. \tag{3.4}$$

By the way, if $S_{\bar{x}\bar{y}}(\omega)$ concerns two different processes, then we call it the **cross-power spectral density function**. If it concerns only one process, then we have the **auto-power spectral density function** $S_{\bar{x}\bar{x}}(\omega)$.

3.2 The Laplace transform versus the Fourier transform

Let's examine a system. This system has input $\bar{u}(t)$, output $\bar{y}(t)$ and an impulse response function $h(t)$. When dealing with systems, people often confuse the Laplace transform with the Fourier transform. These two transforms are, respectively, defined as

$$X(s) = \mathcal{L}\{x(t)\} = \int_{-\infty}^{+\infty} x(t)e^{-st}\,dt \qquad \text{and} \qquad X(\omega) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}\,d\omega. \tag{3.5}$$

The Laplace transform is more general than the Fourier transform. (If you insert the special case $s = j\omega$ in the Laplace transform, you get the Fourier transform.) The Laplace transform $H(s) = Y(s)/U(s)$ of the impulse response function $h(t)$ is called the **transfer function**. It can be used very well to investigate the transient responses of the system. (Think of the final value theorem and such.)

However, the Fourier transform has its qualities as well. The Fourier transform $H(\omega) = Y(\omega)/U(\omega)$ of $h(t)$ is called the **frequency response function** (FRF). It can be used very well to examine the frequency response of the system. Since, in this chapter, we're examining the frequency response of time-invariant processes, we will use the FRF.

3.3 System analysis in the frequency domain

Let's suppose that we know the properties of the stochastic input process $\bar{u}(t)$ which we put into a system. We also know the system dynamics, in the form of the impulse response function $h(t)$ or, alternatively, its Fourier transform $H(\omega)$. Can we then find the properties of the stochastic output process $\bar{y}(t)$?

The answer is simple: yes we can. First of all, we can find the mean $\mu_{\bar{y}}$ of $\bar{y}(t)$. It is given by $\mu_{\bar{y}} = H(0)\mu_{\bar{u}}$. However, usually we assume that the mean is zero. (If not, then we can normalize the signals by subtracting the mean.) If this is the case, then we can find the covariance function for $\bar{u}$ and $\bar{y}$. We have

$$C_{\bar{u}\bar{y}}(\tau) = C_{\bar{u}\bar{u}}(\tau) * h(\tau) = \int_{-\infty}^{+\infty} C_{\bar{u}\bar{u}}(\tau-\theta)h(\theta)\,d\theta. \tag{3.6}$$

The $*$ operator indicates the **convolution integral**, which is defined as shown above. Also,

$$C_{\bar{y}\bar{u}}(\tau) = C_{\bar{u}\bar{y}}(-\tau) = C_{\bar{u}\bar{u}}(\tau) * h(-\tau) \qquad \text{and} \qquad C_{\bar{y}\bar{y}}(\tau) = C_{\bar{u}\bar{u}}(\tau) * h(\tau) * h(-\tau). \tag{3.7}$$

To find the power spectral density function, we can simply take the Fourier transform. And luckily, the convolution integral in the time domain is simply multiplication in the frequency domain. So,

$$S_{\bar{u}\bar{y}}(\omega) = \mathcal{F}\{C_{\bar{u}\bar{y}}(\tau)\} = H(\omega)S_{\bar{u}\bar{u}}(\omega), \qquad S_{\bar{y}\bar{u}}(\omega) = H(-\omega)S_{\bar{u}\bar{u}}(\omega) \quad \text{and} \quad S_{\bar{y}\bar{y}}(\omega) = |H(\omega)|^2 S_{\bar{u}\bar{u}}(\omega). \tag{3.8}$$

The variance of the output process can now be found using

$$\sigma_{\bar{y}}^2 = C_{\bar{y}\bar{y}}(\tau = 0) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{\bar{x}\bar{x}}(\omega)\,d\omega = \frac{1}{\pi}\int_{0}^{+\infty} S_{\bar{x}\bar{x}}(\omega)\,d\omega = \frac{1}{\pi}\int_{0}^{+\infty} |H(\omega)|^2 S_{\bar{u}\bar{u}}(\omega)\,d\omega. \tag{3.9}$$

3.4 White and colored noise in the frequency domain

Previously, we have defined white noise $\bar{w}(t)$. The covariance function was $C_{\bar{w}\bar{w}}(\tau) = W\delta(\tau)$. The power spectral density function now becomes

$$S_{\bar{w}\bar{w}}(\omega) = \int_{-\infty}^{+\infty} C_{\bar{w}\bar{w}}(\tau)e^{-j\omega\tau}\,d\tau = W. \tag{3.10}$$

In real life, this of course isn't possible. (A signal can't have energy at all frequencies.) So instead, we call a signal white noise if $S_{\bar{w}\bar{w}}(\omega) = W$ for $-\omega_1 < \omega < \omega_1$, with ω_1 sufficiently big.

Now let's suppose that we use white noise as the input $\bar{u}(t)$ of a system. The resulting output has a power spectral density function of $S_{\bar{y}\bar{y}}(\omega) = |H(\omega)|^2 W$. The variance of the output can then be found using

$$\sigma_{\bar{y}}^2 = W\left(\frac{1}{2\pi}\int_{-\infty}^{+\infty} |H(\omega)|^2\,d\omega\right). \tag{3.11}$$

The output can be seen as filtered white noise, also known as **colored noise**, with the FRF $H(\omega)$ as the **shaping filter**.

3.5 A system with noise

Let's consider a system with input $\bar{u}(t)$ and output $\bar{x}(t)$. Assume that we don't know the frequency response function $H(\omega)$ of the system. But luckily, we can measure the output. However, the measured output $\bar{y}(t)$ is distorted by a noise $\bar{n}(t)$. Thus, $\bar{y}(t) = \bar{x}(t) + \bar{n}(t)$. The question is, can we find $H(\omega)$? Yes, we can. After some derivation, we can find that

$$H(\omega) = \frac{S_{\bar{u}\bar{y}}(\omega)}{S_{\bar{u}\bar{u}}(\omega)}. \tag{3.12}$$

We can also find information about the noise $\bar{n}(t)$. Its PSD function is given by

$$S_{\bar{n}\bar{n}}(\omega) = S_{\bar{y}\bar{y}}(\omega) - |H(\omega)|^2 S_{\bar{u}\bar{u}}(\omega) = S_{\bar{y}\bar{y}}(\omega) - \frac{|S_{\bar{u}\bar{y}}(\omega)|^2}{S_{\bar{u}\bar{u}}(\omega)}. \tag{3.13}$$

Finally, we can also compare the real output signal $\bar{x}(t)$ to the measured output signal $\bar{y}(t)$. We then find that

$$\frac{S_{\bar{x}\bar{x}}(\omega)}{S_{\bar{y}\bar{y}}(\omega)} = \frac{|H(\omega)^2|S_{\bar{u}\bar{u}}(\omega)}{S_{\bar{y}\bar{y}}(\omega)} = \frac{|S_{\bar{u}\bar{y}}(\omega)^2|}{S_{\bar{u}\bar{u}}(\omega)S_{\bar{y}\bar{y}}(\omega)} = \Gamma_{\bar{u}\bar{y}}(\omega)^2, \qquad \text{where} \qquad \Gamma_{\bar{u}\bar{y}}(\omega) = \sqrt{\frac{|S_{\bar{u}\bar{y}}(\omega)^2|}{S_{\bar{u}\bar{u}}(\omega)S_{\bar{y}\bar{y}}(\omega)}}. \tag{3.14}$$

The function $\Gamma_{\bar{u}\bar{y}}(\omega)$ is called the **coherence** between the system input $\bar{u}(t)$ and the measured output $\bar{y}(t)$. A value of 0 indicates no coherence, while a value of 1 indicates full coherence.