Multivariable stochastic processes

Previously, we have only dealt with single-input single-output systems. But what happens if you insert a stochastic vector into a multi-input system? That's what we'll look at in this chapter. First, we'll look at some multivariable probability theory. After that, we're going to examine the properties of signals as they are passed through a system. Finally, we discuss how we can use the impulse response function.

1 Multivariable probability theory

1.1 Distribution functions of stochastic vectors

Let's examine a **stochastic vector** $\bar{\mathbf{x}}$. This is simply a vector of stochastic variables. So, we have

$$\bar{\mathbf{x}} = \begin{bmatrix} \bar{x}_1 & \bar{x}_2 & \dots & \bar{x}_n \end{bmatrix}^T. \tag{1.1}$$

The probability distribution function and the probability density function simply equal the joint probability distribution/density functions of the variables $\bar{x}_i$. So,

$$F_{\bar{\mathbf{x}}}(\mathbf{x}) = \Pr\{\bar{x}_1 \leq x_1 \wedge \bar{x}_2 \leq x_2 \wedge \dots \wedge \bar{x}_n < x_n\} \qquad \text{and} \qquad f_{\bar{\mathbf{x}}}(\mathbf{x}) = \frac{\partial^n F_{\bar{\mathbf{x}}}(\mathbf{x})}{\partial x_1 \, \partial x_2 \dots \partial x_n}. \tag{1.2}$$

1.2 Properties of stochastic vectors

In practice, we generally can't determine the exact distribution functions. Instead, we'll simply look at important parameters. For example, the **mean** (or **average**) $\mu_{\bar{\mathbf{x}}}$ is defined as

$$\mu_{\bar{\mathbf{x}}} = \mathrm{E}\{\bar{\mathbf{x}}\} = \begin{bmatrix} \mathrm{E}\{\bar{x}_1\} & \mathrm{E}\{\bar{x}_2\} & \dots & \mathrm{E}\{\bar{x}_n\} \end{bmatrix}^T. \tag{1.3}$$

So, applying the expectation operator E to a vector or matrix simply means applying it to every individual element of the vector/matrix. Similarly to the mean, we also have the $n \times m$ **covariance matrix** $C_{\bar{\mathbf{x}}\bar{\mathbf{y}}}$ of two stochastic vectors $\bar{\mathbf{x}}$ (size n) and $\bar{\mathbf{y}}$ (size m). It is defined as

$$C_{\bar{\mathbf{x}}\bar{\mathbf{y}}} = \mathrm{E}\left\{(\bar{\mathbf{x}} - \mu_{\bar{\mathbf{x}}})(\bar{\mathbf{y}} - \mu_{\bar{\mathbf{y}}})^T\right\}. \tag{1.4}$$

In the above equation, we again simply have to take the expectation of every parameter $(\bar{x}_i - \mu_{\bar{x}_i})(\bar{y}_j - \mu_{\bar{y}_j})$ of the matrix $(\bar{\mathbf{x}} - \mu_{\bar{\mathbf{x}}})(\bar{\mathbf{y}} - \mu_{\bar{\mathbf{y}}})^T$ to find $C_{\bar{\mathbf{x}}\bar{\mathbf{y}}}$.

Next to the covariance matrix, we of course also have the **autocovariance matrix** $C_{\bar{\mathbf{x}}\bar{\mathbf{x}}}$. It can be noted that this is a symmetric matrix ($C_{\bar{x}_i\bar{x}_j} = C_{\bar{x}_j\bar{x}_i}$). Also, its diagonal elements are the variances of the individual parameters ($C_{\bar{x}_i\bar{x}_i} = \sigma_{\bar{x}_i}^2$). We can use the autocovariance matrix to find the **correlation matrix** $K_{\bar{\mathbf{x}}\bar{\mathbf{x}}}$. This matrix is defined as

$$K_{\bar{\mathbf{x}}\bar{\mathbf{x}}} = \begin{bmatrix} \frac{C_{\bar{x}_1\bar{x}_1}}{\sigma_{\bar{x}_1}\sigma_{\bar{x}_1}} & \frac{C_{\bar{x}_1\bar{x}_2}}{\sigma_{\bar{x}_1}\sigma_{\bar{x}_2}} & \dots & \frac{C_{\bar{x}_1\bar{x}_n}}{\sigma_{\bar{x}_1}\sigma_{\bar{x}_n}} \\ \frac{C_{\bar{x}_1\bar{x}_2}}{\sigma_{\bar{x}_1}\sigma_{\bar{x}_2}} & \frac{C_{\bar{x}_2\bar{x}_2}}{\sigma_{\bar{x}_2}\sigma_{\bar{x}_2}} & \dots & \frac{C_{\bar{x}_2\bar{x}_n}}{\sigma_{\bar{x}_2}\sigma_{\bar{x}_n}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{C_{\bar{x}_1\bar{x}_n}}{\sigma_{\bar{x}_1}\sigma_{\bar{x}_n}} & \frac{C_{\bar{x}_2\bar{x}_n}}{\sigma_{\bar{x}_2}\sigma_{\bar{x}_n}} & \dots & \frac{C_{\bar{x}_n\bar{x}_n}}{\sigma_{\bar{x}_n}\sigma_{\bar{x}_n}} \end{bmatrix} = \begin{bmatrix} 1 & \frac{C_{\bar{x}_1\bar{x}_2}}{\sigma_{\bar{x}_1}\sigma_{\bar{x}_2}} & \dots & \frac{C_{\bar{x}_1\bar{x}_n}}{\sigma_{\bar{x}_1}\sigma_{\bar{x}_n}} \\ \frac{C_{\bar{x}_1\bar{x}_2}}{\sigma_{\bar{x}_1}\sigma_{\bar{x}_2}} & 1 & \dots & \frac{C_{\bar{x}_2\bar{x}_n}}{\sigma_{\bar{x}_2}\sigma_{\bar{x}_n}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{C_{\bar{x}_1\bar{x}_n}}{\sigma_{\bar{x}_1}\sigma_{\bar{x}_n}} & \frac{C_{\bar{x}_2\bar{x}_n}}{\sigma_{\bar{x}_2}\sigma_{\bar{x}_n}} & \dots & 1 \end{bmatrix}. \tag{1.5}$$

Stochastic vectors can be transformed linearly. For example, we may have $\bar{\mathbf{y}} = A\bar{\mathbf{x}}$. Let's suppose that we know the properties of the vector $\bar{\mathbf{x}}$. The properties of $\bar{\mathbf{y}}$ can then be found using

$$\mu_{\bar{\mathbf{y}}} = A\mu_{\bar{\mathbf{x}}} \qquad \text{and} \qquad C_{\bar{\mathbf{y}}\bar{\mathbf{y}}} = AC_{\bar{\mathbf{x}}\bar{\mathbf{x}}}A^T. \tag{1.6}$$

1.3 Properties of multivariable stochastic processes

We can extend the above properties to stochastic processes. Let's examine the multivariable stochastic processes $\bar{\mathbf{x}}(t)$ and $\bar{\mathbf{y}}(t)$. Once more, we assume that these properties are stationary. In a previous chapter, we defined the covariance function $C_{\bar{x}\bar{y}}(\tau)$ of the signals $\bar{x}(t)$ and $\bar{y}(t)$ as the covariance between $\bar{x}(t)$ and $\bar{y}(t+\tau)$. We do exactly the same to define the covariance function $C_{\bar{\mathbf{x}}\bar{\mathbf{y}}}(\tau)$ of the two processes $\bar{\mathbf{x}}(t)$ and $\bar{\mathbf{y}}(t)$. We thus get

$$C_{\bar{\mathbf{x}}\bar{\mathbf{y}}}(\tau) = \mathrm{E}\left\{(\bar{\mathbf{x}}(t) - \mu_{\bar{\mathbf{x}}}(t))(\bar{\mathbf{y}}(t+\tau) - \mu_{\bar{\mathbf{y}}}(t+\tau))^T\right\}. \tag{1.7}$$

Once we have the covariance function, we can find the power spectral density $S_{\bar{\mathbf{x}}\bar{\mathbf{y}}}(\omega)$ function for multivariable stochastic processes. This is once more simply the Fourier transform of $C_{\bar{\mathbf{x}}\bar{\mathbf{y}}}(\tau)$. So, $S_{\bar{\mathbf{x}}\bar{\mathbf{y}}}(\omega) = \mathcal{F}\left\{C_{\bar{\mathbf{x}}\bar{\mathbf{y}}}(\tau)\right\}$. By the way, when you want to take the Fourier transform of a matrix, you simply transform all the individual elements of the matrix separately.

2 Stochastic processes in systems

2.1 Continuous-time and discrete-time systems

Let's examine a multivariable linear system. We denote the state vector by $\mathbf{x}$, the input vector by $\mathbf{u}$ and the output vector by $\mathbf{y}$. We can write the system in its state space form. This is done for continuous (left) and discrete systems (right) like

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t), \qquad \mathbf{x}[k+1] = \Phi\mathbf{x}[k] + \Gamma\mathbf{u}[k], \tag{2.1}$$

$$\mathbf{y}(t) = C\mathbf{x}(t) + D\mathbf{u}(t), \qquad \mathbf{y}[k] = C\mathbf{x}[k] + D\mathbf{u}[k]. \tag{2.2}$$

Now let's ask ourselves an interesting question. What will happen if we don't put a deterministic input vector $\mathbf{u}$ into the system, but a stochastic input vector $\bar{\mathbf{u}}$? Well, we usually assume that $\bar{\mathbf{u}}$ is a Gaussian vector. And in this case, it can be shown that $\bar{\mathbf{x}}$ and $\bar{\mathbf{y}}$ will be Gaussian vectors as well. How to find their properties will be discussed in the upcoming two sections.

2.2 Properties for continuous-time systems

Let's examine the state equation $\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t)$ of a continuous system. This equation can be solved. We will then find that

$$\mathbf{x}(t) = \Phi(t,t_0)\mathbf{x}(t_0) + \int_{t_0}^{t} \Phi(t,\tau)B\mathbf{u}(\tau)\,d\tau, \tag{2.3}$$

where $\Phi(t,t_0)$ is the **transition matrix**, defined as

$$\Phi(t,t_0) = e^{(t-t_0)A} = I + (t-t_0)A + \frac{(t-t_0)^2A^2}{2!} + \frac{(t-t_0)^3A^3}{3!} + \ldots = \sum_{n=0}^{+\infty}\frac{(t-t_0)^nA^n}{n!}. \tag{2.4}$$

Now let's suppose that we use white noise $\bar{\mathbf{w}}(t)$ as input. We thus have $\mu_{\bar{\mathbf{w}}} = \mathbf{0}$ and $C_{\bar{\mathbf{w}}\bar{\mathbf{w}}}(\tau) = W\delta(\tau)$, where W is the **intensity matrix**. We can use the above equations to find the mean $\mu_{\bar{\mathbf{x}}}(t)$ and the covariance matrix $C_{\bar{\mathbf{x}}\bar{\mathbf{x}}}(t)$ of the resulting stochastic state vector $\bar{\mathbf{x}}(t)$ at time t. We will have $\mu_{\bar{\mathbf{x}}}(t) = \Phi(t,t_0)\mu_{\bar{\mathbf{x}}}(t_0)$ and

$$C_{\bar{\mathbf{x}}\bar{\mathbf{x}}}(t_1,t_2) = \Phi(t_1,t_0)C_{\bar{\mathbf{x}}\bar{\mathbf{x}}}(t_0,t_0)\Phi(t_2,t_0)^T + \int_{t_0}^{\min(t_1,t_2)} \Phi(t_1,\tau)BWB^T\Phi(t_2,\tau)^T\,d\tau. \tag{2.5}$$

Note that we have used the notation $C_{\bar{\mathbf{x}}\bar{\mathbf{x}}}(t_1, t_2)$, instead of the normal notation $C_{\bar{\mathbf{x}}\bar{\mathbf{x}}}(\tau)$. The reason for this is that the stochastic process $\bar{\mathbf{x}}(t)$ is not necessarily stationary. If we simply want to know the covariance matrix of $\bar{\mathbf{x}}(t)$ at time t, then we can insert $t_1 = t_2 = t$. We denote this matrix then as $C_{\bar{\mathbf{x}}\bar{\mathbf{x}}}(t)$. (Note that this is a different matrix function than $C_{\bar{\mathbf{x}}\bar{\mathbf{x}}}(\tau)$.)

When dealing with systems, we usually aren't interested in transient behavior. Instead, it would be nice to know the steady state solution $C_{\bar{\mathbf{x}}\bar{\mathbf{x}},ss}$ of the above equation. By setting $dC_{\bar{\mathbf{x}}\bar{\mathbf{x}}}(t,t)/dt$ to zero, it can be derived that

$$0 = AC_{\bar{\mathbf{x}}\bar{\mathbf{x}},ss} + C_{\bar{\mathbf{x}}\bar{\mathbf{x}},ss}A^T + BWB^T. \tag{2.6}$$

This is the **continuous-time Lyapunov equation**. A unique solution only exists if the matrix A is exponentially stable. (In other words, all eigenvalues are strictly negative.) If this is the case, then the solution is given by

$$C_{\bar{\mathbf{x}}\bar{\mathbf{x}},ss} = \int_{t_0}^{+\infty} e^{\tau A} BWB^T e^{\tau A^T}\, d\tau. \tag{2.7}$$

It is interesting to note that the above equation is equal to equation (2.5) when $t \to \infty$. The covariance matrix of exponentially stable systems thus always converges to the steady state covariance matrix.

2.3 Properties for discrete-time systems

In the previous paragraph, we considered a continuous system. Now, let's examine a discrete system. The state of this system satisfies the **linear difference equation** $\mathbf{x}[k+1] = \Phi\mathbf{x}[k] + \Gamma\mathbf{u}[k]$. Let's suppose that we derived this discrete system from a continuous system. If Δt is the sampling time, then we have

$$\Phi = \Phi(t_{k+1}, t_k) = e^{\Delta t\, A} = I + \Delta t\, A + \frac{\Delta t^2 A^2}{2!} + \frac{\Delta t^3 A^3}{3!} + \ldots = \sum_{n=0}^{+\infty} \frac{\Delta t^n A^n}{n!}, \tag{2.8}$$

$$\Gamma = \Delta t\, B + \frac{\Delta t^2 AB}{2!} + \frac{\Delta t^3 A^2 B}{3!} + \ldots = \sum_{n=1}^{+\infty} \frac{\Delta t^n A^{n-1} B}{n!}. \tag{2.9}$$

Note the similarity between the discrete-time system matrix Φ and the continuous-time transition matrix $\Phi(t, t_0)$. (That's the reason why the same symbol is used for both parameters.) The direct equation for finding $\mathbf{x}[k]$ is now given by

$$\mathbf{x}[k] = \Phi^k \mathbf{x}[0] + \sum_{n=0}^{k-1} \Phi^n \Gamma \mathbf{u}[k - n - 1]. \tag{2.10}$$

Let's suppose that we use white noise $\bar{\mathbf{w}}[k]$ as input. So, we have $\mu_{\bar{\mathbf{w}}} = \mathbf{0}$ and $C_{\bar{\mathbf{w}}\bar{\mathbf{w}}}[k] = W_d \delta[k]$. (By the way, $\delta[k]$ is the **Kronecker delta function**. We have $\delta[k] = 1$ if $k = 0$ and $\delta[k] = 0$ otherwise. Also, W_d is the intensity of the discrete noise.) With this data, the properties of $\mathbf{x}[k]$ can be derived. We find that $\mu_{\bar{\mathbf{x}}}[t] = \Phi^n \mu_{\bar{\mathbf{x}}}[0]$ and

$$C_{\bar{\mathbf{x}}\bar{\mathbf{x}}}[k_1, k_2] = \Phi^{k_1} \bar{\mathbf{x}}[0,0] \left(\Phi^T\right)^{k_2} + \sum_{n=0}^{\min(k_1,k_2)-1} \Phi^n \Gamma W_d \Gamma^T \left(\Phi^T\right)^n. \tag{2.11}$$

Often, we only want to find the covariance matrix of the stochastic variable $\bar{\mathbf{x}}[k]$ at time k. We then simply take $k_1 = k_2 = k$. The resulting matrix is denoted as $C_{\bar{\mathbf{x}}\bar{\mathbf{x}}}[k]$.

Let's suppose that we have some continuous process, and we are turning this into a discrete process. We already know how to find the system matrices Φ and Γ. However, given that we know the continuous noise intensity matrix W, how do we find the discrete noise intensity matrix W_d? It can be shown that, for small time steps Δt, we approximately have $W_d = W/\Delta t$. If we use this intensity matrix, then our discrete system is a good approximation of our non-discrete system.

We remain with the question of how to find the steady state covariance matrix $C_{\bar{\mathbf{x}}\bar{\mathbf{x}},ss}$. This time, it can be shown that it must satisfy

$$C_{\bar{\mathbf{x}}\bar{\mathbf{x}},ss} = \Phi C_{\bar{\mathbf{x}}\bar{\mathbf{x}},ss}\Phi^T + \Gamma W_d \Gamma^T. \tag{2.12}$$

This is the **discrete-time Lyapunov equation**. A unique solution exists if Φ is exponentially stable. (That is, if all eigenvalues λ of Φ satisfy $|\lambda| < 1$.) However, no analytic solution is available. Instead, the solution is usually found using computational/numerical methods.

3 The impulse response function

3.1 Finding the impulse response function

When examining a system, it is always interesting to look at the relation between the input and the output. Let's suppose that this relation is given by the **impulse response matrix** $h_{\bar{\mathbf{y}}\bar{\mathbf{u}}}(t)$ or, in an abbreviated notation, simply $h(t)$. If we denote the Fourier transform of this matrix by $H(\omega)$, then we have

$$\bar{\mathbf{y}}(t) = h(t) * \bar{\mathbf{u}}(t) \qquad \text{and} \qquad \bar{\mathbf{Y}}(\omega) = H(\omega)\bar{\mathbf{U}}(\omega). \tag{3.1}$$

The question remains: how can we find the impulse response function? For that, we can use the equation

$$h(t) = C\Phi(t, t_0)B + D. \tag{3.2}$$

Let's suppose that we have a system of which we do not know the system matrices. However, we are able to experiment with the system. How do we now find the impulse response function? Well, first we set the initial state $\mathbf{x}(t_0)$ to zero. Then, we simply set all inputs to zero, except for one input $u_i(t)$. We put an impulse function on this input. (So, $u_i(t) = \delta(t)$.) The resulting output $\mathbf{y}(t)$ now equals the ith column $\mathbf{h_i}(t)$ of the impulse response matrix $h(t)$. Perform this trick for all inputs/columns i and we have found the impulse response matrix $h(t)$.

There is also a slightly alternative method. It can be shown that putting an impulse function on $u_i(t)$ is equivalent to giving the system an initial condition $\mathbf{x}(t_0) = \mathbf{B_i}$, with $\mathbf{B_i}$ the ith column of B. So, if we apply this trick for all inputs/columns i, then we have again found the impulse response matrix $h(t)$.

3.2 The covariance matrix and the PSD function

Having the impulse response matrix can be very convenient. We can use it to find the covariance matrices between $\bar{\mathbf{u}}(t)$ and $\bar{\mathbf{y}}(t)$. This is done using

$$C_{\bar{\mathbf{u}}\bar{\mathbf{y}}}(\tau) = C_{\bar{\mathbf{u}}\bar{\mathbf{u}}}(\tau) * h(\tau)^T, \qquad C_{\bar{\mathbf{y}}\bar{\mathbf{u}}}(\tau) = C_{\bar{\mathbf{u}}\bar{\mathbf{y}}}(-\tau)^T = h(-\tau) * C_{\bar{\mathbf{u}}\bar{\mathbf{y}}}(\tau), \tag{3.3}$$

$$C_{\bar{\mathbf{y}}\bar{\mathbf{y}}}(\tau) = h(-\tau) * C_{\bar{\mathbf{u}}\bar{\mathbf{u}}}(\tau) * h(\tau)^T. \tag{3.4}$$

If we Fourier transform the above equations to the frequency domain, then we will find the power spectral density function. So,

$$S_{\bar{\mathbf{u}}\bar{\mathbf{y}}}(\omega) = S_{\bar{\mathbf{u}}\bar{\mathbf{u}}}(\omega)H(\omega)^T, \qquad S_{\bar{\mathbf{y}}\bar{\mathbf{u}}}(\omega) = S_{\bar{\mathbf{u}}\bar{\mathbf{y}}}(-\omega)^T = H(-\omega)S_{\bar{\mathbf{u}}\bar{\mathbf{y}}}(\omega), \tag{3.5}$$

$$S_{\bar{\mathbf{y}}\bar{\mathbf{y}}}(\omega) = H(-\omega)S_{\bar{\mathbf{u}}\bar{\mathbf{u}}}(\omega)H(\omega)^T. \tag{3.6}$$

By the way, all the above tricks also work if you use $\bar{\mathbf{x}}$ instead of $\bar{\mathbf{y}}$. But you then of course need to use the impulse response matrix $h_{\bar{\mathbf{x}}\bar{\mathbf{u}}}(t)$ instead of $h_{\bar{\mathbf{y}}\bar{\mathbf{u}}}(t)$.