Describing atmospheric turbulence

The previous chapters only discussed theory. It is time to look at how we can apply this theory to aircraft. First, we need to get a mathematical framework of turbulence. That's what we'll derive in this chapter. We start by examining some basic information about turbulence. We then look at how we can model turbulence as a stochastic process. Finally, we examine possible covariance and PSD functions for turbulence.

1 Basics of turbulence

1.1 Causes of turbulence

Let's examine the atmosphere. This atmosphere is often subject to turbulence. We'll take a look at where this turbulence comes from. First of all, it depends on the **temperature lapse rate** $\lambda = dT/dh$ in the atmosphere. This is the temperature change for every meter which we go up. In the ICAO standard atmosphere, up to 11 km, we have $\lambda = -0.0065$ °C/m.

Now examine a small parcel of air in the atmosphere. When this parcel goes up, its temperature will decrease. The rate of temperature decrease during adiabatic ascent is denoted by β. This parameter mostly depends on how much water is in the air. For dry air, we roughly have $\beta_{dry} = 0.0098$ °/m. For saturated air, this is less, though the exact value strongly depends on the temperature and the pressure.

Let's suppose that $|\lambda| > |\beta|$. (The absolute sign is present to prevent confusion with minus signs and such.) When our parcel of air now goes up, it cools less than the surrounding air. So, its density ρ is lower than the surrounding air. This causes buoyancy, causing our parcel of air to go up faster. We thus have **vertical instability**. (If, however, we have $|\lambda| < |\beta|$, then we have **vertical stability**.) Vertical instability is a common cause of vertical gusts.

Another cause of turbulence is windshear. Let's assume that the wind vector is directed horizontally. We can now distinguish **horizontal windshear** ($\partial V_w/\partial x$ and $\partial V_w/\partial y$), where the wind velocity varies per horizontal position, and **vertical windshear** ($\partial V_w/\partial z$), where the wind velocity varies per vertical position. Windshear causes friction between layers of air, which in turn causes turbulence.

1.2 Types of turbulence

To be able to quantify turbulence, we introduce the **eddy energy equation**

$$\frac{dE}{dt} = S + H + B - D. \tag{1.1}$$

Here, E is the **turbulent kinetic energy**, S is a term relating to vertical windshear and H is a term relating to horizontal windshear. Both S and H are positive. The term B is related to vertical stability. If we have vertical stability, then it is negative. In case of vertical instability, it is positive. Finally, D represents heat dissipation. Although the term is always positive, its exact value depends on E.

Let's examine the above equation for some conditions. We start on the ground. Here, we have a relatively big value for S. However, if we go up, S quickly decreases. The parameter B is positive during the day and negative during clear nights. It doesn't change much with height. So, close to the ground, S is dominant, while a bit higher up, B is dominant.

In clouds, we have saturated air. Such air is vertically unstable. So, B is positive. This is especially the case for rain-producing **cumulonimbus** clouds. Also, once the vertical instability has caused air to go up/down, windshears will be present. So, S and H will be positive too.

The term **clear-air turbulence** concerns turbulence high up, in clear air. It often occurs around jet streams, at altitudes of 10.000 to 12.000 meters. For this kind of turbulence, the horizontal windshear term H is often the most important term.

Finally, there is **mountain wave turbulence**, which occurs in the vincinity of mountains. These mountains perturb the air flow, causing turbulence. This type of turbulence can become very strong.

We can distinguish four degrees of turbulence intensity. In **light** turbulence, objects in the aircraft still remain at rest. In **moderate** turbulence, unsecured objects start to move about. In **severe** turbulence, the aircraft may momentarily be out of control. Finally, in **extreme** turbulence, it is impossible to control the aircraft. Structural damage may very well be present.

2 Modeling turbulence as a stochastic process

2.1 Splitting up the wind velocity

In principle, turbulence is a deterministic process, just like everything else in nature. But, because it is so hard to predict, it is much easier to simply consider it as a stochastic process. In fact, let's consider the 'deterministic' **gust vector** $\mathbf{V_g}(\mathbf{r}, t)$, being the velocity of air with respect to the ground. We generally split it up into two parts. These are the average wind velocity $\mathbf{V}_{g,av}$ and the deviations $\bar{\mathbf{u}}(\mathbf{r}, t)$, which we consider to be stochastic. So,

$$\mathbf{V_g}(\mathbf{r}, t) = \mathbf{V}_{g,av} - \bar{\mathbf{u}}(\mathbf{r}, t). \tag{2.1}$$

(The minus sign is present due to convention.) We hereby declare the average wind velocity $\mathbf{V}_{g,av}$ a matter of navigation/guidance. We will only concern ourselves with the velocity deviation vector $\bar{\mathbf{u}}(\mathbf{r}, t)$. This velocity vector has three components $\bar{u}_1(\mathbf{r}, t)$, $\bar{u}_2(\mathbf{r}, t)$ and $\bar{u}_3(\mathbf{r}, t)$. Each of these components depends on four parameters: the position $\mathbf{r} = [\xi_1, \xi_2, \xi_3]^T$ and the time t.

An important parameter is the covariance matrix of the wind velocity. This matrix can be found using

$$C_{\bar{u}\bar{u}}(\mathbf{r}, \mathbf{t}; \mathbf{r} + \xi, t + \tau) = \mathrm{E}\left\{\bar{\mathbf{u}}(\mathbf{r}, t)\bar{\mathbf{u}}(\mathbf{r} + \xi, t + \tau)^T\right\}. \tag{2.2}$$

We can also find the power spectral density function $S_{\bar{u}\bar{u}}(\mathbf{r}, t; \mathbf{\Omega}, \omega)$. To do this, we simply take the Fourier transform of $C_{\bar{u}\bar{u}}(\mathbf{r}, t; \mathbf{r} + \xi, t + \tau)$. However, this is slightly more difficult now, since this matrix now depends on four parameters. Because of this, the Fourier transform has become

$$S_{\bar{u}\bar{u}}(\mathbf{r}, t; \mathbf{\Omega}, \omega) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} C_{\bar{u}\bar{u}}(\mathbf{r}, t; \mathbf{r} + \xi, t + \tau) e^{-j(\mathbf{\Omega}\xi^T + \omega t)}\, d\xi_1\, d\xi_2\, d\xi_3\, d\tau. \tag{2.3}$$

By the way, Ω is called the **spatial frequency**. It is related to the **wavelength** λ of the turbulence according to $\Omega = 2\pi/\lambda$.

2.2 Simplifying assumptions

Currently, atmospheric turbulence is still a bit too difficult to work with. So we have to make some simplifying assumptions.

- We assume that turbulence is normally distributed. So, $\bar{\mathbf{u}}(\mathbf{r}, t)$ has a normal distribution. Since $\bar{\mathbf{u}}(\mathbf{r}, t)$ has zero mean, we thus only need to know $C_{\bar{u}\bar{u}}(\mathbf{r}, t; \mathbf{r} + \xi, t + \tau)$ to fully describe $\bar{\mathbf{u}}(\mathbf{r}, t)$.
- We assume that turbulence is a stationary process. In fact, we assume that $\bar{\mathbf{u}}(\mathbf{r}, t)$ does not depend on time at all. We thus write $\bar{\mathbf{u}}(\mathbf{r})$, $C_{\bar{u}\bar{u}}(\mathbf{r}; \mathbf{r}+\xi)$ and $S_{\bar{u}\bar{u}}(\mathbf{r}; \mathbf{\Omega})$. (This assumption is called **Taylor's hypothesis**.)
- We assume that turbulence is homogeneous along the flight path. So, the turbulence does not depend on the position. We thus write $C_{\bar{u}\bar{u}}(\xi)$ and $S_{\bar{u}\bar{u}}(\mathbf{\Omega})$. This assumption also implies that turbulence is an ergodic process.

- We assume that turbulence is an isotropic process: the statistical properties are independent of direction. We thus have $\sigma^2_{\bar{u}_1} = \sigma^2_{\bar{u}_2} = \sigma^2_{\bar{u}_3} = \sigma^2$. This assumption seems to hold for high altitudes, but is not so accurate close to the ground.

2.3 The fundamental correlation functions

To see the effect of the assumptions that have been made, we will examine two points **a** and **b** in the atmosphere. We denote the relative position of these points by the vector $\xi = \mathbf{b} - \mathbf{a}$. Now examine the components of the velocities $\bar{\mathbf{u}}(\mathbf{a})$ and $\bar{\mathbf{u}}(\mathbf{b})$ in the direction of ξ in both points. (The so-called **longitudinal** components.) There is a correlation between these velocity components. Due to our assumptions, this correlation only depends on the distance $|\xi|$ and is denoted as $f(|\xi|)$.

In a similar way, we can look at each of the velocity components perpendicular to the vector ξ. (The so-called **lateral** components.) Once more, the correlation between these components only depends on the distance $|\xi|$. We denote this correlation by $g(|\xi|)$. Both **fundamental correlation functions** f and g can be found from the PSD function $S_{\bar{u}\bar{u}}(\mathbf{\Omega})$. Once we have them, we can find the covariance matrix of the turbulence. It is given by

$$C_{ij}(\xi) = \sigma^2 \left(\frac{(f(|\xi|) - g(|\xi|))\,\xi_i \xi_j}{|\xi|^2} + g(|\xi|)\delta_{ij} \right), \tag{2.4}$$

where δ_{ij} is the Kronecker delta function. (It equals 1 if $i = j$ and is zero otherwise.)

Turbulence occurs on many scales. An indication of the scale is the **integral scale of turbulence**. The longitudinal scale L_g and the lateral scale L'_g are, respectively, defined as

$$L_g = \int_0^\infty f(\xi)\,d\xi \qquad \text{and} \qquad L'_g = \int_0^\infty g(\xi)\,d\xi. \tag{2.5}$$

The continuity condition for incompressible fluids imposes a relation between these two scales. This relation is $L_g = 2L'_g$.

3 Finding the covariance and PSD functions for turbulence

3.1 The von Kármán spectra

The question remains what kind of PSD function we should use for turbulence. This is where the difficult mathematical equations come in. The **von Kármán functions** yield spectra that seem to match quite well with theoretical and experimental data on turbulence. So, let's examine it. The longitudinal and lateral spectra $S_{lo}(\Omega)$ and $S_{la}(\Omega)$ are, respectively, given by

$$S_{lo}(\Omega) = 2\sigma^2 L_g \frac{1}{\left(1 + (1.339 L_g \Omega)^2\right)^{5/6}} \qquad \text{and} \qquad S_{la}(\Omega) = \sigma^2 L_g \frac{1 + \frac{8}{3}(1.339 L_g \Omega)^2}{\left(1 + (1.339 L_g \Omega)^2\right)^{11/6}}. \tag{3.1}$$

If we take the inverse Fourier transform, then we find that

$$f(\xi) = \frac{2^{\frac{2}{3}}}{\Gamma\left(\frac{1}{3}\right)} \left(\frac{\xi}{1.339 L_g}\right)^{\frac{1}{3}} K_{\frac{1}{3}}\left(\frac{\xi}{1.339 L_g}\right), \tag{3.2}$$

$$g(\xi) = \frac{2^{\frac{2}{3}}}{\Gamma\left(\frac{1}{3}\right)} \left(\frac{\xi}{1.339 L_g}\right)^{\frac{1}{3}} \left(K_{\frac{1}{3}}\left(\frac{\xi}{1.339 L_g}\right) - \frac{1}{2}\left(\frac{\xi}{1.339 L_g}\right) K_{\frac{2}{3}}\left(\frac{\xi}{1.339 L_g}\right) \right). \tag{3.3}$$

In the above equation, $\Gamma(z)$ denots the Gamma function and $K_m(z)$ denotes that modified Bessel function of the second kind. For reasons of brevity, we won't examine their rather lengthy definitions. However, if the functions f and g are found, then the covariance matrix could be found using equation (2.4).

3.2 The von Kármán spectra applied to aircraft

Previously, we have considered an arbitrary reference frame and seperation vector ξ. Now, we can specify these for an aircraft. Let's use the aircraft stability reference frame. In this case, the turbulence velocity is $\bar{\mathbf{u}} = [\bar{u}_g, \bar{v}_g, \bar{w}_g]^T$, with $\bar{u}_g$ the longitudinal gust velocity (positive backwards), $\bar{v}_g$ the lateral gust velocity (positive to the left) and $\bar{w}_g$ the vertical gust velocity (positive upward). We also choose $\xi = [V\tau, 0, 0]^T$.

The result is that we can express the covariance matrix as a function of time τ again, instead of position ξ. (We simply use $\xi = V\tau$.) Also, we can express the PSD function in the angular frequency ω again, instead of the spatial frequency Ω. (We now use $\omega = V\Omega$.) The relations between the old and the new functions are given by

$$C_{\bar{u}\bar{u},new}(\tau) = C_{\bar{u}\bar{u},old}(\xi = V\tau) \qquad \text{and} \qquad S_{\bar{u}\bar{u},new}(\omega) = \frac{1}{V} S_{\bar{u}\bar{u},old}(\Omega = \omega/V). \tag{3.4}$$

If we apply this to the von Kármán spectra, then we find that

$$S_{\bar{u}_g\bar{u}_g}(\omega) = 2\sigma^2 \frac{L_g}{V} \frac{1}{\left(1 + \left(1.339 \frac{L_g \omega}{V}\right)^2\right)^{5/6}}, \tag{3.5}$$

$$S_{\bar{v}_g\bar{v}_g}(\omega) = S_{\bar{w}_g\bar{w}_g}(\omega) = \sigma^2 \frac{L_g}{V} \frac{1 + \frac{8}{3}\left(1.339 \frac{L_g \omega}{V}\right)^2}{\left(1 + \left(1.339 \frac{L_g \omega}{V}\right)^2\right)^{11/6}}. \tag{3.6}$$

In this special situation, the cross-PSD functions $S_{\bar{u}_g\bar{v}_g}(\omega)$, $S_{\bar{u}_g\bar{w}_g}(\omega)$ and $S_{\bar{v}_g\bar{w}_g}(\omega)$ are zero.

3.3 The Dryden spectral form

There is a problem with the von Kármán spectra. They are not rational functions. Having rational functions would simplify computations. To solve this, the **Dryden spectral form** is introduced. This function is a rational function. And furthermore, it more or less equals the von Kármán spectra on all frequencies except for really high ones.

In the Dryden spectral form, we again have $\xi = [V\tau, 0, 0]^T$. However, this time

$$S_{\bar{u}_g\bar{u}_g}(\omega) = 2\sigma^2 \frac{L_g}{V} \frac{1}{1 + \left(\frac{L_g \omega}{V}\right)^2} \qquad \text{and} \qquad S_{\bar{v}_g\bar{v}_g}(\omega) = S_{\bar{w}_g\bar{w}_g}(\omega) = \sigma^2 \frac{L_g}{V} \frac{1 + 3\left(\frac{L_g \omega}{V}\right)^2}{\left(1 + \left(\frac{L_g \omega}{V}\right)^2\right)^2}. \tag{3.7}$$

To find the covariance matrix, we can again use equation (2.4). But this time, we need to insert

$$f(\xi) = e^{-\frac{\xi}{L_g}} \qquad \text{and} \qquad g(\xi) = e^{-\frac{\xi}{L_g}}\left(1 - \frac{\xi}{2L_g}\right). \tag{3.8}$$

3.4 Generating a turbulence signal

Let's suppose that we have chosen which spectral form to use. We now want to generate a set of turbulence data. So how do we do that? The main idea is that we use the equation

$$S_{\bar{y}\bar{y}}(\omega) = |H(\omega)|^2 S_{\bar{u}\bar{u}}(\omega) \qquad \text{or} \qquad |H(\omega)|^2 = \frac{S_{\bar{y}\bar{y}}(\omega)}{S_{\bar{u}\bar{u}}(\omega)}. \tag{3.9}$$

The output PSD function $S_{\bar{y}\bar{y}}(\omega)$ is known: it is the spectral form which we selected. As input, we usually take white noise, so $S_{\bar{u}\bar{u}}(\omega) = 1$. Now we need to find the function $H(\omega)$ which satisfies the above equation. The solution $H(\omega)$ is called the **forming filter**. Once we have $H(\omega)$, we generate a white noise signal $U(\omega)$ and use $Y(\omega) = H(\omega)U(\omega)$ to form our turbulence signal $Y(\omega)$.

Once we have the forming filter $H(\omega)$, we can also put it in a state space form. This can, however, be quite complicated. So we won't treat this in depth here. Instead, we'll only mention that the forming filters for the Dryden spectral form are

$$H_{\bar{u}_g\bar{w}_1}(\omega) = \frac{\bar{u}_g(\omega)}{\bar{w}_1(\omega)} = \sigma\sqrt{\frac{2L_g}{V}}\frac{1}{1-\frac{L_g}{V}j\omega} \qquad \text{and} \qquad H_{\bar{w}_g\bar{w}_3}(\omega) = \frac{\bar{w}_g(\omega)}{\bar{w}_3(\omega)} = \sigma\sqrt{\frac{L_g}{V}}\frac{1+\sqrt{3}\frac{L_g}{V}j\omega}{\left(1-\frac{L_g}{V}j\omega\right)^2}. \tag{3.10}$$

If we subsitute/generalize $s = j\omega$, then we can put the above equations into state space form. If we define

$$w_g^*(t) = \dot{w}_g(t) - \sigma\sqrt{\frac{3V}{L_g}}w_3(t), \tag{3.11}$$

then we get

$$\dot{u}_g(t) = -\frac{V}{L_g}u_g(t) + \sigma\sqrt{\frac{2V}{L_g}}w_1(t), \tag{3.12}$$

$$\begin{bmatrix}\dot{w}_g(t)\\ \dot{w}_g^*(t)\end{bmatrix} = \begin{bmatrix}0 & 1\\ -\frac{V^2}{L_g^2} & -2\frac{V}{L_g}\end{bmatrix}\begin{bmatrix}w_g(t)\\ w_g^*(t)\end{bmatrix} + \begin{bmatrix}\sigma\sqrt{\frac{3V}{L_g}}\\ (1-2\sqrt{3})\,\sigma\sqrt{\left(\frac{V}{L_g}\right)^3}\end{bmatrix}w_3(t). \tag{3.13}$$

3.5 Finding the quantative parameters close to the ground

When setting up a turbulence model, we do need the parameters $\sigma_{\bar{u}_g}$, $\sigma_{\bar{v}_g}$, $\sigma_{\bar{w}_g}$ and L_g. These parameters are based on experimental data. Tables are available to find the parameters. Using these tables for relatively low altitudes (below 450 m) can be difficult. This is mainly because, for these low altitudes, the homogeneous and isotropic flow assumptions don't hold anymore. To solve this problem, we need some data.

First of all, we need to know the temperature lapse rate λ. Second, we also need an indication of the wind speed. For this, usually the wind speed at a reference height is used. (30 ft/9.15 m is an often-used reference height.) With these two parameters, we can find $\sigma_{\bar{w}_g}$ from tables. The quantities $\sigma_{\bar{u}_g}$ and $\sigma_{\bar{v}_g}$ are a bit harder to find though. This is because they strongly depend on the height h at which we want to know the turbulence properties. A guideline that is often used to find them is

$$\underbrace{\frac{\sigma_{\bar{u}_g}}{\sigma_{\bar{w}_g}} = \frac{\sigma_{\bar{v}_g}}{\sigma_{\bar{w}_g}} = 2.5,}_{0\text{m}\leq h<15\text{m}} \qquad \underbrace{\frac{\sigma_{\bar{u}_g}}{\sigma_{\bar{w}_g}} = \frac{\sigma_{\bar{v}_g}}{\sigma_{\bar{w}_g}} = 1.25 - 0.001h}_{15\text{m}\leq h<250\text{m}} \qquad \text{and} \qquad \underbrace{\frac{\sigma_{\bar{u}_g}}{\sigma_{\bar{w}_g}} = \frac{\sigma_{\bar{v}_g}}{\sigma_{\bar{w}_g}} = 1}_{250\text{m}\leq h}. \tag{3.14}$$

We can then simply insert the values for $\sigma_{\bar{u}_g}$, $\sigma_{\bar{v}_g}$ and $\sigma_{\bar{w}_g}$ in the right positions of equations (3.5), (3.6) and (3.7).

To find the value of L_g, we have to use tables again. All we have to know for this parameter are the temperature lapse rate λ and the height h at which we want to know the turbulence properties.