Asymmetric responses to turbulence

In the previous chapter we examined the symmetric response of an aircraft to turbulence. In this chapter, we'll focus on the asymmetric response. This is a bit more difficult than the symmetric response. Why this is the case will be examined first. After that, the asymmetric force and moment coefficients will be derived. At the end, the asymmetric PSD functions and the equations of motion will be examined.

1 The covariance and PSD function in two-dimensional space

1.1 Deriving the covariance matrix

Previously, we have assumed that the turbulence only varies in a longitudinal direction. This works when we're examining longitudinal motions. But when examining lateral motions, the lateral distance y also needs to be taken into account. So, in this chapter, the turbulence parameters u_g, v_g and w_g depend on x and y. The turbulence covariance matrix is thus given by

$$C_{\bar{\mathbf{u}}\bar{\mathbf{u}}}(x,y) = \begin{bmatrix} C_{u_g u_g}(x,y) & 0 & 0 \\ 0 & C_{v_g v_g}(x,y) & 0 \\ 0 & 0 & C_{w_g w_g}(x,y) \end{bmatrix}. \tag{1.1}$$

Note that, due to the isotropic assumption, the parameters u_g, v_g and w_g are mutually independent. We also have $C_{u_g u_g}(x,y) = \mathrm{E}\{u_g(0,0), u_g(x,y)\}$ and the same for v_g and w_g.

Let's denote the distance to the point $P = (x,y)$ by $r = \sqrt{x^2+y^2}$. When the functions $f(r)$ and $g(r)$ are known, the terms $C_{u_g u_g}(x,y)$ and $C_{v_g v_g}(x,y)$ of the covariance matrix can be found using

$$C_{u_g u_g}(x,y) = \sigma_{u_g}^2 \left(f(r)\left(\frac{x}{r}\right)^2 + g(r)\left(\frac{y}{r}\right)^2 \right) \quad \text{and} \quad C_{v_g v_g}(x,y) = \sigma_{v_g}^2 \left(f(r)\left(\frac{y}{r}\right)^2 + g(r)\left(\frac{x}{r}\right)^2 \right). \tag{1.2}$$

Also, we have $C_{w_g w_g}(x,y) = \sigma_{w_g}^2 g(r)$. However, often the covariance matrices are expressed, not in x and y, but in the dimensionless parameters x/L_g and y/L_g. It could be worthwhile to keep this in mind when reading other texts on atmospheric flight dynamics.

1.2 Deriving the power spectral density function

To derive the PSD function, we simply take the Fourier transform of the covariance matrix. This time, the covariance matrix is a function of two variables. The Fourier transform thus becomes

$$S(\Omega_x L_g, \Omega_y L_g) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} C\left(\frac{x}{L_g}, \frac{y}{L_g}\right) e^{-j(\Omega_x x \Omega_y y)}\, d\frac{x}{L_g}\, d\frac{y}{L_g}. \tag{1.3}$$

Note that, for the PSD matrix, we have also used dimensionless parameters. This time they are $\Omega_x L_g$ and $\Omega_y L_g$.

For the Dryden spectral form, the above integral can be solved. The obtained results are

$$S_{u_g u_g}(\Omega_x L_g, \Omega_y L_g) = \pi \sigma_{u_g}^2 \frac{1 + \Omega_x^2 L_g^2 + 4\Omega_y^2 L_g^2}{\left(1 + \Omega_x^2 L_g^2 + \Omega_y^2 L_g^2\right)^{5/2}}, \tag{1.4}$$

$$S_{v_g v_g}(\Omega_x L_g, \Omega_y L_g) = \pi \sigma_{v_g}^2 \frac{1 + 4\Omega_x^2 L_g^2 + \Omega_y^2 L_g^2}{\left(1 + \Omega_x^2 L_g^2 + \Omega_y^2 L_g^2\right)^{5/2}}, \tag{1.5}$$

$$S_{w_g w_g}(\Omega_x L_g, \Omega_y L_g) = \pi \sigma_{w_g}^2 \frac{3\Omega_x^2 L_g^2 + 3\Omega_y^2 L_g^2}{\left(1 + \Omega_x^2 L_g^2 + \Omega_y^2 L_g^2\right)^{5/2}}. \tag{1.6}$$

There is a relation with the one-dimensional variant $S'_{u_g u_g}(\Omega_x L_g)$ of the PSD function which we've used in earlier chapters. To find it from the above equation, we apply the inverse Fourier transform for the parameter $(\Omega_y L_g)$. We thus have

$$S'_{u_g u_g}(\Omega_x L_g) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{u_g u_g}(\Omega_x L_g, \Omega_y L_g)\, d(\Omega_y L_g). \tag{1.7}$$

The same relation holds for $S'_{v_g v_g}(\Omega_x L_g)$ and $S'_{w_g w_g}(\Omega_x L_g)$.

The turbulence field resulting from the PSD function can be seen as a superpositioning of multiple turbulence fields. Each turbulence field has spatial frequencies Ω_x and Ω_y and corresponding wavelengths $\lambda_x = 2\pi/\Omega_x$ and $\lambda_y = 2\pi/\Omega_y$. The turbulence velocity is now given by

$$u_g = u_{g_{max}} \mathrm{Re}\left(e^{j(\Omega_x x + \Omega_y y)}\right), \tag{1.8}$$

with the same for v_g and w_g. The above function can be seen as a sinusoid wave in two-dimensional space. The direction in which the waves 'run' is then given by $\arctan(\Omega_y/\Omega_x)$.

2 Finding the asymmetric forces and moments

2.1 Asymmetric forces and moments caused by longitudinal gusts u_g

To investigate the aircraft response to turbulence, we need to look at the forces and moments caused by turbulence. We'll do that now. First, we will examine the asymmetric forces and moments caused by longitudinal gusts u_g. For this, we use equation (1.8). In fact, we rewrite it to

$$u_g(x,y) = u_{g_{max}}\cos(\Omega_x x)\cos(\Omega_y y) + u_{g_{max}}\sin(\Omega_x x)\sin(\Omega_y y) = u_{g_1}(x,y) + u_{g_2}(x,y). \tag{2.1}$$

The first part $u_{g_1}(x,y)$ of the above equation is symmetric. It will thus not cause any asymmetric forces and moments. So, we will only examine the asymmetric function $u_{g_2}(x,y)$. To do this, we look at a small strip of the wing. For this small strip, we calculate the change in lift dL. We can then integrate $y\,dL$ over the entire wing to calculate the rolling moment caused by the gust. This gives us the coefficient of rolling motion due to gust

$$C_{l_g} = C_{l_{u_g}}\left(\Omega_y \frac{b}{2}\right)\hat{u}_g, \qquad \text{where} \qquad C_{l_{u_g}}\left(\Omega_y \frac{b}{2}\right) = -\frac{4}{Sb}\int_0^{\frac{b}{2}} c_l c \sin(\Omega_y y) y\, dy. \tag{2.2}$$

(Keep in mind that c_l and c also still depend on y.) The rolling moment due to turbulence C_{l_g} is similar to the rolling moment due to a yawing motion $C_{l_{r_w}}$ caused by the wing. In fact, we can relate the two parameters through $C_{l_{u_g}}\left(\Omega_y \frac{b}{2}\right)$ according to

$$C_{l_{u_g}}\left(\Omega_y \frac{b}{2}\right) = -C_{l_{r_w}} h\left(\Omega_y \frac{b}{2}\right), \qquad \text{where} \qquad h\left(\Omega_y \frac{b}{2}\right) = \frac{b}{2}\frac{\int_0^{\frac{b}{2}} c_l c \sin(\Omega_y y) y\, dy}{\int_0^{\frac{b}{2}} c_l c y^2\, dy}. \tag{2.3}$$

Determining the yawing coefficient due to longitudinal gust C_{n_g} goes in a similar way. We now find that

$$C_{n_g} = C_{n_{u_g}}\left(\Omega_y \frac{b}{2}\right)\hat{u}_g, \qquad \text{where} \qquad C_{n_{u_g}}\left(\Omega_y \frac{b}{2}\right) = -C_{n_{r_w}} h\left(\Omega_y \frac{b}{2}\right). \tag{2.4}$$

The function $h\left(\Omega_y \frac{b}{2}\right)$ is exactly the same as earlier. Finally, the lateral forces due to longitudinal gusts $C_{Y_{u_g}}$ are assumed to be negligible. So, $C_{Y_{u_g}} = 0$.

2.2 Asymmetric forces and moments caused by lateral gusts v_g

Let's examine the asymmetric forces and moments caused by lateral gusts v_g. We can split up v_g in a similar way as u_g. However, v_g is an asymmetric velocity. So this time we need to use the symmetric part $v_{g_1}(x,y) = v_{g_{max}} \cos(\Omega_x x)\cos(\Omega_y y)$ in our calculations. Also, we assume that v_g is approximately constant along the wing. Thus, $\cos(\Omega_y y) \approx 1$. We now define the **gust angle of sideslip** β_g as

$$\beta_g = \frac{v_g}{V} = \frac{v_{g_{max}} \cos(\Omega_x x)}{V}. \tag{2.5}$$

We would like to find the coefficients C_{Y_g}, C_{l_g} and C_{n_g}. Using a derivation similar to the one used in the previous chapter, we can find that

$$C_{Y_g} = \left(C_{Y_{\beta_g}} + C_{Y_{\dot{\beta}_g}} D_b\right)\beta_g, \qquad C_{l_g} = \left(C_{l_{\beta_g}} + C_{l_{\dot{\beta}_g}} D_b\right)\beta_g \quad \text{and} \quad C_{n_g} = \left(C_{n_{\beta_g}} + C_{n_{\dot{\beta}_g}} D_b\right)\beta_g. \tag{2.6}$$

Also, like in the previous chapter, we have

$$C_{Y_{\beta_g}} = C_{Y_\beta}, \qquad C_{l_{\beta_g}} = C_{l_\beta} \qquad \text{and} \qquad C_{n_{\beta_g}} = C_{n_\beta}. \tag{2.7}$$

The other three coefficients can, also analagous to the previous chapter, be approximated using

$$C_{Y_{\dot{\beta}_g}} = C_{Y_{\dot{\beta}}} + C_{Y_r}, \qquad C_{l_{\dot{\beta}_g}} = C_{l_{\dot{\beta}}} + C_{l_r} \qquad \text{and} \qquad C_{n_{\dot{\beta}_g}} = C_{n_{\dot{\beta}}} + C_{n_r}. \tag{2.8}$$

For aircraft with straight wings and a relatively small tailplane, these three derivatives are often negligible. So, for the sake of simplicity, we often simply use $C_{Y_{\dot{\beta}_g}} = C_{l_{\dot{\beta}_g}} = C_{n_{\dot{\beta}_g}} = 0$.

2.3 Asymmetric forces and moments caused by vertical gusts w_g

When examining vertical gusts, we use the symmetric part $w_{g_2}(x,y) = w_{g_{max}} \sin(\Omega_x x)\sin(\Omega_y y)$ of the vertical gust $w_g(x,y)$. The gust angle of attack is still defined as $\alpha_g(x,y) = w_g(x,y)/V$. The coefficients $C_{l_{\alpha_g}}$ and $C_{n_{\alpha_g}}$ are now very similar to the coefficients $C_{l_{u_g}}$ and $C_{n_{u_g}}$. In fact, we have

$$C_{l_g} = C_{l_{\alpha_g}}\left(\Omega_y \frac{b}{2}\right)\alpha_g \qquad \text{and} \qquad C_{n_g} = C_{n_{\alpha_g}}\left(\Omega_y \frac{b}{2}\right)\alpha_g. \tag{2.9}$$

Here, the functions $C_{l_{\alpha_g}}\left(\Omega_y \frac{b}{2}\right)$ and $C_{n_{\alpha_g}}\left(\Omega_y \frac{b}{2}\right)$ are similar to the functions $C_{l_{u_g}}\left(\Omega_y \frac{b}{2}\right)$ and $C_{n_{u_g}}\left(\Omega_y \frac{b}{2}\right)$, respectively. The above coefficients can also be related to the coefficients for a rolling motion, according to

$$C_{l_{\alpha_g}}\left(\Omega_y \frac{b}{2}\right) = C_{l_{p_w}} h\left(\Omega_y \frac{b}{2}\right) \qquad \text{and} \qquad C_{n_{\alpha_g}}\left(\Omega_y \frac{b}{2}\right) = C_{n_{p_w}} h\left(\Omega_y \frac{b}{2}\right). \tag{2.10}$$

Finally, we assume that the side force due to α_g is negligible. So, $C_{Y_{\alpha_g}} = 0$.

2.4 Alternative derivation of v_g coefficients

There is an alternative way to derive the coefficients $C_{Y_{\beta_g}}$, $C_{l_{\beta_g}}$, $C_{n_{\beta_g}}$, $C_{Y_{\dot{\beta}_g}}$, $C_{l_{\dot{\beta}_g}}$ and $C_{n_{\dot{\beta}_g}}$. In this method, we make use of the gust penetration effect, just like we did in the previous chapter. First, a gust β_g hits the wing and the fuselage. (We assume that the aircraft CG and the wing aerodynamic center coincide.) A time $\tau = l_v/V$ later, the gust hits the vertical tailplane. ($l_v \approx x_v - x_{cg}$ is the distance between the wing and the aircraft CG.) However, the gust at the vertical tailplane has a magnitude

$$\beta_{v_g} = \left(1 - \frac{\partial\sigma}{\partial\beta}\right)\beta_g, \tag{2.11}$$

with σ the **sidewash** caused by the wing/fuselage. We can now approximate

$$C_{Y_g} = C_{Y_{\beta_g}} \beta_g + C_{Y_{\dot{\beta}_g}} D_b \beta_g. \tag{2.12}$$

Based on the above data, the coefficients $C_{Y_{\beta_g}}$ and $C_{Y_{\dot{\beta}_g}}$ can be determined. They are

$$C_{Y_{\beta_g}} = C_{Y_{f_\beta}} - C_{Y_{v_\alpha}} \left(\frac{V_v}{V}\right)^2 \frac{S_v}{S} \left(1 - \frac{\partial \sigma}{\partial \beta}\right), \tag{2.13}$$

$$C_{Y_{\dot{\beta}_g}} = C_{Y_{v_\alpha}} \left(\frac{V_v}{V}\right)^2 \frac{S_v}{S} \left(1 - \frac{\partial \sigma}{\partial \beta}\right). \tag{2.14}$$

By the way, $C_{Y_{f_\beta}}$ is the contribution of the fuselage to C_{Y_β}. It is used because the wing hardly effects the coefficient C_{Y_β}. A similar expression as the one above can be derived for C_{l_g} and C_{n_g}. However, for these two parameters, the coefficients are given by

$$C_{Y_{\beta_g}} = C_{l_{w_\beta}} - C_{Y_{v_\alpha}} \left(\frac{V_v}{V}\right)^2 \frac{S_v}{S} \left(\frac{z_v - z_{cg}}{b} \cos\alpha - \frac{x_v - x_{cg}}{\sin}\alpha\right) \left(1 - \frac{\partial \sigma}{\partial \beta}\right), \tag{2.15}$$

$$C_{Y_{\dot{\beta}_g}} = C_{Y_{v_\alpha}} \left(\frac{V_v}{V}\right)^2 \frac{S_v}{S} \left(\frac{z_v - z_{cg}}{b} \cos\alpha - \frac{x_v - x_{cg}}{\sin}\alpha\right) \left(1 - \frac{\partial \sigma}{\partial \beta}\right), \tag{2.16}$$

$$C_{Y_{\beta_g}} = C_{n_{f_\beta}} - C_{Y_{v_\alpha}} \left(\frac{V_v}{V}\right)^2 \frac{S_v}{S} \left(\frac{x_v - x_{cg}}{b} \cos\alpha - \frac{z_v - z_{cg}}{\sin}\alpha\right) \left(1 - \frac{\partial \sigma}{\partial \beta}\right), \tag{2.17}$$

$$C_{Y_{\dot{\beta}_g}} = C_{Y_{v_\alpha}} \left(\frac{V_v}{V}\right)^2 \frac{S_v}{S} \left(\frac{x_v - x_{cg}}{b} \cos\alpha - \frac{z_v - z_{cg}}{\sin}\alpha\right) \left(1 - \frac{\partial \sigma}{\partial \beta}\right). \tag{2.18}$$

3 The PSD function and the asymmetric equations of motion

3.1 The PSD function of force and moment coefficients

We now know how to find the force and moment coefficients that are acting on the aircraft. The next step is to find the PSD functions of them. The method for this is mostly the same for all coefficients. But we're going to demonstrate it on C_{l_g}. Equation (2.2) now implies that

$$S_{C_{l_g} C_{l_g}}(\Omega_x L_g, \Omega_y L_g, B) = C_{l_{u_g}}^2 (\Omega_y L_g B)\, S_{\hat{u}_g \hat{u}_g}(\Omega_x L_g, \Omega_y L_g). \tag{3.1}$$

In the above equation, we have defined another dimensionless coefficient: $B = \frac{b}{2L_g}$. Usually, the coefficient B is known. So then the above equation is two-dimensional. It would, however, be preferable for the equation to be one-dimensional. We can make it one-dimensional using

$$S_{C_{l_g} C_{l_g}}(\Omega_x L_g, B) = C_{l_{rw}}^2 \int_0^\infty h^2 (\Omega_y L_g B)\, S_{\hat{u}_g \hat{u}_g}(\Omega_x L_g, \Omega_y L_g)\, d(\Omega_y L_g) = C_{l_{rw}}^2 I_{\hat{u}_g \hat{u}_g}(\Omega_x L_g, B), \tag{3.2}$$

where the **effective one-dimensional PSD function** $I_{\hat{u}_g \hat{u}_g}(\Omega_x L_g, B)$ is defined as the integral in the above equation. The variance of the force coefficient C_{l_g} can now be found using

$$\mathrm{E}\left\{C_{l_g}^2\right\} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{C_{l_g} C_{l_g}}(\Omega_x L_g, B)\, d(\Omega_x L_g) = \frac{1}{2\pi} C_{l_{rw}}^2 \int_{-\infty}^{+\infty} I_{\hat{u}_g \hat{u}_g}(\Omega_x L_g, B)\, d(\Omega_x L_g). \tag{3.3}$$

3.2 Approximating the one-dimensional PSD function

The above method can be simplified. It can be assumed that the product $c_l c$ has a negligible influence on the value of $h\left(\Omega_y L_g B\right)$. In this case, $h\left(\Omega_y L_g B\right)$ can be solved analytically. We then have

$$h\left(\Omega_y L_g B\right) = \frac{b}{2} \frac{\int_0^{\frac{b}{2}} \sin(\Omega_y y) y \, dy}{\int_0^{\frac{b}{2}} y^2 \, dy} = 3 \frac{\sin(\Omega_y L_g B) - (\Omega_y L_g B) \cos(\Omega_y L_g B)}{(\Omega_y L_g B)^2}. \tag{3.4}$$

Based on this, the function $I_{\hat{u}_g \hat{u}_g}(\Omega_x L_g, B)$, and similarly the function $I_{\alpha_g \alpha_g}(\Omega_x L_g, B)$ as well, can be approximated. This is done using the equations

$$I_{\hat{u}_g \hat{u}_g}(\Omega_x L_g, B) = I_{\hat{u}_g \hat{u}_g}(0, B) \frac{1 + \tau_3^2 \Omega_x^2 L_g^2}{(1 + \tau_1^2 \Omega_x^2 L_g^2)(1 + \tau_2^2 \Omega_x^2 L_g^2)}, \tag{3.5}$$

$$I_{\alpha_g \alpha_g}(\Omega_x L_g, B) = I_{\alpha_g \alpha_g}(0, B) \frac{1 + \tau_6^2 \Omega_x^2 L_g^2}{(1 + \tau_4^2 \Omega_x^2 L_g^2)(1 + \tau_5^2 \Omega_x^2 L_g^2)}. \tag{3.6}$$

It is important to remember that the above equations are approximations. But they do prove to be quite acceptable approximations. The constants τ_1 to τ_6 in the above equation depend on B. Their values can be found in tables.

3.3 The asymmetric equations of motion for an aircraft in turbulence

Let's derive the asymmetric equations of motion of an aircraft, when turbulence is involved. Based on the assumptions that have been made, the relations that have been found and the coefficients that have been calculated, we can find that

$$\begin{bmatrix} C_{Y_\beta} - 2\mu_b D_b & C_L & C_{Y_p} & C_{Y_r} - 4\mu_b \\ 0 & -\frac{1}{2} D_b & 1 & 0 \\ C_{l_\beta} & 0 & C_{l_p} - 4\mu_b K_X^2 D_b & C_{l_r} + 4\mu_b K_{XZ} D_b \\ C_{n_\beta} & 0 & C_{n_p} + 4\mu_b K_{XZ} D_b & C_{n_r} - 4\mu_b K_Z^2 D_b \end{bmatrix} \begin{bmatrix} \beta \\ \varphi \\ \frac{pb}{2V} \\ \frac{rb}{2V} \end{bmatrix} =$$

$$- \begin{bmatrix} 0 & C_{Y_{\delta_r}} & 0 & C_{Y_\beta} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ C_{l_{\delta_a}} & C_{l_{\delta_r}} & -C_{l_{r_w}} & C_{l_\beta} & C_{l_{p_w}} \\ C_{n_{\delta_a}} & C_{n_{\delta_r}} & -C_{n_{r_w}} & C_{n_\beta} & C_{n_{p_w}} \end{bmatrix} \begin{bmatrix} \delta_a \\ \delta_r \\ \hat{u}_g \\ \beta_g \\ \alpha_g \end{bmatrix}. \tag{3.7}$$

Just like in the previous chapter, the above equations of motion can be put in state space form. For that, you would have to use the definition $D_b = \frac{b}{V} \frac{d}{dt}$. Also, the equations can be combined with a state space form of the gust filters. But again, we won't discuss that here.