4

Structures

To construct is to put together *structural elements* to create a structure, a *cohesive whole* that meets previously-determined demands. The structural elements are linked to one another by means of *joints*. The structure is linked to its normally fixed environment through *supports*. In this chapter, we will address a number of types of structural elements, joints, supports and structures. We will consider only two-dimensional structures.

In addition to the user requirements, which relate to the function of the structure, there are also mechanical demands (strength and stiffness), requirements relating to the structure itself (such as rate of construction, availability of the material), design requirements (representation), requirements relating to the physical components of the structure (such as climate control, warmth and sound insulation), and last but not least, economic requirements. Any contradictory requirements have to be weighed against one another wisely. To do so, a methodical approach is needed. Designing a structure is therefore anything but a random process.

As far as the mechanical section of a structure is concerned (strength and stiffness), an attempt must always be made to make the most efficient use of the specific properties of the *structural elements*.

In Section 4.1, we distinguish between a *particle element*, a *line element*, a *surface element*, and a *spatial element*.

Section 4.2 addresses the joints between structural elements, and more particularly the *hinged joint* and the *rigid joint*. We will also look at the total number of unknown *interaction forces*, so that at a later stage we can identify whether or not the forces in a structure can or cannot be calculated using solely the equilibrium equations.

As far as supports are concerned, we will look at the number of *degrees of freedom* (possible movement) in the support, and at the *support reactions* that a support can generate. Section 4.3 looks at *bar supports*, *roller supports*, *hinged supports*, and *fixed supports*.

Many spatial structures can be seen as a system of planar structures constructed from line elements. Investigating such planar frames is therefore certainly worth the effort. Based on matters such as the type of loading, the nature of the joints, and the external appearance, Section 4.4 defines a number of planar frames.

Structures are supported in such a way that all free movement is restricted. This type of structure is referred to as a *kinematically determinate* or *immovable* structure. If there are too few supports, or if they are not applied effectively, the structure, or a part of it, will have a degree of freedom that cannot be restrained. The structure is no longer immmovable. The structure is then said to be *kinematically indeterminate*, or is referred to as a *mechanism*.

If it is possible to define all the support reactions and interaction forces in a structure using solely equilibrium equations, it is called a *statically determinate structure*. If there are too many unknown forces to determine them based on the equilibrium, the structure is said to be *statically indeterminate*. To determine the forces in a statically indeterminate structure, the deformation of the structure must be taken into account, which is beyond the scope of this book.

The last part of the chapter, Section 4.5, looks at the *kinematic/static (in)determinacy* of planar structures.

4.1 Structural elements

As far as structural mechanics is concerned (strength and stiffness), one always tries to make the most efficient use of the specific properties of a limited number of building blocks, or *structural elements*. The way of modelling in structural mechanics allows one to distinguish the following four types of structural elements:

- *Particle element* (see Figure 4.1a)
  All dimensions of the element are negligibly small with respect to those of other elements.
- *Line element* (see Figure 4.1b)
  Two of the dimensions of the element (those of the cross-section) are considerably smaller than the third dimension (the length).
- *Surface element* (see Figure 4.1c)
  One dimension of the element (the thickness) is considerably smaller than the other two dimensions (the length and width).
- *Spatial element* (see Figure 4.1d)
  All the dimensions of the element are of the same order of magnitude as those of other elements and are therefore not negligible.

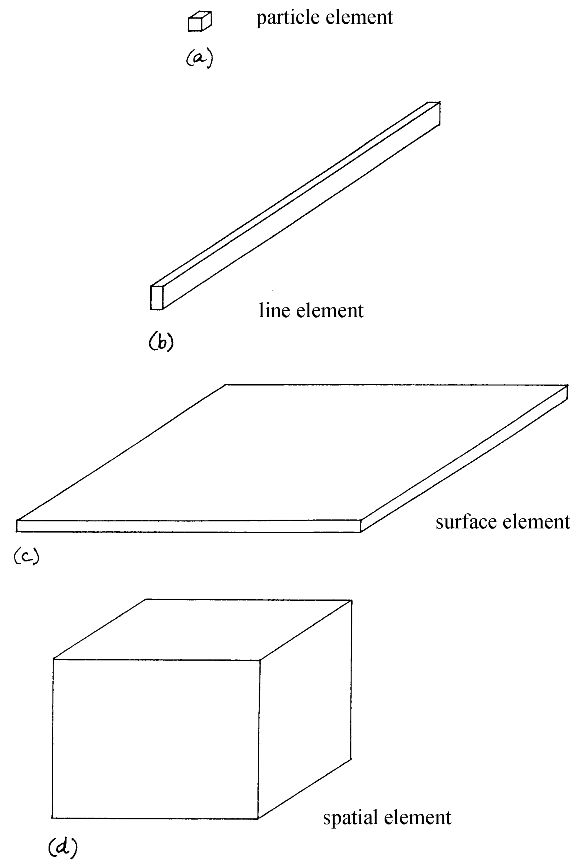


Figure 4.1 Structural elements: (a) particle element, (b) line element, (c) surface element (d) spatial element.

4.1.1 Particle element

A *particle element* (Figure 4.1a) is a zero-dimensional structural element: all dimensions are negligibly small with respect to those of other elements. The dimensions of the element play a subordinate role. This is addressed further in Section 4.1.5. Also, see Section 4.2, in which particle elements are used for modelling hinged and fixed joints.

4.1.2 Line element

A *line element* (Figure 4.1b) is a one-dimensional structural element: two of the dimensions of the element (those of the cross-section) are significantly smaller than the third dimension (the length).

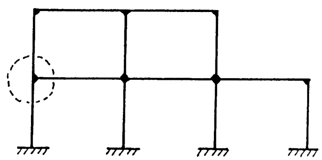


Figure 4.2 The model of a structure made of line elements; the joints between the line elements are particle elements (joints).

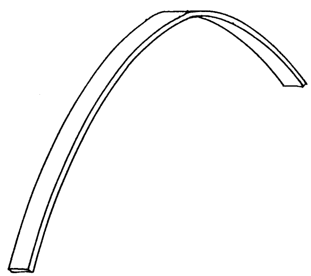


Figure 4.3 A (rigid) curved line element is called an arch.

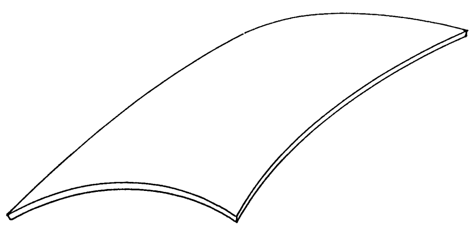


Figure 4.4 A (rigid) curved surface element is called a shell.

By using simplified assumptions in the smallest directions (those of the cross-section), all the properties of the line element can be assigned to a single line, the so-called *axis* of the line element. In mechanics, a bar which in reality is three-dimensional, can often be modelled by a one-dimensional member; the member is depicted by a single line: the *bar axis*.

Figure 4.2 represents the mechanical diagram of a structure constructed from line elements.

Line elements with a straight axis as known by a wide range of names, such as *bar*, *beam*, *joist*, *girder*, *column*, *post* and *member*. The nomenclature sometimes relates to the position of the line element in the structure: horizontal (beam, joist, girder) or vertical (column, post, stay). Hereafter, we will refer to a line element in general as a *member*.

An (inflexible) curved line element is known as an *arch*, see Figure 4.3. A line element without a particular shape is a *cable*: cables adapt to the loading.

4.1.3 Surface element

A *surface element* (Figure 4.1c) is a two-dimensional structural element: one dimension (the thickness) is small with respect to the other two dimensions (the length and width).

The behaviour of this element, which in reality is three-dimensional, can be described sufficiently accurately by means of a two-dimensional model by making simplified assumptions with respect to the thickness. In the two-dimensional model, all the properties of the element are assigned to a plane. This *reference plane* is sometimes also called the *central plane*. In a mechanical diagram, only the reference plane (without thickness) of the surface element is depicted.

With *plates*, the reference plane is a flat plane. With *shells*, the reference plane is curved (see Figure 4.4). If the reference plane does not have its own shape, but adapts to the loading, it is called a *membrane* or *film*. Plates are also given other names, such as *slab*, *floor*, *wall* and *disc*.

4.1.4 Spatial element

A *spatial element* (Figure 4.1d) is a three-dimensional structural element: all the dimensions are of the same order of magnitude as those of other elements. In a more general sense, a spatial element can be defined as an element for which the model of a particle, line, or surface element does not suffice.

4.1.5 Modelling structural elements

It was stated above that the difference between the four kinds of structural elements is the result of the *modelling method*, and that this strongly depends on the information sought by the research or calculation.

To illustrate, refer to the concrete bar structure and its model in Figure 4.2. The model has been created to investigate the mechanical behaviour of the structure as a whole. The lines in the diagram represent the beams and columns, which have been schematised as *line elements*. The beams and columns are rigidly joined to one another. In the model, these joints are represented as *particle elements* (capable of transferring both forces as well as concentrated couples).

In Figure 4.5, the circled joint between the beam and outer column has been elaborated. Further investigation shows that there is a complex interplay of forces in the joint; the concrete transfers the compressive forces, and the reinforcement bars transfer the tensile forces. This type of investigation is critical for detailed modelling of the joint. Can the concrete transfer the compressive forces; how much reinforcement is required for transferring the tensile forces, and where should this reinforcement be placed?

When we are studying the behaviour of a structure as a whole, we can

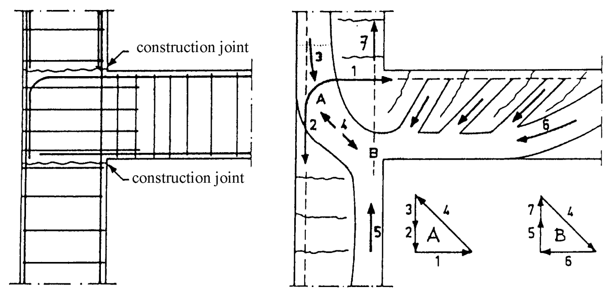


Figure 4.5 In a detailed study, a joint should be modelled as a spatial structural element.

model joints as particle elements. When we are studying detailed behaviour of a joint, it must be modelled as a spatial element.

In principle, all structural elements are three-dimensional, and therefore are spatial elements; modelling them as particle, line, or surface elements always means that some information and accuracy is lost. This is acceptable as long as the model of the structure gives results close enough to the actual structure. If there is too much discrepancy, the model will have to be modified to include more detail.

The justification of the models used below derives from satisfactory results obtained over many years.

4.2 Joints between structural elements

Two bodies can be joined together in a wide variety of ways. For joints between structural elements, in the same plane, there are two kinds:

- *Hinged joints* (hinges);
- *Fixed joints* (entirely rigid or infinitely rigid joints).

In a *hinged joint*, or *hinge*, the joined parts cannot translate with respect to one another, but can rotate freely with respect to one another. In a *rigid joint*, the joined parts cannot translate with respect to one another, nor can they rotate with respect to one another. The forces that the structural elements exert on one another in a joint are referred to as *interaction forces* or *joint forces*.

Hinges will always have a certain amount of resistance to rotation, even if only due to the occurrence of friction. If this resistance is limited, the joint can be idealised as a frictionless hinge. When the resistance to rotation in a joint is very large, the joint tends to be represented as infinitely stiff. The reality will always lie between these two extremes.

Spring joints are joints in which the magnitude of the acting interaction forces is related to the deformation in the joint. These will not be covered here.

4.2.1 Hinged joints

In Figure 4.6a, the bodies (1) en (2) are joined by a hinge at S. In the figure, the *hinge* is depicted as a small open circle. The bodies are able to rotate freely with respect to one another about the hinge S, but cannot translate with respect to one another. The bodies can exert only forces on one another at S; they cannot exert any couple.

Dissecting a body into its joints, and at the same time depicting the forces that are exerted on the body in the joints, is referred to as *isolating* the body; the diagram so formed is called the *free body diagram*.

In Figure 4.6b, both bodies have been isolated from one another and the forces that the bodies exert on one another in the joint are shown. Based on Newton's third law of action and reaction, these *interaction forces* are equal and opposite (see Section 1.4.1). In other words: $S^{(1)} = S^{(2)} = S$.

In the hinged joint shown, there are two unknowns: the magnitude of the *hinge force*[1] S and the direction of its line of action. We could also select the two components S_{h} and S_{v} as unknowns.

A joint comes about by some means of joining. In hinged joints, this could be a pin or axis, perpendicular to the plane shown, about which both bodies can rotate, and through which they can exert forces on one another.

In Figure 4.6c, the pin has also been isolated in S for both bodies (1) and (2). The pin is seen as a *particle*, even though it is shown as a body in

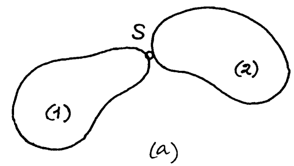


Figure 4.6a (a) Two bodies joined in a hinge at S.

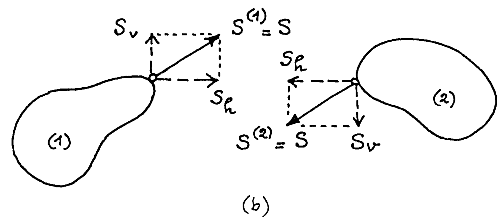


Figure 4.6b The forces acting on each body at the hinged joint S.

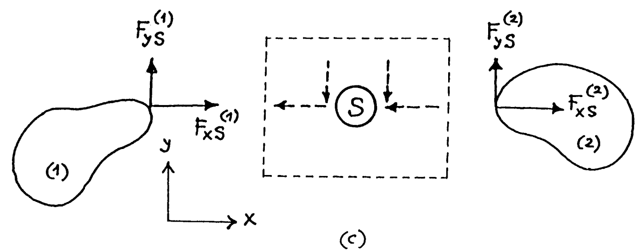
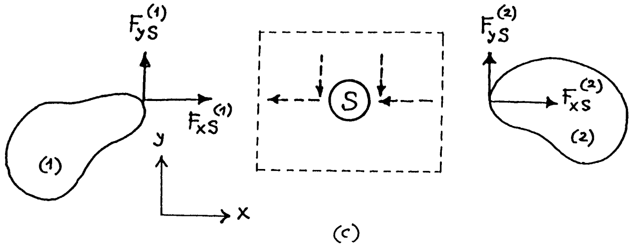


Figure 4.6c In hinged joint S, not only have both bodies been isolated, but so has joint S which should be seen as a particle element. The forces shown are the interaction forces between the two bodies and joint S.

[1] Although we are talking about a hinge force S (singular) in reality it concerns a *pair of forces* (plural).



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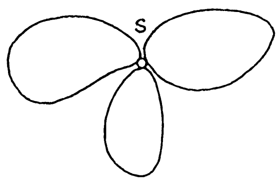

Figure 4.7 Three bodies hinged together at joint S.

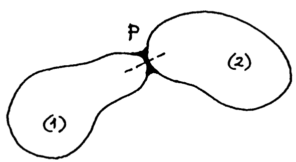

Figure 4.8a Two bodies joined rigidly at P.

the figure, in this case a *circle*. This circle is also known as the *connection* between the bodies (1) and (2).

Imagine $F^{(1)}_{x;\mathrm{S}}$; $F^{(1)}_{y;\mathrm{S}}$ and $F^{(2)}_{x;\mathrm{S}}$; $F^{(2)}_{y;\mathrm{S}}$ are the forces exerted through the connection (pin S) in the xy coordinate system given on body (1) and body (2) respectively.[1] This makes four unknown forces. Based on Newton's third law, equal and opposite forces are exerted on the connection. If the system of bodies is in equilibrium, then each of the parts must be in equilibrium, including the connection. The force equilibrium of the connection therefore gives

$$\sum F_x = -F^{(1)}_{x;\mathrm{S}} - F^{(2)}_{x;\mathrm{S}} = 0,$$

$$\sum F_y = -F^{(1)}_{y;\mathrm{S}} - F^{(2)}_{y;\mathrm{S}} = 0.$$

There are therefore two linear relationships between the four unknown forces $F^{(1)}_{x;\mathrm{S}}$, $F^{(1)}_{y;\mathrm{S}}$, $F^{(2)}_{x;\mathrm{S}}$ and $F^{(2)}_{y;\mathrm{S}}$, so that two of the four unknowns can be eliminated, leaving two independent interaction forces in the hinged joint:

$$F^{(1)}_{x;\mathrm{S}} = -F^{(2)}_{x;\mathrm{S}} \; (= S_{\mathrm{h}}),$$

$$F^{(1)}_{y;\mathrm{S}} = -F^{(2)}_{y;\mathrm{S}} \; (= S_{\mathrm{v}}).$$

The formal approach described here to determine the number of unknown (independent) interaction forces in a hinged joint seems rather complicated if you compare it to the simple approach in Figure 4.6b. The formal approach, however, offers clear benefits if more than two bodies are joined together at the hinge.

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[1] The upper index indicates the body on which the force is exerted.

For example, in Figure 4.7, three bodies are joined at a hinge S. Six interaction forces act on the hinge. The force equilibrium of the hinge gives two linear relationships between these six unknowns, so that we are left with $6 - 2 = 4$ independent interaction forces in S.

4.2.2 Fixed joints

Fixed joints are also referred to as *rigid joints*.[1]

The two bodies (1) and (2) in Figure 4.8a are rigidly joined at P. The fixed joint is depicted in the figure as a thickening at P. The joint at P ensures that the bodies cannot translate nor rotate with respect to one another. The joint could be realised as a pin that, in the plane of the figure, is stuck into both bodies.

Both bodies have been isolated in Figure 4.8b. Three unknown interaction forces P_h, P_v and P_m are exerted in P. Although P_m stands for the two equal and opposite *couples*, it is referred to as a *force* when *generalising*.

If more than two bodies are rigidly connected at a joint, as in Figure 4.9a, the easiest way of finding the number of unknown (independent) interaction forces is the formal approach, in which the joint is also isolated. The joint is seen as a *particle* that in addition to forces can now also transfer *concentrated couples*.

In Figure 4.9b, the bodies (1), (2) and (3) and the *joint* have been isolated. Since it can transfer couples, the connection has been depicted as a *square*.

In the xy coordinate system shown, $F^{(e)}_{x;\mathrm{P}}$, $F^{(e)}_{y;\mathrm{P}}$ and $F^{(e)}_{z;\mathrm{P}}$ are the (generalised) forces that are exerted through the connection at P on body (e) ($e = 1, 2, 3$). Based on Newton's third law, equal and opposite forces are exerted on the connection, making a total of nine unknown forces. If the

1 This is actually an incomplete definition. It is preferable to refer to an *infinitely stiff joint*.

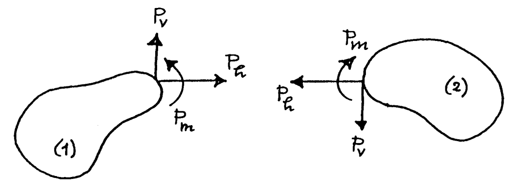


Figure 4.8b The three interaction forces between both bodies isolated at P.

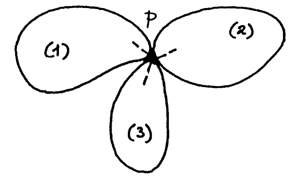


Figure 4.9a Three bodies rigidly conneced at P.

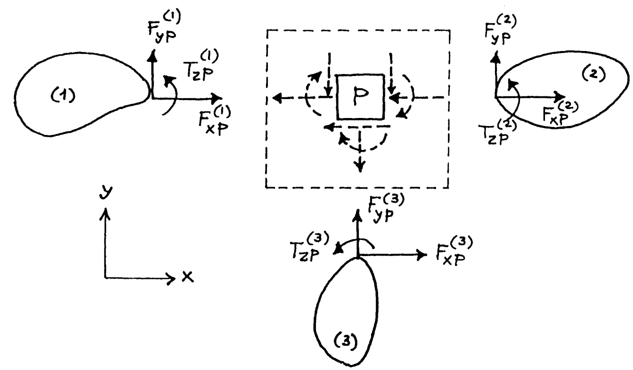


Figure 4.9b The interaction forces between the three bodies and the joint P shown as a particle element.

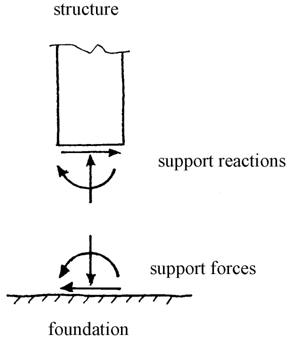


Figure 4.10 The interaction forces in a support are pairs of forces. The forces that act on the foundation are called support forces or support actions, and the equal and opposite forces acting on the structure are called support reactions.

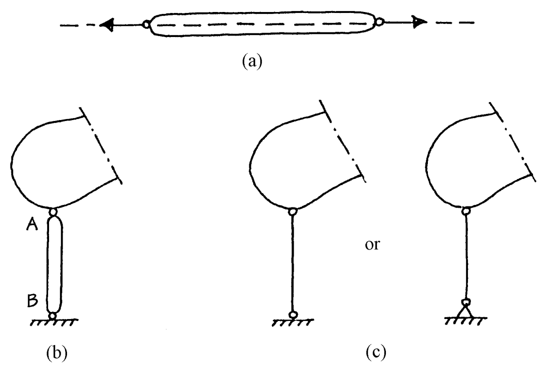


Figure 4.11 (a) A two-force member is a straight bar that is joined at both ends with a hinge to its surroundings and is loaded only by forces at its ends. A two-force member can transfer only forces of which the line of action passes through both hinges. (b) A bar support. (c) Model of a bar support.

system of bodies is in equilibrium, the connection is also in equilibrium. There are three equilibrium equations for the connection: two for the force equilibrium and one for the moment equilibrium. These equilibrium equations give three linear relationships between the nine unknowns, so that $9 - 3 = 6$ independent interaction forces remain at the fixed joint between the three bodies.

4.3 Supports

Most structures are not free-floating, but are joined to a *fixed* environment. The joints between the structure and its fixed environment are called *supports*.

The interaction forces that act in the supports on the structure are known as *support reactions*. They act in the direction in which displacement of the structure is prevented. The forces that the structure exerts on the supports (for example on the foundation) are called *support forces* or *support actions*. The support forces are equal and opposite to the support reactions (see Figure 4.10).

We will look at four types of supports:
- bar supports;
- roller supports;
- hinged supports;
- (fully) fixed supports.

4.3.1 Bar supports

A *two-force member* is a straight bar which is joined to its environment at both ends by a hinge, and is loaded only by forces at the ends. From the moment equilibrium it follows that such members can transfer forces only when the line of action passes through both hinges (see Figure 4.11a).

In a *bar support* the two-force member is used as a *link* between the structure and the immovable environment (see Figure 4.11b). Figure 4.11c is a model of the bar support: the bar support is depicted as a single line between the two hinges. The immovable environment is generally shown by means of a hatched area.

In Figure 4.12, the two-force member has been isolated at hinges A and B. The position of the two-force member (the line joining both hinges) fixes the line of action of the interaction forces F. Only the magnitude of F (with its sign for the correct direction) is unknown.

When the body moves, point A is forced to follow a circle with centre B by the two-force member (see Figure 4.13a). If the displacement remains very small with respect to the length of the two-force member (which is generally the case), then the arc is almost the same as the tangent at A to the circle (Figure 4.13b). Note that the diagram of the structure is much smaller than the actual structure, and also the displacement is strongly magnified in the diagram.

The bar support at A prevents displacement in the direction of the bar. Displacement in the direction perpendicular to the bar is free (Figure 4.13b), as is a rotation of the body about A.

Imagine that in the free displacement of the body, $u_{x;\mathrm{A}}$ and $u_{y;\mathrm{A}}$ are displacements of A in the x and y directions, and that $\varphi_{z;\mathrm{A}}$ is the rotation of the body about A. *Generalising*, the *rotation* is called *a motion*. For a bar support at A (with the bar in the y direction) the *generalised motions* are:

$$
\begin{aligned}
u_{x;\mathrm{A}} &= \text{unknown} && \text{(free motion)},\\
u_{y;\mathrm{A}} &= 0 && \text{(prescribed motion)},\\
\varphi_{z;\mathrm{A}} &= \text{unknown} && \text{(free motion)}.
\end{aligned}
$$

The bar support prevents free motion of point A by exerting forces on it.

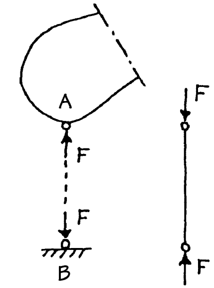


Figure 4.12 The two-force member, isolated from the body and support, with the interaction forces.

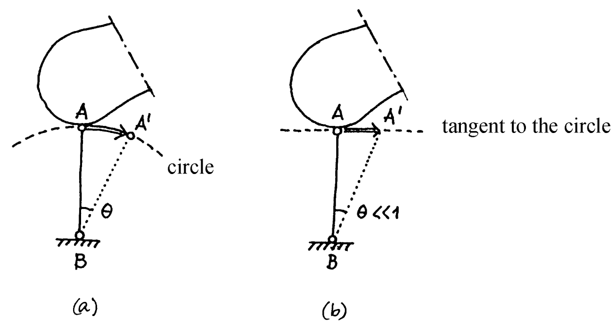
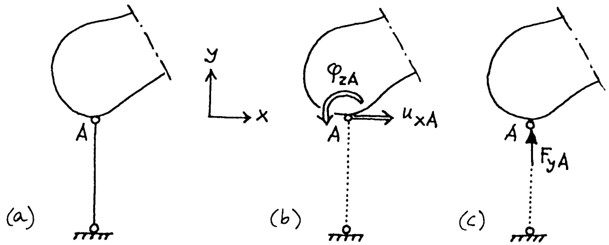


Figure 4.13 (a) If the body moves, the bar AB forces point A to follow a circular path with centre B. (b) If the motion remains small with respect to the length of the bar, the arc can be approximated by the tangent at A to the circle.



bar support degrees of freedom support reaction

Figure 4.14 (a) The bar support has (b) two degrees of freedom (a rotation and a displacement perpendicular to the bar) and gives (c) one support reaction (a force in the direction of the bar).

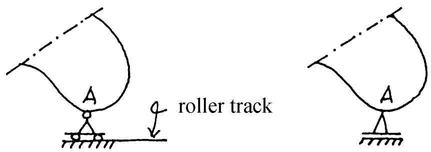
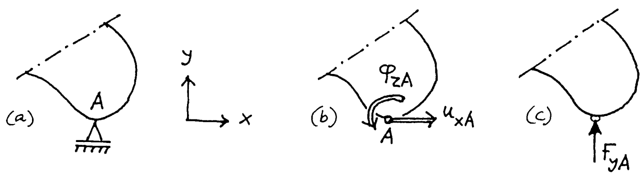


Figure 4.15 Model of a roller support.



roller support degrees of freedom support reaction

Figure 4.16 (a) The roller support has (b) two degrees of freedom (a rotation and a movement along the rolling surface) and generates (c) one support reaction (a force perpendicular to the roller track).

For the *generalised forces* at A in the given coordinate system:

$$F_{x;\mathrm{A}} = 0 \quad \text{(prescribed force)},$$
$$F_{y;\mathrm{A}} = \text{unknown} \quad \text{(free force)},$$
$$T_{z;\mathrm{A}} = 0 \quad \text{(prescribed force)}.$$

The free (freely adjustable) motions are called the *degrees of freedom* at the support; the free (freely adjustable) force is the *support reaction*. A bar support therefore has two degrees of freedom and generates one support reaction.

If a motion is prescribed, the associated force is unknown, and *vice versa*. This is true not only for bar supports but also for all other supports discussed below. The total number of degrees of freedom and support reactions is therefore always three for a support (in a plane). In Figure 4.14 the degrees of freedom and support reactions are shown. Sometimes, motions are depicted by means of open arrows, while forces are depicted by closed arrows.

4.3.2 Roller supports

Figure 4.15 is a schematic representation of a *roller support*. For a roller support at A, the body can move parallel to the so-called *roller track*, and can also rotate freely about A. Only motion of A perpendicular to the roller track is prevented; this is the direction in which the interaction force is exerted.

For the roller support in Figure 4.16, with the roller track parallel to the x axis, the following applies for the motion at A:

$$u_{x;\mathrm{A}} = \text{unknown} \quad \text{(free motion)},$$
$$u_{y;\mathrm{A}} = 0 \quad \text{(prescribed motion)},$$
$$\varphi_{z;\mathrm{A}} = \text{unknown} \quad \text{(free motion)}.$$

The following applies for the forces in A in the coordinate system given:

$$\begin{aligned} F_{x;\mathrm{A}} &= 0 && \text{(prescribed force),} \\ F_{y;\mathrm{A}} &= \text{unknown} && \text{(free force),} \\ T_{z;\mathrm{A}} &= 0 && \text{(prescribed force).} \end{aligned}$$

The roller support therefore has two degrees of freedom and generates one support reaction. Note the parallel with a bar support!

Figure 4.18 shows a steel roller support used in older bridge structures. Due to the continuous sideways movement, the roller can end up askew after a while. To prevent this happening, the roller is provided with a *tooth* structure on its sides (comparable to a cogwheel). In order to prevent displacement in the z direction, a *groove* is sometimes cut into the roller that fits over an open *ridge* in the *rail* and *bearing pedestal*.

This example of a steel roller support provides a good picture of how it works. Roller supports can be made of materials other than steel, but then as *sliding supports*. Examples include supports made of rubber or plastics (neoprene), occasionally in combination with Teflon to reduce friction.

The roller support shown can transfer only compressive forces and no tensile forces. This is not a problem as long as the loading generates only compressive forces in the support. Such a load could be, for example, the ever-present weight of the structure. Generally speaking, the weight of a structure, such as a bridge, is sufficiently large to ensure that the roller support is continuously loaded by compressive forces. If a roller support also has to be able to transfer tensile forces, special structural provisions have to be made.

It is assumed here that a roller support can transfer both tensile and compressive forces.

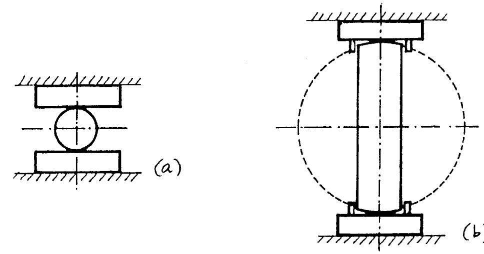


Figure 4.17 (a) An example of a simple roller support. (b) If the roller is large enough and the movements and rotations remain small, a large part of the roller can be omitted. The roller support changes into a bar support.

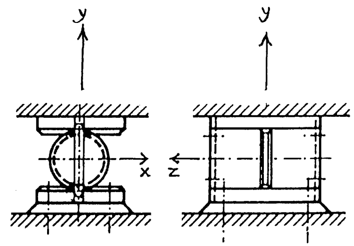
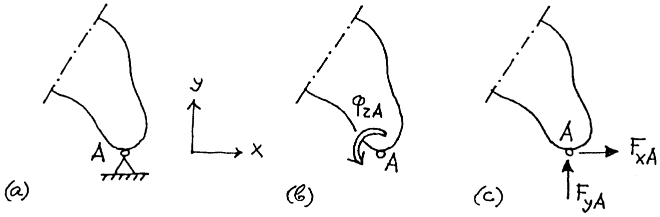


Figure 4.18 A steel roller support.



hinged support degree of freedom support reactions

Figure 4.19 (a) The hinged support has (b) one degree of freedom (a rotation) and generates (c) two support reactions (the two components of a force).

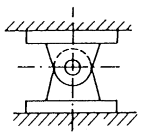


Figure 4.20 Simple example of a hinged support.

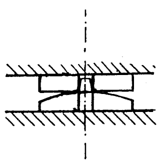


Figure 4.21 A steel hinged support, previously used in smaller bridges. The horizontal movement is prevented by the pin.

4.3.3 Hinged supports

A *hinged support* is a hinge between the structure and its immovable environment (see Section 4.2.1). A hinged support is modelled in Figure 4.19a (the open circle is often omitted). In a hinged support at A, the displacement of the body at A is prevented and the body can only rotate about A. The support cannot transfer a couple, but can transfer a force. The interaction force is unknown with respect to both magnitude and direction.

For the hinged support in Figure 4.19 with the coordinate system shown, the following applies for motion at A:

$$\begin{aligned} u_{x;\mathrm{A}} &= 0 && \text{(prescribed motion)},\\ u_{y;\mathrm{A}} &= 0 && \text{(prescribed motion)},\\ \varphi_{z;\mathrm{A}} &= \text{unknown} && \text{(free motion)}. \end{aligned}$$

and for the forces at A:

$$\begin{aligned} F_{x;\mathrm{A}} &= \text{unknown} && \text{(free force)},\\ F_{y;\mathrm{A}} &= \text{unknown} && \text{(free force)},\\ T_{z;\mathrm{A}} &= 0 && \text{(prescribed force)}. \end{aligned}$$

A hinged support therefore has one degree of freedom (a rotation) and generates two support reactions (the two components of a force).

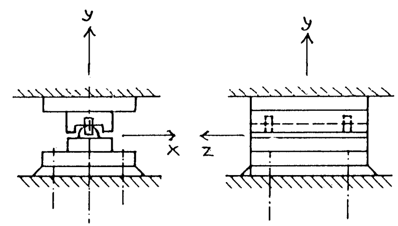


Figure 4.22 A steel hinged support as still found in many bridges today.

Figure 4.20 is a good example of a hinged support. Figure 4.21 shows how a steel hinged support can be used in small bridges. Horizontal motion is prevented by a *pin*. The steel hinged support in Figure 4.22 can transfer large forces and is an example of what is used in larger bridges. Like roller supports, hinged supports can be made from materials other than steel, or from a combination of materials Although the supports in Figures 4.21 and 4.22 can transfer only compressive forces, it is assumed below that hinged supports can also transfer tensile forces.

4.3.4 Fixed supports

A fixed support is an infinitely stiff or rigid joint between a body and its environment, see also Section 4.2.2. Figure 4.23a is a model of a fixed support (the dotted line is generally omitted). At A, the fixed support prevents both the displacement and rotation of the body. In fixed supports, all motion is prescribed: fixed supports therefore have no degrees of freedom. A fixed support has three support reactions, see Figure 4.23: two forces and a so-called *fixed-end moment*.

The balcony (cantilever beam) in Figure 4.24a is an example of a fixed supported structure. Another example is the support in Figure 4.24b of a concrete column on a concrete foundation, constructed as a single, monolithic whole.

In many cases, a fixed support will not fully prevent rotation. Such a support is incomplete and is referred to as a *spring support* if the magnitude of the rotation is related to the magnitude of the fixed-end moment. We will always refer to a *fully fixed support* below.

4.3.5 Free support

Frequently, a beam, such as a floor beam, is placed directly on the masonry or concrete. Here, a roller support or hinged support described above is not

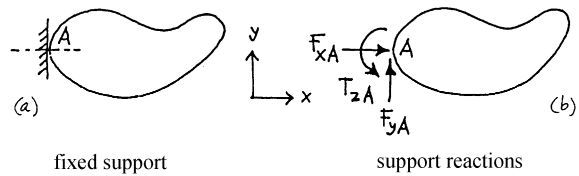


Figure 4.23 (a) The fixed support has no degrees of freedom and generates (b) three support reactions (the two components of a force and a fixed-end moment).

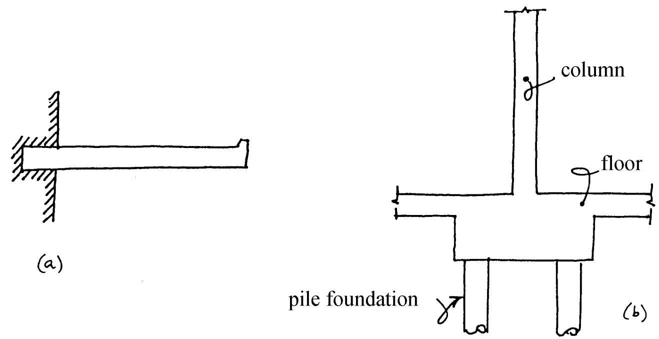


Figure 4.24 Examples of fixed supports: (a) a balcony and (b) a concrete column that forms a single monolithic whole with the concrete foundation.

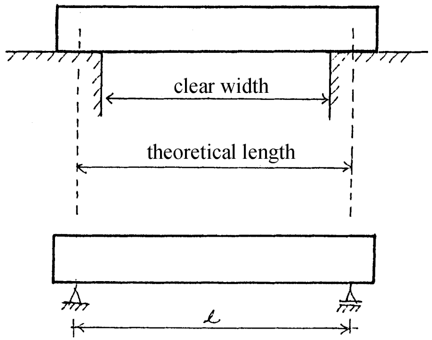


Figure 4.25 The simply supported beam modelled as a beam with a hinged and roller support.

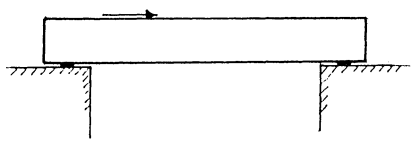


Figure 4.26 A beam supported on rubber blocks and loaded by a horizontal force.

used. Sometimes, the function of the roller is fulfilled by a slide layer of steel felt, Teflon, or other suitable material.

In practice, this sort of beam is often referred to as *freely supported* or simply supported, and is generally modelled as a beam on a hinge and a roller (see Figure 4.25).

In the event of vertical loading, it is arbitrary on which side the roller or hinge is placed. The model of a freely supported beam must however be performed with the necessary reserve if it relates to support reactions as a result of a horizontal load. For example, in the beam in Figure 4.26, that is supported on rubber blocks at both ends and which is loaded by a horizontal brake force, the model of a *free support* leads to incorrect (horizontal) support reactions.

4.4 Planar structures

A spatial structure can often be viewed as a system of planar structures composed of line elements. It is therefore certainly worth investigating the properties of such planar structures in more detail. Based amongst other things on the nature of the joints and the external appearance, various types of planar structures can be distinguished.

4.4.1 Modelling structures

In mechanics, a structure is a three-dimensional cohesive whole of structural elements that has to be able to resist external influences (the loads).

In many cases, structures appear to have been designed and built in such a way that the loads are transferred to the foundation via certain planes. In such cases, the three-dimensional structure can be modelled as a system of so-called *planar structures* (or two-dimensional structures). This is illustrated using two examples.

The first example is the bridge in Figure 4.27a. The loading by the traffic is transferred from the plane of the road deck to the vertical walls. These walls are in practice the spanning structure and transfer the load via the supports to the abutments, which subsequently transfer it to the foundation.

Surface elements (plates) can be used for the road deck and the walls; and together they form a so-called *trough bridge*. If the transverse measurements of the bridge are small compared to the span, the bridge can be modelled as a line element, or in other words, a bar with a U-section.

In order to limit the use of material and thereby reduce the self-weight that has to be carried, the surface elements can be replaced by planar structures made of line elements, as has been done in Figure 4.27b for the vertical walls.

The second example is the apartment building in Figure 4.28a. The structure consists of only surface elements. The vertical floor loading is transferred to the vertical walls and from there is transferred to the foundation. The horizontal wind loading is also distributed across the floors via the walls to the foundation.

Figure 4.28b represents the same building, but now all the horizontal and vertical surface elements in the main load-bearing structure have been replaced by planar structures made up of beams and columns. Although the structure now consists of only line elements, the transfer of forces is mostly unchanged and occurs through the same planes as in Figure 4.28a.

These examples illustrate that spatial structures can be composed of planar structures that consist of line elements. It is therefore certainly worth the effort of further investigating these types of planar structures.

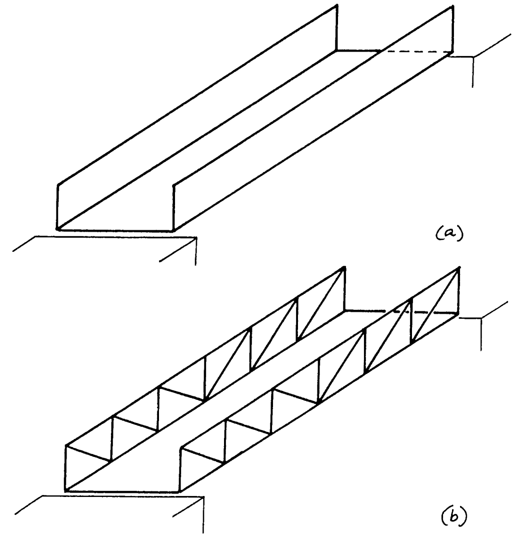


Figure 4.27 (a) A trough bridge, composed of surface elements; (b) the same bridge with the walls replaced by trusses.

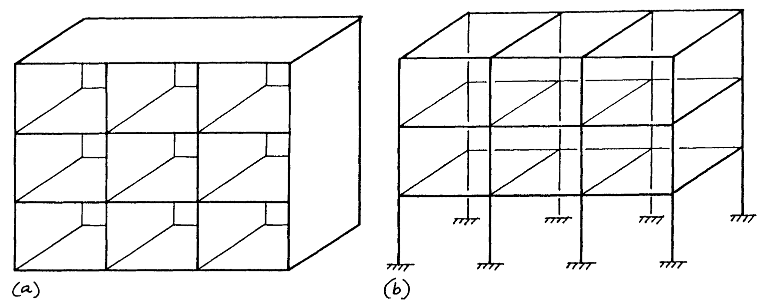
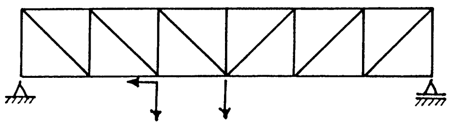
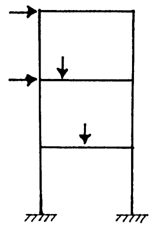


Figure 4.28 (a) An apartment building constructed of only surface elements; (b) the same building with the main load-bearing structure constructed of beams and columns.



(a) planar truss



(b) frame

Figure 4.29 (a) A truss with by definition solely hinged joints and (b) a frame with by definition exclusively rigid joints.

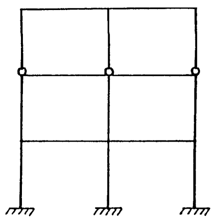


Figure 4.30 If there are also hinged joints in a frame, these have to be clearly indicated by means of open circles.

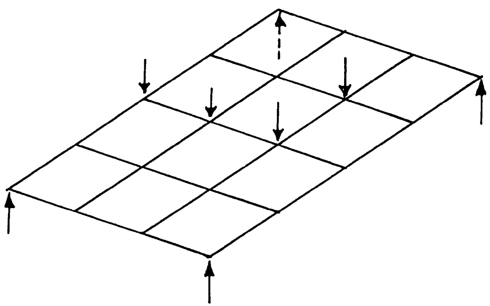


Figure 4.31 A beam grillage.

4.4.2 Planar trusses and frames

Planar *trusses* and *frames* are planar structures that are loaded *in their plane* (see Figure 4.29).

The difference between a truss and a frame is determined by the nature of the joints in the connections.

- in a truss, the bars are joined together by *hinges at all the connections*;[1]
- in frames, *all the joints are fixed and entirely stiff.*

The truss in Figure 4.29a appeared in the bridge in Figure 4.27b. The open circles, which represent the hinged joints, are generally omitted as in a truss all the joints are by definition hinged. The structure in Figure 4.29b is a frame. You will recognise part of the building in Figure 4.28b here, with the vertical floor loading and the horizontal wind loading. Sometimes the stiffnesses of the joints are accentuated by thickenings in the connections, but generally they are omitted. If there are also hinged joints in a frame, they have to be clearly depicted by means of open circles. This is the case in Figure 4.30, which could represent a building made of concrete, on which a steel floor was placed at a later stage.

4.4.3 Beam grillages

Beam grillages are planar structures that are loaded *normal to their plane*, see Figure 4.31. A beam grillage consists of two cooperative beam layers: beams and girders. The beams and girders are generally placed in two mutually perpendicular directions.

Beams grillages are often used as floor structures in bridges and buildings. Lock doors are also sometimes built as a system of beams and girders. A

[1] In Chapter 9, which addresses calculations related to trusses, another demand is covered, namely that the load has to be exerted only at connections.

façade made of posts and girders (columns and beams), with perpendicular wind loading, can sometimes also be seen as a beam grillage.

Calculating the forces and deformations in a beam grillage is in fact a three-dimensional problem. For information about the spatial character, refer to Section 3.3.4, examples 1 and 2.

4.4.4 Frames

Frames are planar, bent beams structures that are *loaded in the plane of the structure*. Such structures are often used to cover a space (warehouse, sports arena, and so forth).

Figure 4.32 shows a number of simple examples of *frames*. In Figure 4.33, both fixed supports have been replaced by hinged supports, so that the structure is now referred to as a two-hinged frame. If the structure with hinged supports itself consists of two parts joined by a hinge, this is referred to as a three-hinged frame (see Figure 4.34). If the beam structure is not bent but arched, then the structure in Figure 4.35a is called a two-hinged arch, and the structure in Figure 4.35b a three-hinged arch.

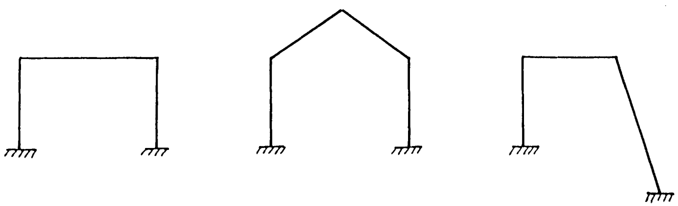


Figure 4.32 Examples of fixed frames.

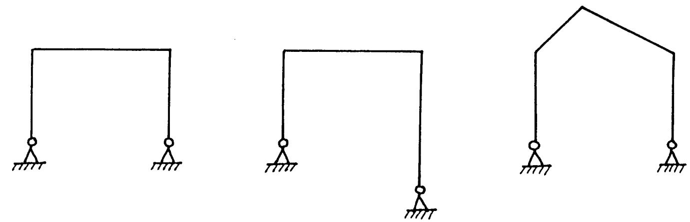


Figure 4.33 Examples of two-hinged frames.

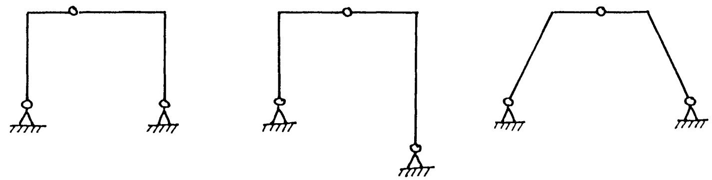


Figure 4.34 Examples of three-hinged frames.

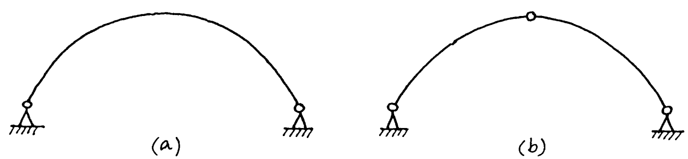


Figure 4.35 (a) A two-hinged arch and (b) a three-hinged arch.

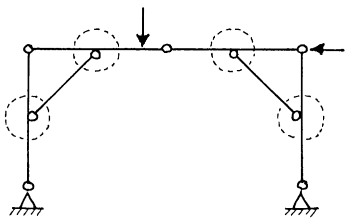


Figure 4.36 A shored frame.

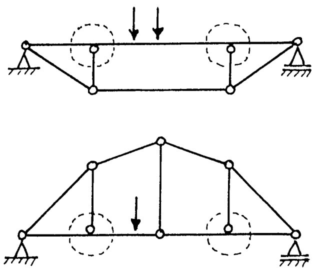


Figure 4.37 Examples of trussed beams.

4.4.5 Special structures

It will be clear that a wide range of planar structures can be constructed using line elements. Two types of structure not mentioned in the earlier categories are shown here. The structure in Figure 4.36 is called a *shored frame*. The structures in Figure 4.37 go by the name of trussed beams.

Although the structures in Figures 4.36 and 4.37 include hinges at all the connections, none of these structures are trusses. A characteristic of a truss is that *all the ends* of the members that merge in a connection are hinged together. This is not the case in the circled connections. Here the hinge is attached to the *outside* of a so-called *continuous beam*, and is *not fitted internally* in the beam.

4.5 Kinematic/static (in)determinate structures

Structures are supported in such a way that all free movements are prevented. This type of structure is known as *kinematically determinate* or *immovable*. If there are too few supports, or if they are not applied effectively, the structure, or part of the structure, will have a certain freedom of movement that is not resisted. The structure is then no longer immovable. This type of structure is called *kinematically indeterminate*, or is called a *mechanism*. If it is possible to calculate all the support reactions and interaction forces for a kinematically determinate structure using only equilibrium equations, it is called a statically determinate structure. If there are too many unknown forces to be able to determine them from the equilibrium, the structure is said to be *statically indeterminate*. In order to determine the forces in a statically indeterminate structure, the deformation of the structure must be taken into account.

4.5.1 Kinematically (in)determinate supported rigid structures

A *dimensionally stable structure* or *self-contained structure* is a structure that, isolated from its supports, retains its shape. If we neglect the deformations that occur, a self-contained structure can be seen as a rigid body. In a plane, a rigid body has *three degrees of freedom*: two components of a translation and one rotation, see Figure 4.38a.

In Figure 4.38b, the block is supported by a bar (two-force member) and is free to move in the direction perpendicular to the bar (on the condition that the movements remain small, see Section 4.3.1) and can rotate about A. The bar support at A reduces the three degrees of freedom of the body to two.

The freedom of movement can be limited further with a second bar support, for example at B (see Figure 4.38c). The movement that the body can now perform, with (minor) movement at A and B perpendicular to the bars, can be interpreted as a rotation about the so-called (*instantaneous*) *centre of rotation* RC, which is located on the intersection of the two bars.[1] With two bar supports the number of degrees of freedom of the block has been reduced to one.

The last degree of freedom, the rotation about RC, can be removed with a third bar support, for example at C (see Figure 4.38d). The three bars now prevent all possible movement. However the body is pulled or pushed, it remains where it is. This is referred to as the body having an *immovable* or *kinematically determinate* support.

Three bar supports (at least) are required for an immovable or kinematically

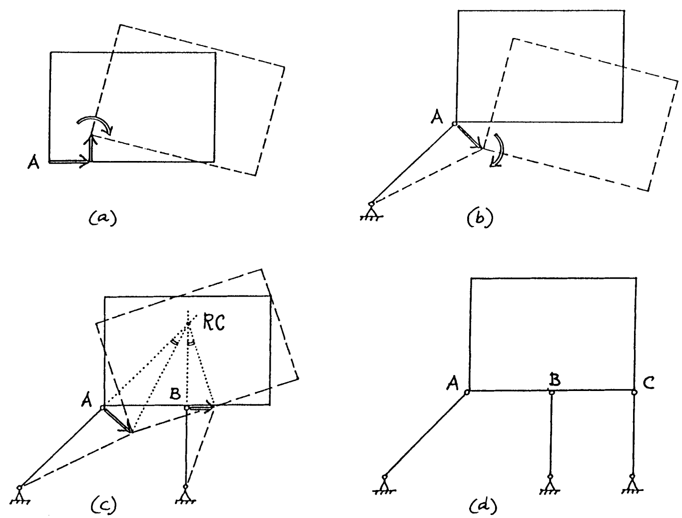


Figure 4.38 In a plane, a rigid body has three degrees of freedom; (b) due to the bar support at A this number is reduced to two; (c) two bar supports act as a hinged support at the centre of rotation RC, the point of intersection of the two bars; there is only one degree of freedom left: the rotation about RC; (d) with three bar supports, the body is immovable or kinematically determinate.

[1] The fact that the centre of rotation RC is a fixed point is true only if the rotation is still small. When considering Figure 4.42, one should not be confused by the fact that the displacements have been drawn to a large scale with respect to the dimensions of the structure.

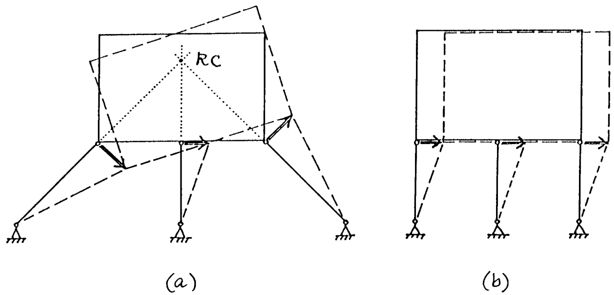


Figure 4.39 Examples of movable or kinematically indeterminate supports: the support permits (a) a rotation about RC and (b) a movement perpendicular to the supporting bars.

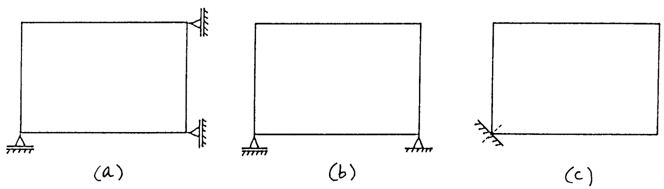


Figure 4.40 Examples of immovable or kinematically determinate supports with (a) three rollers, (b) a roller and a hinge, and (c) a fixed support.

determinate support of a rigid body. The bars may not all intersect in one point, or all be parallel, as is shown in Figure 4.39. In the support in Figure 4.39a, all bars intersect at the rotation centre RC, allowing the body to rotate. This support is *movable* or *kinematically indeterminate*. The support in Figure 4.39b, in which all bars are parallel to one another (intersect at a point at infinite distance), is also kinematically indeterminate, as the body is free to move in the direction perpendicular to the bar supports.

The similarities between bar supports and roller supports were repeatedly pointed out in Section 4.2. Figure 4.38c also shows that two bar supports act as a hinged support at the rotation centre RC, the intersection of the two bars.

An immovable support of a rigid body is therefore also possible with three roller supports, as in Figure 4.40a, or with a roller and hinged support, as in Figure 4.40b. It should be clear that a fixed support of a rigid body, as in Figure 4.40c, also is an immovable support.

4.5.2 Statically (in)determinate supported rigid structures

Instead of investigating the freedom of movement of a body, it is possible to determine also how many support reactions would be needed to keep the body in equilibrium under all imaginable loading conditions.

The support *reactions* adapt to the loading (the *action*) until equilibrium is reached. The unknown support reactions must therefore meet the conditions of the three equilibrium equations (in a plane) that apply to a rigid body. With three support reactions, there is an equal number of unknowns as equilibrium equations and, with the exception of a number of special cases which will be addressed later on, the support reactions can be deduced directly from the equilibrium. The support is then referred to as *statically determinate*.

An example is the rectangular block in Figure 4.41a, supported by three bars (two-force members). The resultant of the loading on the block is the force R, with components R_x and R_y (not shown).

In Figure 4.41b the block has been isolated from its supports and the unknown support reactions F_1, F_2, and F_3 are shown. For equilibrium, the following have to apply:

$$\sum F_x = -\tfrac{1}{2}\sqrt{2} \cdot F_1 + R_x = 0,$$

$$\sum F_y = -\tfrac{1}{2}\sqrt{2} \cdot F_1 - F_2 - F_3 + R_y = 0,$$

$$\sum T_z|\mathrm{D} = -F_3 \cdot b + R \cdot a = 0.$$

To keep the equation for the moment equilibrium transparent, it has been related to the intersection D of the lines of action of F_1 and F_2; here a is the perpendicular distance from D to the line of action of R.

In matrix notation, the three equilibrium equations are

$$\begin{bmatrix} \frac{1}{2}\sqrt{2} & 0 & 0 \\ \frac{1}{2}\sqrt{2} & 1 & 1 \\ 0 & 0 & b \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} = \begin{bmatrix} R_x \\ R_y \\ R \cdot a \end{bmatrix}.$$

We could wonder whether this system of three linear equations with three unknowns has a unique solution under all imaginable loading (i.e. for all possible values of R and a).

One way to find out is simply to try to solve the set of equations; we find that

$$F_1 = R_x\sqrt{2},$$

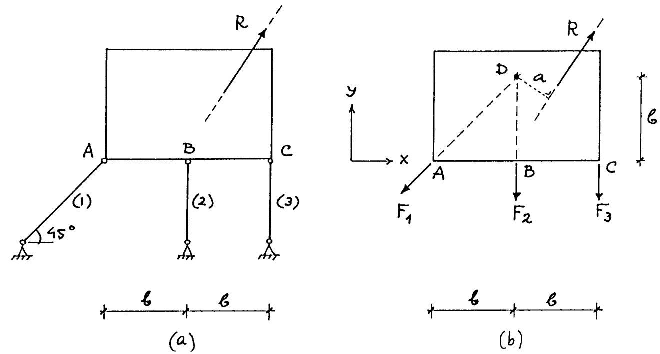


Figure 4.41 (a) A block supported by three bars; (b) the isolated block.

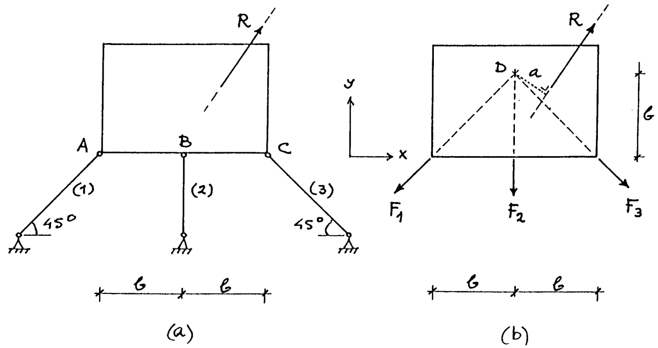


Figure 4.42 (a) A block supported by three bars; (b) the support is kinematically indeterminate as no moment equilibrium about D is possible.

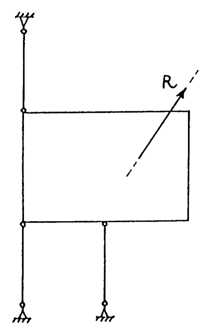


Figure 4.43 The support is kinematically indeterminate as no horizontal force equilibrium is possible.

$$F_2 = -R_x + R_y - \frac{a}{b}R,$$

$$F_3 = +\frac{a}{b}R.$$

The values of F_1 to F_3 exist for all values of a and R. The support is therefore *kinematically determinate* (equilibrium is possible with any arbitrary loading) and *statically determinate* (the support reactions can be determined from the equilibrium).

A more general answer is found in linear algebra: there is a unique solution if the determinant of the coefficient matrix is not equal to zero. This is indeed the case in this example:

$$\text{Det} = \tfrac{1}{2}\sqrt{2} \cdot b \neq 0.$$

The set of equations cannot be solved if the determinant of the coefficient matrix is zero. The figures in the coefficient matrix are determined by the manner in which the body is supported. The fact that the determinant is zero means that, from a physical perspective, the support is kinematically indeterminate.

In order to illustrate this, the bar support (3) in Figure 4.42 has been placed at an angle. With this type of support, the three equilibrium conditions can be represented by

$$\begin{bmatrix} \frac{1}{2}\sqrt{2} & 0 & 0 \\ \frac{1}{2}\sqrt{2} & 1 & +\frac{1}{2}\sqrt{2} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} = \begin{bmatrix} R_x \\ R_y \\ R \cdot a \end{bmatrix}.$$

The determinant of the coefficient matrix is now zero.

In the last equilibrium equation, the moment equilibrium about D, the condition $R \cdot a \neq 0$ cannot be met. Neither can the support reactions for $R \cdot a = 0$ be determined. The method of support in Figure 4.42 allows a rotation about D and is therefore *kinematically indeterminate*. This is in line with what we determined in the previous section for a support on three bars that pass through a single point.

The support in Figure 4.43, on three parallel bars, is also kinematically indeterminate. No force equilibrium is possible in the direction normal to the bars and the block is able to move in that direction. With less than three support reactions, there are more equilibrium equations than unknowns. Here the support is also kinematically indeterminate: the conditions for all three equilibrium equations cannot be met for arbitrary loading. In Figure 4.44 in case (a) moment equilibrium is not possible and a rotation occurs about A. In case (b) horizontal force equilibrium is not possible, and the block will move horizontally.

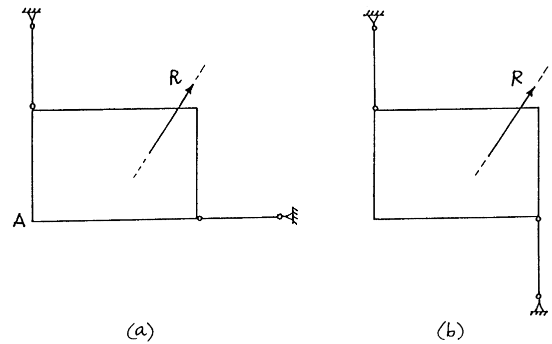


Figure 4.44 With less than three support reactions, the support is kinematically indeterminate: (a) moment equilibrium is not possible about A; (b) horizontal force equilibrium is not possible.

With more than three bar supports, such as in Figure 4.45, which do not all pass through a single point and are not all parallel, the support is immovable or kinematically determinate. The number of unknown support reactions is now larger than the available number of equilibrium equations and a unique solution is impossible. In fact, there is an infinity of solutions that satisfy the equilibrium equations. An immovable support of a rigid body or self-contained structure with more than three support reactions is therefore referred to as being *statically indeterminate* (or *hyperstatic*). In a statically indeterminate support, the support reactions cannot be deduced directly from the equilibrium, and the deformation of the structure will also have to be included in the consideration.

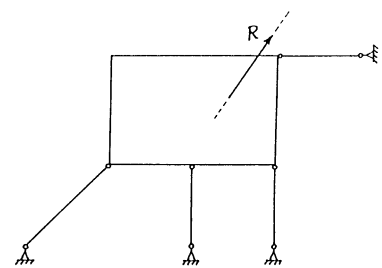


Figure 4.45 If the support of a rigid body or self-contained structure has more than three support reactions, then the support is statically indeterminate.

To summarise, for a rigid body or self-contained structure with r support reactions:

$r < 3$ the support is kinematically indeterminate (movable);

$r \geq 3$ the support is kinematically determinate (immovable), unless all the support reactions pass through a single point or are parallel to one another.

$r \geq 3$ is therefore a necessary, but not sufficient condition for kinematically determinate support of a rigid body or self-contained structure.

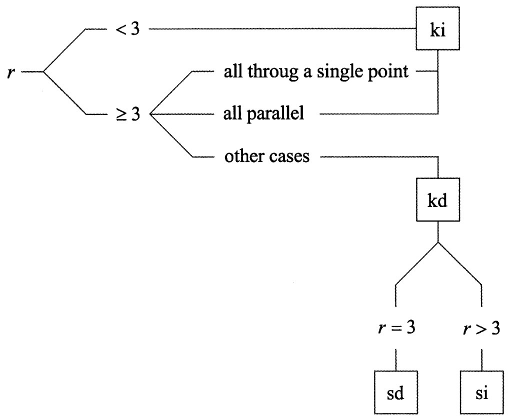


Figure 4.46 The relationship between the number of bar supports r (support reactions) and the **k**inematic/**s**tatic (**in**)**d**eterminacy of the support of a rigid body or self-contained structure.

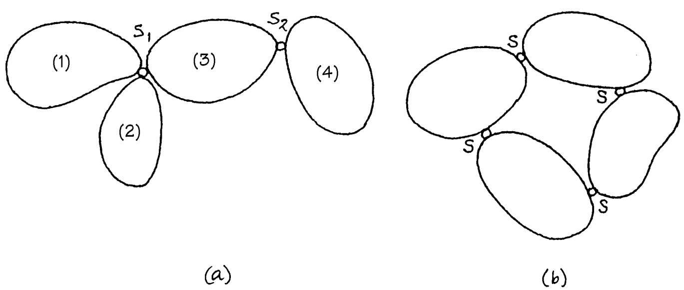


Figure 4.47 Compound structures: (a) with 6 degrees of freedom; (b) with 4 degrees of freedom.

If the support is kinematically determinate, the following distinctions are also possible:

$r = 3$ the support is statically determinate: all the support reactions follow directly from the equilibrium;

$r > 3$ the support is statically indeterminate: the three equilibrium equations are not enough to determine all the support reactions.

The statements above are summarised in Figure 4.46.

4.5.3 Kinematically/statically (in)determinate supported compound structures

So far, we have looked only at *dimensionally stable structures* or *self-contained structures*. In this section, we will look at *dimensionally unstable structures* or *compound structures*. Isolated from its supports, compound structures are unable to retain their shape, as the composite parts can move with respect to one another.

Figure 4.47 shows two examples of compound structures, without their supports. The structures consist of a number of rigid (or self-contained) parts (sub-structures), which are capable of rotating with respect to one another at the hinged joints S. For immovable or kinematically determinate supports, more than three bar supports (support reactions) are now required. The immovable support of the compound structure in Figure 4.47a needs at least six bar supports. Body (1) can be fixed with three bars (Figure 4.48a). Here, S_1 has become a hinged support for the bodies (2) and (3). For each of these bodies, one bar is sufficient to fix them (Figure 4.48b). Now only body (4) can still rotate around S_2, which can be prevented with a sixth bar (Figure 4.48c). To achieve an immovable support, more than six bar supports could also be used of course; six is the minimum required.

The number of bar supports (support reactions) required for an immovable support can also be deduced, as in Section 4.5.2, by analysing the equilibrium and comparing the number of unknowns (support reactions and interaction forces) with the number of equilibrium equations available.

Imagine

r = number of support reactions,

v = number of interaction forces,

e = number of equilibrium equations,

and

$$n = r + v - e.$$

n is equal to the difference between the number of unknowns $(r + v)$ and the number of available equilibrium equations e.

If all the support reactions have been applied *effectively*, the following two cases can be distinguished:

- $n < 0$ – *the structure is kinematically indeterminate*
  The total number of unknown forces $r + v$ is smaller than the number of available equilibrium equations e. This means that the equilibrium equations cannot be solved for arbitrary loading. The structure may move under certain loads. The number of support reactions r is too small to remove all the degrees of freedom. The support is therefore kinematically indeterminate (movable). A kinematically indeterminate structure is also referred to as a *mechanism*.
  The negative value of n is equal to the number of *degrees of freedom* (movement possibilities) of the structure (or the mechanism).
- $n \geq 0$ – *the structure is kinematically determinate*
  For an immovable support (or kinematically determinate structure) it would seem that $n \geq 0$.

For kinematically determinate structures $(n \geq 0)$, one can distinguish between two cases:

- $n = 0$ – *the structure is statically determinate*
  The number of unknown forces $r + v$ is equal to the number of available

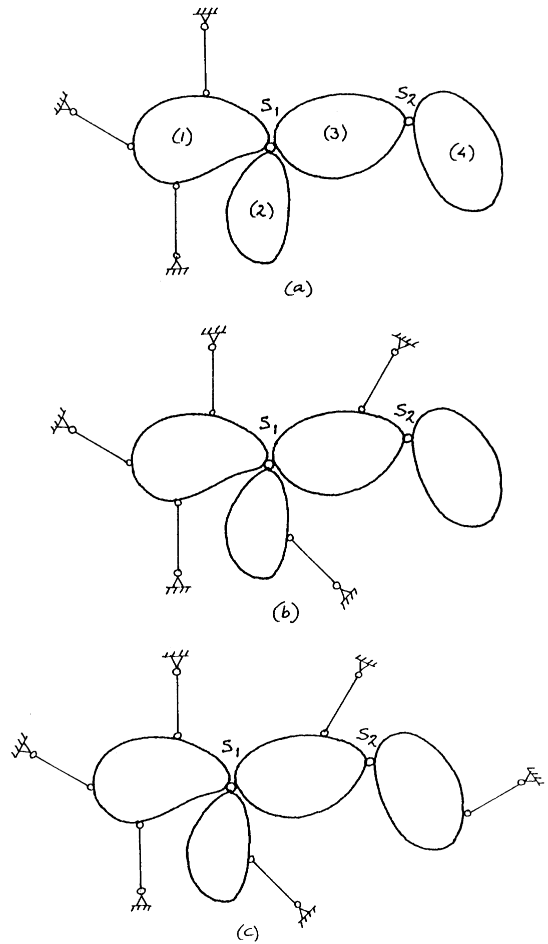


Figure 4.48 Each effectively applied bar support reduces the number of degrees of freedom by one.

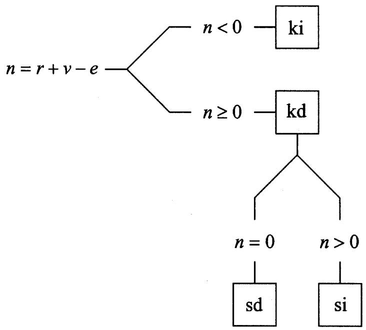


Figure 4.49 Summary of the <u>**k**</u>inematic/<u>**s**</u>tatic (<u>**i**</u>n)<u>**d**</u>eterminacy of a structure.

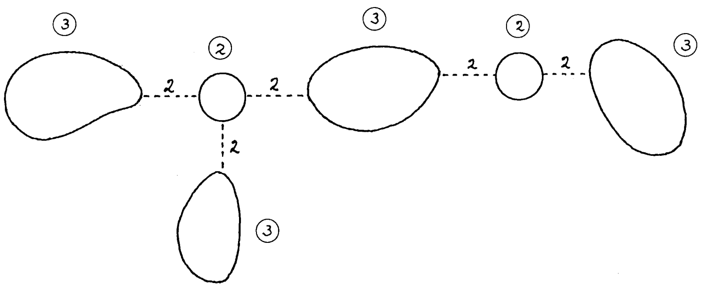


Figure 4.50 When isolating a compound structure, it falls apart into a number of (self-contained) sub-structures and a number of hinged joints.

equilibrium equations. All the support reactions and interaction forces can be determined on the basis of the equilibrium. The structure is statically determinate.

- $n > 0$ – *the structure is statically indeterminate*
  The number of unknowns is greater than the number of available equilibrium equations. An infinity of solutions satisfy the equilibrium equations. The structure is statically indeterminate. The value of n is called the *degree of static indeterminacy*.

Figure 4.49 provides a summary of these statements.

For the compound structure in Figure 4.47a, we will now determine with how many (*effectively placed*) support reactions the structure can be supported in an immovable way. To do so, it will be assumed that the self-contained sub-structures do not directly exert forces on one another, but that they do so via the *joints*. When isolated, the compound structure therefore falls apart into a number of sub-structures and a number of joints (see Figure 4.50).

There are two interaction forces at every joint between a sub-structure and a (hinged) joint. In the figure, the connections are shown by dotted lines, and the number of interaction forces is shown. There are a total of five joints, which brings the total number of unknown interaction forces to

$$v = 5 \times 2 = 10.$$

Each self-contained sub-structure gives three equilibrium equations (force equilibrium and moment equilibrium) and each hinged joint gives two equilibrium equations (only force equilibrium). These numbers are included in the circles in Figure 4.50.

With four sub-structures and two joints, the total number of available equilibrium equations becomes

$$e = 4 \times 3 + 2 \times 2 = 16.$$

Without support reactions ($r = 0$) this gives

$$n = r + v - e = 0 + 10 - 16 = -6.$$

The compound structure in Figure 4.47a therefore has six *degrees of freedom*.

The *minimum* number of required support reactions r for a kinematically determinate structure follows from the condition $n = 0$:

$$n = r + v - e = 0 \Rightarrow r = e - v = 16 - 10 = 6.$$

For an immovable support, six *effectively applied* bar supports (support reactions) are therefore sufficient. This is in line with what was found earlier (see Figure 4.48c): for, each bar support removes one degree of freedom.

We have frequently used the phrase *effectively applied* bar supports. In Figure 4.48c, all the bars have been applied effectively. If in Figure 4.48 the bar supports were not placed effectively, for example by using all the bars to support body (1), the structure would remain kinematically indeterminate even though the condition $n \geq 0$ is met.

The condition $n \geq 0$ for a kinematically determinate structure is not a sufficient condition, as it is always possible to apply the supports (ineffectively) so that the structure remains kinematically indeterminate. One must be aware of this.

An example of the above is the structure in Figure 4.51, in which three hinges are on a straight line. Imagine that the hinged joint S and both bodies are isolated. The hinged joint gives two equilibrium equations and each body gives three. The total number of available equilibrium equations is

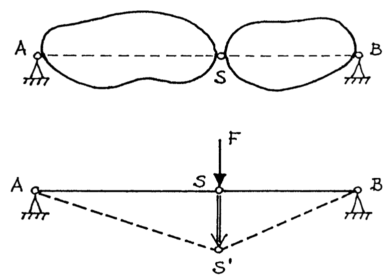


Figure 4.51 This structure, with three hinges in line at A, S and B, allows (minor) movement at S, and is therefore kinematically indeterminate.

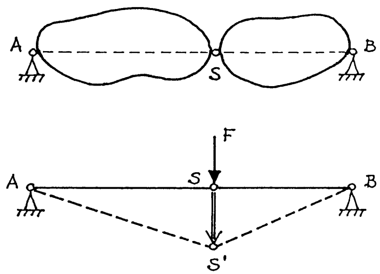


Figure 4.51 This structure, with three hinges in line at A, S and B, allows (minor) movement at S, and is therefore kinematically indeterminate.

then

$$e = 2 + 2 \times 3 = 8.$$

The number of interaction forces between joint S and both bodies is

$$v = 4.$$

A hinged support can provide two support reactions. For two hinged supports (A and B), it therefore applies that

$$r = 4.$$

This gives

$$n = r + v - e = 4 + 4 - 8 = 0.$$

Further investigation shows however that in case of a load normal to the line through the three hinges, the conditions for moment equilibrium cannot be met. For example, a vertical force at S can never create an equilibrium with the horizontal(!) support reactions at A and B. For such a load, the structure in S will allow (minor) movement. The structure is therefore kinematically indeterminate, even though $n = 0$.

4.5.4 Static (in)determinacy of a frame

A frame is a structure constructed of members that are connected to one another at *rigid* or *hinged* joints. In order to be able to determine the static (in)determinacy for a kinematically determinate frame, we use the procedure based on a consideration of equilibrium from the previous section: all members and joints in the structure are isolated. Joints are also assumed at the supports.

This is illustrated in Figure 4.52a; all bars and joints have been isolated in Figure 4.52b.

Each member gives three equilibrium equations (force equilibrium and moment equilibrium).

Two different types have to be distinguished as far as the joints are concerned:

- Joints on which only forces can be exerted (fully hinged[1] joints); they are shown as *circles*, and give two equilibrium equations (force equilibrium).
- Joints on which both couples and forces can be exerted (rigid[2] and incompletely hinged[3] joints); they are shown as *squares*, and give three equilibrium equations (force equilibrium and moment equilibrium).

The number of equilibrium equations that the members and joints introduce is included as a circled value in Figure 4.52b.

The four bars therefore give $4 \times 3 = 12$ equilibrium equations, the two fully hinged joints give $2 \times 2 = 4$ equilibrium equations while the other two joints give $2 \times 3 = 6$ equilibrium equations. The total number of available equilibrium equations is therefore

$$e = 12 + 4 + 6 = 22.$$

The connections between the members and the joints are shown in Figure 4.52b by means of dashed lines; also the number of interaction forces is shown. Here, we have to distinguish between the following:

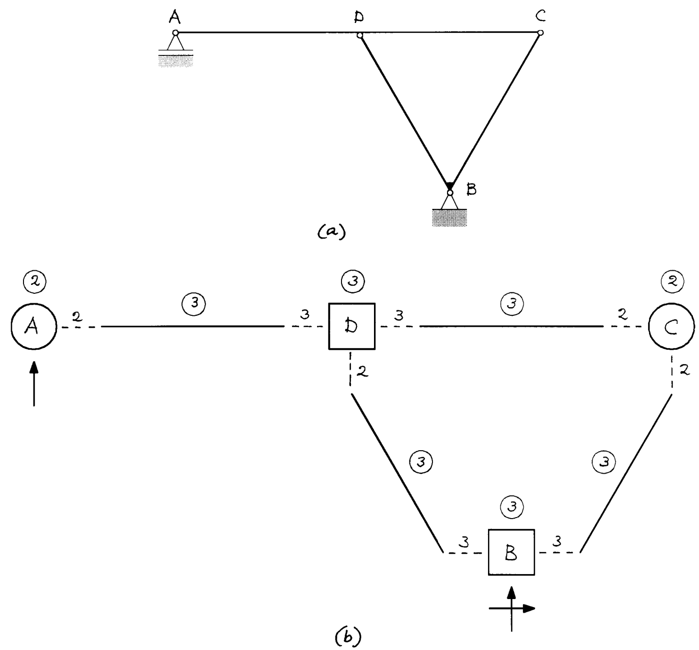


Figure 4.52 (a) A structure of which in (b) all the members and joints have been isolated. Joints have also been assumed at the supports. Joints only subject to forces are shown as circles; joints that can also be subject to couples are shown as squares.

1 All the bars that meet at the joint are connected to the joint by a hinge.
2 All the bars that meet at the joint are connected to the joint rigidly.
3 Of all the bars that meet at the joint, some are connected to the joint by a hinge, and some are connected rigidly.

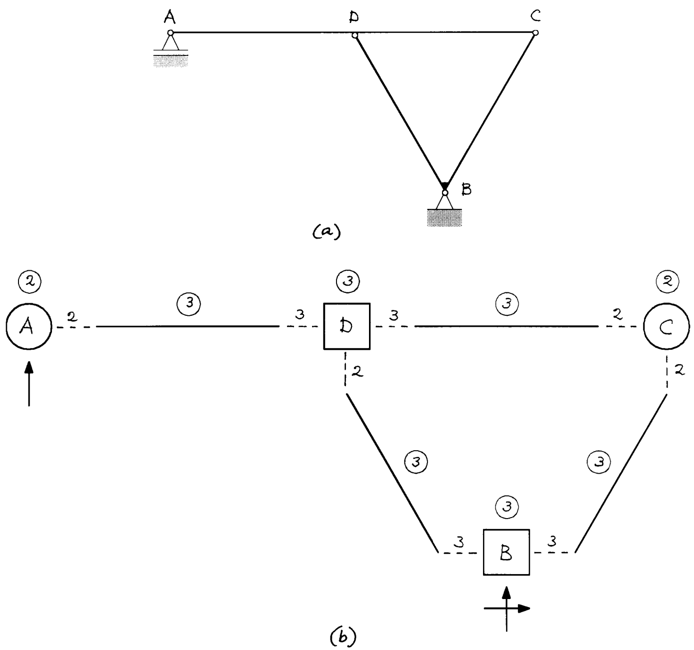


Figure 4.52 (a) A structure of which in (b) all the members and joints have been isolated. Joints have also been assumed at the supports. Joints only subject to forces are shown as circles; joints that can also be subject to couples are shown as squares.

- Hinged connections between the end of the member and the connection – two interaction forces are acting here, and
- Rigid connections between the end of the bar and the joint – three interaction forces are acting here.

With four hinged connections between member and joint and four rigid connections, the total number of interaction forces is

$$v = 4 \times 2 + 4 \times 3 = 20.$$

In Figure 4.52b the support reactions that can act on joints A and B are also shown. The roller support provides one support reaction, and the hinged support provides two. The total number of support reactions is therefore

$$r = 1 + 2 = 3.$$

For the difference n of the number of unknown forces (support reactions and interaction forces) and the number of available equilibrium equations, we arrive at

$$n = r + v - e = 3 + 20 - 22 = 1.$$

This means that the structure is statically indeterminate to the first degree: there is one unknown too many to be able to derive all the support reactions and interaction forces directly from the equilibrium.

By isolating the structure into all its smallest parts (members and joints) the procedure used can be laborious and prone to calculation errors. The static indeterminacy can often be found more quickly and with fewer calculations by releasing the structure into a number of larger parts. This is illustrated with help of the frame in Figure 4.53a.

With the section in Figure 4.53b, the structure falls apart into two self-contained parts. There are three equilibrium equations available per part.

Together both parts give $e = 2 \times 3 = 6$ equilibrium equations. The section was introduced across three members. Three unknown interaction forces are acting in each section. The total number of interaction forces is therefore: $v = 3 \times 3 = 9$. The hinged support gives two support reactions and the roller support gives one. Together that makes $r = 2 + 1 = 3$ unknown support reactions. The numbers of interaction forces and support reactions are shown in the figure.

The degree of static indeterminacy (the number of unknowns too many) is therefore

$$n = r + v - e = 3 + 9 - 6 = 6.$$

More generally speaking we can say that

$$n = r + v - 3s$$

in which

r = number of support reactions,

v = number of interaction forces in the section(s) applied,

s = number of rigid sections (sub-structures).

In this way, we find in Figure 4.53c that

$$n = (3 + 4 \times 3) - 3 \times 3 = 6.$$

And for the three sections in Figure 4.53d

$$n = (3 + 7 \times 3) - 3 \times 6 = 6.$$

A condition for an accurate result is that the section(s) has/have to be applied in such a way that the sub-structures are *singly-cohesive*. This means

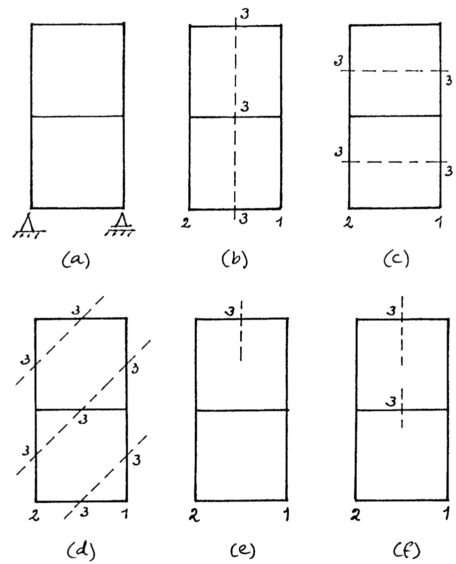


Figure 4.53 (a) An internally statically indeterminate frame; (b) to (d) with the sections shown, all the sub-structures are singly-cohesive; (e) with the section shown, the frame remains multiply-cohesive; (f) with these two sections, the frame becomes singly-cohesive.

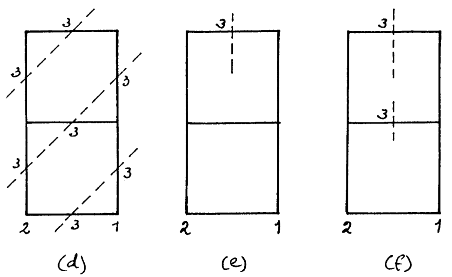


Figure 4.53 Internally statically indeterminated structure; (d) with the sections shown, all the sub-structures are singly-cohesive; (e) with this section, the frame remains multiply-cohesive; (f) with these two sections, the frame becomes singly-cohesive.

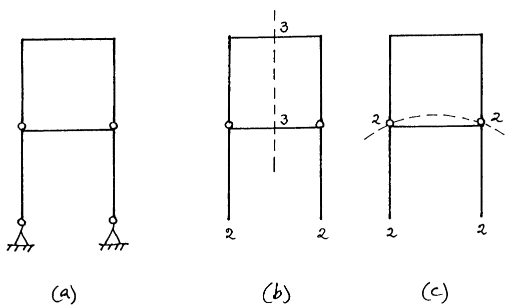


Figure 4.54 (a) A frame structure; (b) with the section shown the sub-structures are not self-contained; (c) with a section across the hinges, the sub-structures are self-contained.

that the cohesion in the sub-structure has to be such that for an arbitrary section across any member, the sub-structure has to fall apart into two new self-contained (or rigid) parts.

For example, it is not possible to determine the static indeterminacy for the section in Figure 4.53e. The structure is not singly-cohesive, as the extra 'cut', in Figure 4.53f does not make the structure fall apart into two new self-contained (or rigid) parts. In contrast, the static indeterminacy can be found for the two 'cuts' in Figure 4.53f. This structure is singly-cohesive, as each extra 'cut' over any member makes the structure fall apart into two new parts.

The degree of static indeterminacy is

$$n = (3 + 2 \times 3) - 3 \times 1 = 6.$$

Note that the support reactions of the six-fold *statically indeterminate structure* in Figure 4.53 can be found directly from the equilibrium equations. The support of the structure therefore is *statically determinate*. A statically indeterminate structure for which one can find the support reactions directly from the equilibrium is also said to be *internally statically indeterminate*.

If, as in Figure 4.54a, there are hinged joints in a structure, you have to be aware whether the parts into which you split the structure are self-contained and retain their shape. In that respect, the section in Figure 4.54b is not effective. You should choose the section across the hinges here, see Figure 4.54c. The degree of static indeterminacy is

$$n = (4 + 4) - 3 \times 2 = 2.$$

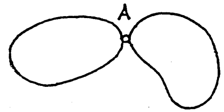
There is no simple recipe to determine the degree of static indeterminacy quickly. The approach depends on the insight into how forces are transferred within structures; this insight develops with experience.

4.6 Problems

Joints between structural elements (Section 4.2)

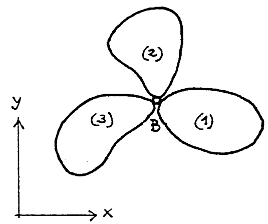
4.1 Two bodies are joined in hinge A.

Question:
How many (independent) interaction forces are there at A?



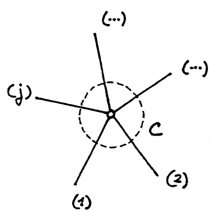
4.2 Three bodies (1), (2), and (3) are connected at B by a hinge. The bodies exert forces on one another via joint B. The joint is modelled as a particle element.

Question:
a. Isolate the bodies at B, draw all the interaction forces acting between the bodies and joint B, and name them in the xy coordinate system shown.
b. How many equilibrium equations are available for the joint?
c. How many independent interaction forces are there at B?



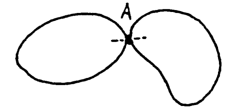
4.3 In joint C, j bars are connected by a hinge.

Question:
Derive the relationship between the number of joined bars j and the number of independent interaction forces i at C.



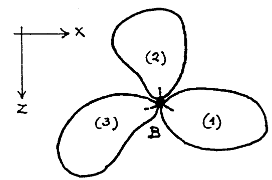
4.4 In A, two bodies are connected rigidly.

Question:
How many (independent) interaction forces are there at A?



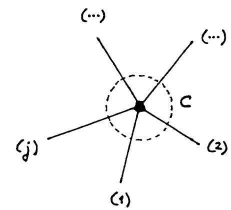
4.5 The three bodies (1), (2), and (3) are rigidly connected at B. The bodies exert forces on one another via joint B. The joint is modelled as a particle element.

Question:
a. Isolate the bodies at B, draw all the interaction forces between the bodies and joint B, and name them in the xz coordinate system shown.
b. How many equilibrium equations are there for the joint?
c. How many independent interaction forces are there at B?

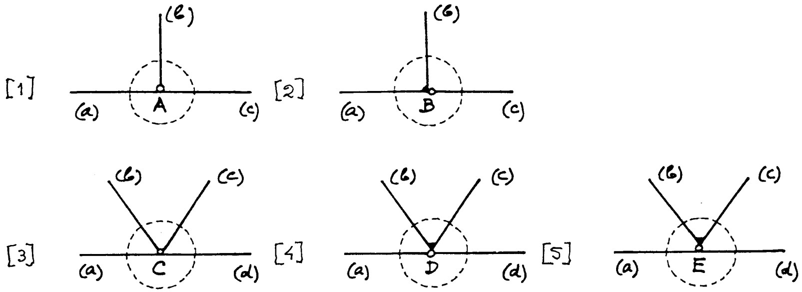


4.6 At joint C, j bars are rigidly connected.

Question:
For joint C, derive the relationship between the number of independent interaction forces i and the number of rigidly joined bars j.



4.7: 1–5 A number of bars are connected in a variety of ways at a joint.



Question:

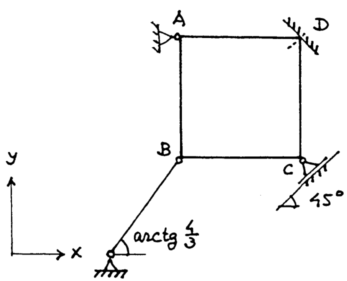
a. Determine the number of connection forces between each of the bar ends and the joint.
b. Determine the number of equilibrium equations available for the joint.
c. Determine the number of independent connection forces at the joint.

Supports (Section 4.3)

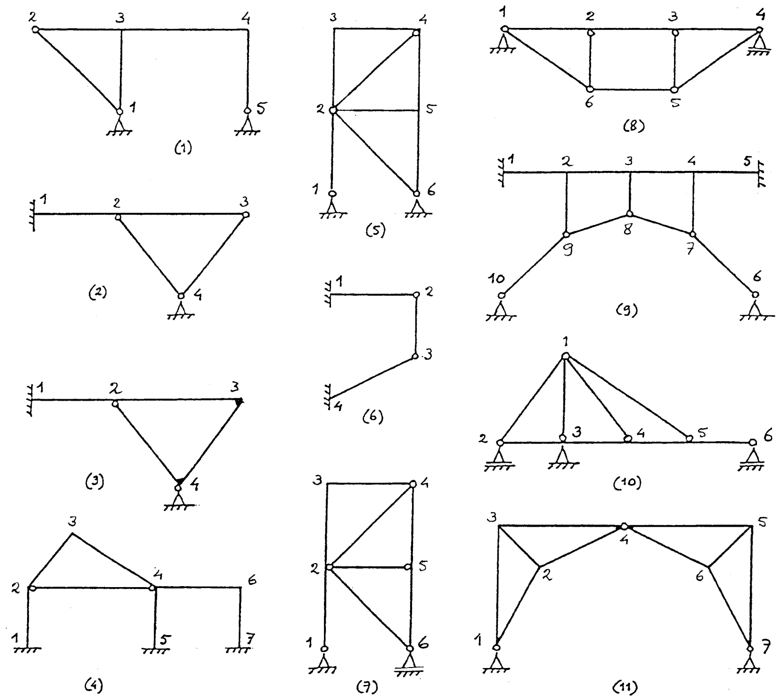
4.8 A square block ABCD is supported at its four corners as shown. If the block is loaded, it will deform.

Question:

a. How many displacements, and in which directions, do the supports at the corners permit?
b. How many displacements, and in which directions, are prescribed at the corners by the supports?
c. How many and which forces can develop freely at the supports?
d. How many and which forces are prescribed in the corners by the supports?



4.9: 1–11 A number of structures are shown.

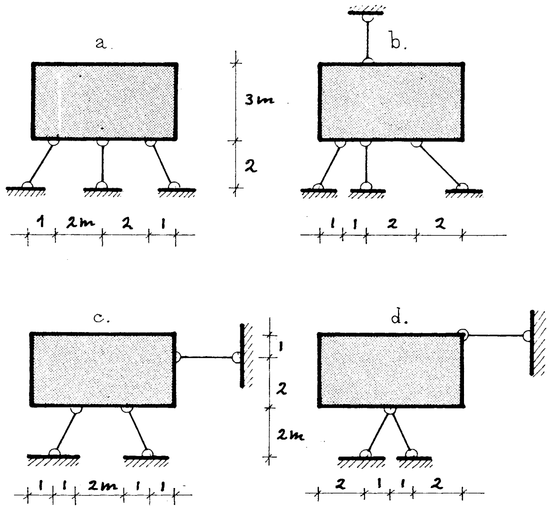


Question:

a. What types of support are being used?
b. How many and which support reactions will these supports supply?

Kinematic/static (in)determinacy of structures (Section 4.5)

4.10 A block is supported in four different ways.



Question:
Which support method is not effective?

4.11: 1–16 A rectangular block is supported in a variety of ways.

Question:

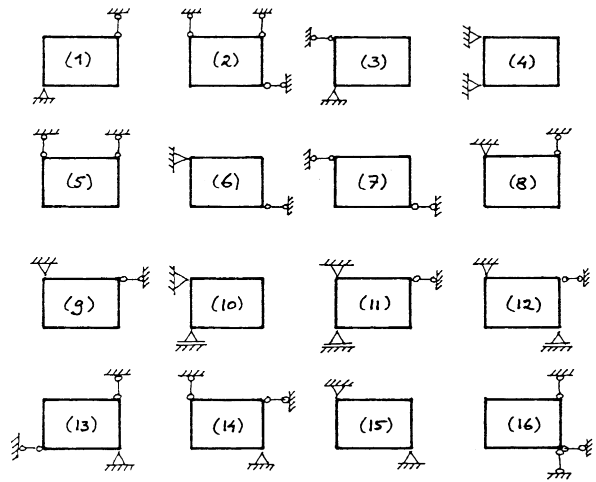
a. Determine whether the support is kinematically determinate (kd) or kinematically indeterminate (ki).

Do so in two different ways:
- first investigate the freedom of movement for the method of support given, and
- secondly count the number of support reactions present, and if there are enough, determine whether the support reactions are situated properly (and can form an equilibrium system with an arbitrary loading).

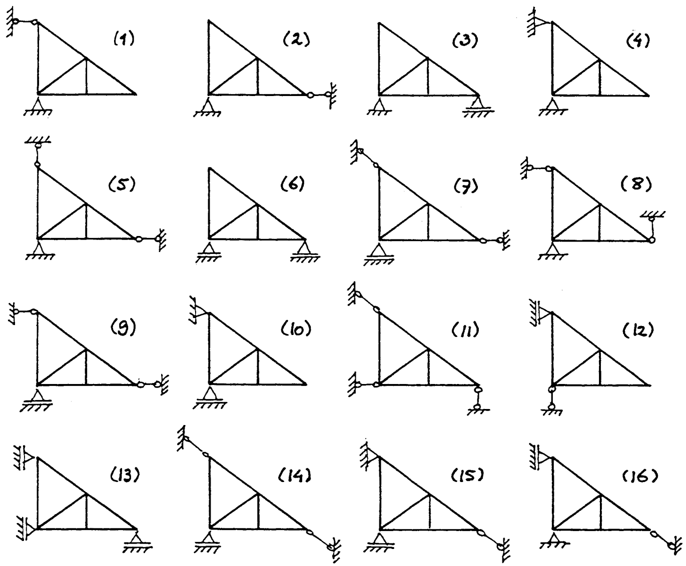
b. If the support is kinematically indeterminate, give the number of degrees of freedom v.

c. If the support is kinematically determinate, indicate whether the support is statically determinate (sd) or statically indeterminate (si).

d. If the support is statically indeterminate, give the degree of static indeterminacy n.



4.12: 1–16 A rigid truss is supported in a variety of ways.



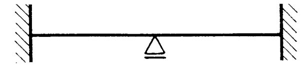
Question:

a. For each of the cases, determine whether the support is kinematically determinate (kd) or kinematically indeterminate (ki).
b. If the support is kinematically indeterminate, give the number of degrees of freedom v.
c. If the support is kinematically determinate, indicate whether the support is statically determinate (sd) or statically indeterminate (si).
d. If the support is statically indeterminate, give the degree of static indeterminacy n.

4.13 Which of the following statements is true for the beam shown?

The beam is:

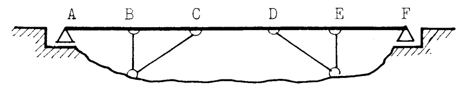
a. Kinematically determinate.
b. Statically determinate.
c. Statically indeterminate to the fourth degree.
d. Statically indeterminate to the seventh degree.



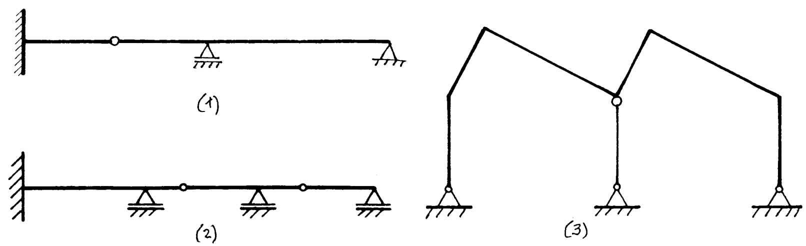
4.14 A bridge beam is resting on roller supports at A and F, and on bar supports at B, C, D, and E.

Question:
What is the degree of static indeterminacy of this structure?

a. 2.
b. 3.
c. 4.
d. 5.



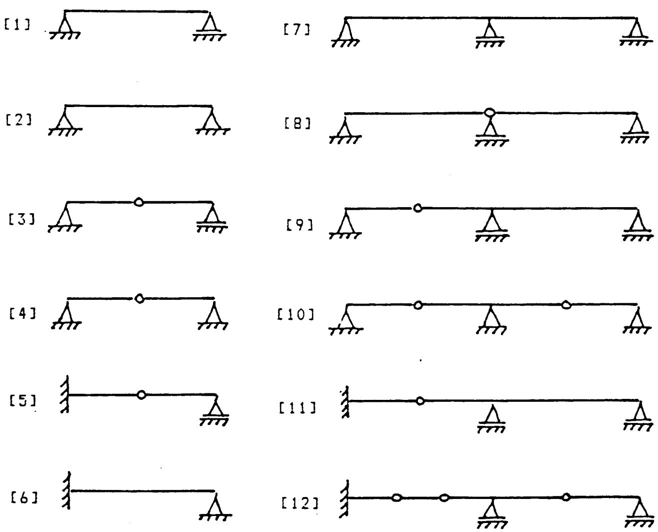
4.15: 1–3



Which statement applies to the structure shown? The structure is:

a. Statically determinate.
b. Statically indeterminate to the first degree.
c. Statically indeterminate to the second degree.
d. Statically indeterminate to the third degree.
e. Statically indeterminate to the fifth degree.
f. Statically indeterminate to the sixth degree.

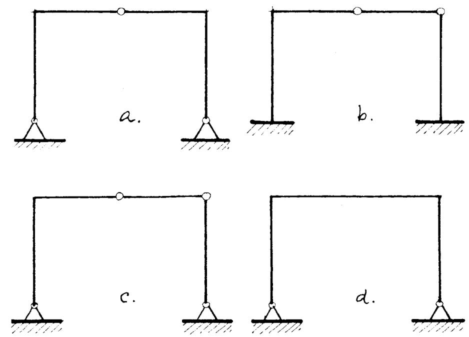
4.16: 1–12 A number of beams and hinged beams are supported as shown.



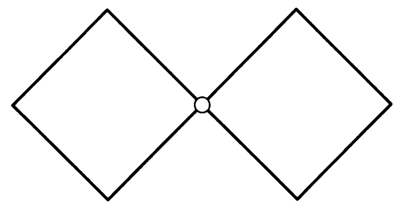
Questions:

a. Is the structure kinematically determinate or indeterminate? If the structure is kinematically indeterminate, show the movement (displacements) that can occur freely. If the structure is kinematically determinate, go to question b.
b. Is the structure statically determinate or indeterminate? If the structure is statically indeterminate, give the degree of static indeterminacy n.

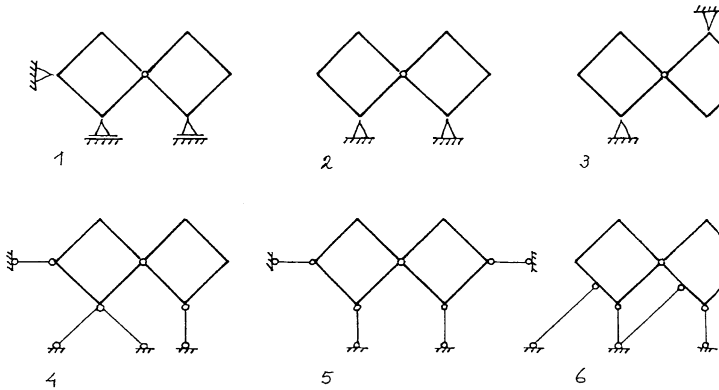
4.17 *Question*:
Which structure is kinematically determinate and statically indeterminate?



4.18 *Question*:
Show that for a kinematically determinate support of the two blocks connected by a hinge, four support reactions are required.



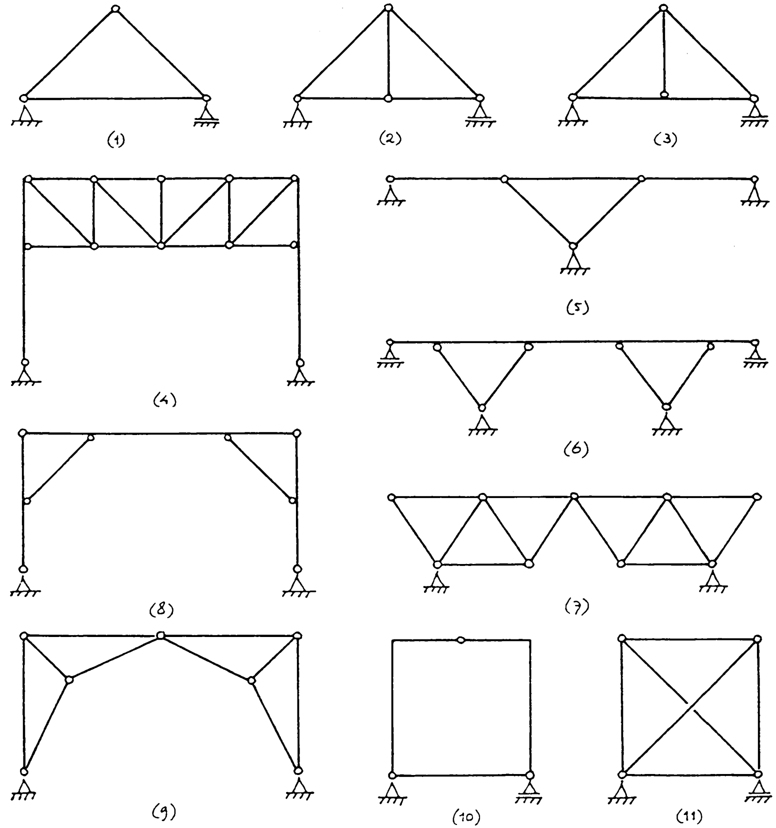
4.19: 1–6 Two square blocks are connected by a hinge and supported in a variety of ways.



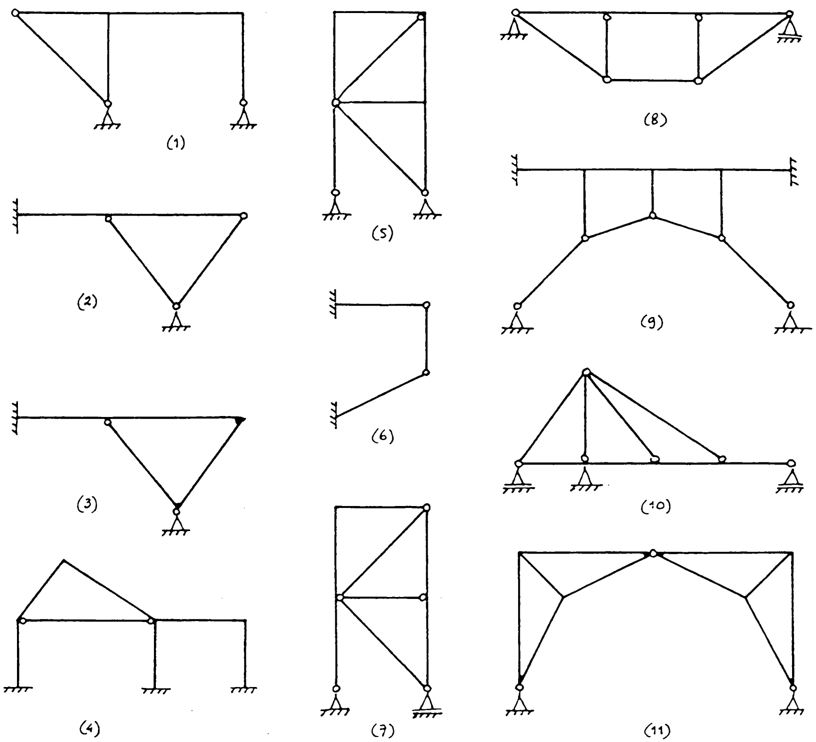
Question:
Determine whether the number of two-force members for the given method of support is sufficient.

4.20: 1–11 A number of kinematically determinate structures are shown.
Question:

a. Is the structure a truss or not?
b. Is the structure statically determinate or indeterminate?
c. If the structure is statically indeterminate, give the degree of static indeterminacy.



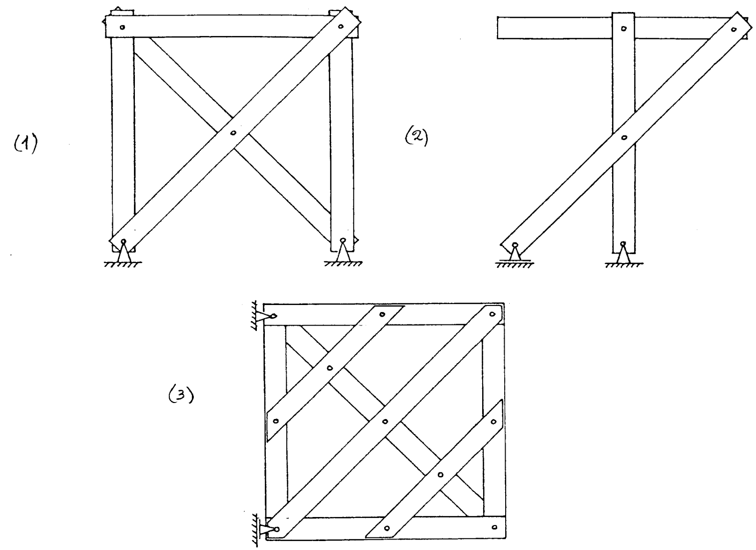
4.21: 1–11 A number of kinematically determinate bar structures are shown.



Question:

a. Is the structure statically determinate or indeterminate?
b. If the structure is statically indeterminate, indicate the degree of static indeterminacy.

4.22: 1–3 The structures shown are constructed from a number of planks. All the joints are hinges.



Question:

a. Is the structure a truss?
b. Is the structure statically determinate or indeterminate?
c. If the structure is statically indeterminate, give the degree of static indeterminacy.