3

Statics of a Rigid Body

So far, we have discussed the behaviour of a particle with negligibly small dimensions, for which one can assume that all forces act on the particle at the same point. In this chapter, we will show that for the equilibrium of a rigid body with certain dimensions, the point of application (or actually the lines of action) of the various forces are of critical importance.

The equilibrium of a particle demands that the resultant of all forces be zero. This condition is also necessary for a body, but is not sufficient. Forces on a body can together form a *couple* that will try to turn it. In this chapter, we will define the *moment of a couple* as well as the *moment of a force*. Equilibrium demands that a body does not rotate. In addition to the *force equilibrium* of a body, if it is to be in equilibrium, it must also be in *moment equilibrium*.

In the first instance, in order to keep the discussion simple, we will look only at coplanar forces. Section 3.1 addresses compounding and resolving forces and moments, while Section 3.2 looks at the equilibrium of a body in a plane.

When considering equilibrium, we can consider forces as sliding vectors. In the spatial discussion in Section 3.3, we will talk about the fact that moments of a force and of a couple are vectors. The chapter ends with Section 3.4, in which we look at equilibrium equations for a body in space.

The discussions relate to rigid bodies. In reality, there are no rigid bodies, as all solids are deformable. Most construction material deforms so little, however, that for equilibrium of a body, it can often be considered as non-deformable.[1]

3.1 Coplanar forces and moments

3.1.1 Motion of a rigid body

If several forces act on a body with particular dimensions, they can have various points of application. For the motion (the equilibrium) of a body, it is certainly important *where* the forces act. For example, with a billiard ball, it makes a difference whether one strikes the ball on the left or on the right. And if you want to lift the construction element in Figure 3.1, it makes a great difference whether you lift it from one of the upper corners or from the middle. Only in the latter case, on the basis of symmetry, can you expect the construction element not to rotate.

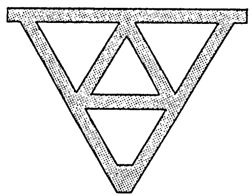


Figure 3.1 An element from the so-called "nabla beam" over the Haringvliet sluices, part of the Delta works in the Netherlands.

The movement of a rigid body differs from that of a particle in the sense that we also have to take the *rotation* of the body into consideration.

If we investigate the free motion of a rigid body, under the action of forces with zero resultant, there is a particular point that moves with uniform speed in a straight line (or is and stays at rest). This point about which the body

1 There are exceptions. For example, a *stability investigation* – an investigation into the *reliability of the equilibrium* – investigates how the distribution of forces changes as a result of deformation of the structure. In such cases, one has to relate the equilibrium to the deformed geometry, however small the deformations might be, and the structure may no longer be considered rigid. This topic falls outside the scope of this book.

can perform further rotations is called the *mass centre*, MC.[1]

Without addressing the theory, we will cover four examples of how a rigid body, which originally is at rest, starts to move if it is subject to forces. In order to keep the discussion simple, we will confine our attention to cases in which all the forces are coplanar.

1. The body is subject to two equal and opposite forces with the same line of action, see Figure 3.2a. The state of movement does not change: if the body is at rest it remains at rest. The two forces are *in equilibrium*. The equilibrium is not influenced by the location of the points of application of the forces on their common line of action. The forces can be moved along their lines of action without any effect on the motion.

2. The body is subject to a force of which the line of action passes through the mass centre MC (see Figure 3.2b).
   The mass centre MC will move in a straight line as if it were a particle in which the entire mass is concentrated. No rotation occurs: the body performs a *translation*. The effect of the force does not change when the point of application is chosen elsewhere on the line of action of the force.

3. The body is subject to two equal and opposite forces with parallel lines of action (see Figure 3.2c).
   The mass centre MC remains at rest, but the body starts to *rotate* about an axis through MC perpendicular to the plane in which the forces are applied. For the progression of the movement, it now matters whether the forces *maintain their direction*, or *turn with* the body.

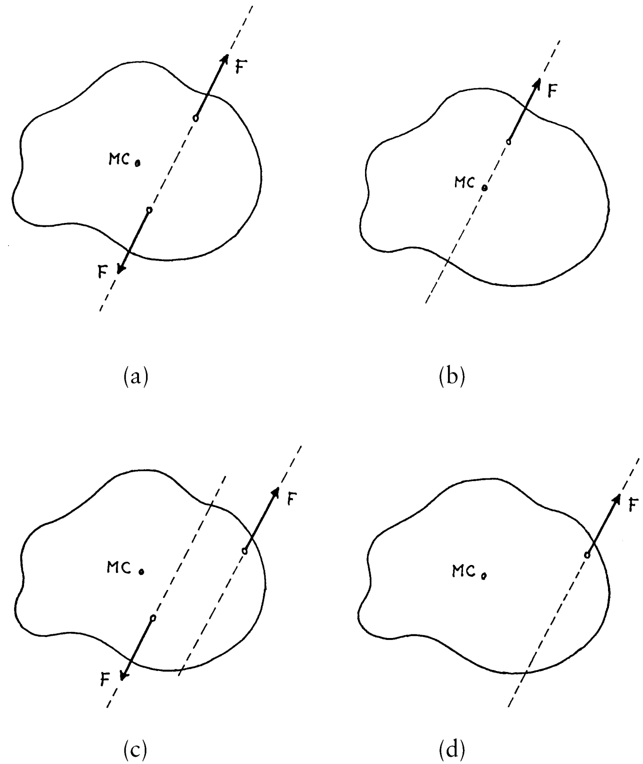
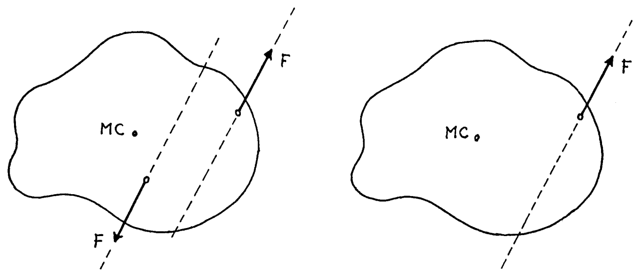


Figure 3.2 Equilibrium or motion of a body subject to forces: (a) equilibrium, (b) translation, (c) rotation about MC; (d) rotation and translation.

[1] Since in a homogenous gravitational field the *centre of gravity* and *mass centre* of a body are the same, both names are often used interchangeably.



(c) (d)

Figure 3.2 Equilibrium or motion of a body subject to forces: (c) rotation about MC; (d) rotation and translation.

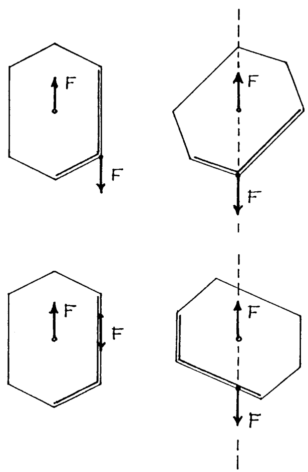


Figure 3.3 Forces that maintain their direction during rotation are fixed vectors: the final position depends on the location of the points of application.

If the forces turn with the body, it does not matter where they are exerted on their lines of action and they can be moved along their lines of action. If the forces maintain their direction, such as forces resultant from the gravitational field, they cannot be moved along their lines of action.

In Figure 3.3 this is illustrated by a plate subject to a pair of forces. One of the forces acts on the middle of the plate, the other on a point on the edge. Under the influence of these forces, the plate will move to a state of equilibrium, in which the lines of action coincide. The final position depends on where the forces are applied.

If one limits oneself to the so-called *instantaneous movement* of the body, or in other words the movement immediately after the application of the forces when the rotations are still very small, then the difference noted disappears, and the forces may be moved along their lines of action. The difference also disappears if one investigates the equilibrium of a body at rest, a situation without rotation.

4. A force acts on a body, and the line of action does not pass through the mass centre MC (see Figure 3.2d).
   The mass centre will start to move as if the force were applied directly to MC, and the body will also rotate about MC. The body experiences both a *translation* and a *rotation*.

Conclusion: *For the equilibrium (or instantaneous movement) of a rigid body, it does not matter at which point of its line of action a force is applied. The force on a rigid body can therefore be seen as a sliding vector. Although physically impossible, one can therefore also allow a force to "apply itself" to a point outside the body.*

Note: In investigating the *deformation* or *phenomena inside a body*, one cannot move a force along its line of action, and the force must be considered as a fixed vector.

In the bar in Figure 3.4, one can clearly see what happens if one changes the points of application of the two equal and opposite forces $F_1 = F$ and $F_2 = F$, with a common line of action. As far as the equilibrium is concerned, it is irrelevant where F_1 and F_2 are applied, while it certainly makes a difference to what happens "*internally*" and for the *deformation of the bar*: the upper bar is loaded by a tensile force and will lengthen, while the lower bar is loaded by compression, and will shorten.

3.1.2 Graphical composition of non-parallel forces

In the previous section it was stated that when considering the (instantaneous) movement and equilibrium of a rigid body, one can shift the forces along their lines of action. This means that it is possible to determine the resultant R of the two forces F_1 and F_2 in Figure 3.5 graphically by shifting them both to the intersection of their lines of action, and then applying the parallelogram rule. The resultant R is an *imaginary* force that with respect to the equilibrium of the body has the same effect as the two forces F_1 and F_2 together. We say that R is *statically equivalent* to F_1 and F_2.

Besides the magnitude and direction of the resultant, we also find the location of its line of action ℓ. It is pointless talking about the point of application, only its line of action is fixed.

The magnitude and direction of the resultant can also be determined in a force polygon (see Figure 3.6). The line of action is determined by realising that it has to pass through the intersection of the lines of action of the forces to be compounded. Note that here the line of action of the resultant is entirely outside the body!

If several forces have to be compounded together, this can be done in phases by first determining the resultant of two forces, then compounding it with the third force, and so forth. This procedure is shown in Figure 3.7a.

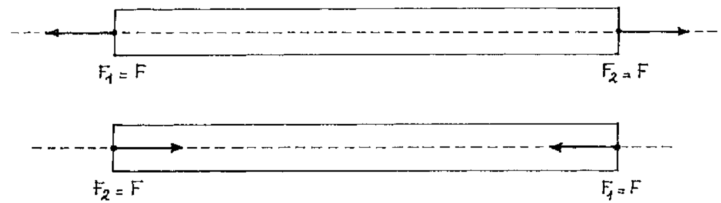


Figure 3.4 When considering equilibrium, one can shift forces along their lines of action. This is not permitted for considerations of what happens "internally".

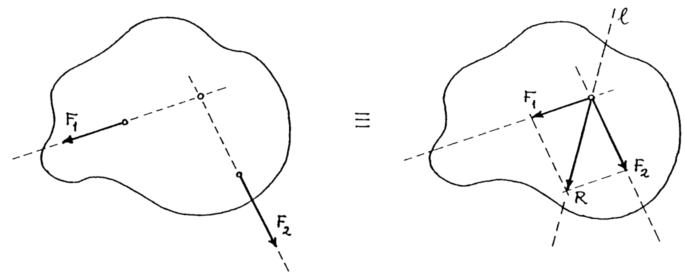


Figure 3.5 Compounding two forces using the parallelogram rule.

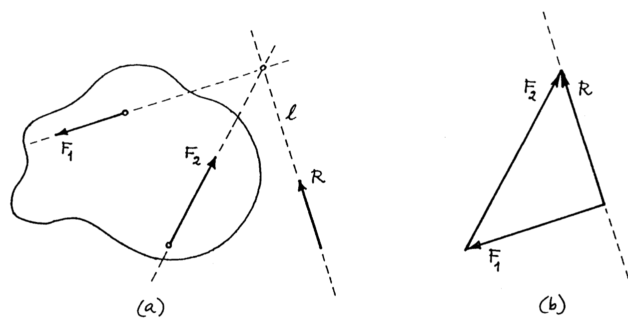


Figure 3.6 Compounding two forces using (a) line of action figure and (b) force polygon.

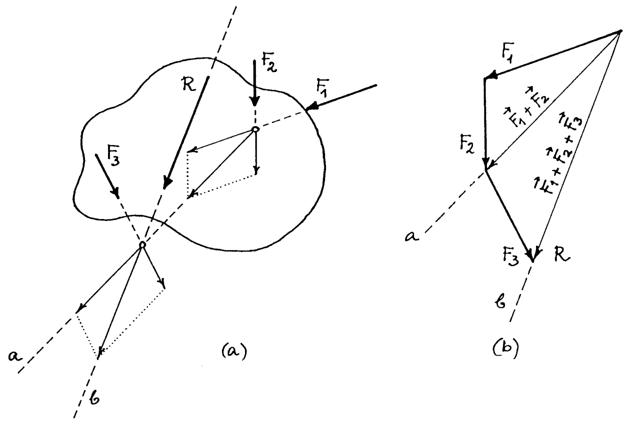


Figure 3.7 Compounding several forces using (a) a line of action figure and (b) force polygon.

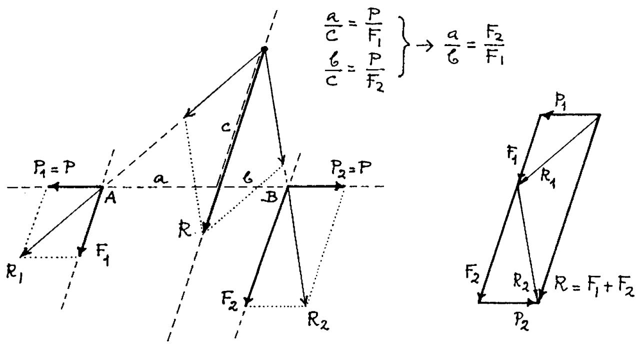


Figure 3.8 Compounding two parallel forces F_1 and F_2 graphically by adding the equilibrium system $P_1 = P$ and $P_2 = P$. Here we use for the forces the visual notation.

The magnitude and direction of the resultant can also be found quickly using a *force polygon*, as in Figure 3.7b. The force polygon does not provide information about the location of the line of action, however. To find the line of action, one would have to revert to Figure 3.7a. This figure is referred to as the *line of action figure*.

For more than two forces, using the line of action figure becomes laborious, and the analytical approach is clearly preferable (see Section 3.1.7). To determine the magnitude and direction of the resultant, the force polygon can still be useful.

3.1.3 Graphical composition of parallel forces

If the forces F_1 and F_2 are almost parallel, or parallel, one can determine the magnitude and direction of the resultant R graphically in a force polygon, although the graphical construction of its line of action (the line of action figure) becomes difficult as the intersection of the lines of action is far away or even at infinity.

In Figure 3.8, F_1 and F_2 are two parallel forces. The body on which the forces act is not shown. A graphical construction of the line of action is possible by having two equal yet opposite forces $P_1 = P$ and $P_2 = P$ apply to point A on the line of action of F_1, and to point B on the line of action of F_2, with AB as their common line of action. The magnitude of P can be chosen arbitrarily.

Since P_1 and P_2 together form an equilibrium system, the combined effect of the forces F_1, F_2, P_1 and P_2 is equal to that of only F_1 and F_2.

If R_1 is the resultant of F_1 and P_1, and R_2 is the resultant of F_2 and P_2, then the line of action of the resultant of all the forces, that is the resultant R of F_1 and F_2, passes through the intersection of the lines of action of R_1 and R_2.

From the graphical construction, one can see that the line of action of the resultant R of two parallel forces F_1 and F_2, acting in the same direction, is between their lines of action, nearer the larger force, and such that the distances a and b to the lines of action of F_1 and F_2 respectively are reversed proportionally to the magnitudes of these forces (see Figure 3.9a):

$$\frac{a}{b} = \frac{F_2}{F_1}.$$

If the two parallel forces F_1 and F_2 have opposite directions, then the resultant R has the same direction as the larger of the two forces, and the line of action of R is outside the lines of action of F_1 and F_2 on the side of the larger force. Now too, the distances a and b from the line of action of R to the lines of action of F_1 and F_2 are reversed proportionally to the magnitude of these forces (see Figure 3.9b).

In conclusion, for the resultant R of two parallel forces F_1 and F_2:

- R is in the direction of the larger force;
- the line of action of R is closer to the larger force;
- R is between F_1 and F_2 if these forces are in the same direction;
- R is outside F_1 and F_2 if these forces have opposite directions.

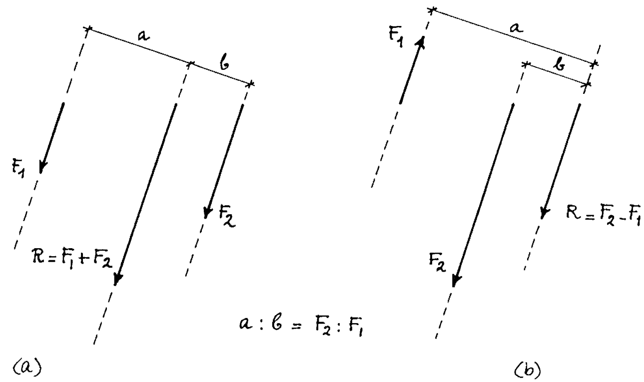


Figure 3.9 The resultant R of two parallel forces F_1 and F_2, in (a) the same and (b) opposite directions.

3.1.4 Moment of a couple

Figure 3.10 shows the special case of two equal and opposite parallel forces $F_1 = F$ and $F_2 = F$. If we want to graphically compound these forces in the way described above, using two equal and opposite additional forces $P_1 = P$ and $P_2 = P$, with the common line of action AB, we again find two equal and opposite parallel forces R_1 and R_2 (see Figure 3.11).

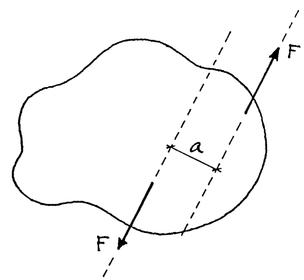
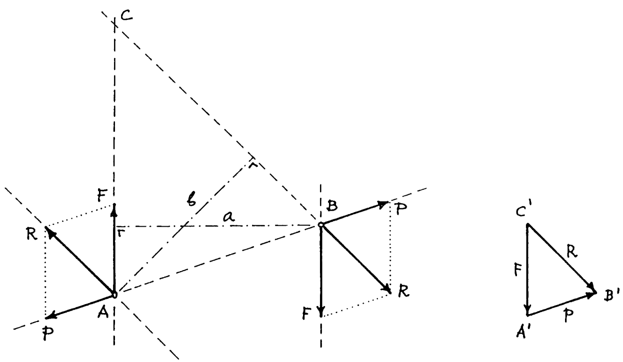


Figure 3.10 The pair of forces F forms a couple. a is the couple arm. The product Fa is the moment of the couple.

It is impossible to compound the pair of forces F into a single force. We call such a pair of forces a couple. The product of the magnitude of F of the forces and the distance a between the lines of action is called the *moment of the couple*. As symbol for this quantity we use the letter T:

$$T = Fa.$$



line of action figure

force polygon

Figure 3.11 The result of a couple does not change if one replaces it by another couple with the same moment and the same direction of rotation: $Fa = Rb$.

a is referred to as the *couple arm* and is always measured perpendicularly to the lines of action.

The two forces $R_1 = R$ and $R_2 = R$ also form a couple. Here the moment of the couple is

$$T = Rb,$$

b is the couple arm.

Since $P_1 = P$ and $P_2 = P$ form an equilibrium system, the effect of the couple caused by the forces R with arm b is equal (*statically equivalent*) to the effect of the couple formed by the forces F with arm a. The moment of the couple is therefore the same for both:

$$T = Fa = Rb.$$

This can also be derived from line of action figure in Figure 3.11.

Consider triangle ABC; its area is

$$\text{area ABC} = \tfrac{1}{2}a \cdot \text{AC} = \tfrac{1}{2}b \cdot \text{BC},$$

so that

$$\frac{a}{b} = \frac{\text{BC}}{\text{AC}}.$$

Triangle ABC, from the line of action figure, is geometrically similar to force triangle A′B′C′, so that the corresponding sides are proportional:

$$\frac{\text{BC}}{\text{AC}} = \frac{\text{B}'\text{C}'}{\text{A}'\text{C}'} = \frac{R}{F}.$$

On combining these two equations we deduce that

$$\frac{a}{b} = \frac{R}{F},$$

which is equivalent to

$$Fa = Rb.$$

Conclusion: *The effect of a couple on the equilibrium of a body does not change if you replace it by another couple with the* same moment *and* the same direction of rotation.

The magnitude of the moment of a couple determines the state of rotation of the body. In addition to a *magnitude*, the moment also has a *direction of rotation*. The sign for the direction of rotation is linked to the coordinate system, see the sign convention in Section 1.3.2.

In the xy coordinate system shown, the moment of the couple in Figure 3.12a is

$$T_z = -Fa = -12 \text{ kNm}.$$

The letter T is given the index z, which indicates the normal of the plane in which the couple acts.

In Figure 3.12b the moment of the same couple is in another coordinate system:

$$T_y = +Fa = +12 \text{ kNm}.$$

In Figure 3.12c, the couple is represented by a curved arrow. In this *visual notation* the arrow indicates the direction of rotation of the moment and includes a value. The same conventions apply as for the visual notation of

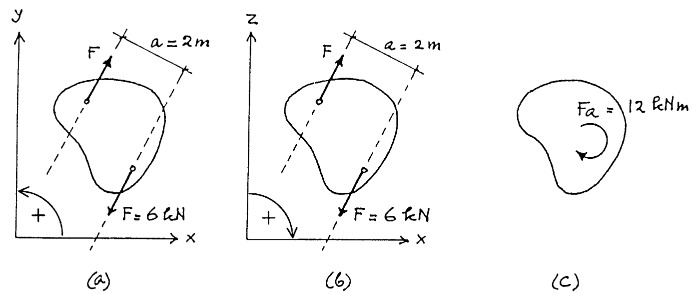


Figure 3.12 Three times the same couple: (a) $T_z = -12$ kNm, (b) $T_y = +12$ kNm, (c) the couple using visual notation.

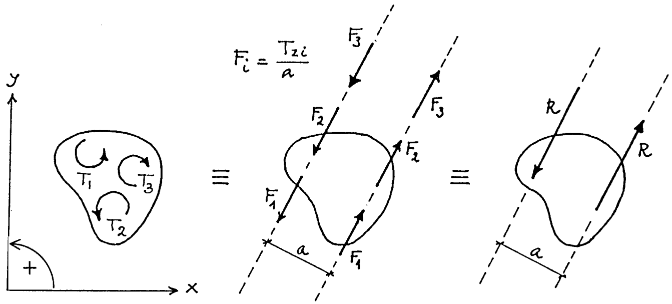


Figure 3.13 The moment of the resultant couple is found by adding the moments of the couples to be compounded (algebraically).

a force (see Section 1.3.6).

Couples can be compounded in many different ways. In Figure 3.13, the couples operating in the xy plane, T_1, T_2 and T_3, have been compounded by replacing them by equal couples of which the forces have common lines of action. If $T_1 = 12$ kNm, $T_2 = 6$ kNm and $T_3 = 10$ kNm, and the distance a between the lines of action is 4 m, then, in the coordinate system shown,

$$T_{z;1} = +T_1 = +12 \text{ kNm} = F_1 a \Rightarrow F_1 = T_{z;1}/a = +3 \text{ kN},$$

$$T_{z;2} = +T_2 = +6 \text{ kNm} = F_2 a \Rightarrow F_2 = T_{z;2}/a = +1.5 \text{ kN},$$

$$T_{z;3} = -T_3 = -10 \text{ kNm} = F_3 a \Rightarrow F_3 = T_{z;3}/a = -2.5 \text{ kN}.$$

Note: The force F_3 has the value 2.5 kN and acts opposite to the direction shown in Figure 3.13.

The moment of the resultant couple is

$$T_z = Ra = (F_1 + F_2 + F_3) \cdot a = \sum_{i=1}^{3} T_{z;i} = 8 \text{ kNm}.$$

If all the couples are exerted in the same plane, the moment of the resultant couple is found by compounding the couple moments simply by adding them together.

The example shows that the couples form an equilibrium system if the sum of their moments is zero (because $R = 0$).

3.1.5 The moment of a force about a point

The *moment of a force* about a point A is defined as the product of magnitude F of the force and the perpendicular distance a from point A to the line of action of the force. The sign of the moment is plus or minus, depending on whether the force F turns the body in the positive or negative direction of rotation about A.

For Figure 3.14, the moment of force F with respect to A is seen as positive as F causes a rotation about A in the positive direction of rotation in the xy plane:

$$T_z|\mathrm{A} = +Fa = +(10\ \mathrm{kN})(4\ \mathrm{m}) = +40\ \mathrm{kNm}.$$

The same force F causes the body to rotate about B in the negative direction of rotation. The moment of F about B is therefore negative:

$$T_z|\mathrm{B} = -Fb = -(10\ \mathrm{kN})(5\ \mathrm{m}) = -50\ \mathrm{kNm}.$$

The moment of the force F about a point C located on its line of action, is zero:

$$T_z|\mathrm{C} = 0.$$

For a force, in contrast to a couple, one has to specify the point about which the moment is being calculated. Here, this is done by including the point in question, after a vertical line, in the expression for the moment.

Figure 3.15 shows a single force F acting at point B. Now introduce two equal and opposite forces $F_1 = F$ and $F_2 = F$ acting at point A. Since F_1 and F_2 together form an equilibrium system, the single force F at B is statically equivalent to the three forces F at B and F_1 and F_2 at A. F at B and $F_2 = F$ at A together form a couple with moment Fa. The force $F = 7$ kN at B is therefore statically equivalent with a force $F = 7$ kN at B and a couple with moment $Fa = 21$ kNm.

Conclusion: *The moment of a force F about a point A is equal to the moment of the couple one has to add when moving the force parallel to its a line of action to A.*

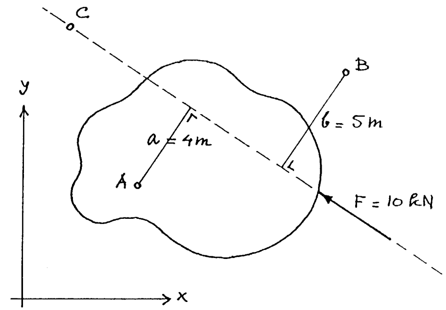


Figure 3.14 The moment of a force with respect to a point is defined as the product of the magnitude of the force and the perpendicular distance to its line of action.

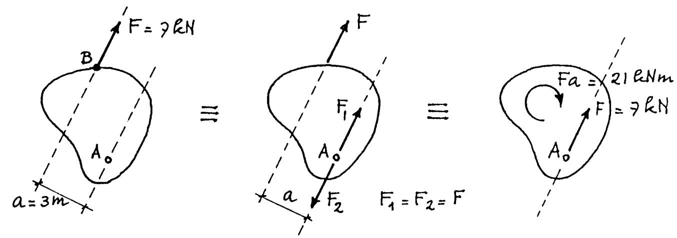


Figure 3.15 The moment of the force F about point A is equal to the moment of the couple that one has to add to the force if one shifts it parallel to its line of action through A.

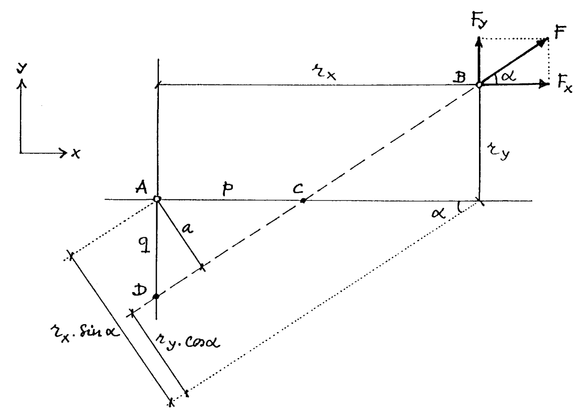


Figure 3.16 The moment of the force F about point A is equal to the sum of the moments about A of its components: $T_z|\text{A} = Fa = F_y r_x - F_x r_y$.

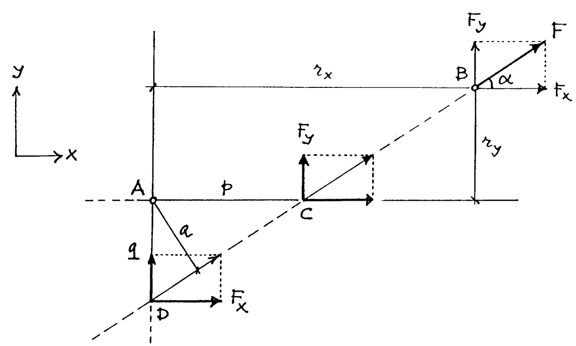


Figure 3.17 The moment of a force does not change if the force is shifted along its line of action: $T_z|\text{A} = Fa = F_y p = F_x q$.

The moment of a force F in the xy plane about a point A in the same plane can be calculated in a variety of ways (see Figure 3.16). The components of F are

$$F_x = F\cos\alpha,$$

$$F_y = F\sin\alpha.$$

If the force is applied in a point B, then

$$r_x = x_\text{B} - x_\text{A},$$

$$r_y = y_\text{B} - y_\text{A}.$$

From the figure one can derive

$$a = r_x \sin\alpha - r_y \cos\alpha.$$

For the moment of F about A applies

$$T_z|\text{A} = Fa = F(r_x \sin\alpha - r_y \cos\alpha) = F_y r_x - F_x r_y.$$

$F_y r_x$ is the moment of the component F_y about A, and $-F_x r_y$ is the moment of the component F_x about A. This shows that the moment of a force F about a point A is equal to the sum of the moments about A of its components.

Since the moment of a force does not change if the force is moved along its line of action, it is sometimes useful to shift the force to point C or D (see Figure 3.17). In this case, the moment of F about A is

$$T_z|\text{A} = Fa = F_y p = F_x q.$$

Example

The moment about A of the force at B in Figure 3.18 can now be calculated as follows:

- Force multiplied by the distance to its line of action:

$$T_z|\text{A} = -(2\sqrt{5}\text{ kN})(2\sqrt{5}\text{ m}) = -20\text{ kNm}.$$

- Force in B resolved into its components:

$$T_z|\text{A} = -(4\text{ kNm})(3\text{ m}) - (2\text{ kNm})(4\text{ m}) = -20\text{ kNm}.$$

- Force shifted to C:

$$T_z|\text{A} = -(2\text{ kN})(10\text{ m}) = -20\text{ kNm}.$$

- Force shifted to D:

$$T_z|\text{A} = -(4\text{ kN})(5\text{ m}) = -20\text{ kNm}.$$

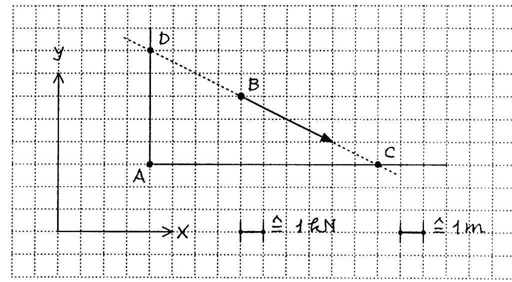


Figure 3.18 The moment about A of the force at B can be calculated in various ways.

3.1.6 Moment theorems

In Figure 3.19, R is the resultant of the forces F_1 and F_2:

$$R_x = F_{x;1} + F_{x;2},$$

$$R_y = F_{y;1} + F_{y;2}.$$

In order to be able to determine the moment of F_1 and F_2 about an arbitrary point A, both forces are shifted to the intersection of their lines of action. In the previous section, it was shown that the moment of a force about an arbitrary point is equal to the sum of the moments of its components about that point. Therefore, for the moment of F_1 and F_2 about A it is true that:

$$\begin{aligned}\sum T_z|\text{A} &= (T_z|\text{A due to } F_1) + (T_z|\text{A due to } F_2)\\ &= (F_{y;1}r_x - F_{x;1}r_y) + (F_{y;2}r_x - F_{x;2}r_y)\\ &= (F_{y;1} + F_{y;2})r_x - (F_{x;1} + F_{x;2})r_y\end{aligned}$$

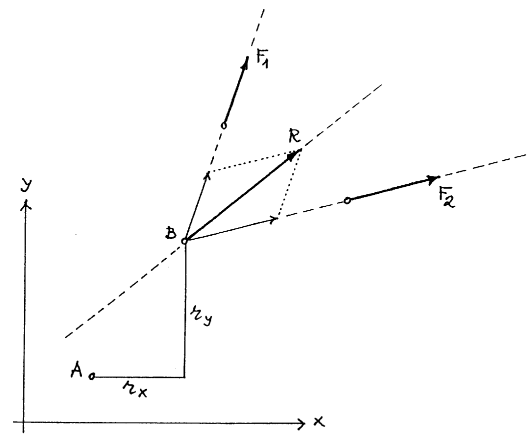


Figure 3.19 The sum of the moments of F_1 and F_2 about an arbitrary point A is equal to the moment of the resultant R about that point A. This is known as Varignon's First Theorem.

$$= R_y r_x - R_x r_y$$

$$= T_z|\text{A due to } R.$$

Conclusion: *If two forces* F_1 *and* F_2 *have a resultant* R*, the sum of the moments of* F_1 *and* F_2 *about an arbitrary point A is equal to the moment of the resultant* R *about that point A. This is called* Varignon's First Moment Theorem.[1] *The theorem also applies if* F_1 *and* F_2 *have parallel lines of action.*

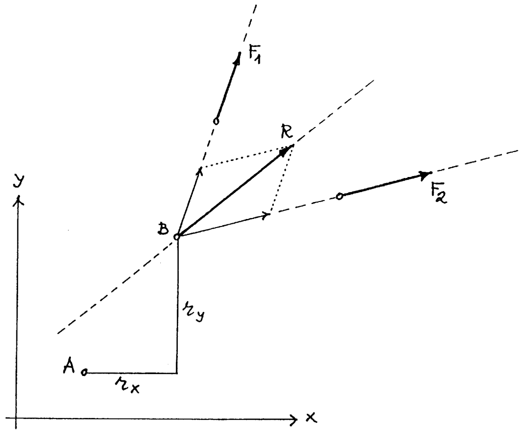


Figure 3.19 The sum of the moments of F_1 and F_2 about an arbitrary point A is equal to the moment of the resultant R about that point A. This is known as Varignon's First Theorem.

If the two forces F_1 and F_2 together form a couple, the sum of the moments of F_1 and F_2 is independent of the point with respect to which the moment is determined. This sum of moments is equal to the moment of the couple. This is known as the *Varignon's Second Moment Theorem.*

Varignon's momentary theorems can be applied repeatedly if several forces act in the same plane. This results in the following *General Moment Theorem*:
The sum of the moments of a number of forces distributed in a plane, about an arbitrary point A in that plane, is *either* equal to the moment of the resultant force about that point or equal to the moment of the resultant couple.

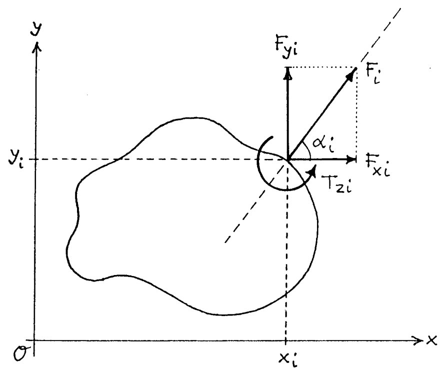


Figure 3.20 A body loaded by couples $T_{z;i}$ and forces F_i, with components $F_{x;i}$ and $F_{y;i}$, at points i ($i = 1, 2, 3, \ldots$).

3.1.7 Compounding forces and moments analytically

Compounding coplanar forces and couples analytically is now relatively simple. Each of the forces F_i ($i = 1, 2, \ldots$) can be resolved into the components $F_{x;i}$ and $F_{y;i}$, and for each of these forces, we can now determine the moment about an arbitrary point A. In fact, this means that all the forces are shifted to point A with addition of a couple (see Section 3.1.5). If we place the origin O of the coordinate system at A, and x_i and y_i are the

[1] Pujol Varignon (1654–1722) was a French mathematician.

coordinates of the point of application of force F_i (or of another point on the line of action of F_i), then (see Figure 3.20)

$$R_x = \sum F_{x;i} = \sum F_i \cos\alpha_i,$$

$$R_y = \sum F_{y;i} = \sum F_i \sin\alpha_i,$$

$$\sum T_z|\mathrm{O} = \sum\{(F_{y;i}x_i - F_{x;i}y_i) + T_{z;i}\}.$$

The sum of the moments also includes the moments of the (concentrated) couples $T_{z;i}$ that may be applied on the body.

For the (instantaneous) movement or the equilibrium of a rigid body, one may replace the force system by a single resultant force R at O together with a couple $\sum T_z|\mathrm{O}$ (see Figure 3.21a).

The resultant force R at O can be compounded with the couple $\sum T_z|\mathrm{O}$ into a single force R by shifting it parallel to itself to a line of action at a perpendicular distance a from O (see Figure 3.21b):

$$a = \frac{\sum T_z|\mathrm{O}}{R}.$$

The line of action of R can also be found as follows. Imagine that (x, y) is an arbitrary point on the line of action of R (see Figure 3.21b). According to the moment theorem,

$$\sum T_z|\mathrm{O} = R_y x - R_x y.$$

The values for $\sum T_z|\mathrm{O}$, R_x and R_y are known, while those of x and y are unknown. This expression therefore also provides the equation for the line of action of R. The line of action of R intersects the x axis at

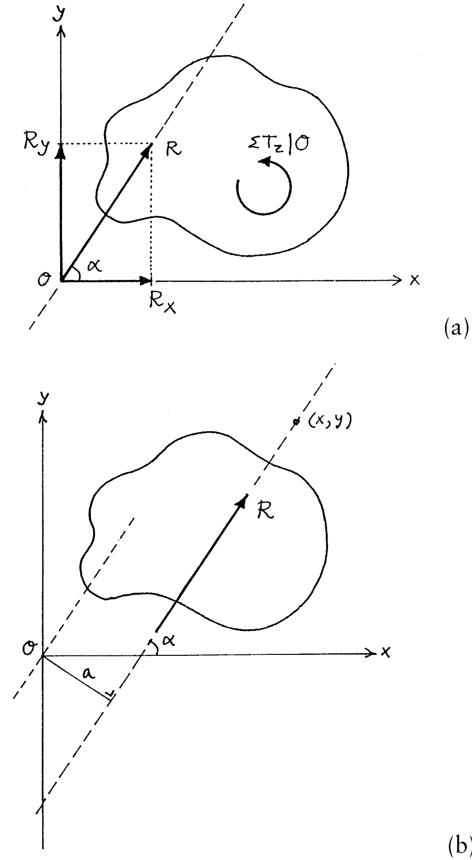


Figure 3.21 (a) The resultant force R at O and the associated couple $\sum T_z|\mathrm{O}$ are statically equivalent to (b) a force R at a distance $a = (\sum T_z|\mathrm{O})/R$ from O.

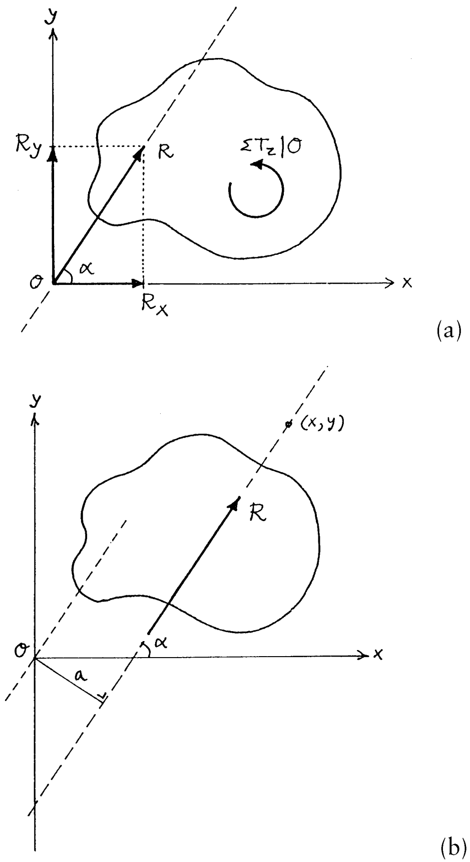


Figure 3.21 (a) The resultant force R at O and the associated couple $\sum T_z|\mathrm{O}$ are statically equivalent to (b) a force R at a distance $a = (\sum T_z|\mathrm{O})/R$ from O.

$$x = \frac{\sum T_z|\mathrm{O}}{R_y}; \quad y = 0,$$

and the y axis at

$$x = 0; \quad y = -\frac{\sum T_z|\mathrm{O}}{R_x}.$$

A special case is when $R = 0$ and $\sum T_z|\mathrm{O} \neq 0$. In this case, there is no resultant force, while there is a resultant couple. When also $\sum T_z|\mathrm{O} = 0$, then there is equally no resultant couple and the forces together form an equilibrium system.

To summarise, with respect to the resultant of a system of forces and couples, one can distinguish the following cases:

- $R \neq 0$ and $\sum T_z|\mathrm{O} \neq 0$
  There is a resultant force, and the line of action does not pass through O.
- $R \neq 0$ and $\sum T_z|\mathrm{O} = 0$
  There is a resultant force of which the line of action passes through O.
- $R = 0$ and $\sum T_z|\mathrm{O} \neq 0$
  There is no resultant force, but there is a resultant couple.
- $R = 0$ and $\sum T_z|\mathrm{O} = 0$
  The forces and couples together form an equilibrium system.

Example

Three forces and a couple are exerted on the triangular block in Figure 3.22a. The magnitude and the direction of the forces can be found in the diagram, as can the direction of couple T. The magnitude of the couple is 30 kNm.

Question:
Determine the magnitude, direction, and line of action of the resultant force on the block.

Solution:
For convenience sake, the units (kN and/or m) are not always shown in the interim calculations. For the components of the resultant force R applies

$$R_x = \sum_{i=1}^{3} F_{x;i} = -10 + 30 + 0 = +20 \text{ kN},$$

$$R_y = \sum_{i=1}^{3} F_{y;i} = 0 + 20 - 40 = -20 \text{ kN}$$

so that

$$R = \sqrt{R_x^2 + R_y^2} = \sqrt{20^2 + (-20)^2} = 20\sqrt{2} \text{ kN}.$$

The magnitude and direction of R and of its components can of course also be determined graphically by using a force polygon (see Figure 3.22b).

The moment about O of the three forces and the couple is

$$\begin{aligned}\sum T_z|\text{O} = {} & +10 \times 6 && \text{(for } F_1\text{)} \\ & +(20 \times 6 - 30 \times 3) && \text{(for } F_2\text{, resolved into its components)} \\ & -40 \times 4 && \text{(for } F_3\text{)} \\ & +30 && \text{(for } T\text{)} \\ = {} & -40 \text{ kNm.} \end{aligned}$$

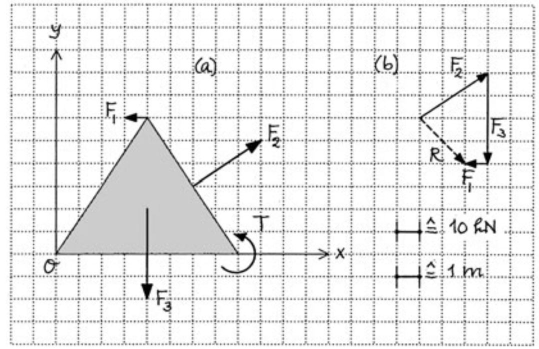


Figure 3.22 (a) A triangular block subject to three forces and a couple; (b) the resultant force R on the block, determined using a force polygon.

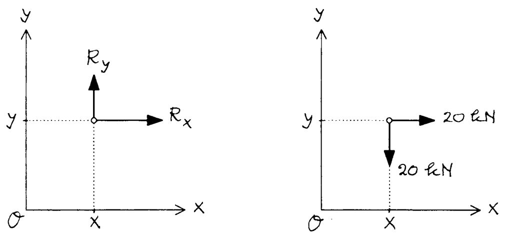


Figure 3.23 The resultant force R at a point (x, y) of its line of action: (a) in its components R_x and R_y and (b) in components as they act in reality.

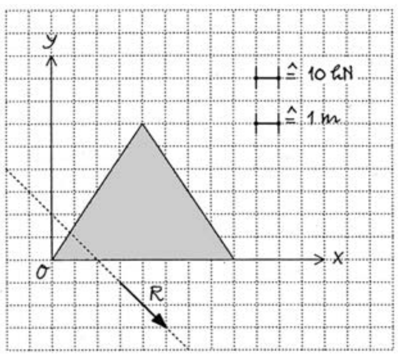


Figure 3.24 The resultant R and its line of action.

The resultant R must have the same moment about O as the three forces and the couple. Imagine (x, y) is a point on the line of action of R, with components R_x and R_y (see Figure 3.23a). Then

$$\sum T_z|\mathrm{O} = -40\ \mathrm{kNm} = R_y x - R_x y.$$

With $R_x = +20$ kN and $R_y = -20$ kN, this gives the following equation for the line of action of the resultant R:

$$-40\ \mathrm{kNm} = (-20\ \mathrm{kN})x - (+20\ \mathrm{kN})y \Rightarrow x + y = 2\ \mathrm{m}.$$

Of course it is also possible to depict R_x and R_y as in Figure 3.23b, according to the actual magnitude and direction. This figure immediately gives the expression shown above fore the line of action of R. Figure 3.24 shows the resultant R with its line of action.

Note: If one performs the calculation using a picture, all the unknown quantities that are related to the coordinate system in that picture have to be shown *positively*. In Figure 3.23a that would be x, y, R_x and R_y, in Figure 3.23b this only relates to x and y.

3.1.8 Resolving a force along given lines of action graphically

A force F, with given magnitude, direction, and line of action, can be resolved along three given lines of action a, b and c, which do not intersect in one point, into the forces F_a, F_b and F_c (see Figure 3.25a).

Here

$$\vec{F} = \vec{F}_\mathrm{a} + \vec{F}_\mathrm{b} + \vec{F}_\mathrm{c},$$

so that

$$(\vec{F} - \vec{F}_a) = (\vec{F}_b + \vec{F}_c).$$

$(\vec{F} - \vec{F}_a)$ and $(\vec{F}_b + \vec{F}_c)$ are equal and therefore have the same line of action. The line of action of $(\vec{F} - \vec{F}_a)$ passes through the intersection A of the lines of action of $\vec{F}$ and $\vec{F}_a$. The line of action of $(\vec{F}_b + \vec{F}_c)$ passes through the intersection S_{ab} of the lines of action b and c. Therefore, AS_{bc} is the line of action of both $(\vec{F} - \vec{F}_a)$ and $(\vec{F}_b + \vec{F}_c)$. $\vec{F}$ at A can now be resolved into $\vec{F}_a$ with line of action a and $(\vec{F}_b + \vec{F}_c)$ with line of action AS_{bc}. Subsequently $(\vec{F}_b + \vec{F}_c)$ at S_{bc} can be resolved into $\vec{F}_b$ and $\vec{F}_c$. This is shown graphically in Figure 3.25b in a single force polygon.

The order in which $\vec{F}$ is resolved is irrelevant. In Figure 3.26 $\vec{F}$ is first resolved at B into $\vec{F}_b$ and $(\vec{F}_a + \vec{F}_c)$ and subsequently $(\vec{F}_a + \vec{F}_c)$ at S_{ac}, the intersection of the lines of action a and c, is resolved into $\vec{F}_a$ and $\vec{F}_c$. The force polygon now has a different shape, as the forces were resolved in a different order, but the result is the same.

The name *Culmann*[1] is associated with this graphical method in the literature.

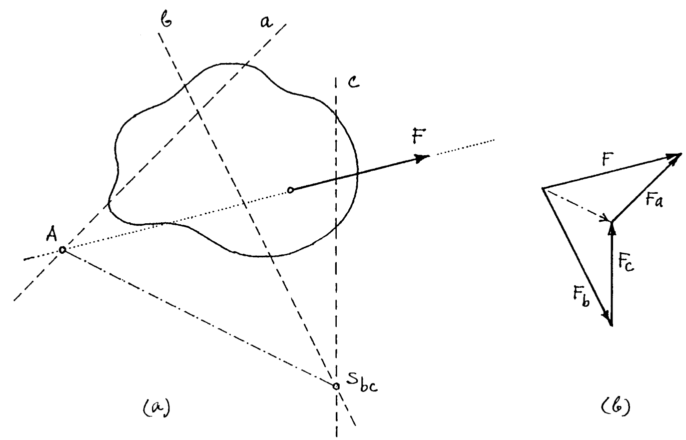


Figure 3.25 Resolving the force F graphically along three given lines of action; (a) line of action figure and (b) force polygon.

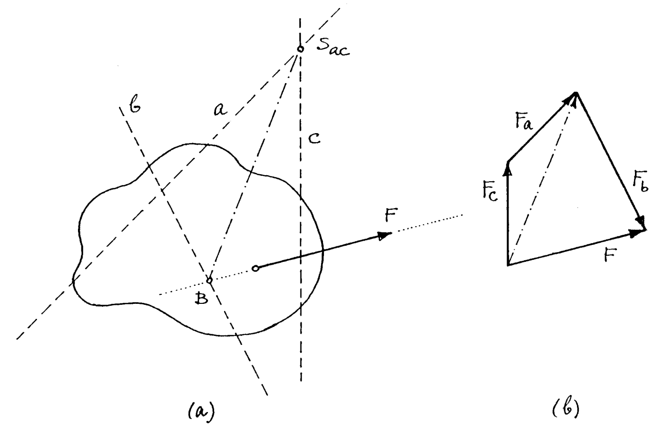


Figure 3.26 Resolving the force F graphically along three given lines of action; (a) line of action figure and (b) force polygon.

3.1.9 Resolving a force along given lines of action analytically

Resolving F into three forces F_a, F_b and F_c along given lines of action a, b, and c, can of course also be done analytically. Of the many possible methods, the method below is based on Varignon's first moment theorem:

[1] Karl Culmann (1821–1881), a German engineer, was involved in the design and construction of important railway bridges and was especially known for his graphical methods for calculating structures.

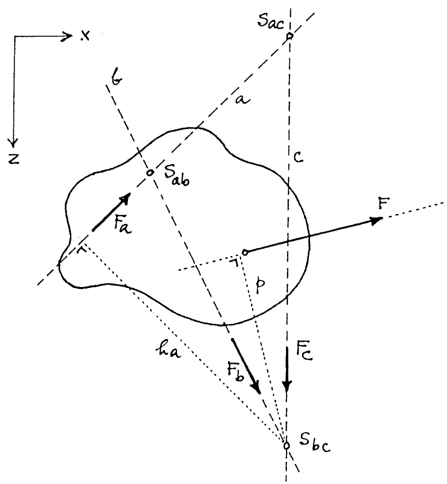


Figure 3.27 Analytically resolving force F along three given lines of action.

the moment of F about an arbitrary point is equal to the sum of the moments of F_a, F_b and F_c about that same point.

In Figure 3.27, the directions of the as yet unknown forces F_a, F_b, and F_c have been assumed. In addition, a coordinate system has been assumed in order to be able to indicate the sign of the moments (the direction of rotation).

If the moment theorem is applied with respect to S_{bc}, the intersection of the lines of action of F_b and F_c, then these forces do not contribute to the sum of the moments, and one can determine F_a directly:

$$\sum T_y|S_{bc} = -F \cdot p = -F_a \cdot h_a \Rightarrow F_a = \frac{p}{h_a}F.$$

Note: The signs are related to the xz coordinate system shown.

By applying the moment theorem in the same way with respect to S_{ac} and S_{ab} respectively, we also find F_a and F_c directly.

Since the direction of rotation of F about S_{ab} is opposite to that of F_c about S_{ab} the value of F_c will be negative. This means that the force F_c works opposite to the direction assumed in Figure 3.27.

The analytical approach can also be used for resolving a couple into three forces along given lines of action.

Example
The block in Figure 3.28a is subject to the three forces F_a, F_b and F_c, along given lines of action a, b and c. The resultant is the couple T with the direction shown in the figure.

Question:
Determine the three forces if $T = 80$ kNm.

Solution:
In Figure 3.28b an assumption was made with respect to the directions of the forces. In the coordinate system given

$$\sum T_z|\mathrm{A} = -\tfrac{4}{5}F_\mathrm{a} \times (4\ \mathrm{m}) = -T = -80\ \mathrm{kNm} \ \Rightarrow \ F_\mathrm{a} = +25\ \mathrm{kN},$$

$$\sum T_z|\mathrm{B} = +F_\mathrm{b} \times (4\ \mathrm{m}) = -T = -80\ \mathrm{kNm} \ \Rightarrow \ F_\mathrm{b} = -20\ \mathrm{kN}.$$

The minus sign in the latter answer shows that the force F_b acts in the opposite direction to that assumed in Figure 3.28b.

From $\sum T_z|\mathrm{C} = -80$ kNm we can derive F_c directly. Finding the location of C, the intersection of the lines of action, takes some calculation. The force F_c is therefore easier to find since the resultant force is zero:

$$\sum F_x = \tfrac{3}{5}F_\mathrm{a} + F_\mathrm{c} = \tfrac{3}{5} \times (25\ \mathrm{kN}) + F_\mathrm{c} = 0 \Rightarrow F_\mathrm{c} = -15\ \mathrm{kN}.$$

Apparently the direction of F_c was also falsely assumed. Figure 3.28c shows the forces as they are actually exerted on the block. It would indeed not be difficult to determine the correct directions of the forces prior to making the calculation.

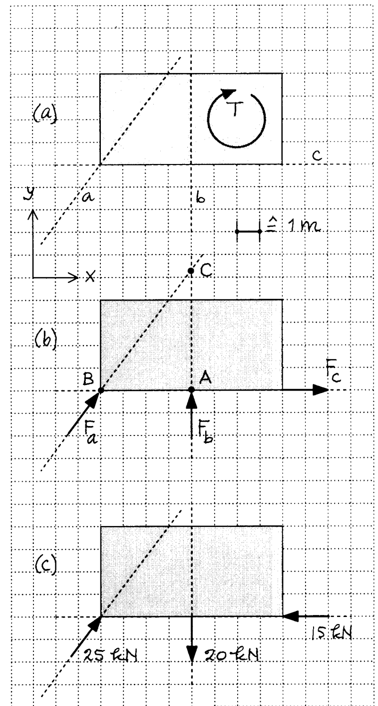


Figure 3.28 Resolving a couple into three forces along given lines of action: (a) the couple T and the lines of action a, b and c; (b) the assumed directions of the forces F_a, F_b and F_c; (c) the forces as they have to act on the block in reality if they are to be statically equivalent to the couple.

3.2 Equilibrium of a rigid body in a plane

For the (instantaneous) motion of a rigid body, the system of forces exerted on it can be replaced by a single force at an arbitrary point and a couple. When considering the motion of the body, it is preferable to choose the mass centre as that point, as the motion can then be split into a *translation* due to the force, and a *rotation* due to the couple (see Section 3.1.1).

From the above, it follows that a rigid body is in equilibrium if for all the forces exerted on it, the resultant force and the resultant couple are zero. The *equilibrium conditions* for a rigid body, only subject to forces in the xy plane, are:

$$\sum F_x = 0,$$
$$\sum F_y = 0,$$
$$\sum T_z = 0.$$

The summation symbol means that all contributions of the forces acting on the body have to be added.

The first two equations stand for the *force equilibrium* in respectively the x and y direction, and express that there is no resultant force. The third equation stands for the *moment equilibrium*, and expresses that the forces together do not form a resultant couple. Here, the moment with respect to an arbitrary point has to be determined for all the forces, and the moments have to be added together.

If (concentrated) couples are applied to the body, schematically represented by curved arrows, their moments of course also have to be included in the moment summation. The equations for the force equilibrium are not influenced by these couples.

For particles (with negligibly small dimensions), the force equilibrium is a necessary and sufficient condition for equilibrium. For rigid bodies (with finite measurements) the force equilibrium is a necessary but not sufficient condition for equilibrium; since a body can rotate, another condition is required, namely the moment equilibrium.

3.2.1 Equilibrium equations

In a plane, the equilibrium of a body is assured if it meets two conditions for the force equilibrium and one condition for the moment equilibrium:

$$\sum F_x = 0,$$
$$\sum F_y = 0,$$
$$\sum T_z = 0.$$

These *equilibrium equations* in a plane can be replaced by three arbitrary linear combinations, on the condition that these combinations are independent. Three of these combinations are mentioned separately below:

1. The condition of force equilibrium in two mutually perpendicular directions can be replaced by the condition that of all the forces, the sum of the components in two arbitrary directions is zero.
2. The equilibrium can also be described by three moment conditions with respect to three points A, B and C that are not in a straight line:

$$\sum T_z|\mathrm{A} = 0,$$
$$\sum T_z|\mathrm{B} = 0,$$
$$\sum T_z|\mathrm{C} = 0.$$

That these three equations are sufficient to ensure equilibrium can be shown as follows (see Figure 3.29). Each system of coplanar forces and couples can be replaced by either a *resultant force*, or a *resultant couple*. If $\sum T_z|\mathrm{A} = 0$, there is no resultant couple. There could still be a resultant force of which the line of action must pass through A. If $\sum T_z|\mathrm{B} = 0$, the line of action of the resultant force must also pass through B. If C is not located on AB (the line of action of the resultant force), and $\sum T_z|\mathrm{C} = 0$, the resultant force can only be zero.

3. The equilibrium can also be formulated by two moment conditions with respect to two points A and B and an equation for the force equilibrium in a direction that is not perpendicular to AB:

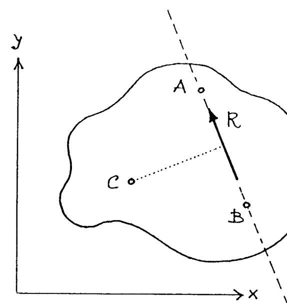


Figure 3.29 The relationships $\sum T_z|\mathrm{A} = 0$ and $\sum T_z|\mathrm{B} = 0$ imply that there is no resultant couple and that, if there is a resultant force R, its line of action is along AB.

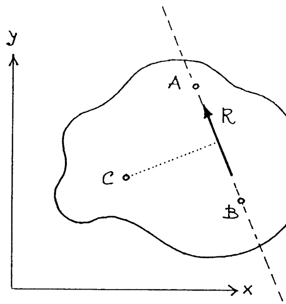


Figure 3.29 The relationships $\sum T_z|\mathrm{A} = 0$ and $\sum T_z|\mathrm{B} = 0$ imply that there is no resultant couple and that, if there is a resultant force R, its line of action is along AB.

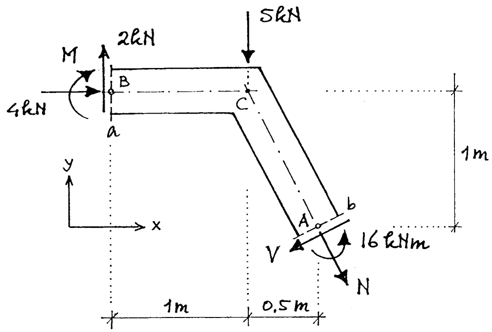


Figure 3.30 Corner joint in a frame; the three unknown section forces M, V and N can be deduced from the equilibrium.

$$\sum T_z|\mathrm{A} = 0,$$

$$\sum T_z|\mathrm{B} = 0,$$

$$\sum F_x = 0 \text{ (the } x \text{ direction may not be perpendicular to AB).}$$

The relationships $\sum T_z|\mathrm{A} = 0$ and $\sum T_z|\mathrm{B} = 0$ imply that there is no resultant couple and that, if there is a resultant force, its line of action coincides with AB (see Figure 3.29). The resultant force is zero if the condition for force equilibrium is met in the direction AB, or in another direction that is not perpendicular to AB.

The equilibrium conditions can therefore be formulated in various ways. For a manual calculation, one always has to look for equilibrium equations that are as simple as possible in order to limit the amount of calculation. When using a computer for the calculation, the systematics and the general applicability of the set up of the calculation (the program) are more important than the number of calculations involved and the laborious character of the calculations.

Example

Figure 3.30 shows the corner joint of a frame. The joint is loaded at C by a vertical force of 5 kN. So-called section forces act on the cross-sectional planes a and b. They act in the centre lines shown. The system is in equilibrium.

Question:

Determine the three unknown section forces M, V and N (with the correct sign for the directions shown).[1]

[1] M (bending moment), V (shear force) and N (normal force) are section forces. Their nomenclature and sign conventions will be revealed in Chapter 10.

Solution:
The two unknown section forces V and N are determined using the two equations for the force equilibrium. For the coordinate system shown,

$$\sum F_x = +(4\text{ kN}) - \tfrac{2}{5}\sqrt{5} \times V + \tfrac{1}{5}\sqrt{5} \times N = 0,$$

$$\sum F_y = +(2\text{ kN}) - (5\text{ kN}) - \tfrac{1}{5}\sqrt{5} \times V - \tfrac{2}{5}\sqrt{5} \times N = 0.$$

These are two equations with two unknowns. The solution is

$$V = +\sqrt{5}\text{ kN} \quad\text{and}\quad N = -2\sqrt{5}\text{ kN}.$$

It would also be possible to construct a closed force polygon and to derive the forces from there. This is shown in Figure 3.31. The force of $2\sqrt{5}$ kN on the line of action of N is active in an opposite direction to that shown in Figure 3.30. That is why there is a minus sign in the expression for N.

M is found using the equation for the moment equilibrium about an arbitrary point. If A is selected, the contribution of V and N to the moment is zero, and M can be found even if V and N are still unknown:

$$\sum T_z|\text{A} = -M - (4\text{ kN})(1\text{ m}) - (2\text{ kN})(1.5\text{ m}) + \\ +(5\text{ kN})(0.5\text{ m}) + (16\text{ kNm}) = 0 \Rightarrow M = 11.5\text{ kNm}.$$

If M had been calculated first, one would be able to derive V directly afterwards from, for example, the moment equilibrium about C:

$$\sum T_z|\text{C} = -(11.5\text{ kNm}) - (2\text{ kN})(1\text{ m}) - V \times \left(\tfrac{1}{2}\sqrt{5}\text{ m}\right) + (16\text{ kNm}) = 0.$$

This again gives $V = +\sqrt{5}$ kN. As such, there are several ways to derive the unknown forces.

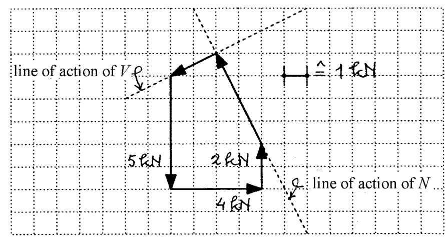


Figure 3.31 The closed force polygon represents the force equilibrium for the corner joint in the frame.

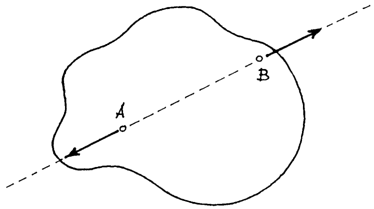


Figure 3.32 A body subject to two forces at two points.

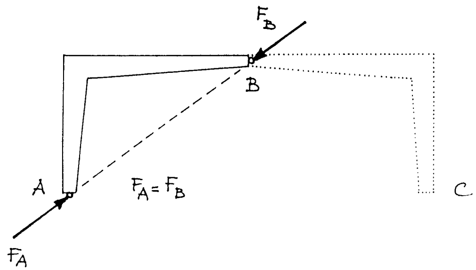


Figure 3.33 The left section of the three-hinged frame ABC, as an example of a body subject to two forces at two points.

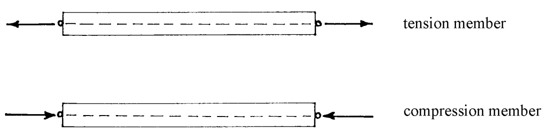


Figure 3.34 Two-force members are straight bars that can transfer forces only along their so-called bar axes.

The various guises of the equilibrium equations offer an important opportunity for performing *control calculations*. Checking results is necessary not only for manual calculations, but also for computer calculations.

3.2.2 Particular cases of equilibrium

In the analysis of the transfer of forces in structures, certain equilibrium systems are quite common. For a good insight into the behaviour of a structure, it is important to be able to quickly recognise three more or less particular cases of equilibrium. They are covered below.

1. A body subject to two forces at two points (see Figure 3.32).

A body subject to two forces can be in equilibrium only if both forces:

- have the same line of action,
- have the same magnitude, and
- have opposite directions.

If these three conditions are not all met, the two forces together form either a resultant force or a resultant couple, and the system will not be in equilibrium.

Figure 3.33 shows the left part AB of a so-called *three-hinged frame*. The foundation exerts a force F_A at A on AB, while the right part BC of the frame exerts a force F_B at B on AB. If we neglect the weight of the frame, the part AB of the frame is in equilibrium only if both forces F_A and F_B are equal and opposite, with AB as the common line of action.

Certain construction elements are intentionally designed to this type of force transfer. These are straight bars, only subject to a force at both ends (see Figure 3.34). Such bars, which can transfer forces only along their so-called bar axis, are called *two-force members*. Depending on whether they are loaded by tensile or compressive forces, they are also referred to as tension members or compression members.

When analysing structures, one must be able to recognise two-force members quickly. Structures made solely of two-force members are called *trusses*. Figure 3.35 is an example of a truss. In this truss, bar CD is loaded by compression. Calculating the forces in a truss is covered in detail in Chapter 9. It can be noted at this stage that, in a joint, the bars that come together exert forces on one another on the basis of the law of action and reaction. It is therefore possible for several forces to be exerted concurrently on the end of a bar. For example, the two compression forces on the ends of bar CD in Figure 3.35 are in fact the resultants of several forces.

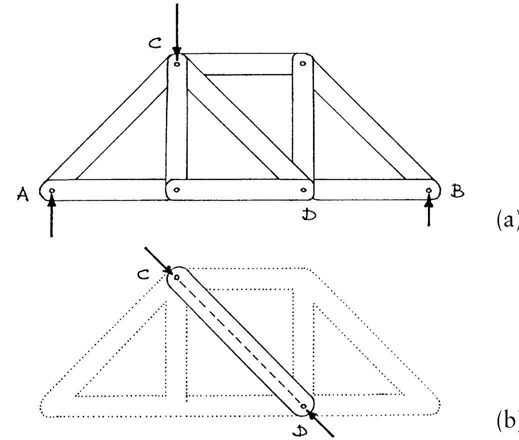


Figure 3.35 (a) The truss as a structure of two-force members; (b) the forces at the ends of compression member CD are the resultants of several forces.

Example

Two forces are exerted on the body in Figure 3.36a: F_A is exerted on A, F_B is exerted on B. Of F_A, only the horizontal component of 28 kN is given. The body is in equilibrium.

Question:
The magnitude and direction of F_B.

Solution:
If two forces are exerted on a body, the body can only be in equilibrium if the two forces have a common line of action, an equal magnitude and an opposite direction. In vector notation: $\vec{F}_A = -\vec{F}_B$. From the moment equilibrium about A, it follows that the common line of action of F_A and F_B is along AB (see Figure 3.36b). In that case, the horizontal component of F_B is $(4/5)F_B$. From the horizontal force equilibrium, it follows that:

$$\sum F_x = -(28\text{ kN}) + \tfrac{4}{5}F_B = 0 \Rightarrow F_B = 35\text{ kN}.$$

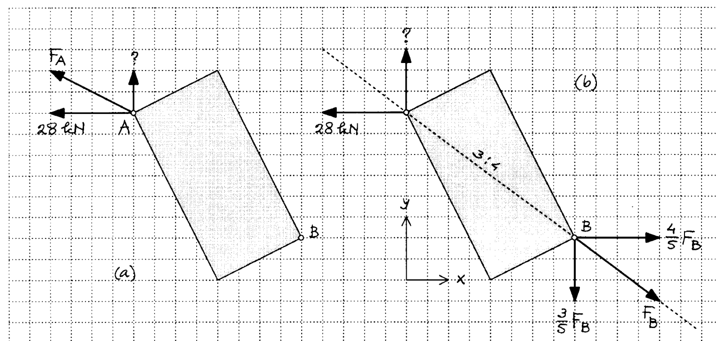


Figure 3.36 The body, subject to two forces F_A and F_B at A and B, is in equilibrium if these forces are equal and opposite and have the common line of action AB.

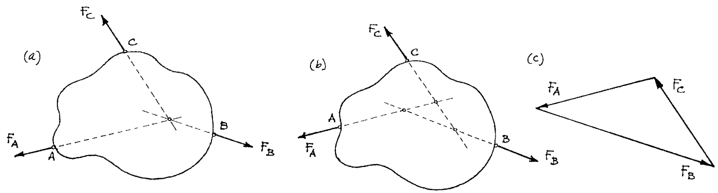


Figure 3.37 A body subject to three forces at three points: (a) moment equilibrium exists; (b) there is no moment equilibrium; (c) the closed force polygon shows that both bodies are in force equilibrium.

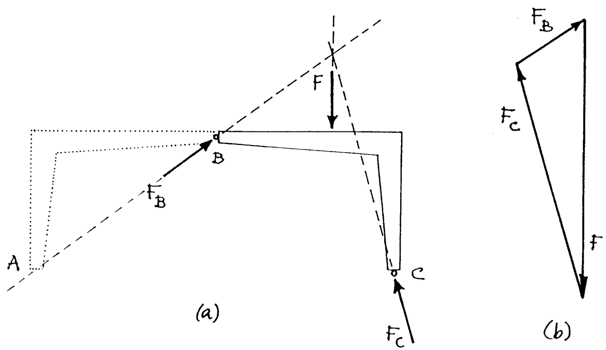


Figure 3.38 The right part of the three-hinged frame ABC as an example of a body subject to three forces at three points: (a) moment equilibrium exists because the lines of action pass through a single point and (b) there is force equilibrium because the forces form a closed force polygon.

2. A body subject to forces at three points (see Figure 3.37).

A body that is subject to three forces can be in equilibrium only if

- the three forces are coplanar,
- the forces form a closed force polygon (*force equilibrium*),[1] and
- their lines of action pass through a single point (*moment equilibrium*).

The same closed force polygon (c) is applicable for both bodies (a) and (b) in Figure 3.37: there is therefore force equilibrium in both cases. In case (a) there is moment equilibrium. This is easily checked by determining the moment of the three forces about the intersection of the three lines of action: none of the forces contribute to the sum of the moments. There is no moment equilibrium in case (b). The system of forces forms a resultant couple. The magnitude of the couple is determined by deriving the sum of the moments about the intersection of two lines of action.

Figure 3.38 shows the right-hand part BC of the *three-hinged frame*, mentioned earlier. This part of the frame is loaded by the vertical force F shown. In addition, the left frame part AB is exerting a force F_B at B on BC and the foundation is exerting a force F_C at C on BC. Moment equilibrium is only possible if the lines of action of the three forces F, F_B and F_C pass through a single point. The force equilibrium exists if the three forces form a closed force polygon.

Example

The block in Figure 3.39, loaded by two forces in C, is kept in equilibrium by the three forces A_h, A_v and B_v.

Question:

Determine these three forces and check the moment equilibrium and the

[1] It should be noted that three forces in space can only form a closed force polygon if they are acting in the same plane. The first condition is therefore actually superfluous as a result of the second.

force equilibrium graphically.

Solution:
The three unknown forces are determined using the three equilibrium equations:

$$\sum F_x = A_h + (4\text{ kN}) = 0,$$

$$\sum F_y = A_v + B_v - (6\text{ kN}) = 0,$$

$$\sum T_z|A = +B_v \times (6\text{ m}) - (6\text{ kN})(8\text{ m}) - (4\text{ kN})(3\text{ m}) = 0.$$

The first equation gives $A_h = -4$ kN, the third gives $B_v = 10$ kN, and the second equation gives $A_v = -4$ kN.

A_v can also be found directly from the moment equilibrium about B:

$$\sum T_z|B = -A_v \times (6\text{ m}) - (6\text{ kN})(2\text{ m}) - (4\text{ kN})(3\text{ m}) = 0 \Rightarrow A_v = -4\text{ kN}.$$

The fact that A_h and A_v are negative means that they act in a direction opposite to the directions given in Figure 3.39.

In Figure 3.40a, the forces are depicted as they act on the block in reality. The block is subject to forces at three points:

- the resultant of the two forces at C, with line of action c,
- the force B_v at B, with line of action b, and
- the resultant of the forces A_h and A_v at A, with line of action a.

Graphical check of the moment equilibrium (see Figure 3.40a):
For a body subject to three forces, the lines of action of the three forces have to pass through a single point. This condition is met.

Graphical check of the force equilibrium (see Figure 3.40b):
There is force equilibrium if all the forces acting on the block form a closed force polygon. This is the case.

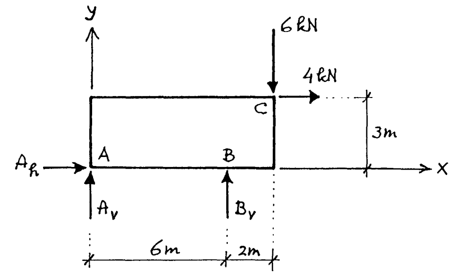


Figure 3.39 A block, loaded by two forces at C, is kept in equilibrium by the three forces A_h, A_v and B_v.

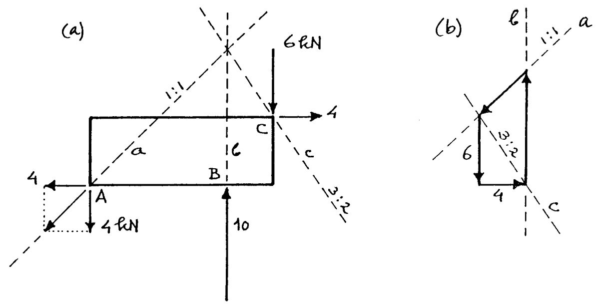


Figure 3.40 (a) Forces are exerted on the block at three points of which the lines of action pass through a single point, so that there is moment equilibrium; (b) since all the forces exerted on the block form a closed force polygon there is also force equilibrium.

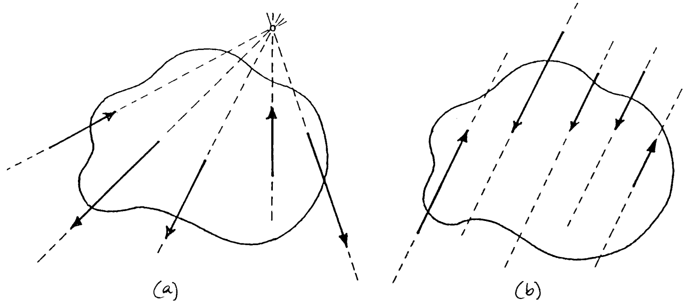


Figure 3.41 A body subject to several forces. (a) All the lines of action pass through a single point: there is moment equilibrium. (b) All the lines of action are parallel: there is force equilibrium in the direction perpendicular to these lines of action.

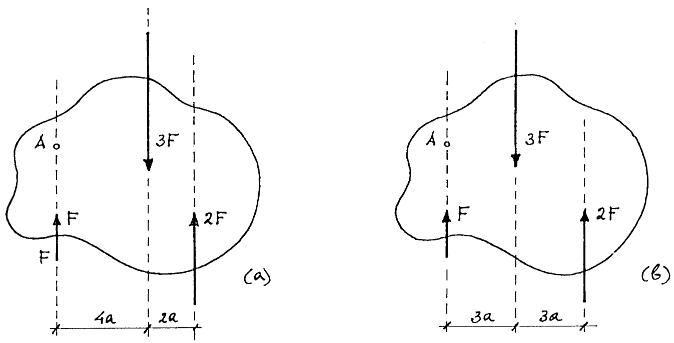


Figure 3.42 Both bodies are in force equilibrium; (a) there is moment equilibrium; (b) there is no moment equilibrium: the forces together form a couple (anti-clockwise) with magnitude $3Fa$.

3. A body subject to several forces of which the lines of action all pass through a single point (Figure 3.41a) or are all parallel (Figure 3.41b).

If for all the forces on a body the lines of action intersect at a single point, the moment equilibrium of the body is assured (see Figure 3.41a). The force equilibrium needs further investigation.

If for all the forces on a body the lines of action are parallel, the force equilibrium is assured in the direction perpendicular to the lines of action (see Figure 3.41b).

The force equilibrium in other directions, and the moment equilibrium needs further investigation.

Example
In Figure 3.42, both bodies are in force equilibrium. If one investigates the moment equilibrium by determining the sum of the moments of the forces, for example about A, it turns out that in case (a) the system is in moment equilibrium, while it is not in moment equilibrium in case (b). In case (a) the forces form an equilibrium system. In case (b), the forces form a couple acting anti-clockwise with magnitude $3Fa$. Nothing can be said about the sign associated with the direction of the couple until a coordinate system is chosen.

3.3 Forces and moments in space

So far, we looked at the equilibrium of a body only in the simple case in which all the forces and couples act in one plane. The moment was taken about a point in the same plane. In this section we look at the general three dimensional case. Here we have to define the concept of moment of forces and couples more generally.

3.3.1 Moment of a force about a point

Imagine a force $\vec{F}$ in space, with point of application B (see Figure 3.43). The moment $\vec{T}$ of this force about point A is now defined as the *vector product* (cross product) of the *position vector* $\vec{r}$, from A to B, and the *force vector* $\vec{F}$:

$$\vec{T} = \vec{r} \times \vec{F}.$$

The vector product of two vectors $\vec{r}$ and $\vec{F}$ is a vector with magnitude $rF\sin\theta$ and perpendicular to both $\vec{r}$ and $\vec{F}$. Here r and F are the magnitudes of $\vec{r}$ and $\vec{F}$ respectively, and θ is the smaller angle between the vectors $\vec{r}$ and $\vec{F}$ when both are drawn outwards from the same point.

There are two useful rules for finding the *direction* of the moment vector $\vec{T}$. The first is that it corresponds to the direction in which a *corkscrew* (with a right-hand screw) moves when the handle is turned from the first vector $\vec{r}$ to the second vector $\vec{F}$ through the angle θ (that is the direction of the rotation that the moment will cause about A) (see Figure 3.44). If necessary, the vectors will have to be shifted to the intersection of their lines of action. An alternative for finding the direction of $\vec{T}$ is the so-called *right-hand rule*: if one bends the fingers of the right hand to form a fist in the direction of the rotation that $\vec{F}$ would cause about A, then the thumb points in the direction of the moment vector.

In Figure 3.44, the vectors $\vec{r}$ and $\vec{F}$ are in the xy plane. The moment vector $\vec{T}$ is then parallel to the z axis. The figure also shows the perpendicular line AC from point A to the line of action of $\vec{F}$. The length of line segment AC is $r\sin\theta$, and the magnitude of the vector product is therefore equal to the product of the magnitude of the force and the distance from point A to the line of action of the force.[1] This corresponds to the definition of the

[1] Note that $\vec{T}$ is again independent of the location of the point of application B on the line of action of $\vec{F}$.

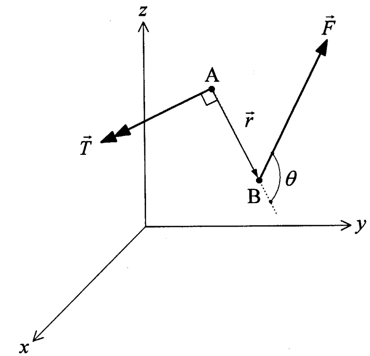


Figure 3.43 The moment of the force $\vec{F}$ about a point A is defined as the vector product $\vec{T} = \vec{r} \times \vec{F}$; the moment vector $\vec{T}$ is perpendicular to the plane through $\vec{r}$ and $\vec{F}$.

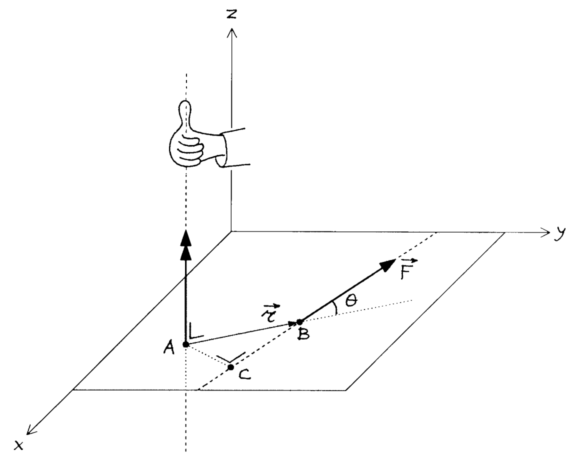


Figure 3.44 The direction of moment vector $\vec{T}$ is determined by the corkscrew rule or the right-hand rule.

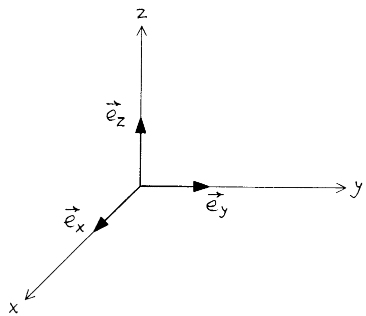


Figure 3.45 The unit vectors $\vec{e}_x$, $\vec{e}_y$ and $\vec{e}_z$.

moment of a force about a point, as given in Section 3.1.5. In that section, the limitation to forces and points in the same plane was essential. Here the definition is more general.

In order to distinguish a *moment vector* from a *force vector* in an illustration, the moment vector is often given a double arrow point.

The vector product can also be effectively described by defining vectors according to components. For example:

$$\vec{r} = r_x\vec{e}_x + r_y\vec{e}_y + r_z\vec{e}_z,$$

$$\vec{F} = F_x\vec{e}_x + F_y\vec{e}_y + F_z\vec{e}_z.$$

For the vector products of the mutually perpendicular unit vectors as shown in Figure 3.45, the following relationships apply on the basis of the definition of a vector product:

$$\vec{e}_x \times \vec{e}_y = -\vec{e}_y \times \vec{e}_x = \vec{e}_z,$$

$$\vec{e}_y \times \vec{e}_z = -\vec{e}_z \times \vec{e}_y = \vec{e}_x,$$

$$\vec{e}_z \times \vec{e}_x = -\vec{e}_x \times \vec{e}_z = \vec{e}_y,$$

and

$$\vec{e}_x \times \vec{e}_x = \vec{e}_y \times \vec{e}_y = \vec{e}_z \times \vec{e}_z = 0.$$

The components of $\vec{T} = \vec{r} \times \vec{F}$ are therefore

$$T_x = r_y F_z - r_z F_y,$$

$$T_y = r_z F_x - r_x F_z,$$

$$T_z = r_x F_y - r_y F_x.$$

This definition for the components of the moment vector $\vec{T}$ is a generalisation of the definition of T_z as given in Section 3.1.5.

For the moment vector $\vec{T}$ and its components T_x, T_y and T_z it is again preferable to mention the point A about which the moment was determined, such as $\vec{T}|A$, $T_x|A$, and so forth.

An alternative notation for the moment vector $\vec{T}$ is

$$\vec{T} = \vec{r} \times \vec{F} = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix}.$$

The components of $\vec{T}$ are found by developing the *determinant*.

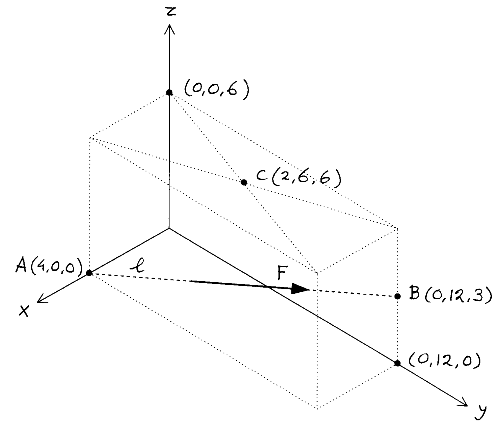


Figure 3.46 The line of action ℓ of force $F = 65$ kN passes through the points A(4, 0, 0) and B(0, 12, 3). The question relates to the moment of the force about point C(2, 6, 6). The coordinates are expressed in metres.

Example
For the force $F = 65$ kN in Figure 3.46, the line of action ℓ passes through the points A(4, 0, 0) and B(0, 12, 3). The coordinates are expressed in metres.

Question:
Determine the moment of the force about point C(2, 6, 6).

Solution:
The units used are kN and m; they are not always shown in interim calculations.

First the components F_x, F_y and F_z are determined (see Section 2.2.1). Vector $\overline{AB}$ (pointing from A to B) has the same direction as the force $\vec{F}$. If $\overline{AB}$ is hereafter referred to as $\vec{d}$, then

$$\vec{d} = d_x\vec{e}_x + d_y\vec{e}_y + d_z\vec{e}_z = (-4\vec{e}_x + 12\vec{e}_y + 3\vec{e}_z) \text{ m},$$

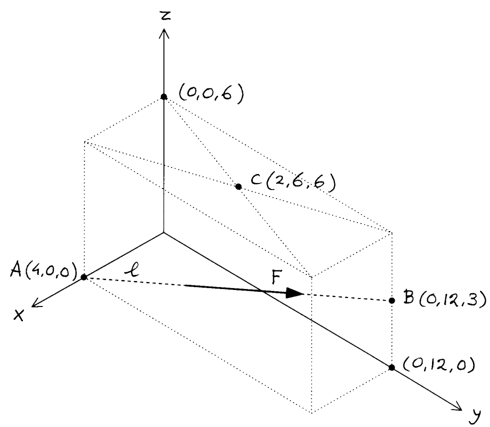


Figure 3.46 The line of action ℓ of force $F = 65$ kN passes through the points A(4, 0, 0) and B(0, 12, 3). The question relates to the moment of the force about point C(2, 6, 6). The coordinates are expressed in metres.

and

$$d = |\vec{d}| = \sqrt{(-4)^2 + 12^2 + 3^2} = 13 \text{ m}.$$

Since the direction cosines of $\vec{F}$ and $\vec{d}$ are equal

$$\cos\alpha_x = \frac{F_x}{F} = \frac{d_x}{d} \Rightarrow F_x = F\frac{d_x}{d} = 65 \times \tfrac{-4}{13} = -20 \text{ kN},$$

$$\cos\alpha_y = \frac{F_y}{F} = \frac{d_y}{d} \Rightarrow F_y = F\frac{d_y}{d} = 65 \times \tfrac{12}{13} = +60 \text{ kN},$$

$$\cos\alpha_z = \frac{F_z}{F} = \frac{d_z}{d} \Rightarrow F_z = F\frac{d_z}{d} = 65 \times \tfrac{3}{13} = +15 \text{ kN}.$$

$\vec{F}$ can now be defined according to its components:

$$\vec{F} = (-20\vec{e}_x + 60\vec{e}_y + 15\vec{e}_z) \text{ kN}.$$

Imagine $\vec{F}$ is exerted at point A, then

$$\vec{r} = \overline{\text{CA}} = r_x\vec{e}_x + r_y\vec{e}_y + r_z\vec{e}_z = (+2\vec{e}_x - 6\vec{e}_y - 6\vec{e}_z) \text{ m},$$

and for the moment of $\vec{F}$ with respect to C

$$\vec{T}|\text{C} = \vec{r} \times \vec{F} = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix} = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ +2 & -6 & -6 \\ -20 & +60 & +15 \end{vmatrix}.$$

This gives the following components:

$$T_x|\text{C} = r_yF_z - r_zF_y = (-6 \text{ m})(+15 \text{ kN}) - (-6 \text{ m})(+60 \text{ kN})$$
$$= +270 \text{ kNm},$$

$$T_y|\mathrm{C} = r_z F_x - r_x F_z = (-6\text{ m})(-20\text{ kN}) - (+2\text{ m})(+15\text{ kN})$$
$$= +90\text{ kNm},$$
$$T_z|\mathrm{C} = r_x F_y - r_y F_x = (+2\text{ m})(+60\text{ kN}) - (-6\text{ m})(-20\text{ kN})$$
$$= 0\text{ kNm}.$$

To show that the moment of force $\vec{F}$ with respect to C is independent of the point of application on its line of action, the following represents an example in which $\vec{F}$ is exerted at B. In that case

$$\vec{r} = \overline{\mathrm{CB}} = (-2\vec{e}_x + 6\vec{e}_y - 3\vec{e}_z)\text{ m}.$$

The result of

$$\vec{T}|\mathrm{C} = \vec{r} \times \vec{F} = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix} = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ -2 & +6 & -3 \\ -20 & +60 & +15 \end{vmatrix}$$

does indeed give the same values:

$$T_x|\mathrm{C} = r_y F_z - r_z F_y = (+6\text{ m})(+15\text{ kN}) - (-3\text{ m})(+60\text{ kN})$$
$$= +270\text{ kNm},$$
$$T_y|\mathrm{C} = r_z F_x - r_x F_z = (-3\text{ m})(-20\text{ kN}) - (-2\text{ m})(+15\text{ kN})$$
$$= +90\text{ kNm},$$
$$T_z|\mathrm{C} = r_x F_y - r_y F_x = (-2\text{ m})(+60\text{ kN}) - (+6\text{ m})(-20\text{ kN})$$
$$= 0\text{ kNm}.$$

Figure 3.47 shows the components of the moment vector in C. The moment vector $\vec{T}$ lies in the horizontal plane through C. Further consideration shows

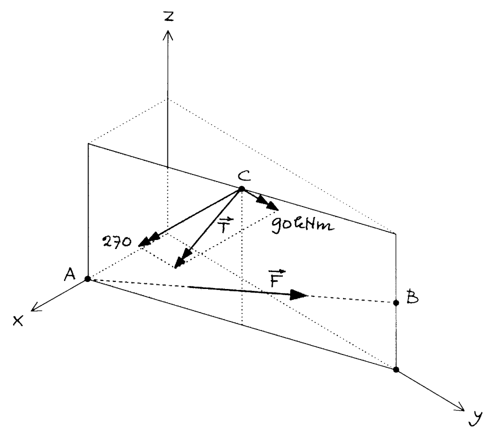


Figure 3.47 The components of vector $\vec{T}$ for the moment of $\vec{F}$ about point C. The moment vector $\vec{T}$ is perpendicular to plane ABC.

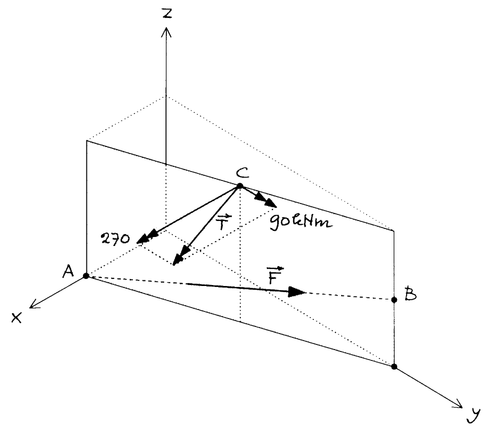


Figure 3.47 The components of vector $\vec{T}$ for the moment of $\vec{F}$ about point C. The moment vector $\vec{T}$ is perpendicular to plane ABC.

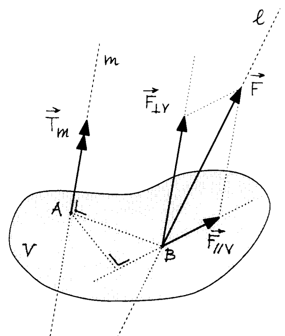


Figure 3.48 The moment of the force $\vec{F}$ about line m is defined as the moment $\vec{T}_m$ of the projection $\vec{F}_{//V}$ of $\vec{F}$ on a plane V perpendicular to m about point A, the intersection of V and m.

that $\vec{T}$ is indeed perpendicular to plane ABC. The magnitude of the resultant moment about C is

$$T|\mathrm{C} = \sqrt{270^2 + 90^2} = 90\sqrt{10} = 284.6 \text{ kNm}.$$

3.3.2 Moment of a force about a line

Figure 3.48 shows a force $\vec{F}$ with line of action ℓ, and a line m. The lines ℓ and m will generally cross one another and not be perpendicular. Imagine V is an arbitrary plane perpendicular to m. The lines ℓ and m intersect the plane V at B and A respectively. In Figure 3.48, it has been assumed that $\vec{F}$ is applied at B. As shall become clear in a moment, $\vec{F}$ may also be applied elsewhere on ℓ.

$\vec{F}$ can be resolved into a component $\vec{F}_{\perp V}$ perpendicular to plane V and so parallel to m, and a component $\vec{F}_{//V}$ in plane V. If $\vec{F}$ is not applied at B, $\vec{F}_{//V}$ is the projection of $\vec{F}$ on V. The line of action of $\vec{F}_{//V}$ is the projection of the line of action ℓ of $\vec{F}$ on V. Wherever one places the plane V perpendicular to m, the line of action of $\vec{F}_{//V}$ always remains the same.

The moment $\vec{T}_m$ of the force $\vec{F}$ about line m has now been defined as the moment of the projection $\vec{F}_{//V}$ of $\vec{F}$ on a plane V perpendicular to m with respect to the intersection A of V and m.

For the components of $\vec{T}|\mathrm{A}$, the moment of $\vec{F}$ about point A in a xyz coordinate system, we have earlier derived that

$$T_x|\mathrm{A} = r_y F_z - r_z F_y,$$

$$T_y|\mathrm{A} = r_z F_x - r_x F_z,$$

$$T_z|\mathrm{A} = r_x F_y - r_y F_x.$$

Here one recognises the moment about three lines through A, parallel to the

x, y and z axis respectively.

Comment:
For a moment about the origin O of the coordinate system or a moment about one of the coordinate axes, the point O is generally omitted in the representation of the moment.

Example
The curved beam AB in Figure 3.49 is loaded at B by a force of which the components are defined with respect to magnitude and direction in the figure.

Question:
Find the moment about the x, y and z axis respectively of the force(s) at B.

Solution:

$$T_x = +(25 \text{ kN})(3 \text{ m}) - (50 \text{ kN})(1 \text{ m}) = +25 \text{ kNm},$$

$$T_y = +(40 \text{ kN})(1 \text{ m}) - (25 \text{ kN})(2 \text{ m}) = -10 \text{ kNm},$$

$$T_z = +(50 \text{ kN})(2 \text{ m}) - (40 \text{ kN})(3 \text{ m}) = -20 \text{ kNm}.$$

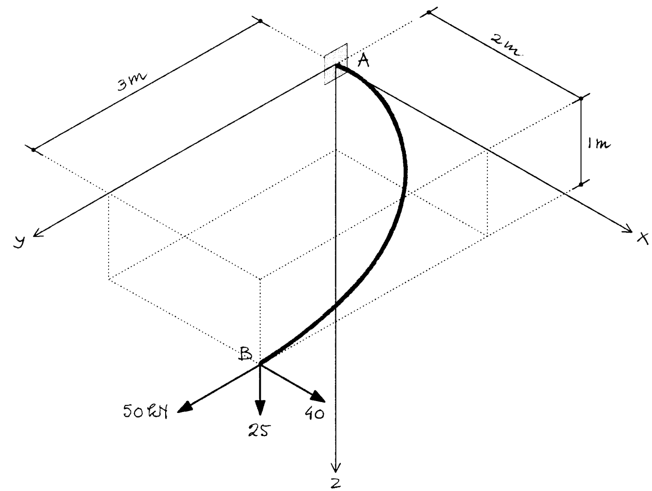


Figure 3.49 A curved beam AB is loaded at B by the three components of a force.

3.3.3 Moment of a couple

Two parallel forces that are equal and opposite form a couple (see Section 3.1.4). Figure 3.50 shows two forces $\vec{F}_1 = \vec{F}$ and $\vec{F}_2 = -\vec{F}$, forming a couple in space.[1]

For the moment of the couple about a point A we have

$$\vec{T}|\text{A} = \vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times F_2 = \vec{r}_1 \times \vec{F} + \vec{r}_2 \times (-\vec{F}) = (\vec{r}_1 - \vec{r}_2) \times \vec{F} = \vec{r} \times \vec{F}.$$

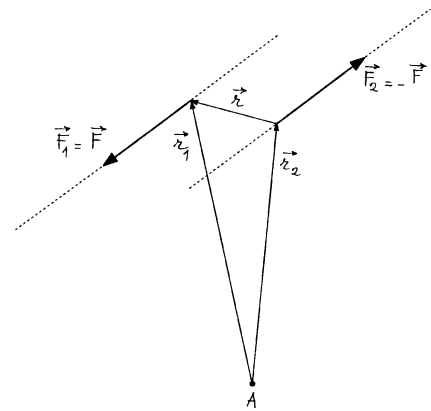


Figure 3.50 The moment of the couple about a point A is $\vec{T} = \vec{r} \times \vec{F}$. This moment is independent of the location of point A.

[1] There is no resultant force, for $\vec{F}_1 + \vec{F}_2 = 0$.

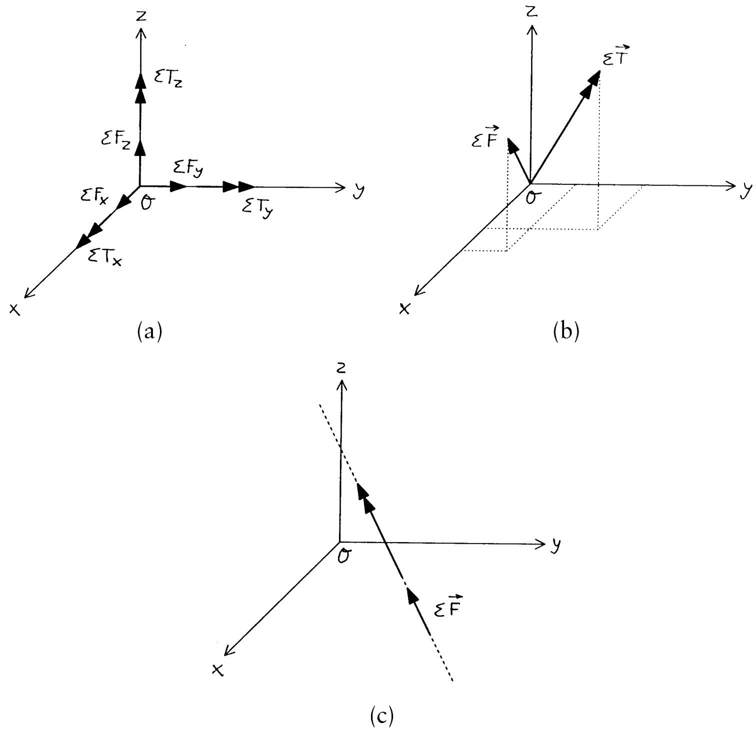


Figure 3.51 (a) The components of a resultant force $\sum \vec{F}$ in O and a resultant couple $\sum \vec{T}$. (b) The resultant force vector and the resultant moment vector need not necessarily have the same direction. (c) By shifting the resultant force $\sum \vec{F}$ parallel to itself one can provide that the resultant moment vector has the same direction as the force vector. The combination of a force and a moment of which the vectors have the same direction is called a *screw*.

The moment of the couple is equal to the moment of the one force about an arbitrary point on the line of action of the other force. The moment is independent of the location of point A about which it was originally determined. This means that the moment of a couple is a *free vector*. The moment vector of the couple is perpendicular to the plane in which the couple acts.

3.3.4 Compounding forces and couples

Compounding forces and couples in space is analytically relatively simple. Each of the forces F_i $(i = 1, 2, \ldots, n)$ can be resolved into the components $F_{x;i}$; $F_{y;i}$; $F_{z;i}$ and for each of these forces, one can determine the moment with respect to an arbitrary point A. In fact, this means that all the forces with the addition of a couple, are shifted to that point A (see Section 3.1.5).

If we place the origin O of the coordinate system at A, and x_i, y_i, z_i are the coordinates of the point of application of force F_i (or of another point on the line of action of F_i), then:

$$\sum F_x = \sum_{i=1}^{n} F_{x;i}, \qquad \sum T_x = \sum_{i=1}^{n} \{(y_i F_{z;i} - z_i F_{y;i}) + T_{x;i}\},$$

$$\sum F_y = \sum_{i=1}^{n} F_{y;i}, \qquad \sum T_y = \sum_{i=1}^{n} \{(z_i F_{x;i} - x_i F_{z;i}) + T_{y;i}\},$$

$$\sum F_z = \sum_{i=1}^{n} F_{z;i}, \qquad \sum T_z = \sum_{i=1}^{n} \{(x_i F_{y;i} - y_i F_{x;i}) + T_{z;i}\}.$$

The moment sum also includes the moments of any (concentrated) couples T_i that act on the body.

$\sum F_x$, $\sum F_y$ and $\sum F_z$ are the components of the resultant force $\sum \vec{F}$ in O while $\sum T_x$, $\sum T_y$ and $\sum T_z$ are the components of a resultant couple $\sum \vec{T}$ (see Figure 3.51a). The resultant force vector $\sum \vec{F}$ at O and the resultant moment vector $\sum \vec{T}$ need not necessarily have the same direction (see Figure 3.51b).

By shifting the resultant force $\sum \vec{F}$ parallel to itself one can provide that the resultant force vector and moment vector have the same direction (see Figure 3.51c). The combination of a force and a moment of which the vectors have the same direction is called a *screw*.[1]

The following represents three examples that relate to the determination of the resultant of a number of forces and/or couples.

Example 1
A flat slab of 6×5 m^2 in the horizontal xy plane is loaded by six vertical forces (see Figure 3.53a). The grid lines are 1 m apart.

Question:
Determine the resultant force R as to magnitude and direction and the location at which it acts on the slab.

Solution:
The units used are kN and m. The units are omitted in the interim calculations.

The x and y components of all the forces given are zero, as are their moments about the z axis, therefore

$$\sum F_x = 0,$$

$$\sum F_y = 0,$$

$$\sum T_z = 0.$$

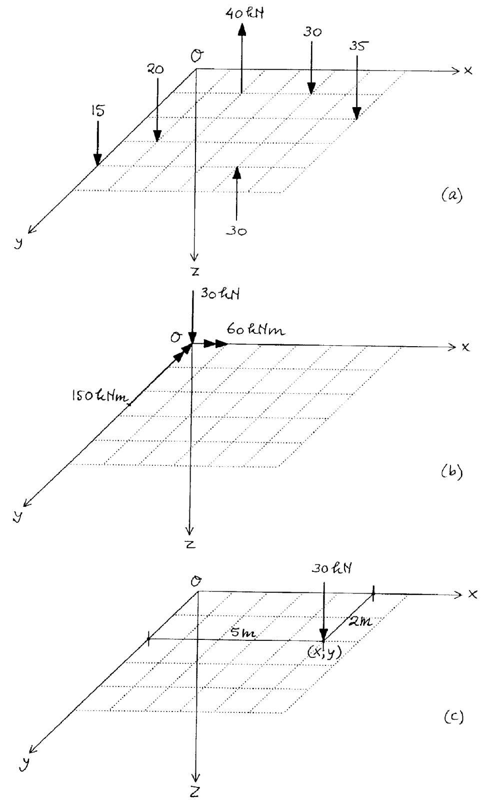


Figure 3.52 (a) A flat slab of 6×5 m^2 in the horizontal xy plane is loaded by six vertical forces. The grid lines are 1 m apart. (b) The system of forces is statically equivalent with a vertical force $R = 30$ kN at O pointing downwards, together with two couples of 150 kNm and 60 kNm of which the moment vectors are along the x and y axis respectively, or (c) with only a force $R = 30$ kN at the point ($x = 5$ m, $y = 2$ m).

[1] Reducing a system of forces and couples into a screw is an interesting academic problem, but is of little practical use and therefore not covered in further detail.

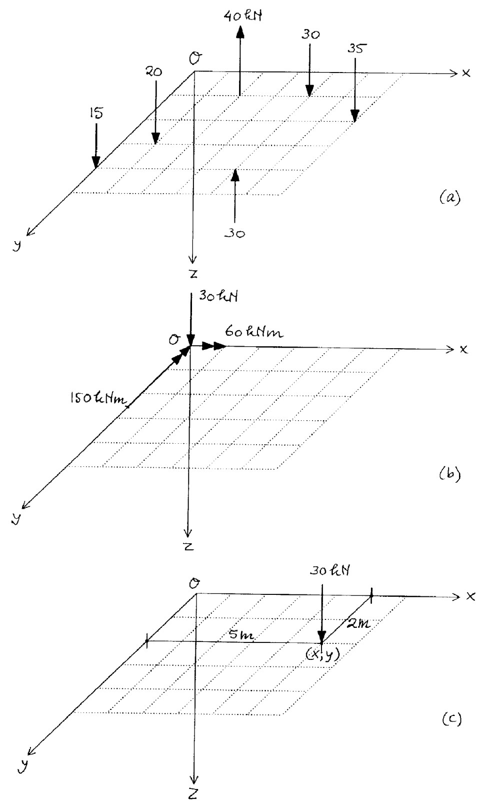


Figure 3.52 (a) A flat slab of $6 \times 5\ \mathrm{m}^2$ in the horizontal xy plane is loaded by six vertical forces. The grid lines are 1 m apart. (b) The system of forces is statically equivalent with a vertical force $R = 30$ kN at O pointing downwards, together with two couples of 150 kNm and 60 kNm of which the moment vectors are along the x and y axis respectively, or (c) with only a force $R = 30$ kN at the point ($x = 5$ m, $y = 2$ m).

In addition,

$$\begin{aligned}
\sum F_z &= +15 + 20 - 40 + 30 - 30 + 35 = +30\ \text{kN},\\
\sum T_x &= -40 \times 1 + 30 \times 1 + 35 \times 2 + 20 \times 3 + 15 \times 4 - 30 \times 4\\
&= +60\ \text{kNm},\\
\sum T_y &= 15 \times 0 - 20 \times 1 + 40 \times 2 - 30 \times 4 + 30 \times 4 - 35 \times 6\\
&= -150\ \text{kNm}.
\end{aligned}$$

So the system of forces can be replaced by a downward force $R = 30$ kN at O, together with two couples of 150 kNm and 60 kNm of which the moment vectors are along the x and y axis respectively (see Figure 3.53b).

Since the moment vectors are perpendicular to force R, they can be eliminated by shifting R to another point of application. Imagine (x, y) is the new point of application (see Figure 3.53c). We can find (x, y) from the condition that $R = 30$ kN has to generate the same moment about the x and y axis as all the forces together, so that

$$\sum T_x = Ry = 60\ \text{kNm} \quad \Rightarrow \quad y = \frac{60\ \text{kNm}}{R} = \frac{60\ \text{kNm}}{30\ \text{kN}} = 2\ \text{m},$$

$$\sum T_y = -Rx = -150\ \text{kNm} \quad \Rightarrow \quad x = \frac{150\ \text{kNm}}{R} = \frac{150\ \text{kNm}}{30\ \text{kN}} = 5\ \text{m}.$$

Example 2
In Figure 3.53a, a flat slab of $6 \times 5\ \text{m}^2$ in the horizontal xy plane is loaded by five vertical forces. The distance between the grid lines is 1 m.

Question:
Determine the resultant of this set of forces.

Solution:
The units used are kN and m. The units are omitted in the interim calculations.

When determining the resultant of this system of parallel forces, only the force sum in the z direction and the moment sum about the x and y axis are relevant:

$$\sum F_y = +25 - 15 + 20 - 45 + 15 = 0\ \text{kN},$$

$$\begin{aligned}\sum T_x &= 25 \times 0 + 15 \times 1 - 20 \times 3 - 15 \times 4 + 15 \times 4 + 45 \times 4 \\ &= +75\ \text{kN},\end{aligned}$$

$$\begin{aligned}\sum T_z &= +25 \times 6 - 15 \times 2 + 20 \times 5 - 45 \times 3 + 15 \times 1 \\ &= +100\ \text{kN}.\end{aligned}$$

There is no resultant force, but there is a resultant couple T of which the moment vector is in the xy plane (see Figure 3.53b). Its magnitude is

$$T = \sqrt{75^2 + 100^2} = 125\ \text{kNm}.$$

The resultant couple acts in a plane perpendicular to the moment vector.

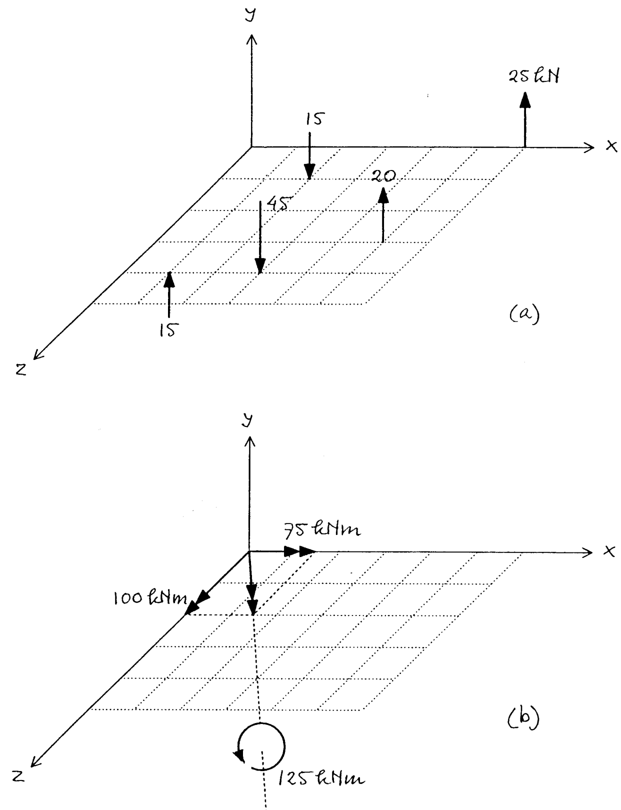


Figure 3.53 (a) A flat slab of $6 \times 5\ \text{m}^2$ in the horizontal xz plane is loaded by five vertical forces. The grid lines are 1 m apart. (b) There is no resultant force, but there is a resultant couple of which the moment vector is in the xz plane. The resultant couple acts in a plane perpendicular to the moment vector.

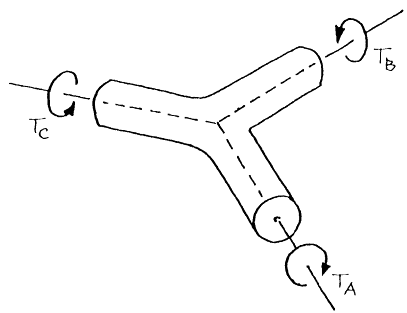


Figure 3.54 A junction of three coplanar tubes that are rigidly connected at equal angles of 120°. The tubes are loaded (*by torsion*) by the couples T_A, T_B and T_C.

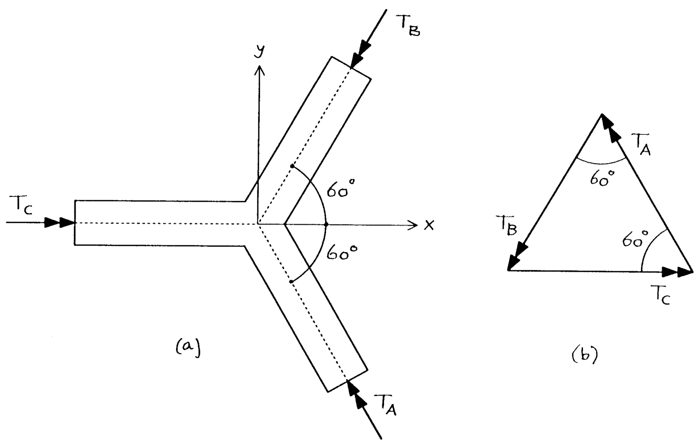


Figure 3.55 (a) The couples acting on the junction represented by their moment vectors. (b) If there is no resultant couple, the three moment vectors must form a closed polygon.

Example 3

Figure 3.54 shows a junction of three coplanar tubes that are rigidly connected at equal angles of 120°. The tubes are loaded (by *torsion*) by the couples T_A, T_B and T_C. The resultant couple on the junction is zero.

Question:
How large are the couples T_A and T_B if $T_C = 75$ Nm?

Solution:
In Figure 3.55a, the couples are represented by their moment vectors. The three vectors are in the xy plane, the plane in which the tubes are located. The resultant moment on the junction is zero if the three vectors form a closed polygon, analogous to the closed force polygon for force equilibrium. The equilateral triangle in Figure 3.55b gives

$$T_A = T_B = T_C = 75 \text{ Nm}.$$

This can of course also be determined analytically. If there is no resultant couple, then

$$\sum T_x = -\tfrac{1}{2}T_A - \tfrac{1}{2}T_B + T_C = -\tfrac{1}{2}T_A - \tfrac{1}{2}T_B + (75 \text{ Nm}) = 0,$$

$$\sum T_y = +\tfrac{1}{2}T_A\sqrt{3} - \tfrac{1}{2}T_B\sqrt{3} = 0.$$

The result of these two equations is again

$$T_A = T_B = 75 \text{ Nm}.$$

3.4 Equilibrium of a rigid body in space

Generalising the equilibrium equations for a rigid body is relatively simple. After all, equilibrium demands that both the resultant force and the resultant moment about an arbitrary point A are zero. This means that the following requirements have to be met by the forces and moments exerted on a rigid body at rest:

$$\sum F_x = 0,$$

$$\sum F_y = 0,$$

$$\sum F_z = 0,$$

$$\sum T_x|\mathrm{A} = 0,$$

$$\sum T_y|\mathrm{A} = 0,$$

$$\sum T_z|\mathrm{A} = 0.$$

The first three equations state that there is *force equilibrium* in the x, y and z directions respectively, and that the body is therefore not subject to translation acceleration. The latter three equations define that there is *moment equilibrium* at A about lines parallel to respectively the x, y and z axis, and that the body is not subject to rotational acceleration.

The following examples address the equilibrium of a body in space.

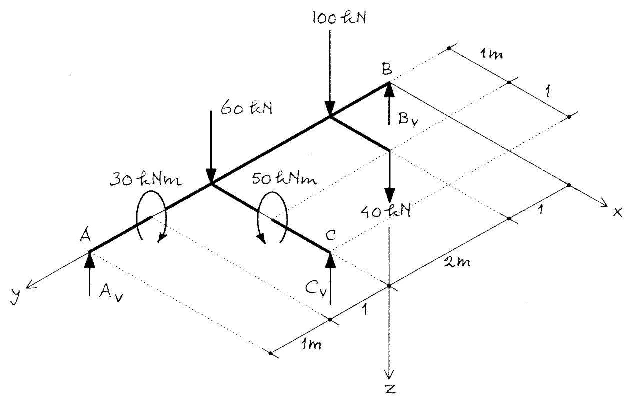


Figure 3.56 A structure consisting of a system of mutually perpendicular beams in the horizontal xy plane that is loaded perpendicularly to its plane by a number of forces and couples. The unknown forces A_v, B_v and C_v have to be derived from the equilibrium.

Example 1

The structure in Figure 3.56 consists of a number of mutually perpendicular beams in the horizontal xy plane that are loaded at the locations shown by three vertical forces of respectively 40, 60 and 100 kN and by two couples of 30 and 50 kNm. The structure is kept in equilibrium by the three vertical forces A_v, B_v and C_v.

Question:
Determine these three unknown forces.

Solution:
Since all the forces are parallel to the z axis,

$$\sum F_x = 0 \quad \text{and} \quad \sum F_y = 0.$$

The moment vectors of both couples are in the xy plane, so that in addition

$$\sum T_z = 0.$$

To determine the three unknown forces, we can use the following three equilibrium equations:

$$\sum F_z = 0, \quad \sum T_x = 0 \quad \text{and} \quad \sum T_y = 0.$$

By choosing the equilibrium equations carefully, and by applying them in a carefully chosen order, it is sometimes possible to cut back on the amount of calculation needed.

C_v is derived directly from $\sum T_y = 0$:

$$\sum T_y = -(30 \text{ kNm}) - (40 \text{ kN}) \times (1 \text{ m}) + C_v \times 2 = 0$$

$$\Rightarrow C_v = +35 \text{ kN}.$$

Next, we find A_v directly from $\sum T_x = 0$:

$$\sum T_x = -A_v \times (5\text{ m}) + (60\text{ kN}) \times (3\text{ m}) + (50\text{ kNm})$$
$$-C_v \times (3\text{ m}) + (100\text{ kN}) \times (1\text{ m}) + (40\text{ kN}) \times (1\text{ m}) = 0$$
$$\Rightarrow A_v = +53\text{ kN},$$

after which B_v follows directly from $\sum F_z = 0$:

$$\sum F_z = +\{(100 + 40 + 60)\text{ kN}\} - A_v - B_v - C_v = 0$$
$$\Rightarrow B_v = +112\text{ kN}.$$

Figure 3.57 shows the forces A_v, B_v and C_v as they act on the structure in reality.

To check, one could also have a look at the moment equilibrium at a point other than the origin, such as point A:

$$\sum T_x|\text{A} = -\{(60 - 35)\text{ kN}\} \times (2\text{ m}) + (50\text{ kNm}) -$$
$$-\{(100 + 40)\text{ kN}\} \times (4\text{ m}) + (112\text{ kN}) \times (5\text{ m}) = 0.$$

The moment equilibrium is also met about a line through A parallel to the x axis.

Example 2
In Figure 3.58, a cube with edge length a and weight G is kept in equilibrium by the six forces F_1 to F_6. For the angle α between the lines of action of the forces applies $\tan\alpha = 3/4$.

Question:
Determine the six forces F_1 to F_6 if $a = 1$ m and $G = 24$ kN.

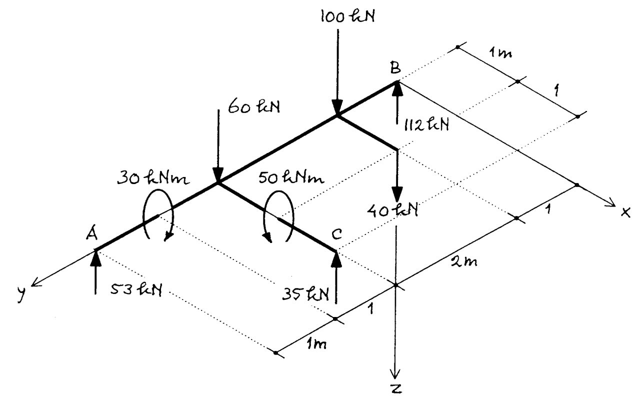


Figure 3.57 The forces A_v, B_v and C_v as they are actually acting on the structure.

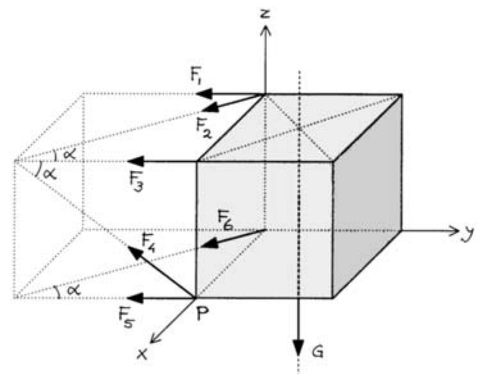


Figure 3.58 A cube with edge length a and weight G is kept in equilibrium by six forces F_1 to F_6. For the angle α between the lines of action of the forces applies $\tan\alpha = 3/4$.

Solution:
When writing down the equilibrium equations, it can sometimes be useful to project all the forces on the three coordinate planes (see Figure 3.59). In doing so, the forces F_2, F_4 and F_6 are resolved into components according to the coordinate directions. Using Figure 3.59 one finds

$$\sum F_x = \tfrac{3}{5}F_2 + \tfrac{3}{5}F_6 = 0, \tag{a}$$

$$\sum F_y = F_1 + \tfrac{4}{5}F_2 + F_3 + \tfrac{4}{5}F_4 + F_5 + \tfrac{4}{5}F_6 = 0, \tag{b}$$

$$\sum F_z = \tfrac{3}{5}F_4 - G = 0, \tag{c}$$

$$\sum T_x = \left(F_1 + \tfrac{4}{5}F_2 + F_3\right) \cdot a - G \cdot \tfrac{1}{2}a = 0, \tag{d}$$

$$\sum T_y = \tfrac{3}{5}F_2 \cdot a - \tfrac{3}{5}F_4 \cdot a + G \cdot \tfrac{1}{2}a = 0, \tag{e}$$

$$\sum T_z = -\left(F_3 - \tfrac{4}{5}F_4 + F_5\right) \cdot a = 0. \tag{f}$$

Equation (c) gives

$$F_4 = \tfrac{5}{3}G = 40 \text{ kN}.$$

Using this, one finds from equation (e)

$$F_2 = \tfrac{5}{6}G = 20 \text{ kN}$$

and then from equation (a)

$$F_6 = -\tfrac{5}{6}G = -20 \text{ kN}.$$

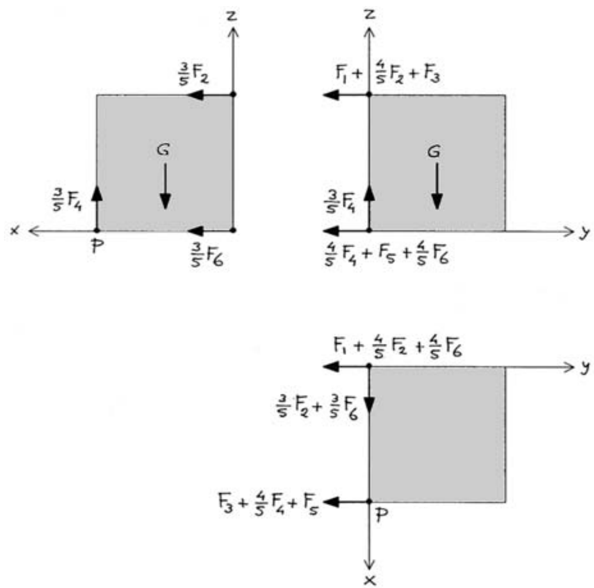


Figure 3.59 All the forces acting on the cube projected on the three coordinate planes.

Determining the forces F_1, F_2 and F_3 from the three remaining equations (b), (d) and (f) demands some arithmetic. Sometimes one can reduce the amount of calculation by looking at the moment equilibrium about another point. Also here:

$$\sum T_z|\text{P} = -\left(F_1 + \tfrac{4}{5}F_2 + \tfrac{4}{5}F_6\right) \cdot a = 0 \qquad \text{(g)}$$

so that

$$F_1 = 0.$$

Equation (d) now gives

$$F_3 = -\tfrac{1}{6}G = -4 \text{ kN}.$$

Finally, equation (f) gives

$$F_5 = -\tfrac{7}{6}G = -28 \text{ kN}.$$

Figure 3.60 depicts the forces (in kN) as they are acting on the cube in reality. The forces F_3, F_5 and F_6 act in directions opposite to those shown in Figure 3.58.

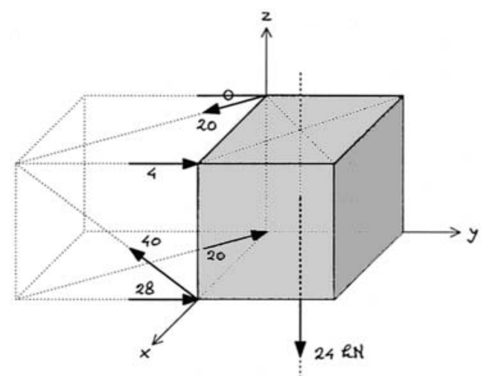


Figure 3.60 The forces (in kN) as they are acting on the cube.

By using alternative equilibrium equation (g), equation (b) for the force equilibrium in y direction was not used, and can be used as a check. With the forces expressed in kN this gives

$$\begin{aligned}\sum F_y &= F_1 + \tfrac{4}{5}F_2 + F_3 + \tfrac{4}{5}F_4 + F_5 + \tfrac{4}{5}F_6\\ &= 0 + \tfrac{4}{5} \times 20 - 4 + \tfrac{4}{5} \times 40 - 28 - \tfrac{4}{5} \times 20 = 0.\end{aligned}$$

The conditions for force equilibrium in y direction are met.

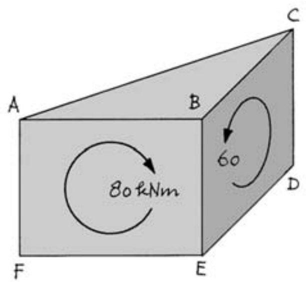


Figure 3.61 The cube, which has been halved diagonally, is subject to a couple of 80 kNm in plane ABEF and a couple of 60 kNm in plane BCDE. The body is kept in equilibrium by a couple on the diagonal plane ACDF.

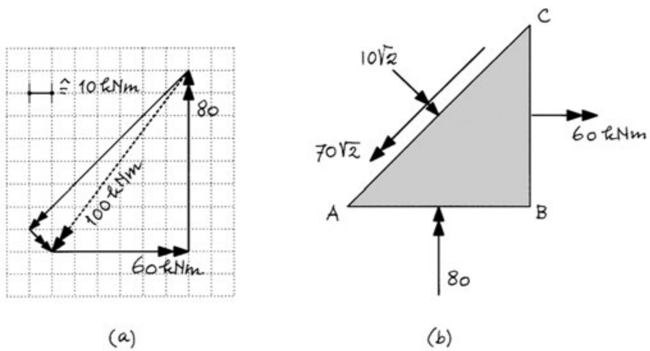


Figure 3.62 (a) If there is moment equilibrium, the moment vectors form a closed polygon. (b) Top view of the diagonally-halved cube with the moment vectors acting on it.

Example 3

The cube that has been halved diagonally in Figure 3.61 is subject to a couple of 80 kNm in plane ABEF and a couple of 60 kNm in plane BCDE. The directions are shown in the figure. The body is kept in equilibrium by a couple on the diagonal plane ACDF.

Question:
Determine the magnitude of that couple and resolve it into a component in plane ACDF and a component perpendicular to plane ACDF.

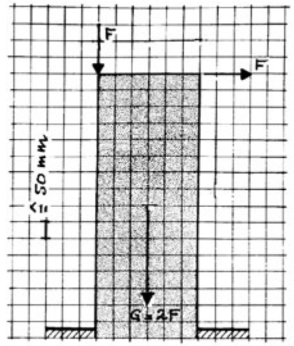
Solution:
There is moment equilibrium if the moment vectors form a closed polygon. The polygon in Figure 3.62a shows that a couple of 100 kNm is acting on plane ACDF. Of this couple, the moment vector has a component perpendicular to plane ACDF of $10\sqrt{2}$ kNm and a component along plane ACDF of $70\sqrt{2}$ kNm. Figure 3.62b shows the top view for the halved cube, with all the moment vectors that act on it.

When interpreting these results, one should remember that the moment vector is perpendicular to the plane on which the couple is exerted. The component of the couple that *is acting in the diagonal plane* has a moment vector perpendicular to that plane and is $10\sqrt{2}$ kNm. The component of the couple *that is acting perpendicular to the diagonal plane* has its moment vector in that plane and is $70\sqrt{2}$ kNm.

3.5 Problems

Compounding forces graphically (Sections 3.1.2 and 3.1.3)

3.1 The line of action of the resultant of the two forces F and the weight $G = 2F$ of the block intersect the right-hand side of the block at a distance a from the top.

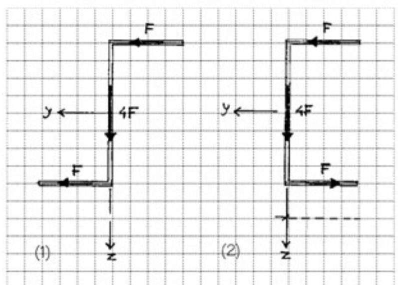


Question:
How large is a?

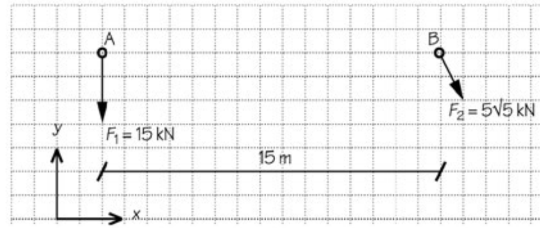
3.2: 1–2 The forces shown are exerted in the web and flange of a thin-walled profile. Length scale: 1 square ≡ 25 mm.

Questions:

a. Using a force polygon determine the magnitude and direction of the resultant of these forces.
b. How large are the components of the resultant in the yz coordinate system shown?
c. Using a line of action figure, determine the location of the line of action of the resultant; where does this line of action intersect the y axis?



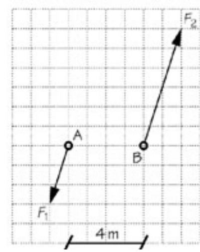
3.3 The forces F_1 and F_2 are exerted on a body at points A and B. The body is not shown. Force scale: 1 square ≡ 5 kN. Length scale: 1 square ≡ 1 m.



Question:
Using a force polygon, determine the magnitude and direction of the resultant of both forces graphically, and in a line of action figure determine the location of the line of action.
Hint: use additional forces at A and B of magnitude 15 kN.

3.4 The two parallel forces F_1 and F_2 are exerted on a body at A and B. The body is not shown. Force scale: 1 square ≡ 10 kN. Length scale: 1 square ≡ 1 m.

Question:
Determine graphically (using a force polygon), the magnitude and direction of the resultant of both forces, and (using a line of action figure) the location of the line of action.
Hint: use additional forces at A and B of magnitude 40 kN.

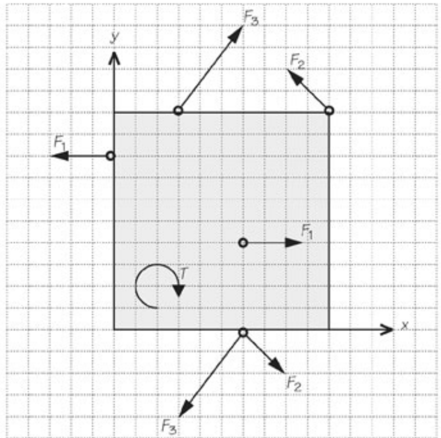


Moment of a couple (Section 3.1.4)

3.5 A block is subject to four couples in the xy plane. Force scale: 1 square $\equiv$ 1 kN. Length scale: 1 square $\equiv$ 1 m.

Question:
Find in the xy coordinate system the moment of:

a. the couple formed by the pair of forces F_1;
b. the couple formed by the pair of forces F_2;
c. the couple formed by the pair of forces F_3;
d. the couple T if T = 10 kNm;
e. the resultant couple.

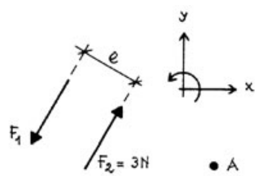


Moment of a force about to a point (Section 3.1.5)

3.6 F_1 and F_2 are statically equivalent to a couple.

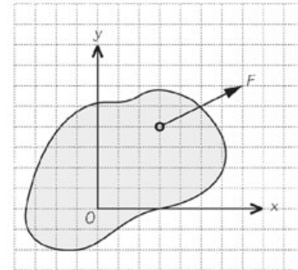
Questions:

a. How large is the distance e between both forces if the moment of F_1 about A is +2160 Nmm and the moment of F_2 about A is −1620 Nmm.
b. How large is the distance e between both forces if the moment of F_1 about a point B is +2160 Nmm and the moment of F_2 about the same point B is +1620 Nmm.



3.7 A force F is exerted on the body at A. Force scale: 1 square $\equiv$ 1 kN. Length scale: 1 square $\equiv$ 1 m.

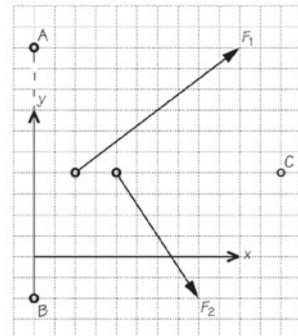
Question:
In four ways (!), calculate the moment of F with respect to the origin O of the xy coordinate system shown.



3.8 Find the forces F_1 and F_2 have magnitudes 250 and 180 kN respectively. Length scale: 1 square $\equiv$ 0.5 m.

Question:
The moment about A, B, and C respectively of:

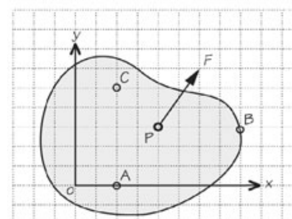
a. F_1;
b. F_2;
c. the resultant of F_1 and F_2.



3.9 For rigid bodies, a force may be shifted parallel to its line of action with the addition of a couple. Force scale: 1 square $\equiv$ 1 kN. Length scale: 1 square $\equiv$ 1 m.

Question:
How large is the moment of that couple if the force F at P is shifted respectively to:

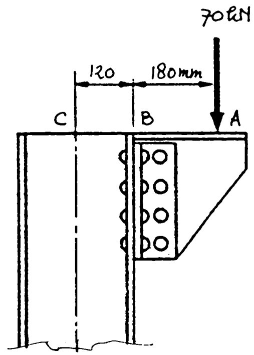
a. A.
b. B.
c. C.
d. O.



3.10 A console in a column is loaded at A by a vertical force of 70 kN.

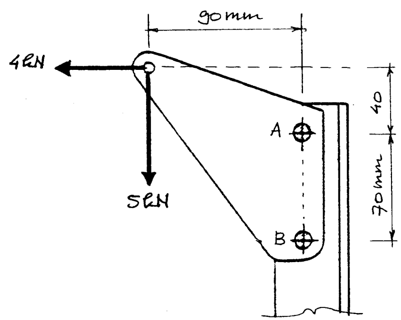
Questions:

a. Replace the force at A by a force at B and a couple.
b. Replace the force at A by a force at C and a couple.



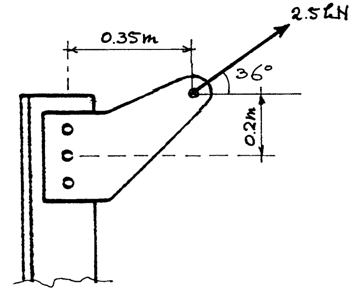
3.11 A console is subject to a horizontal force of 4 kN and a vertical force of 5 kN. In order to calculate the forces on the bolts A and B, the load is shifted to a point exactly halfway between A and B with the addition of a couple.

Question:
Determine the magnitude and direction of the couple.

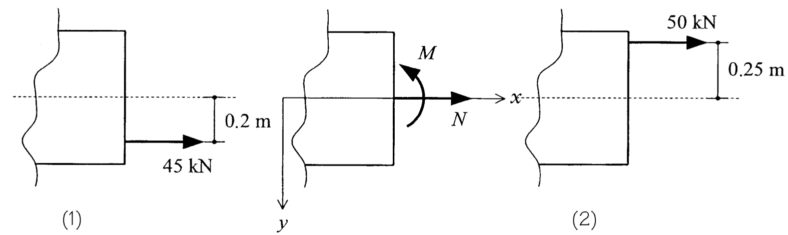


3.12 The console shown is fixed to a column by three bolts. In order to calculate the bolted connection, the load on the console is replaced by a horizontal and a vertical force at the point of the middle bolt, together with a couple.

Question:
The magnitude and direction of the forces and of the couple.



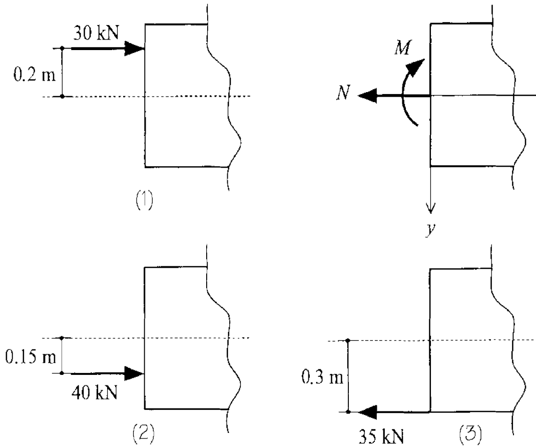
3.13: 1–2 In the left-hand and right-hand figures, a cross-section is subject to an eccentrically-applied tensile force. This force is statically equivalent to a normal force N and a bending moment M. The positive directions of N and M are shown in the middle figure.



Question:
Determine N and M, with the correct sign. Also depict N and M as they act in reality, and include their values.

Comment: N (normal force) and M (bending moment) are so-called section forces. Their nomenclature and sign conventions will be discussed further in Chapter 10.

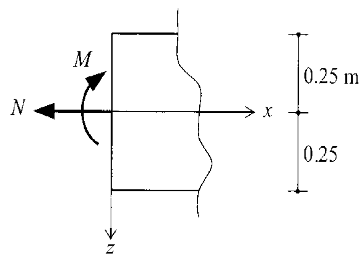
3.14: 1–3 As problem 3.13.



3.15 The section forces $N = 150$ kN and $M = 21$ kN are acting in a cross-section. They can be replaced by a single force acting at a distance e_z from the x axis, whereby it is assumed that e_z is positive if this force is acting on the positive side of the x axis ($z > 0$).

Questions:

a. Depict N and M as they are acting on the cross-section in reality and include their values.
b. Determine e_z with the correct sign.
c. Depict the force that is statically equivalent to N and M. Include its magnitude, direction and point of application.



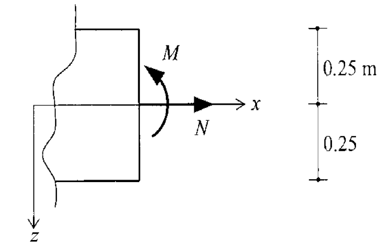
3.16: 1–2 As problem 3.15, but with

1. $N = -1$ kN and $M = +150$ Nm.
2. $N = +42$ kN and $M = -10.5$ kNm.

3.17 The section forces $N = -35$ kN and $M = +10.5$ kNm are acting in a cross-section. They can be replaced by a single force acting at a distance e_z from the x axis, whereby it is assumed that e_z is positive if this force is acting on the positive side of the x axis ($z > 0$).

Questions:

a. Depict N and M as they are acting in the section in reality and include their values.
b. Determine e_z with the correct sign.
c. Depict the force that is statically equivalent to N and M. Include its magnitude, direction and point of application.



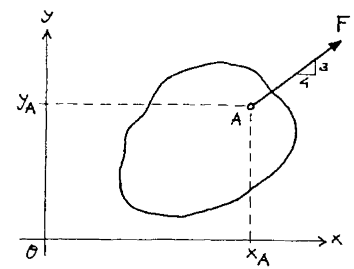
3.18: 1–2 As problem 3.17, but with

1. $N = -25$ kN and $M = -20$ kNm.
2. $N = +42$ kN and $M = -10.5$ kNm.

3.19 A is subject to a force $F = 100$ kN. The moment of this force about O is $T_z|O = 300$ kNm.

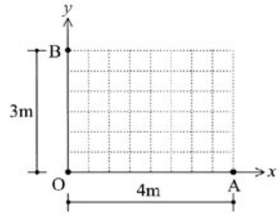
Question:

Where does the line of action intersect the x axis and the y axis respectively?



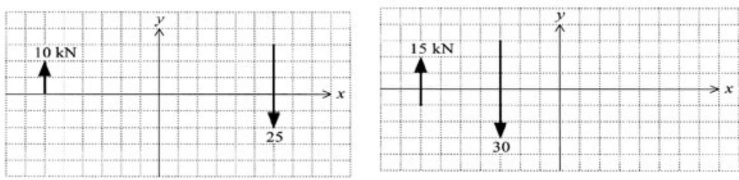
3.20 For a force F the line of action passes through the points A and B. The moment of F about O is $T_z|O = 6$ kNm.

Question:
Determine the components F_x and F_y.



Compounding forces and couples analytically (Section 3.1.7)

3.21: 1–2 The two forces are equivalent to a single force R. Force scale: 1 square ≡ 5 kN. Length scale: 1 square ≡ 1 m.

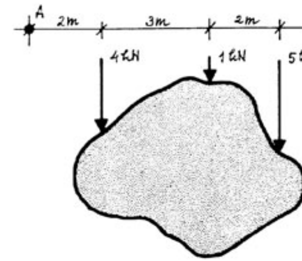


(1) (2)

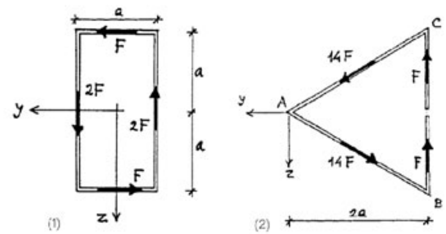
Question:
Where does the line of action R intersect the x axis?
a. $x = -1$ m,
b. $x = +1$ m,
c. $x = +6$ m,
d. $x = +14$ m.

3.22 The resultant of the three parallel forces exerted on the body is R.

Question:
Determine the distance of the line of action of R to point A.

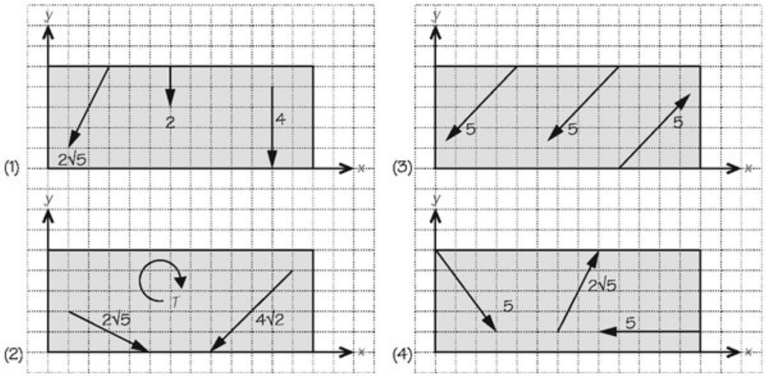


3.23: 1–2 The forces shown act on a thin-walled cross-section.



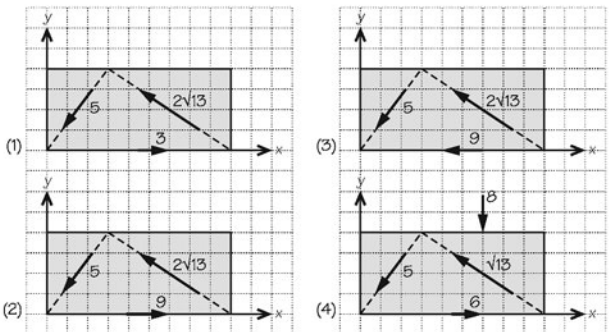
Question:
Determine the line of action, magnitude, and direction of the resultant of these forces.

3.24: 1–4 A number of forces act on a block. In case (2), there is also a couple $T = 36$ kNm. Force scale: 1 square ≡ 1 kN. Length scale: 1 square ≡ 1 m.



Question:
Determine the resultant.

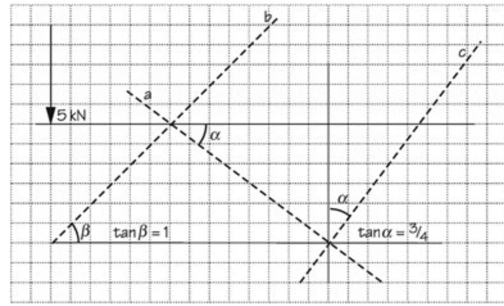
3.25: 1–4 A block is subject to three forces. The forces are not drawn to scale; the values are shown in kN. Length scale: 1 square ≡ 1 m.



Question:
Determine the resultant.

Resolving a force (couple) along three given lines of action (Sections 3.18 and 3.19)

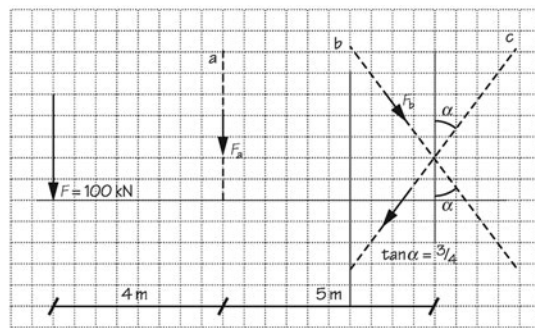
3.26 The force F is replaced by the three forces F_a, F_b and F_c with given lines of action a, b and c.



Question:
Determine the forces F_a, F_b and F_c:

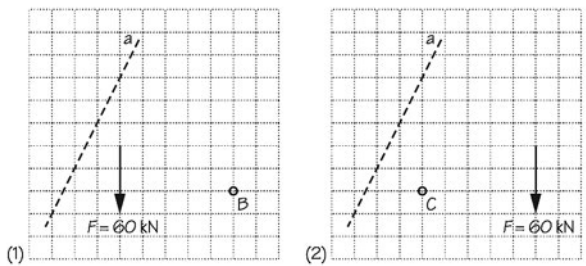
a. graphically;
b. analytically.

3.27 Force F is resolved into the components F_a, F_b and F_c with given lines of action a, b and c.



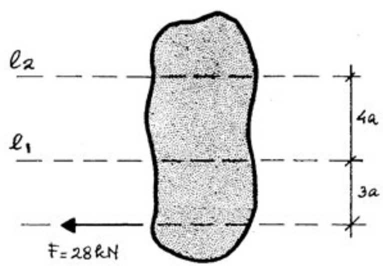
Question:
Find the magnitudes and directions of F_a, F_b and F_c.

3.28: 1–2 The force $F = 60$ kN is replaced by a force along line of action a and a force through point B, respectively point C.



Question:
Determine the magnitudes and directions of these forces.

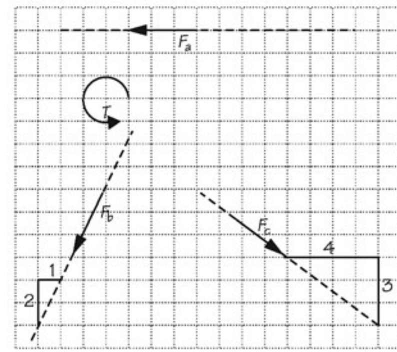
3.29 The force $F = 28$ kN is resolved into two parallel forces F_1 and F_2 with lines of action ℓ_1 and ℓ_2.



Question:
Determine the magnitudes and directions of the forces F_1 and F_2.

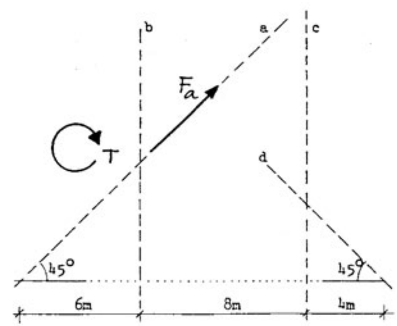
3.30 A couple $T = 110$ kNm is resolved into the forces F_a, F_b and F_c with given lines of action a, b and c. Length scale: 1 square ≡ 1 m.

Question:
Determine F_a, F_b and F_c.



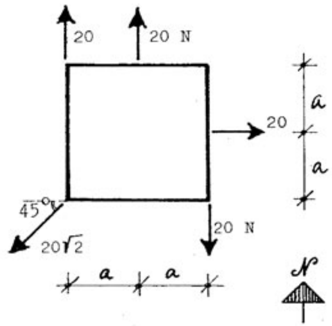
3.31 The couple $T = 60$ kNm is the resultant of four forces F_a, F_b, F_c and F_d with given lines of action a, b, c, and d. The magnitude and direction of the force F_a is given: $F_a = 30\sqrt{2}$ kN.

Question:
Determine F_b, F_c and F_d.



Equilibrium of a rigid body in a plane (Section 3.2)

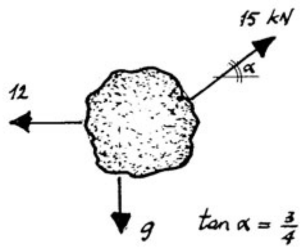
3.32 A block is subject to the forces shown in the horizontal plane. The north direction is shown.



Question:
Which of the following statements about the forces exerted on the block is true?

a. They comply with the three equilibrium conditions.
b. They form a couple together.
c. Their resultant points south-west.
d. Their resultant points east.

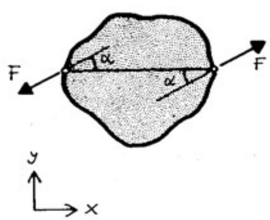
3.33 The three forces shown are exerted on the body.



Question:
Which statement about the body is true?

a. There is moment equilibrium.
b. There is force equilibrium.
c. There is no equilibrium.
d. There is equilibrium.

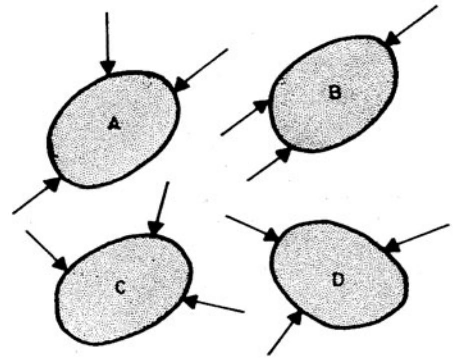
3.34 A body is subject to two parallel forces F.



Question:
Which statement is true?

a. $\sum F_x \neq 0$.
b. $\sum F_y \neq 0$.
c. $\sum T_z \neq 0$.
d. The body is in equilibrium.

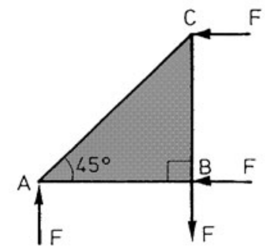
3.35 For two of the bodies shown, the equilibrium depends on the magnitude of the forces. For the other two, it is absolutely certain that they are not in equilibrium (the weights of the bodies are neglected).



Question:
Which of the two bodies are definitely not in equilibrium?

a. A and B.
b. A and C.
c. A and D.
d. B and C.

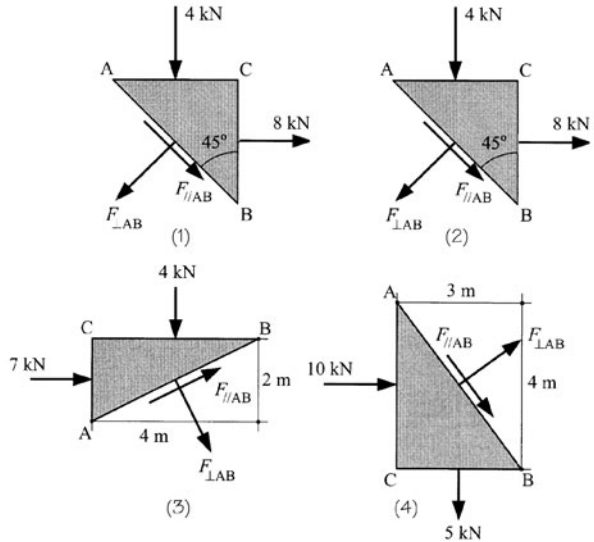
3.36 A triangular plate ABC is subject to four forces each with magnitude F (in the plane of the plate) that are *not* in equilibrium with one another. A fifth force is required to ensure equilibrium.



Question:
The line of action of the fifth force passes through:

a. A.
b. B.
c. C.
d. None of the points A, B and C.

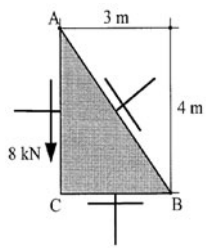
3.37: 1–4 The forces shown act on the edges of the triangular plate ABC. Their points of application are in the middle of the edges. The system is in equilibrium.



Question:
Determine $F_{//AB}$ and $F_{\perp AB}$, with the correct sign. Also depict how the forces are acting in reality and include their values.

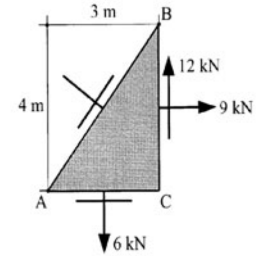
3.38 Of the six forces that act on the middle of the edges of the triangular plate ABC, one is known.

Question:
Which of the remaining five forces can be determined?

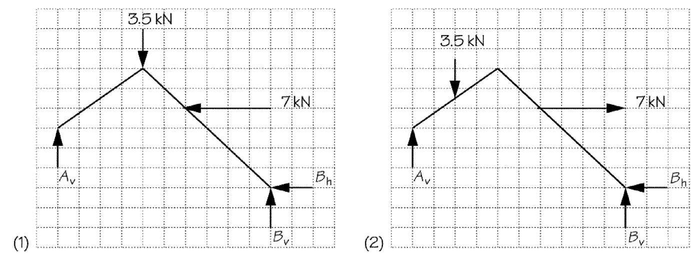


3.39 Of the six forces that act on the middle of the edges of the triangular plate ABC, three are given.

Question:
Determine the other three forces.



3.40: 1–2 A roof structure, loaded by the forces shown of 7 kN and 3.5 kN, is kept in equilibrium by the forces A_v, B_v and B_h. Length scale: 1 square $\equiv$ 0.5 m.

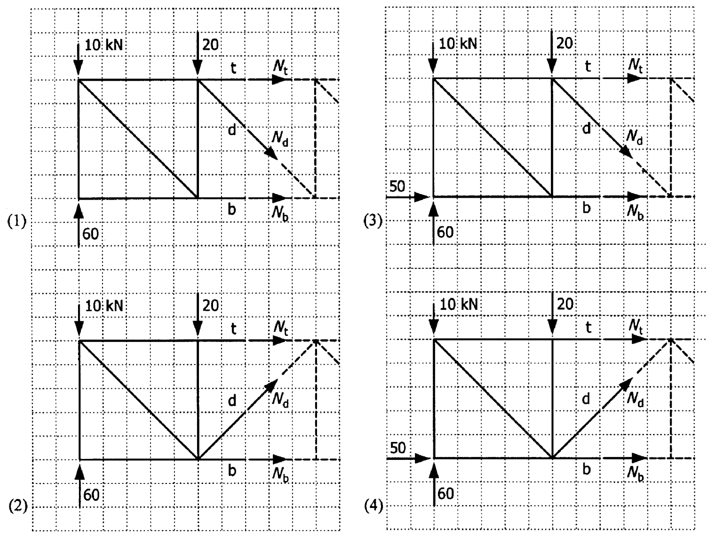


Question:
Determine the forces A_v, B_v and B_h.

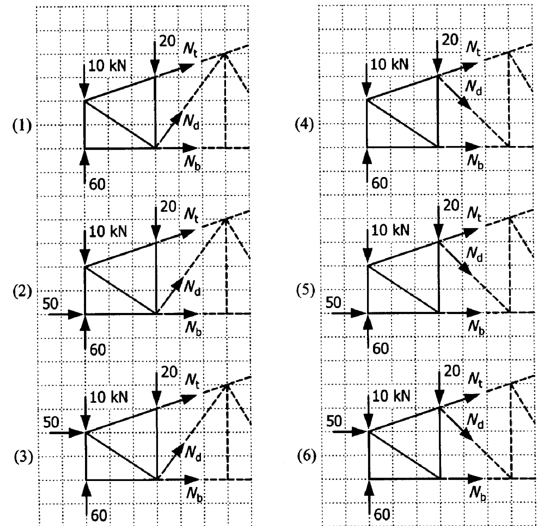
3.41: 1–4 The part isolated (cut) from a so-called *truss* shown in the figure is in equilibrium. The truss is subject to the forces shown. The values are in kN. Length scale: 1 square ≡ 1 m.

Question:
Determine the forces:
a. N_t in top chord member t;
b. N_d in diagonal member d;
c. N_b in bottom chord member b.

Comment: Trusses and calculating the truss forces N are covered in further detail in Chapter 9.



3.42: 1–6 The part isolated from a so-called *truss* shown in the figure is in equilibrium under the influence of the forces shown. The values are shown in kN. Length scale: 1 square ≡ 1 m.

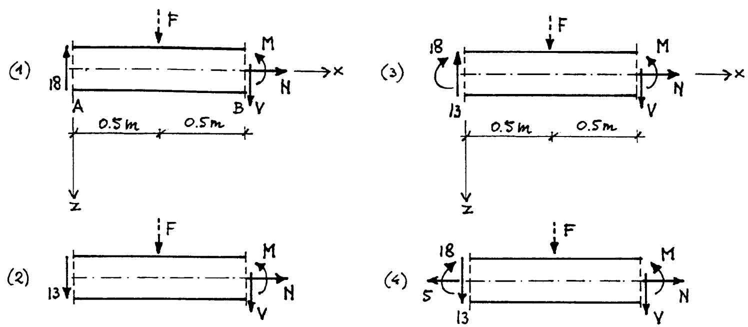


Question:
Determine the forces:
a. N_t in top chord member t;
b. N_d in diagonal member d;
c. N_b in bottom chord member b.

3.43: 1–4 A segment AB of length 1 m is isolated (cut away) from a *beam*. The section forces shown act in cross-section A. The forces are shown in kN and the couples (so-called *bending moments*) in kNm. The segment is in equilibrium.

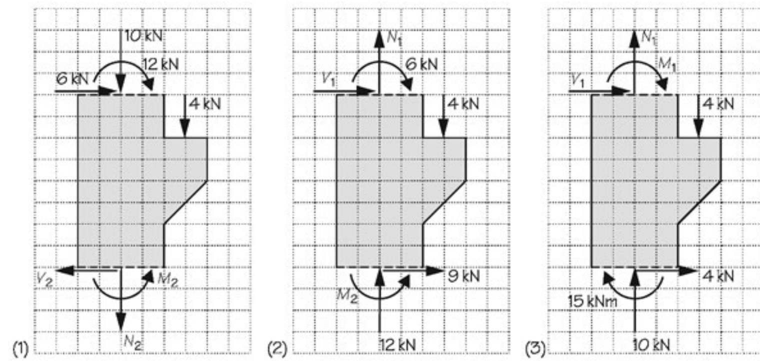
Question:
Determine the section forces N, V and M in cross-section B if:

a. the beam is not loaded between A and B;
b. the beam is loaded in the middle of AB by a vertical force F of 10 kN.



Comment: N (normal force), M (bending moment) and V (shear force) are so-called section forces. Their nomenclature and sign conventions are covered in further detail in Chapter 10.

3.44: 1–3 The body shown has been cut away from a column with console. The section forces N, V and M act at the central axis of the column. The console is subject to a force of 4 kN. The body is in equilibrium. Length scale: 1 square ≡ 0.1 m.

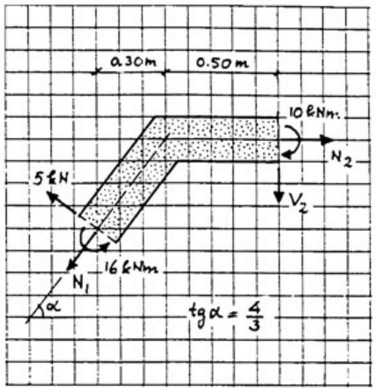


Question:
Determine the unknown section forces.

3.45 The section forces shown act on the cross-sections of the corner isolated from a portal frame. They act at the centre lines. The corner joint is in equilibrium. There is no loading between the cross-sections.

Questions:

a. determine the (normal) force N_1;
b. determine the (normal) force N_2;
c. determine the (shear) force V_2.



3.46 The section forces shown act on the cross-sections of the corner isolated from a portal frame. They act at the centre lines. The corner is additionally loaded by a vertical force F of 4 kN.

Questions:

a. determine the (normal) force N_1;
b. determine the (normal) force N_2;
c. determine the (shear) force V_2.
d. Which of the three forces N_1, N_2 and V_2 is independent of the magnitude of the vertical load F on the corner? Provide reasoning for the answer.