16 Influence Lines

In many structures, the support reactions and section forces depend not only on the magnitudes of the loads, but also on their placement. This is particularly true for bridges, where an important part of the load consists of moving vehicles. Other examples are assembly halls with crane runways or warehouses in which the placement of loads can change.

If we want to choose the dimensions of a structural element to check it for strength and rigidity, it is important to know the location at which the load or set of loads generate the most severe effects. Important tools for finding the most unfavourable placement of loads are the so-called *influence lines*. Influence lines are graphic representations of the magnitude of a support reaction or section force at a fixed location due to a single point load with variable position.[1]

In this chapter we look at how to determine influence lines for forces in statically determinate structures, and how to use them.

There are two methods for determining influence lines: directly from the equilibrium equations (Section 16.1), or by means of virtual work (Section 16.2). We will demonstrate both methods by means of examples.

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[1] There are also influence lines for displacements and rotations. We do not cover those here.

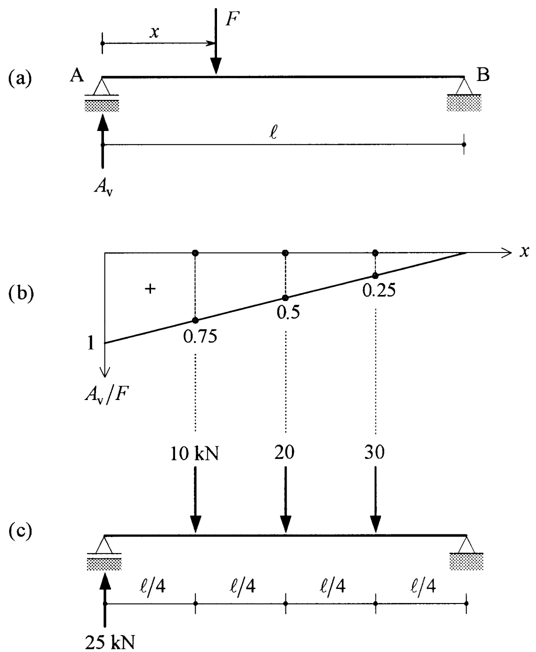


Figure 16.1 (a) Simply supported beam with (b) the influence line for the vertical support reaction at A and (c) the support reaction A_V due to a set of loads.

Finally, we will show how to use influence lines to determine the placement of loads to have the maximum effect (Section 16.3).

16.1 Influence lines using equilibrium equations

Here, for a simply supported beam, we will derive the influence lines for a support reaction, bending moment and shear force directly from the equilibrium equations. As you will notice, this method will become already laborious for a hinged beam.

16.1.1 Simply supported beam

Example 1 – Influence line for a support reaction
The principle of influence lines is discussed on the basis of the simply supported beam AB in Figure 16.1a. The beam is loaded by a moving point load F.

Question:
How does the vertical support reaction A_V at A change as the point load moves from A to B?

Solution:
Assume A_V acts in the direction indicated in Figure 16.1a. If the point load is placed at a distance x from A, it follows from the moment equilibrium about B that

$$A_V = F\frac{\ell - x}{\ell}.$$

If the position x of the point load is assumed to be variable, and we draw A_V/F as a function of x (see Figure 16.1b), we obtain the *influence line for the support reaction* A_V. It is the convention to plot the positive values of A_V/F in the (positive) direction of F.

For influence lines, one does not plot A_v, but rather A_v/F as a function of the position of the point load. We also can interpret the influence line as the variation of the support reaction due to a moving unit load (e.g. $F = 1$ kN).

The magnitude of the support reaction, or rather the *influence* A_v/F, can be read from the influence line at the position of the load. In this way, using the influence line, the support reaction can be quickly derived if the beam is subject to a set of loads. For example, for the case in Figure 16.1c:

$$A_v = +0.75 \times (10\text{ kN}) + 0.5 \times (20\text{ kN}) + 0.25 \times (30\text{ kN}) = +25\text{ kN}.$$

From the influence line we can see that at the position of the force of 10 kN the influence is 0.75, or in other words: the contribution of this force to A_v is:

$$+0.75 \times (10\text{ kN}) = +7.5\text{ kN}.$$

The total support reaction is found by superposing all the individual contributions.

Example 2 – Influence line for a bending moment
We will now determine the bending moment M_C at midspan C for the beam in Example 1 (see Figure 16.2a).

Solution:
Assume the bending moment M_C is positive if it causes tension at C at the underside of the beam. The magnitude of M_C follows from the moment equilibrium of AC (or CB) about C. It should be remembered that it makes a difference in the equilibrium equations whether the load is to the left or right of the cross-section under consideration.

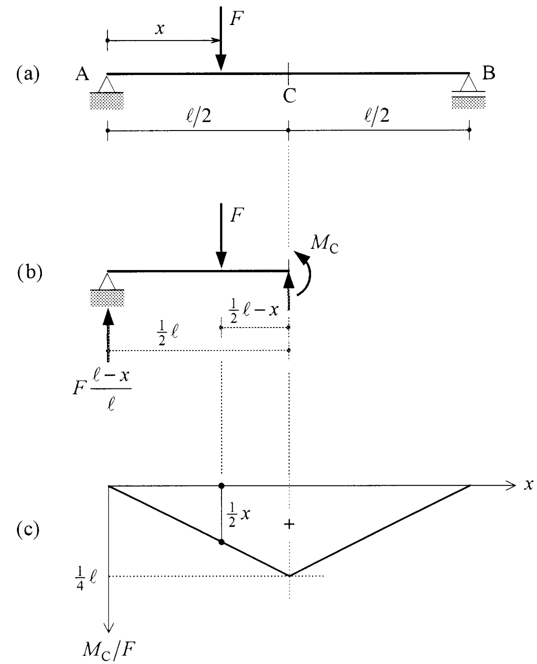


Figure 16.2 (a) Simply supported beam with (b) the isolated left-hand segment and (c) the influence line for the bending moment at C.

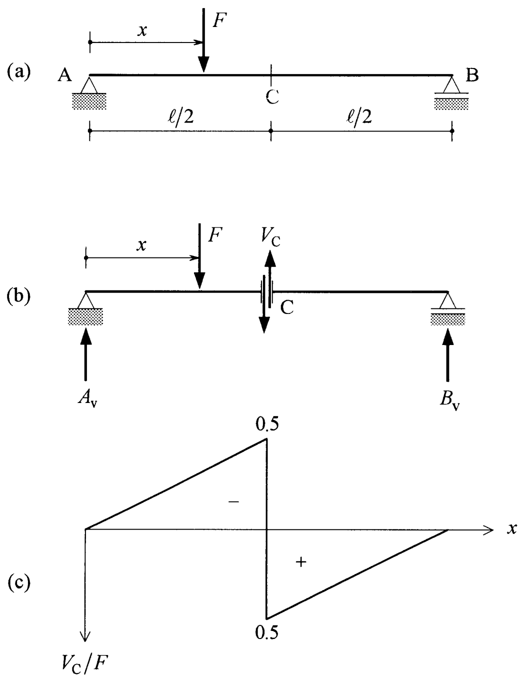


Figure 16.3 (a) Simply supported beam with (b) the positive direction assumed for shear force V_C and (c) the influence line for the shear force at C.

If the load is to the left of C ($0 \le x < \frac{1}{2}\ell$) then the bending moment at C equals (see Figure 16.2b)

$$M_C = +F\frac{\ell - x}{\ell} \cdot \tfrac{1}{2}\ell - F \cdot \left(\tfrac{1}{2}\ell - x\right) = +\tfrac{1}{2}Fx.$$

If the load is to the right of C ($\frac{1}{2}\ell < x \le \ell$) then the bending moment at C equals

$$M_C = +F\frac{\ell - x}{\ell} \cdot \tfrac{1}{2}\ell = +\tfrac{1}{2}F(\ell - x).$$

The variation of M_C/F as a function of x, as shown graphically in Figure 16.2c, is referred to as the *influence line for the bending moment at C*.

The influence line has its maximum in the middle. This means that the bending moment at C is a maximum when the point load F is at midspan:

$$M_C = \left(+\tfrac{1}{4}\ell\right) \times F = +\tfrac{1}{4}F\ell.$$

Example 3 – Influence line for a shear force

For the same beam we will now determine the influence line for the shear force V_C at C (see Figure 16.3a).

Solution:

The direction assumed for V_C is shown in Figure 16.3b. The shear force V_C can be found from the vertical force equilibrium of AC (or CB).

If the load is to the left of C ($\frac{1}{2}\ell < x \le \ell$) the equilibrium of AC gives

$$V_C = A_v - F = F\frac{\ell - x}{\ell} - F = -F\frac{x}{\ell}.$$

If the load is to the right of C ($\frac{1}{2}\ell < x \leq \ell$) then the shear force at C is equal to the support reaction A_v at A:

$$V_C = A_v = F\frac{\ell - x}{\ell}.$$

This determines the *influence line for the shear force at C*. The influence line is shown in Figure 16.3c.

If the point load is to the left of C the influence line is negative. The shear force acts in the direction opposite to the one assumed in Figure 16.3b.

The shear force at C is a maximum if the point load F is directly to the left or right of C:

$$V_C = (\pm 0.5) \times F = \pm\tfrac{1}{2}F.$$

A change of sign occurs at C.

16.1.2 Hinged beam

Figure 16.4 shows the influence lines for a hinged beam. The positive direction of the support reaction B_v at B is shown in Figure 16.4a. For the bending moment and the shear force, the positive directions according to the xz coordinate system are given in Figure 16.4f.

We do not address the calculation of each of these influence lines in detail here. They can be found by, for various positions of the point load, using the equilibrium equations to calculate the magnitude of the various quantities. In this case, that leads to a fair amount of work as there are so many positions of the load to be considered.

It is preferable to investigate a number of characteristic positions of the point load (e.g. just above the support loads, or at the hinges) and to remember that the influence line between certain points has to be linear.

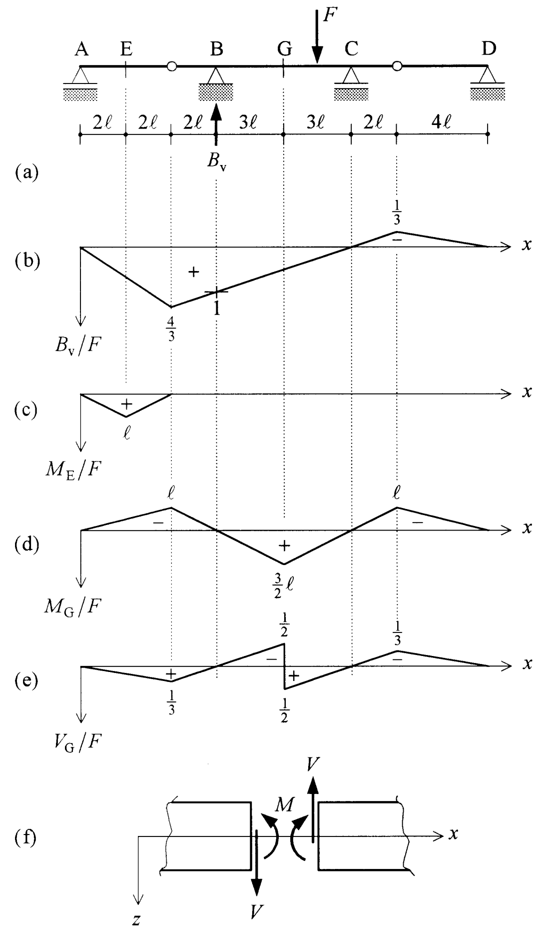


Figure 16.4 (a) Hinged beam with influence lines for (b) the vertical support reaction at B, (c) the bending moment at E, (d) the bending moment at G and (e) the shear force at G; (f) positive directions for bending moment and shear force.

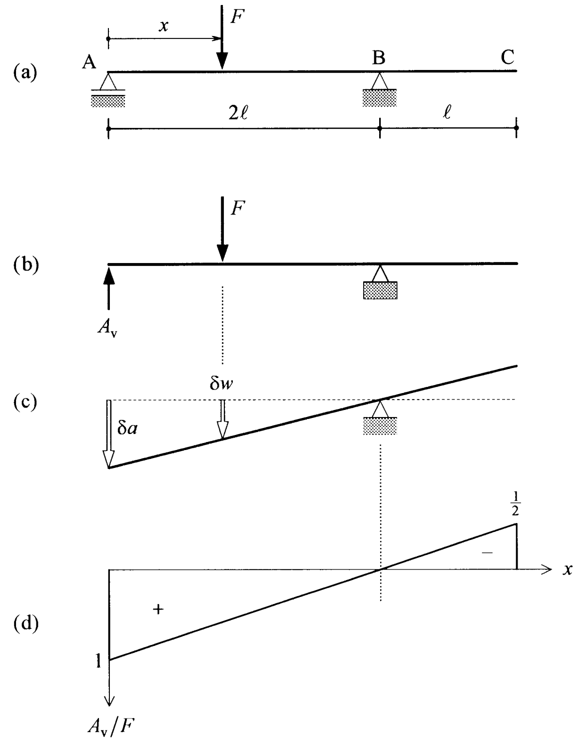


Figure 16.5 (a) Beam with (b) mechanism for determining the vertical support reaction at A, (c) virtual displacement for which A_V performs negative work and (d) the influence line for A_V.

In the next section we introduce an alternative method, based on virtual work.

16.2 Influence lines using virtual work

In Chapter 15 we showed that the virtual work equation offers an alternative formulation of equilibrium equations. When determining influence lines, we can replace the equilibrium equations by a single virtual work equation.

The alternative method, based on *virtual work*, provides the shape of the influence line more quickly with less calculation. It is necessary, however, to take three rules into account. We describe the method, and the rules, below.

16.2.1 Simply supported beam

Example 1 – Influence line for a support reaction
To determine the influence line for the vertical support reaction A_V at A using virtual work, we convert the beam in Figure 16.5a into a mechanism by removing the roller support at A. The unknown support reaction A_V is applied to the mechanism as a load (see Figure 16.5b).

We now apply a virtual displacement to the mechanism by displacing A over a distance δa so that A_V performs *negative work* (first rule) for the directions assumed in Figure 16.5b (see Figure 16.5c). Assume that the point of application of F undergoes a virtual displacement δw and that δw *is positive in the direction of F* (second rule).

For equilibrium, the virtual work is zero:

$$\delta A = -A_V \delta a + F\delta w = 0$$

so that

$$A_{\mathrm{v}} = F\frac{\delta w}{\delta a} \quad \text{or} \quad \frac{A_{\mathrm{v}}}{F} = \frac{\delta w}{\delta a}.$$

The virtual displacement δw (positive in the direction of F) is dependent on the position of the point load. It turns out that A_{v}/F, the ordinate of the influence line, is proportional to the virtual displacement δw. This means that on a certain scale (δa) the deflection line of the mechanism due to virtual displacement is identical with the influence line we are looking for. The influence line for A_{v} is shown in Figure 16.5d. The value of the ordinate at A is equal to 1 as there $\delta w = \delta a$.

The fact that the influence line and the deflection line of the mechanism are the same shape means that the signs of both quantities δw and δa are the same. This is a consequence of the fact that the virtual displacement has been chosen in such a way that the quantity we are looking for (A_{v}) performs negative work (rule 1).

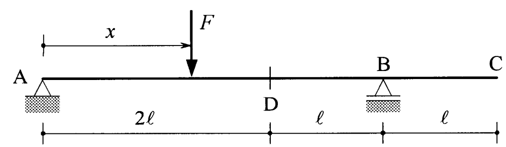
If, for the scale factor δa, we chose *a displacement that is equal to the unit of length* – this is known as a *unit displacement* – (third rule), then the influence line is identical with the deflection line of the mechanism.

Conclusion: *If the force sought performs* negative work *(rule 1) over a* unit displacement *(rule 3), then the influence line is equal to the deflection line of the mechanism. The influence line is positive where the displacement is in the direction of F (segment AB) and negative where the displacement is in the opposite direction of F (segment BC) (rule 2).*

Example 2 – Influence line for a bending moment
The second example relates to the influence line for the bending moment at cross-section D of the beam in Figure 16.6a.

Again we convert the beam into a *mechanism* by introducing a hinge at D. The action of the bending moment at D is replaced by the pair of moments M_{D} that are applied to the mechanism at either side of the hinge (see Figure 16.6b). The direction of M_{D} can be chosen arbitrarily.

(a)



(b)

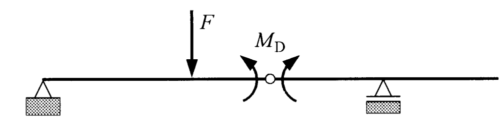


Figure 16.6 (a) Beam with (b) mechanism for determining the bending moment at D.

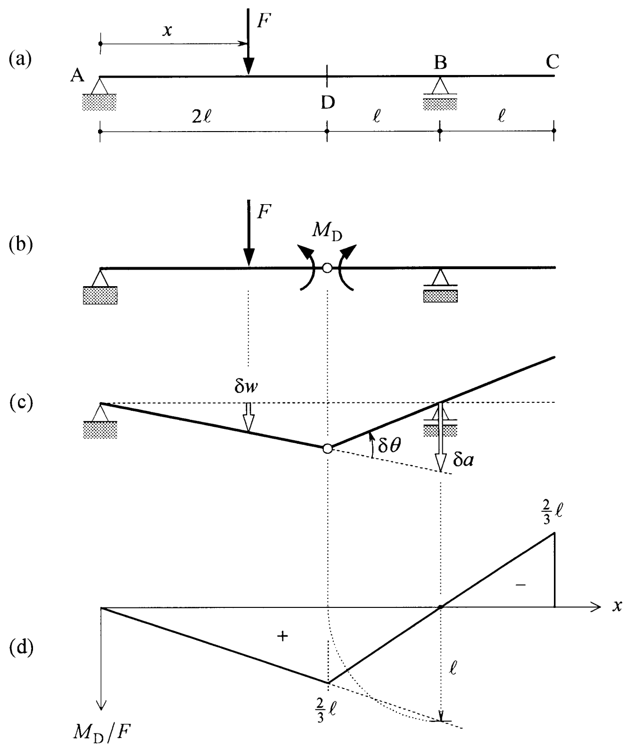


Figure 16.6 (a) Beam with (b) mechanism for determining the bending moment at D. (c) Virtual displacement for which M_D performs negative work and (d) the influence line for M_D.

We apply a virtual displacement to the mechanism by rotating beam segments AD and DBC at D through an angle $\delta\theta$ with respect to one another, but in such a way that M_D *performs negative work* (rule 1) (see Figure 16.6c). Assume that load F is displaced by a distance δw, and that δw *is positive in the direction of* F (rule 2).

For equilibrium, the virtual work is zero:

$$-M_D\delta\theta + F\delta w = 0$$

so that

$$\frac{M_D}{F} = \frac{\delta w}{\delta\theta}.$$

The bending moment M_D is proportional to the displacement δw. The influence line for M_D therefore has the same shape as the deflection line of the mechanism. The scale factor is the angle $\delta\theta$.

For a virtual displacement, δw and $\delta\theta$ are infinitesimally small, but their ratio is finite. The influence line can therefore be drawn on an enlarged scale. $\delta\theta$ can be defined as (see also Section 15.4.2):

$$\delta\theta = \frac{\delta a}{\ell}.$$

If we select δa equal to ℓ, it is said that the angle $\delta\theta$ has the *orthogonal unit value* – this is also referred to as a *unit rotation* (rule 3). In that case the influence line is exactly the same as the deflection line of the mechanism (see Figure 16.6d).

Conclusion: *If we apply a* unit rotation *to the hinge, so that the bending moment performs* negative work, *then the influence line for that bending moment is the same as the deflection line of the mechanism.*

Arcs can be used to construct an orthogonal unit angle as shown in Figure 16.7. In that figure, two arcs are drawn, but one arc is actually sufficient, as appears from the plot of the influence line in Figure 16.6d.

Example 3 – Influence line for a shear force

The third example relates to the influence line for the shear force in cross-section D of the beam in Figure 16.8a.

The procedure is identical to that in the previous examples. The beam is transformed into a mechanism by introducing a *slide joint* or *shear force hinge* at D and replacing the action of the shear force at D by the pair of forces V_D that are applied on the mechanism at either side of the slide joint (see Figure 16.8b). The direction of V_D can be chosen arbitrarily.

Subsequently, the mechanism is subjected to a virtual displacement by displacing segments AD and DBC at D over a distance δu with respect to one another, but in such a way that V_D *performs negative work* (rule 1) (see Figure 16.8c). In the deformed mechanism, beam segments AD and DBC remain parallel to one another.

Assume load F moves over a distance δw, and δw *is positive in the direction of F* (rule 2).

For equilibrium, the virtual work is zero:

$$-V_D\delta u + \delta w = 0$$

so that

$$\frac{V_D}{F} = \frac{\delta w}{\delta u}.$$

The shear force V_D is proportional to the deflection δw. The influence line for the shear force at D is the same shape as the deflection line of the deformed mechanism. The scale factor is the displacement δu.

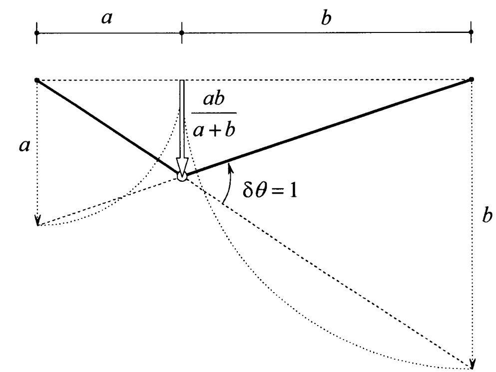


Figure 16.7 The construction of an angle $\delta\theta$ with orthogonal unit value.

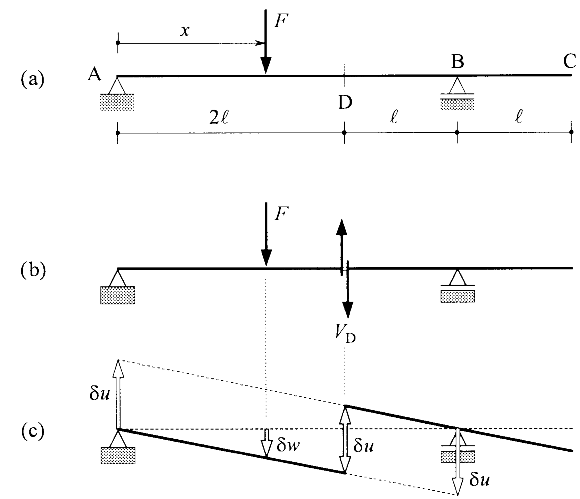


Figure 16.8 (a) Beam with (b) mechanism for determining the shear force at D, (c) virtual displacement for which V_D performs negative work.

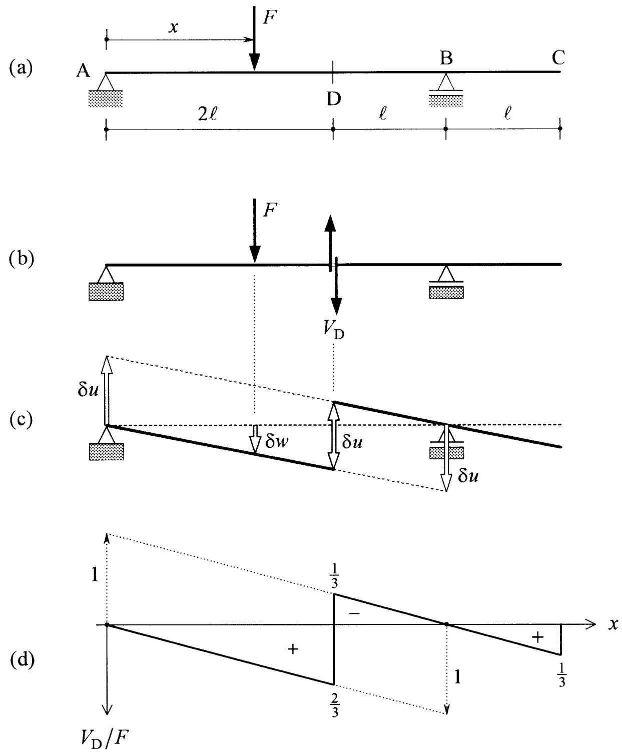


Figure 16.8 (a) Beam with (b) mechanism for determining the shear force at D, (c) virtual displacement for which V_D performs negative work. (d) The influence line for V_D.

If we choose δu as a *unit length displacement* (rule 3), then the influence line is identical to the deflection line of the mechanism (see Figure 16.8d).

Conclusion: *If the shear force performs* negative work *over a* unit displacement, *then the influence line is the same as the deflection line of the mechanism. Where the displacement is in the direction of F (segments AD and BC), the influence line is positive and the shear force acts in the assumed direction. The influence line is negative where the displacement is in the direction opposite to that of F (segment DB); here the shear force acts opposite to the direction assumed.*

16.2.2 General procedure for the method of virtual work

Determining the influence line for a force quantity[1] using the method of virtual work requires the same procedure each time:

- Convert the structure into a mechanism by creating a joint (hinge) that cannot transfer the force quantity in question.
- Allow the force quantity to act on the mechanism as a load.
- Apply a virtual displacement to the mechanism such that the force quantity performs negative work.
- Select a unit displacement or unit rotation for the displacement or rotation respectively over which the force performs work. The deflection line of the deformed mechanism is the influence line.
- The force quantity in question is positive when the displacement is in the direction of the force quantity and negative when it is opposite to the direction of the force quantity.

[1] Generalisation for support reactions and section forces.

16.2.3 Hinged beam

On the basis of a few examples relating to hinged beams, we demonstrate that the method of virtual work provides a quick and easy way of plotting influence lines.

Example 1 – Influence line for a fixed-end moment
The hinged beam in Figure 16.9a has hinges at S_1 and S_2, and is fixed at D. We will determine the influence line for the fixed-end moment M_D at D.

The influence line is found by changing the fixed-end support at D into a hinged support, and there applying a unit rotation such that M_D performs negative work.[1] The deformed mechanism in Figure 16.9b is then the influence line in Figure 16.9c.

The influence line shows that the fixed-end moment for the direction assumed for M_D in Figure 16.9b has a maximum positive value when load F is at S_1:

$$M_D = +2Fa.$$

The most negative fixed-end moment occurs when the load is at A, the end of the overhang:

$$M_D = -4Fa.$$

The zeros in the influence lines allow us to check: the fixed-end moment is always zero when the load is placed at one of the supports B, C or D.

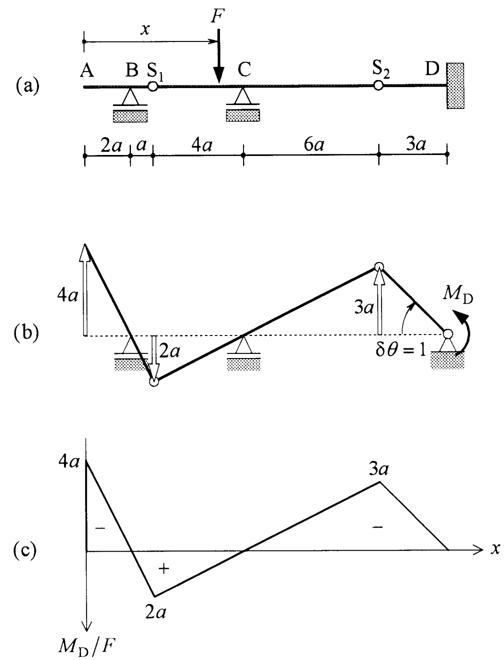


Figure 16.9 (a) Hinged beam with (b) mechanism for determining fixed-end moment M_D at D and (c) the influence line for M_D.

[1] Note: the unit rotation is applied at D and not at S_2!

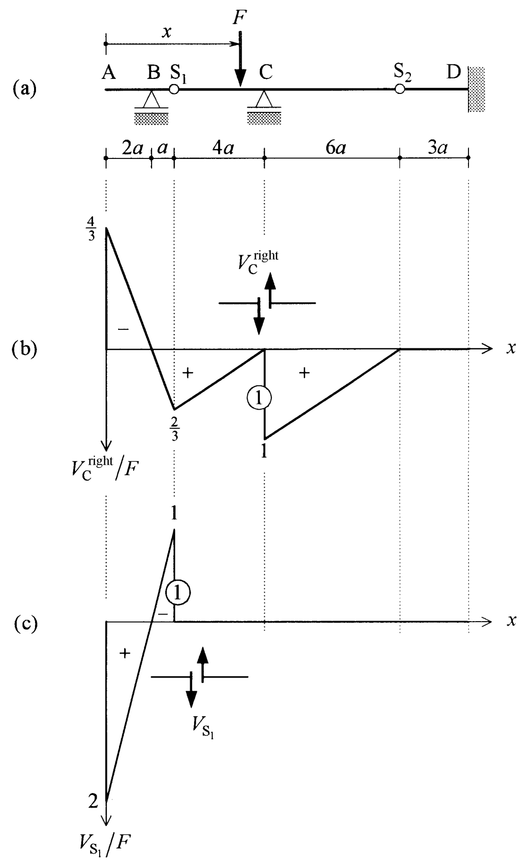


Figure 16.10 (a) Hinged beam with (b) the influence line for the shear force directly to the right of C and (c) the influence line for the shear force at S_1.

Example 2 – Influence line for shear forces
The beam from Example 1 is again shown in Figure 16.10a.

Figure 16.10b shows the influence line for the shear force V_C^{right} directly to the right of C. Figure 16.10c shows the influence line for the shear force V_{S_1} at hinge S_1.

The influence lines are found by introducing a slide joint directly to the right of C, and at S_1, respectively, and there applying a unit displacement such that the shear force performs negative work. The deformed mechanism is then the influence line we are looking for.

The (assumed) positive direction of the shear force is shown separately in the figures.

The mechanisms are not shown separately. However, a ① shows where in the mechanism a unit displacement was applied.

In the influence line for V_C^{right} (Figure 16.10b) the paths through S_1C and CS_2 are parallel. After all, in the mechanism, segments S_1C and CS_2 can displace only with respect to one another, and cannot rotate with respect to one another.

In the influence line for V_{S_1} (Figure 16.10c) the paths through ABS_1 and S_1C are not parallel, as in accordance with the mechanism the segments ABS_1 and S_1C can rotate with respect to one another due to the hinge at S_1.

Example 3 – Various influence lines
Figures 16.11b to 16.11e show various influence lines for the hinged beam in Figure 16.11a. The positive direction of the support reaction B_v at B is shown in Figure 16.11a. For the bending moment and the shear force, the positive directions are related to the xz coordinate system in Figure 16.11f.

These influence lines are also shown in Section 16.1.2. There, we did not address the amount of arithmetic needed for a direct calculation from the equilibrium equation.

The method of virtual work gives the same result with far less effort. If the correct mechanism is selected, and the virtual displacement is applied in such a way that the required quantity performs negative work over a unit displacement or unit rotation, the deformed mechanism is the influence line we are looking for.

The mechanisms are not shown separately. However, a ① shows where in the mechanism a unit displacement or unit rotation was applied.

It is recommended to check the influence line by calculating the value and the sign at a few relevant points from the equilibrium equations.

16.3 Working with influence lines

We have found that the method of virtual work provides the easiest way to find influence lines.

In this section we address working with influence lines, and we do not discuss how they are found. Using a number of examples, we show how to determine the force quantity in question (support reaction, section force) using an influence line for a *set of loads* and a *uniformly distributed load*.

Influence lines are often used for determining the *most unfavourable placement of the load*, the placement where the load has the most severe effect on the quantity in question. We also provide a number of examples of this.

16.3.1 Calculating forces using a given influence line

Example 1 – Set of loads

From an influence line, we can read off the influence of a point load with a variable position on a certain quantity (support reaction, section force) at a

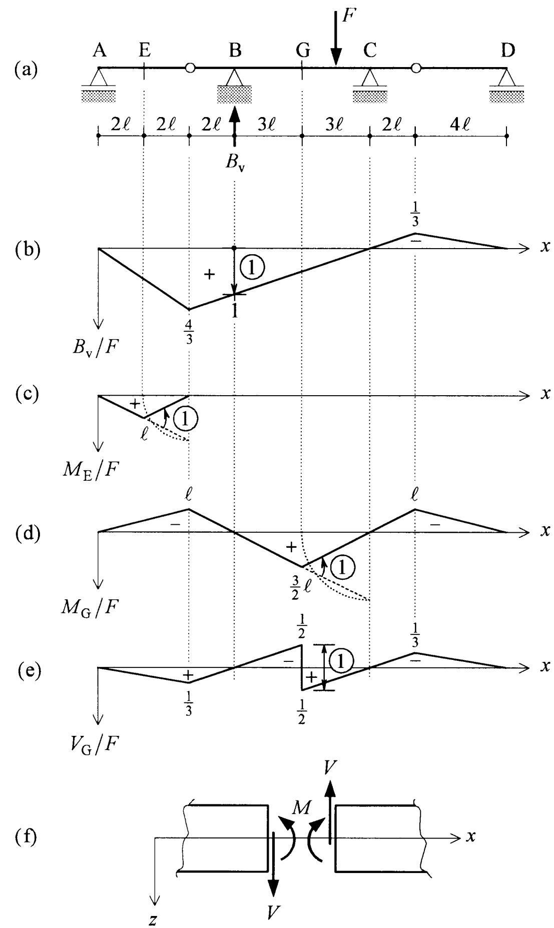


Figure 16.11 (a) Hinged beam with influence lines for (b) the vertical support reaction at B, (c) the bending moment at E, (d) the bending moment at G and (e) the shear force at G; (f) positive directions for bending moment and shear force.

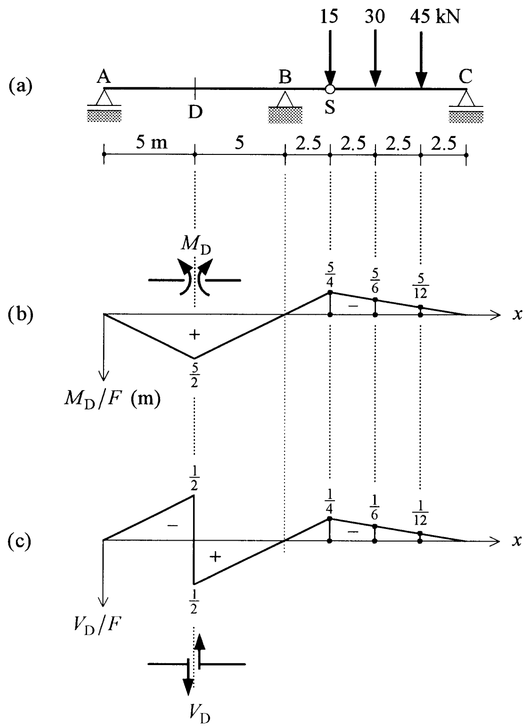


Figure 16.12 (a) Hinged beam with set of loads and the influence lines for (b) the bending moment at D and (c) the shear force at D.

fixed location. We can also say that the influence line gives the variation of a certain quantity due to a movable unit load. The value of the quantity is found at the position of the point load.

Figure 16.12 shows a hinged beam with the influence lines for the bending moment and the shear force at D. The positive directions of M_D and V_D are shown with the influence lines.

The hinged beam is loaded in field BC by a set of forces. The length of the beam, and the positions and magnitudes of the forces are shown in the figure. For each of the point loads, the influence line shows the associated influence quantity at the position of the load. The influence quantity for the bending moment (M_D/F) has the dimension of length. Figure 16.12b therefore includes the values in metres. The influence quantity for the shear force (V_D/F) is dimensionless.

The influence lines give the following for the bending moment:

$$M_D = -\left(\tfrac{5}{4}\text{ m}\right) \times (15\text{ kN}) - \left(\tfrac{5}{6}\text{ m}\right) \times (30\text{ kN}) - \left(\tfrac{5}{12}\text{ m}\right) \times (45\text{ kN})$$
$$= -62.5\text{ kNm}$$

and for the shear force

$$V_D = -\tfrac{1}{4} \times (15\text{ kN}) - \tfrac{1}{6} \times (30\text{ kN}) - \tfrac{1}{12} \times (45\text{ kN}) = -12.5\text{ kN}.$$

The correctness of these values can be checked by considering the equilibrium equations.

Example 2 – Uniformly distributed load

In Figure 16.13a, the beam in the previous example is loaded along its entire length by a uniformly distributed load $q = 80$ kN/m. Here too we will determine the bending moment and the shear force at D.

First we calculate the bending moment. The influence quantity M_D/F, which is a function of x, is hereafter for simplicity denoted by $f(x)$. The contribution dM_D to the bending moment M_D of the distributed load q over a small length dx is found by multiplying the small resulting force $q\,dx$ by the associated value of $f(x)$ of the influence line:

$$dM_D = f(x) \cdot q\,dx.$$

The bending moment at D due to the distributed load between $x = x_1$ and $x = x_2$ is found by summing up all the contributions, or in other words, by integrating:

$$M_D = \int_{x_1}^{x_2} f(x) q\,dx.$$

Since the distributed load is constant here, q can be taken outside the integration symbol:

$$M_D = q \int_{x_1}^{x_2} f(x)\,dx.$$

The integral represents the area of the influence line between x_1 and x_2 (see Figure 16.13b).

The bending moment due to a uniformly distributed load q is therefore equal to the load q, multiplied by the area of the influence line for the part where the load is acting. The signs have to be taken into account when determining the magnitude of the areas.

The bending moment at D due to the uniformly distributed full load is found from the influence line in Figure 16.13b:

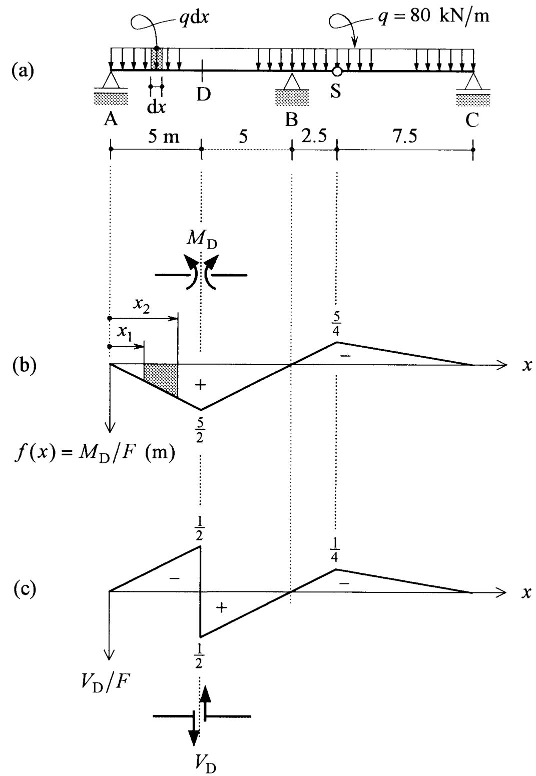


Figure 16.13 (a) Hinged beam with uniformly distributed full load and the influence lines for (b) the bending moment at D and (c) the shear force at D.

$$M_{\mathrm{D}} = \left\{+\tfrac{1}{2} \times (10\ \mathrm{m}) \times \left(\tfrac{5}{2}\ \mathrm{m}\right) - \tfrac{1}{2} \times (10\ \mathrm{m}) \times \left(\tfrac{5}{4}\ \mathrm{m}\right)\right\} \times (80\ \mathrm{kN/m})$$
$$= 500\ \mathrm{kNm}.$$

In the same way, the shear force at D is found from the area of the influence line in Figure 16.13c. Since it is immediately clear that the total area of the influence line over field AB is zero, we have only to determine the area over field BC:

$$V_{\mathrm{D}} = -\tfrac{1}{2} \times (10\ \mathrm{m}) \times \tfrac{1}{4} \times (80\ \mathrm{kN/m}) = -100\ \mathrm{kN}.$$

16.3.2 Most unfavourable placements of loads

If the load consists of a single point load, the influence line shows directly where the force has the maximum effect. Also for a uniformly distributed load the most unfavourable placement is rather easy to find. With a set of loads, however, this is no longer the case, and several positions will have to be investigated.

Here we look at a case in which we can rather easily calculate the most unfavourable position for a set of loads. The second example relates to a uniformly distributed load.

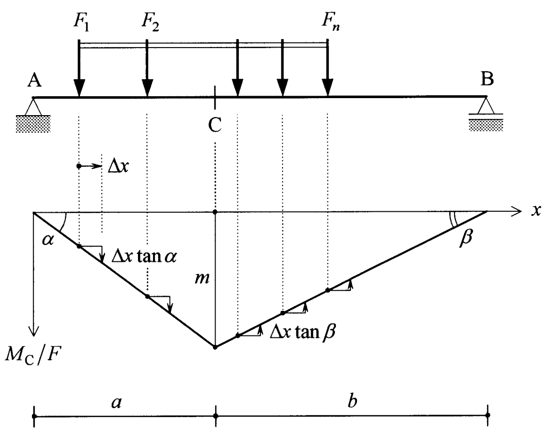


Figure 16.14 Beam with the influence line for the bending moment at C, with a set of loads that moves over a distance Δx.

Figure 16.14 shows the influence line for the bending moment at C for beam AB. The maximum value is m. The beam is subject to a set of loads. Part of the loads is in field AC; another part is in field CB. If the set of loads moves over a distance Δx, the ordinate of the influence line at the position of the loads in field AC increases by

$$\Delta x \tan\alpha = \Delta x \frac{m}{a}.$$

Therefore, the bending moment at C increases by

$$\Delta M_C = \sum_{AC} F_i \cdot \Delta x \frac{m}{a} = R^{(AC)} \frac{m}{a} \Delta x.$$

$R^{(AC)}$ is the resultant of all the loads in field AC.

At the same time, the ordinate of the influence line at the position of each of the loads in field CB decreases by

$$\Delta x \tan \beta = \Delta x \frac{m}{b}.$$

This changes the bending moment at C by

$$\Delta M_C = -\sum_{CB} F_i \cdot \Delta x \frac{m}{b} = -R^{(CB)} \frac{m}{b} \Delta x.$$

$R^{(CB)}$ is the resultant of all the loads in field CB.

Due to the displacement of the set of loads by a distance Δx the total increase of the bending moment at C is

$$\Delta M_C = \left(\frac{R^{(AC)}}{a} - \frac{R^{(CB)}}{b} \right) \cdot m \Delta x = (q^{(AC)} - q^{(CB)}) \cdot m \Delta x.$$

Here, $q^{(AC)} = R^{(AC)}/a$ and $q^{(CB)} = R^{(CB)}/b$ can be seen as the average loads in fields AC and CB respectively. As long as $q^{(AC)}$ is larger than $q^{(CB)}$ the bending moment increases if the set of loads moves in the positive x direction.

If one of the loads passes the location C, the average field loads change. The bending moment M_C is a maximum for that load at C for which $(q^{(AC)} - q^{(CB)})$ is zero or changes sign.

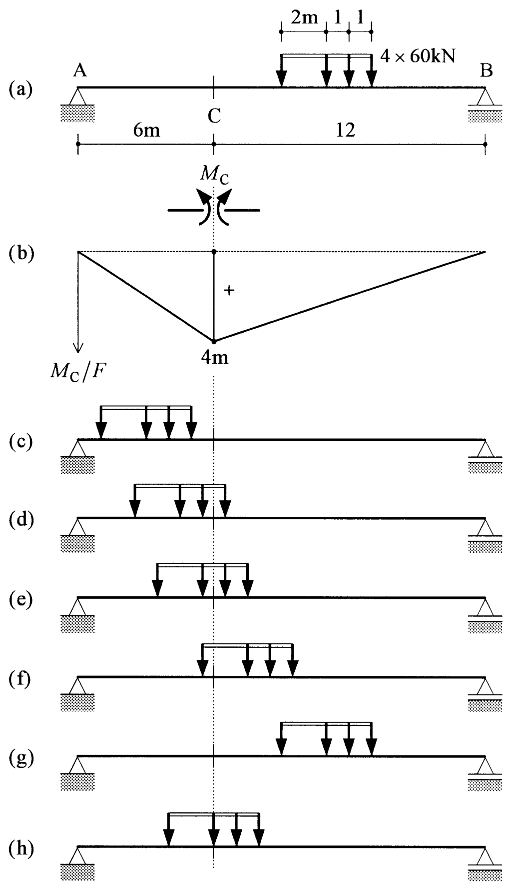


Figure 16.15 (a) Beam with a set of loads; (b) influence line for the bending moment at C; (c) to (g) positions of the set of loads to be investigated; (h) most unfavourable position of the set of loads.

Example 1 – Most unfavourable placement of a set of loads

The beam in Figure 16.15a carries a moving set of loads consisting of four forces of 60 kN for which the mutual distances are shown in the figure. Figure 16.15b shows the influence line for the bending moment at C. We will calculate the maximum bending moment at C due to the set of loads.

Solution:

The maximum bending moment at C occurs when the placement of the set of loads is such that the average loads of fields AC and CB are zero or change sign. Figures 16.15c to 16.15g show five consecutive positions by moving a load from field AC to BC. The average field loads are shown in Table 16.1 for each of the positions.

For the change from position (e) to position (f) in Figure 16.15, a change in sign occurs in $(q^{(AC)} - q^{(CB)})$. Figure 16.15h therefore gives the most unfavourable position of the set of loads in relation to the bending moment at C.

Table 16.1

Position load	$q^{(AC)}$ **(kN/m)**	$q^{(CB)}$ **kN/m)**	$q^{(AC)} - q^{(CB)}$
Figure 16.15c	$\frac{4\times 60}{6} = 40$	0	> 0
Figure 16.15d	$\frac{3\times 60}{6} = 30$	$\frac{1\times 60}{12} = 5$	> 0
Figure 16.15e	$\frac{2\times 60}{6} = 20$	$\frac{2\times 60}{12} = 10$	> 0
Figure 16.15f	$\frac{1\times 60}{6} = 10$	$\frac{3\times 60}{12} = 15$	< 0
Figure 16.15g	0	$\frac{4\times 60}{12} = 20$	< 0

The maximum bending moment is

$$M_C = \left(\tfrac{8}{3} + 4\tfrac{44}{12} + \tfrac{40}{12}\right)\text{(m)} \times (60\text{ kN}) = 820\text{ kNm}.$$

The first term between brackets includes the influence values to be found from the influence line at the position of each of the point loads.

Example 2 – Most unfavourable placement of a uniformly distributed load

The hinged beam in Figure 16.16a is a model of a bridge. The traffic load on the bridge, consisting of a large number of vehicles in a line, is modelled as a uniformly distributed moving load of 90 kN/m. Figure 16.16b shows the influence line for the shear force at E. We will determine the maximum shear force at E in an absolute sense.

Solution:

Since gaps may appear in traffic jams, the uniformly distributed load is sometimes interrupted. The maximum positive shear force occurs when the uniformly distributed load is present in all fields where the influence line is positive as indicated in Figure 16.16c. This gives

$$\begin{aligned} V_E &= \left\{\tfrac{1}{2} \times (30\text{ m}) \times \left(+\tfrac{1}{3}\right) + \tfrac{1}{2} \times (20\text{ m}) \times \left(+\tfrac{2}{3}\right)\right\} \times (90\text{ kN/m}) \\ &= 1050\text{ kN}. \end{aligned}$$

The maximum negative shear force is found for the load in Figure 16.16d:

$$\begin{aligned} V_E &= \left\{\tfrac{1}{2} \times (10\text{ m}) \times \left(-\tfrac{1}{3}\right) + \tfrac{1}{2} \times (30\text{ m}) \times \left(-\tfrac{1}{3}\right)\right\} \times (90\text{ kN/m}) \\ &= -600\text{ kN}. \end{aligned}$$

The maximum shear force in an absolute sense is therefore 1050 kN and occurs with the load shown in Figure 16.16c.

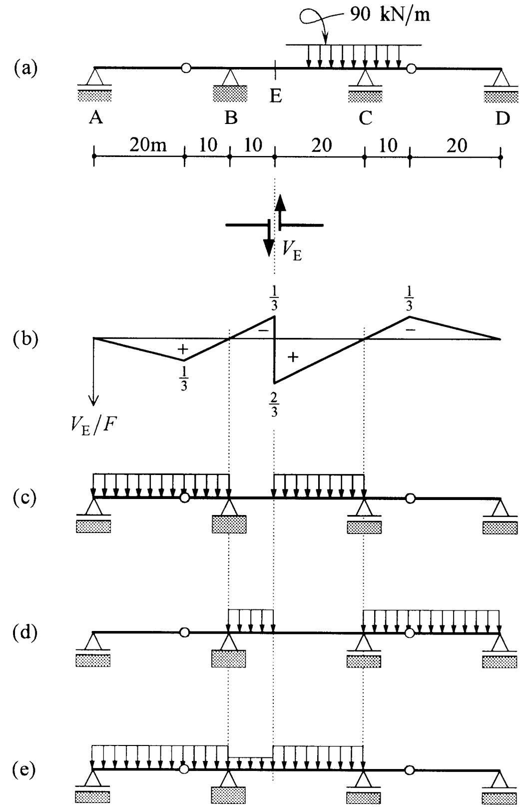


Figure 16.16 (a) Bridge modelled as a hinged beam with uniformly distributed movable load; (b) influence line for the shear force at E; (c) the load that causes the maximum positive shear force; (d) the load that causes the maximum negative shear force; (e) for railway bridges, trains are an uninterrupted load.

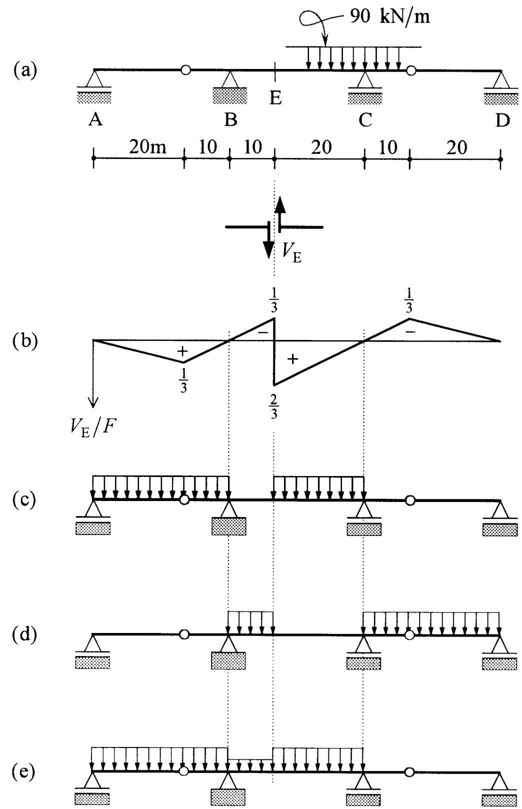


Figure 16.16 (a) Bridge modelled as a hinged beam with uniformly distributed movable load; (b) influence line for the shear force at E; (c) the load that causes the maximum positive shear force; (d) the load that causes the maximum negative shear force; (e) for railway bridges, trains are an uninterrupted load.

The fact that the positive shear force is predominant is clear from the influence line: the positive area under the influence line is larger than the negative area.

In contrast to bridges for standard traffic, loads for trains on railway bridges are uninterrupted loads, that may consist partly of empty carriages, for which one then assumes a lesser load. For a railway bridge, the maximum shear force at E in an absolute sense occurs for the load given in Figure 16.16e. Assume the uniformly distributed load is again 90 kN/m, but now with a minimum of 15 kN/m for the empty carriages. The maximum shear force is then

$$\begin{aligned} V_E &= \left\{\tfrac{1}{2} \times (30\text{ m}) \times \left(+\tfrac{1}{3}\right) + \tfrac{1}{2} \times (20\text{ m}) \times \left(+\tfrac{2}{3}\right)\right\} \times (90\text{ kN/m}) + \\ &\quad + \tfrac{1}{2} \times (10\text{ m}) \times \left(-\tfrac{1}{3}\right) \times (15\text{ kN/m}) \\ &= 1025\text{ kN}. \end{aligned}$$

It is up to the reader to check that neither a distributed load over the entire length AD nor a distributed load over BD are predominant.

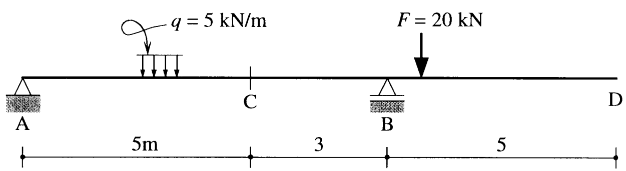
16.4 Problems

General comment: If the influence line of a quantity X is requested, the positive direction of this quantity must be stated beforehand. You must then indicate the corresponding direction of X on the influence line by means of plus and minus signs (or deformation symbols).

Influence lines using virtual work (Sections 16.2 and 16.3)

16.1: 1–5 A point load $F = 20$ kN and a uniformly distributed load $q = 5$ kN/m can move across a beam. The same questions are asked for the following five quantities X:

1. $X =$ bending moment at B.
2. $X =$ bending moment at C.
3. $X =$ shear force directly to the left of B.
4. $X =$ shear force directly to the right of B.
5. $X =$ shear force at C.

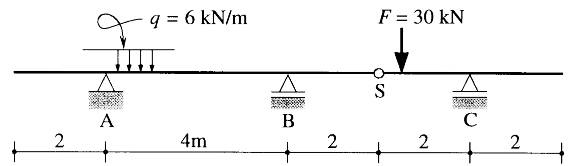


Questions:

a. Draw the influence line for X.
b. Where must the load F be placed so that X is a maximum? Determine this maximum value.
c. Where must the load F be placed so that X is a minimum? Determine this minimum value.
d. Using the influence line, determine the value of X when the uniformly distributed load q acts only on CD.
e. Where must the uniformly distributed load q be placed so that X is a maximum? Determine this maximum value.
f. Where must the uniformly distributed load q be placed so that X is a minimum? Determine this minimum value.

16.2: 1–6 A point load $F = 30$ kN and a uniformly distributed load $q = 6$ kN/m can move over the hinged cantilever beam. The same questions are asked for six different quantities X:

1. $X =$ vertical support reaction at A.
2. $X =$ vertical support reaction at B.
3. $X =$ shear force directly to the left of B.
4. $X =$ shear force directly to the right of B.
5. $X =$ bending moment in the middle of AB.
6. $X =$ bending moment at B.

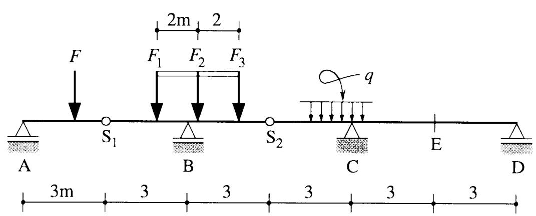


Questions:

a. Draw the influence line for X.
b. Where must the load F be placed so that X is a maximum? Determine this maximum value.
c. Where must the load F be placed so that X is a minimum? Determine this minimum value.
d. Using the influence line, determine the value of X when the uniformly distributed load q acts over the entire length of the beam.
e. Where must the uniformly distributed load q be placed so that X is a maximum? Determine this maximum value.
f. Where must the uniformly distributed load q be placed so that X is a minimum? Determine this minimum value.

16.3: 1–9 A point load F, a set of loads F_1; F_2; F_3 and a uniformly distributed load q can move across the hinged beam in Figure 16.3. Use $F = 40$ kN, $F_1 = 20$ kN, $F_2 = 50$ kN, $F_3 = 30$ kN and $q = 8$ kN/m. The same questions are asked for nine different quantities X:

1. $X =$ vertical support reaction at B.
2. $X =$ vertical support reaction at C.
3. $X =$ vertical support reaction at D.
4. $X =$ bending moment at B.
5. $X =$ bending moment at C.
6. $X =$ bending moment at E.
7. $X =$ shear force directly to the left of C.
8. $X =$ shear force directly to the right of C.
9. $X =$ shear force at S_2.

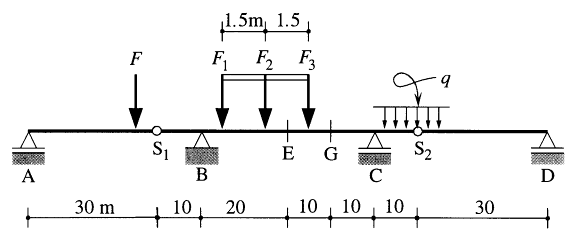


Questions:

a. Draw the influence line for X.
b. Where must the load F be placed so that X is a maximum? Determine this maximum value.
c. Where must the load F be placed so that X is a minimum? Determine this minimum value.
d. Using the influence line, determine the value of X due to the set of loads when F_1 is at S_1.
e. Using the influence line, determine the value of X due to the set of loads when F_2 is at S_2.
f. Using the influence line, determine the value of X when the uniformly distributed load q acts over the entire length of the beam.
g. Where must the uniformly distributed load q be placed so that X is a maximum? Determine this maximum value.
h. Where must the uniformly distributed load q be placed so that X is a minimum? Determine this minimum value.

16.4: 1–10 A point load F, a set of loads F_1; F_2; F_3 and a uniformly distributed load q can move across the hinged beam in Figure 16.4. Use $F = F_1 = F_2 = F_3 = 200$ kN and $q = 24$ kN/m. The same questions are asked for 10 different quantities X:

1. $X =$ vertical support reaction at A.
2. $X =$ vertical support reaction at B.
3. $X =$ bending moment at B.
4. $X =$ bending moment at E.
5. $X =$ bending moment at G.
6. $X =$ shear force at S_1.
7. $X =$ shear force at E.
8. $X =$ shear force at G.
9. $X =$ shear force directly to the left of C.
10. $X =$ shear force directly to the right of C.



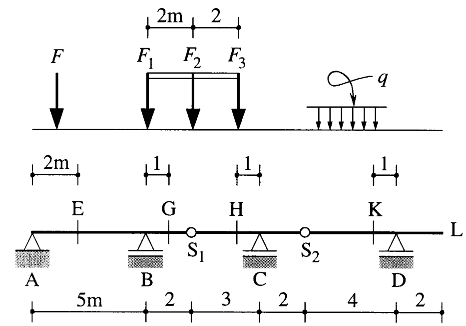
Questions:

a. Draw the influence line for X.
b. Where must the load F be placed so that X is a maximum? Determine this maximum value.
c. Where must the load F be placed so that X is a minimum? Determine this minimum value.

d. Using the influence line, determine the value of X due to the set of loads when F_1 is at S_1.
e. Using the influence line, determine the value of X due to the set of loads when F_2 is at E.
f. Using the influence line, determine the value of X when the uniformly distributed load q acts only on BD.
g. Using the influence line, determine the value of X when the uniformly distributed load q acts over the entire length of the beam.
h. Where must the uniformly distributed load q be placed so that X is a maximum? Determine this maximum value.
i. Where must the uniformly distributed load q be placed so that X is a minimum? Determine this minimum value.

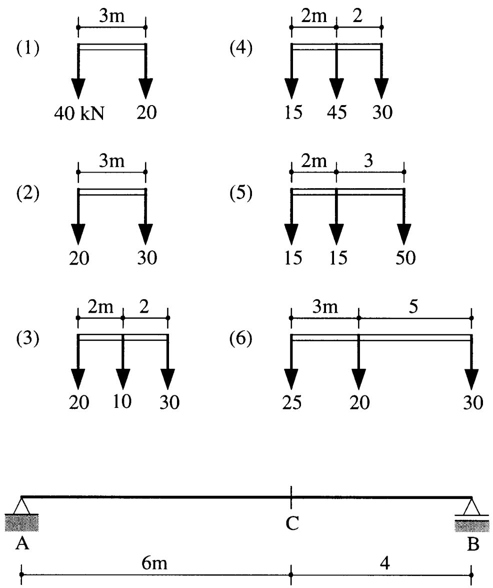
16.5: 1–18 A point load F, a set of loads F_1; F_2; F_3 and a uniformly distributed load q can move across the hinged beam in Figure 16.5. Use $F = 80$ kN, $F_1 = 30$ kN, $F_2 = 50$ kN, $F_3 = 20$ kN and $q = 18$ kN/m. The same questions are asked for 18 different quantities X:

1. X = vertical support reaction at A.
2. X = vertical support reaction at B.
3. X = vertical support reaction at C.
4. X = vertical support reaction at D.
5. X = shear force at E.
6. X = shear force at G.
7. X = shear force at S_1.
8. X = shear force at H.
9. X = shear force directly to the left of C.
10. X = shear force directly to the right of C.
11. X = shear force at S_2.
12. X = shear force at K.
13. X = bending moment at B.
14. X = bending moment at C.
15. X = bending moment at E.
16. X = bending moment at G.
17. X = bending moment at H.
18. X = bending moment at K.



Questions:
a. Draw the influence line for X.
b. Where must the load F be placed so that X is a maximum? Determine this maximum value.
c. Where must the load F be placed so that X is a minimum? Determine this minimum value.
d. Using the influence line, determine the value of X due to the set of loads when F_1 is at E.
e. Using the influence line, determine the value of X due to the set of loads when F_2 is at G.
f. Using the influence line, determine the value of X when the uniformly distributed load q acts over the entire length of the beam.
g. Using the influence line, determine the value of due to a uniformly distributed load q between S_1 and S_2 and between K and L.
h. Where must the uniformly distributed load q be placed so that X is a maximum? Determine this maximum value.
i. Where must the uniformly distributed load q be placed so that X is a minimum? Determine this minimum value.

16.6: 1–6 Given the same simply supported beam and six different sets of loads.



Questions:

a. Determine the influence line for the support reaction at A.
b. Determine the maximum value of the support reaction at A due to the set of loads.
c. Determine the influence line for bending moment at C.
d. Determine the maximum value of the bending moment at C due to the set of loads.
e. Determine the influence line for the shear force at C.
f. Determine the maximum value for the shear force at C due to the set of loads.

16.7: 1–6 Given the same simply supported beam and six different sets of loads. Answer the same questions as in problem 16.6.