15

Virtual Work

In this chapter we deal with the virtual work equation: an often used alternative for the equilibrium equations.

In Section 15.1, we first introduce the concepts *work* and *strain energy*. Performing work can be seen as a mechanical process of a body exchanging energy with its environment. To illustrate the concept, we have included a separate section on strain energy. The strain energy concept plays an important role in calculations that are based on energy considerations, but is not covered further in this chapter.

The concept of work returns in the *virtual work equation*. In Sections 15.2 to 15.4 we show for a particle, a rigid body and a mechanism respectively that the virtual work equation is equivalent to the equilibrium equations. Support reactions and section forces can be derived directly from the equilibrium equations, but also with the *principle of virtual work*. We provide a number of examples in Section 15.5.

The virtual work equation is especially useful for determining the influence lines for support reactions and section forces in statically determinate bar type structures; this is covered in Chapter 16.

15.1 Work and strain energy

In this section we look at the concepts work and strain energy.

15.1.1 Work

If the point of application of force $\vec{F}$ in Figure 15.1a undergoes an infinitesimal displacement $\mathrm{d}\vec{u}$ along path s, this is referred to as the force performing an (infinitesimal) amount of *work* $\mathrm{d}A$, defined as the *inner product* of the vectors $\vec{F}$ and $\mathrm{d}\vec{u}$:

$$\begin{aligned} \mathrm{d}A &= \vec{F} \cdot \mathrm{d}\vec{u} \\ &= F_x\,\mathrm{d}u_x + F_y\,\mathrm{d}u_y + F_z\,\mathrm{d}u_z. \end{aligned}$$

Work is a *scalar quantity*.

The inner product of two vectors can also be calculated as the product of the magnitude (modulus) of both vectors and the cosine of the enclosed angle:

$$\mathrm{d}A = |\vec{F}| \cdot |\mathrm{d}\vec{u}| \cdot \cos\alpha = F \cdot \mathrm{d}u \cdot \cos\alpha.$$

This can be seen as the product of the force and the component of the displacement in the direction of the force, F and $\mathrm{d}u \cos\alpha$ respectively (see Figure 15.1b). It can also be seen as the product of the displacement and the component of the force in the direction of the displacement, $\mathrm{d}u$ and $F \cos\alpha$ respectively (see Figure 15.1c).

Note that the force F does not perform any work if it is normal to the displacement $\mathrm{d}u$ (in that case $\alpha = \pm\pi/2$ and $\cos\alpha = 0$).

If the point of application of the force moves a finite distance along path s,

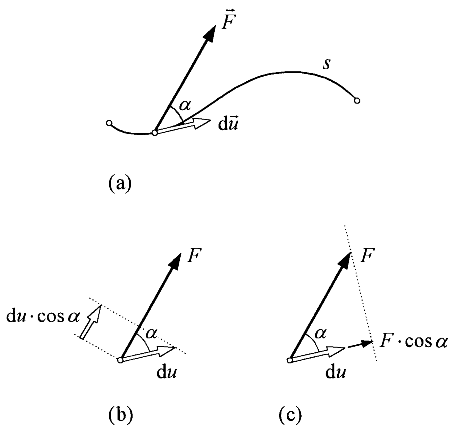

Figure 15.1 Work is defined as the inner product of force $\vec{F}$ and displacement $\mathrm{d}\vec{u}$:
$\mathrm{d}A = \vec{F} \cdot \vec{u} = |\vec{F}| \cdot |\mathrm{d}\vec{u}| \cdot \cos\alpha = F \cdot \mathrm{d}u \cos\alpha = F \cos\alpha \cdot \mathrm{d}u.$

the total amount of work is equal to the sum of the contributions of all the infinitesimal displacements. Mathematically this corresponds to integrating over the path length s:

$$A = \int_s \vec{F} \cdot \mathrm{d}\vec{u}.$$

The magnitude and direction of the force F may depend on the location on the route. If $\vec{F}$ is constant (for a vector that means constant in magnitude and direction), then $\vec{F}$ can be excluded from the integration symbol. The total amount of work performed is then (see Figure 15.2):

$$A = \int_s \vec{F} \cdot \mathrm{d}\vec{u} = \vec{F} \int_s \mathrm{d}\vec{u} = \vec{F} \cdot \vec{u}.$$

In this case, the total amount of work A depends only on the position of the starting and end points and not on the shape of the route followed.

In full

$$A = F_x u_x + F_y u_y + F_z y_z = F u \cos\alpha.$$

Note that no work is performed if $\vec{F}$ and $\vec{u}$ are normal to one another (in that case $\alpha = \pm\pi/2$ and $\cos\alpha = 0$).

Forces that are constant in magnitude and direction include gravitational forces.

The dimension of work is force multiplied by distance. The applicable SI unit is the *joule*, denoted as J:

$$\mathrm{J} = \mathrm{N} \cdot \mathrm{m} = \mathrm{kg} \cdot \mathrm{m}^2/\mathrm{s}^2.$$

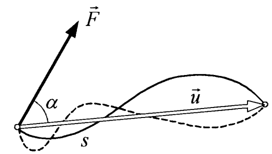

Figure 15.2 If $\vec{F}$ is constant, the total amount of work to be performed depends only on the location of the starting point and end point, and not on the route followed.

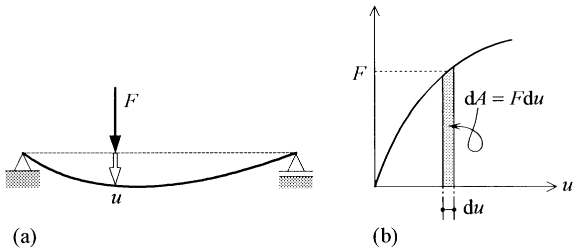

Figure 15.3 (a) A beam subject to bending with (b) the associated load-displacement diagram.

15.1.2 Strain energy

Consider the simply supported beam in Figure 15.3a, with a point load F. Due to a load, the beam will bend. The sag at the concentrated load is u. The relationship between the load F and displacement u can be shown in a *load-displacement diagram* (see Figure 15.3b). The shape of the diagram depends on the properties of the material. The shape is not important at this stage.

If with an increasing load the displacement u increases by an amount $\mathrm{d}u$, the force F performs work

$$\mathrm{d}A = F\,\mathrm{d}u.$$

When the load and displacement have reached their final value, the total amount of work performed is:

$$A = \int_0^u F\,\mathrm{d}u.$$

The total amount of work performed is equal to the area under the load-displacement diagram.

Performing work can be seen as a *mechanical process of energy exchange* between a body and its environment. If the load performs positive work, energy is extracted from the environment and transferred to the structure. If there is no exchange of heat from the structure to its environment, and the structure is and remains at rest (the energy added is not converted into kinetic energy), then the energy transferred is absorbed as *strain energy*. Strain energy is the energy required to deform the structure.

The work performed A is equal to the increase in strain energy E_v:

$$A = \Delta E_\mathrm{v}.$$

In general it is assumed that the strain energy is zero in the undeformed state. In that case, it holds for the deformed state that

$$A = E_\text{V}.$$

The strain energy stored in the beam is equal to the area under the load displacement diagram (see Figure 15.4).

The SI unit for strain energy is *joule*.

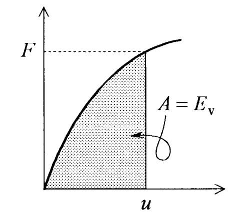

Figure 15.4 The strain energy E_V stored in the beam is equal to the total amount of work performed A, and is equal to the area under the load-displacement diagram.

15.2 Virtual work equation for a particle

In this section, we show that the virtual work equation for a particle is just another form of the equilibrium equations.

When particles are compelled to follow a particular path, the *virtual displacements* that are in line with the (limited) degree of freedom of the particle are known as *kinematically admissible virtual displacements*. These displacements are subject to special demands: they must be geometrically linear. Physically this can be translated into the demand that the virtual displacements must be very small.

15.2.1 Equilibrium

Assume a particle subject to forces (see Figure 15.5). The particle is in equilibrium if the equations for the force equilibrium in the x, y and z directions respectively are satisfied:

$$\sum F_x = 0,$$
$$\sum F_y = 0,$$
$$\sum F_z = 0.$$

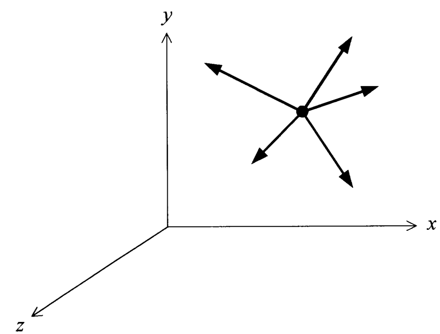

Figure 15.5 A particle subject to forces.

These three equilibrium equations can also be formulated otherwise.

Assume G is a new quantity, defined as follows:

$$G = \lambda_1 \sum F_x + \lambda_2 \sum F_y + \lambda_3 \sum F_z.$$

In this equation, $\lambda_1; \lambda_2; \lambda_3$ are *arbitrary quantities* that *cannot all equal zero concurrently*.

The demand that $G = 0$ for each arbitrary combination of $\lambda_1; \lambda_2; \lambda_3$ (not all equal to zero concurrently) is equivalent to the three equations for the force equilibrium. For example, the combination could be

$$\lambda_1 \neq 0; \;\; \lambda_2 = 0; \;\; \lambda_3 = 0$$

in which case

$$G = \lambda_1 \sum F_x + 0 \times \sum F_y + 0 \times \sum F_z = \lambda_1 \sum F_x$$

and G can be equal to zero only if

$$\sum F_x = 0.$$

Likewise, the combination $\lambda_1 = 0; \lambda_2 \neq 0; \lambda_3 = 0$ leads to $\sum F_y = 0$. The combination $\lambda_1 = 0; \lambda_2 = 0; \lambda_3 \neq 0$ leads to $\sum F_z = 0$.

15.2.2 Virtual work equation

The quantities that are to be chosen arbitrarily $\lambda_1; \lambda_2; \lambda_3$ can also be considered to be arbitrary (imagined) displacements $u_x; u_y; u_z$ (see Figure 15.6), so that

$$G = u_x \sum F_x + u_y \sum F_y + u_z \sum F_z.$$

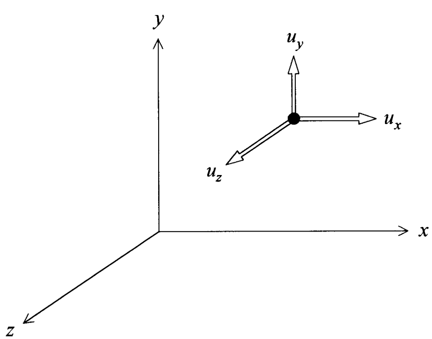

Figure 15.6 The displacement components of a particle.

In this case, G can be interpreted as the (imagined) work A done by the forces acting on the particle.

Since we are not talking about actual but rather *imagined displacements*, of *arbitrary magnitude*, they are referred to as *virtual displacements* and they are denoted by δu_x; δu_y; δu_z (see Figure 15.7). In mathematics, δ is known as the *variation symbol*. A virtual displacement therefore stands for a *variation* of the displacement.

The work due to the virtual displacements is known as *virtual work*. This is denoted by δA:

$$\delta A = \delta u_x \sum F_x + \delta u_y \sum F_y + \delta u_z \sum F_z.$$

If the displacement of a particle in equilibrium is varied, the virtual work done by the forces acting on it is zero. The converse is also true: if one varies the displacement of a particle, and the virtual work is zero, then the particle is in equilibrium.

Conclusion: *A particles is in equilibrium only if the virtual work performed by the forces acting on it is zero for any virtual displacement*:

$$\delta A = \delta u_x \sum F_x + \delta u_y \sum F_y + \delta u_z \sum F_z = 0.$$

This is known as the *principle of virtual work*.

The principle of virtual work combines the three independent equilibrium equations into one virtual work equation. The virtual work equation is just another form of the equilibrium equations.

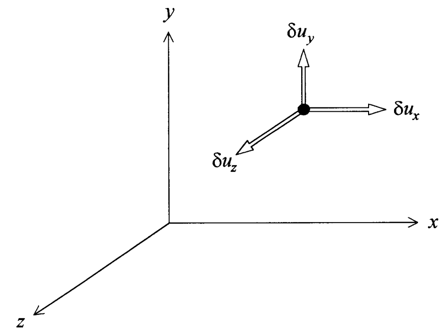

Figure 15.7 The virtual displacements of a particle.

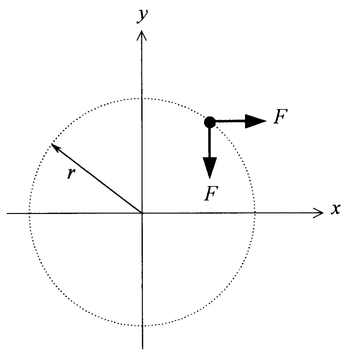

Figure 15.8 A particle that is compelled to follow a circular path with radius r is loaded by two forces F.

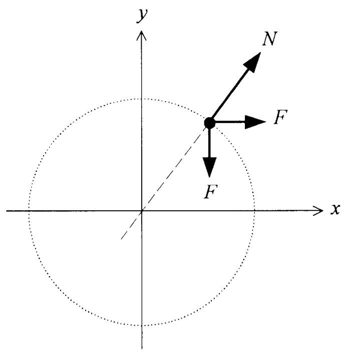

Figure 15.9 The isolated particle with all the forces acting on it.

15.2.3 Kinematically admissible virtual displacements

A particle is compelled to follow a circular path with diameter r in the xy plane. The path can be defined as

$$f(x, y) = x^2 + y^2 - r^2 = 0.$$

There is no friction. The particle is loaded by a horizontal and vertical force F, as shown in Figure 15.8.

Using the principle of virtual work, we now look for the positions on the circle at which the particle is in equilibrium.

In Figure 15.9 the particle has been isolated. Since there is no friction, the interaction force N is normal to the circular path.

If we are not interested in the interaction force N between the particle and its path, we can choose a virtual displacement along the prescribed path. Since N is normal to the path, it does not appear in the virtual work equation.

A virtual displacement that is conform with the (limited) freedom of movement of the particle is referred to as a *kinematically admissible virtual displacement*.

The virtual work δA due to a kinematically admissible virtual displacement is

$$\delta A = +F\delta u_x - F\delta u_y. \tag{a}$$

δu_x and δu_y are the virtual displacements in the x and y directions respectively.

For the path it holds that

$$f(x, y) = x^2 + y^2 - r^2 = 0. \tag{b}$$

After the virtual displacement, it applies that

$$f(x+\delta u_x, y+\delta u_y) = (x+\delta u_x)^2 + (y+\delta u_y)^2 - r^2 = 0. \tag{c}$$

If we combine equations (c) and (b) we find the following relationship between δu_x and δu_y:

$$2x\delta u_x + (\delta u_x)^2 + 2y\delta u_y + (\delta u_y)^2 = 0. \tag{d}$$

Since this equation is determined by the geometry of the prescribed path, it is known as a *geometric equation*.

During the variation of the displacement, the forces do not change direction. The same is true for N. Since N does not perform work, the virtual displacement has to occur along the tangent of the prescribed path, as shown in Figure 15.10. This means that the geometric relationship between δu_x and δu_y has to be linear. Ignoring the quadratic (higher order) terms in the geometric equation (d) can be physically interpreted as a demand that the virtual displacements have to be small.

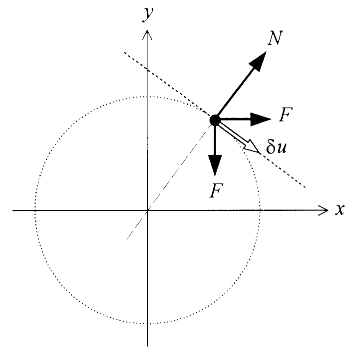

Figure 15.10 During the change in displacement the forces do not change direction. Since N performs no work the virtual displacement δu has to take place along the tangent to the prescribed path.

If we remove the quadratic terms in (d)[1] we find

$$2x\delta u_x + 2y\delta u_y = 0$$

or

$$\delta u_y = -\frac{x}{y}\delta u_x. \tag{e}$$

When the particle is in equilibrium, $\delta A = 0$. With (a) and (e) the virtual work equation becomes

[1] This is referred to as the linearisation of the geometric equation (d).

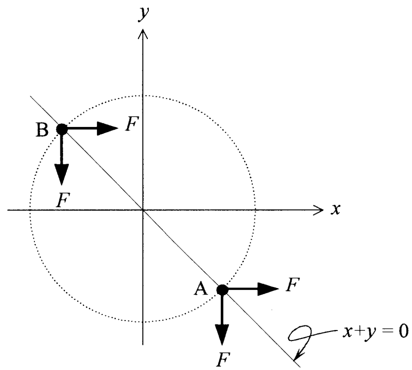

Figure 15.11 There are two locations on the circular path where the particle is in equilibrium under the influence of forces F: A and B.

$$\delta A = +F\delta u_x - F\delta u_y = F\delta u_x + F\frac{x}{y}\delta u_x = \left(1 + \frac{x}{y}\right) F\delta u_x = 0.$$

Since $F \neq 0$ and $\delta u_x \neq 0$ the solution is

$$1 + \frac{x}{y} = 0 \quad \text{or} \quad x + y = 0.$$

The equilibrium positions are therefore on the line $x + y = 0$. This leads to two solutions: the particle under the influence of forces F is in equilibrium at either A or B (see Figure 15.11).

Comment: The principle of virtual work says nothing about the *state of the equilibrium*. It cannot be used to discover that the equilibrium at A is reliable (stable) and that the equilibrium at B is unreliable (unstable).[1]

15.2.4 Virtual displacements in a mathematical sense

The fact that the geometric equations between the varied displacements have to be linear means, from a mathematical perspective, that we have to consider the so-called *first-order variation*. The first-order variation of a function $f(x, y)$ is defined as

$$\delta f(x, y) = \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y$$

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[1] If the particle at B loses its equilibrium as a result of a small disruption it will become increasingly far removed from the original equilibrium position due to forces F: the equilibrium at B is unreliable (unstable equilibrium). At A the forces F compel the particle to return to its original equilibrium position after disruption: the equilibrium at A is reliable (stable equilibrium). Examining the reliability of equilibrium (*stability investigation*) is beyond the scope of this book.

or, with $\delta x = \delta u_x$ and $\delta y = \delta u_y$,

$$\delta f(x, y) = \frac{\partial f}{\partial x}\delta u_x + \frac{\partial f}{\partial y}\delta u_y.$$

For the function

$$f(x, y) = x^2 + y^2 - r^2 = 0$$

that describes the circular path of the particle we find

$$\begin{aligned}\delta f(x, y) &= \frac{\partial}{\partial x}(x^2 + y^2 - r^2)\delta u_x + \frac{\partial}{\partial y}(x^2 + y^2 - r^2)\delta u_y \\ &= 2x\delta u_x + 2y\delta u_y = 0.\end{aligned}$$

This leads directly to the expression (e) we are looking for:

$$\delta u_y = -\frac{x}{y}\delta u_x.$$

The method used here is far simpler than the approach in Section 15.2.3, where we first derived the geometric equation (d) and then *linearised* it (by removing all the non-linear terms).

15.3 Virtual work equation for a rigid body

For rigid bodies, the complete equilibrium equations can also be replaced by a single virtual work equation. Deriving the virtual work equation for a rigid body shows again that the geometrical relationship between the virtual displacements has to be linear.

15.3.1 Equilibrium

In a plane[1] there are three equilibrium equations, two for the force equilibrium:

$$\sum F_x = 0,$$
$$\sum F_y = 0$$

and one for the moment equilibrium:

$$\sum T_z = 0.$$

There is equilibrium when all three conditions are satisfied.

The requirement

$$G = \lambda_1 \sum F_x + \lambda_2 \sum F_y + \lambda_3 \sum T_z = 0$$

for all arbitrary combinations of λ_1; λ_2; λ_3 (not all concurrently zero) is an alternative for the three equilibrium equations:

- The combination $\lambda_1 \neq 0$; $\lambda_2 = 0$; $\lambda_3 = 0$ leads to $\sum F_x = 0$.
- The combination $\lambda_1 = 0$; $\lambda_2 \neq 0$; $\lambda_3 = 0$ leads to $\sum F_y = 0$.
- The combination $\lambda_1 = 0$; $\lambda_2 = 0$; $\lambda_3 \neq 0$ leads to $\sum T_z = 0$.

Assume a number of forces are acting on the body (see Figure 15.12):

F_{xi}; F_{yi} are the components of F_i at point i with coordinates x_i; y_i;
F_{xj}; F_{yj} are the components of F_j at point j with coordinates x_j; y_j; etc.

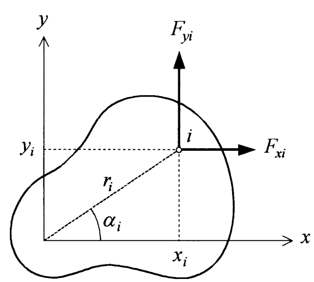


Figure 15.12 The components of force F at point i.

[1] For simplicity we will look only at rigid bodies in the xy plane.

To keep the picture simple, Figure 15.12 includes only the components of the force at point i.

The body is in equilibrium when the following condition is satisfied:

$$G = \lambda_1 \sum_i F_{xi} + \lambda_2 \sum_i F_{yi} + \lambda_3 \sum_i (x_i F_{yi} - y_i F_{xi}) = 0$$

for each arbitrary choice of λ_1; λ_2; λ_3 (not all concurrently zero).

15.3.2 Displacement of a point on a rigid body

The displacement of a rigid body in a plane is defined by the displacement of, for example, point O to O′ and a rotation about O. Assume that the components of the *translation* (displacement) are u_{x0}; u_{y0} and the *rotation* is φ_{z0}.

Instead of using its coordinates x_i; y_i, we can define the location of an arbitrary point i also by its angle α_i and the radius r_i. The displacement of point i is (see Figure 15.13)

$$u_{xi} = u_{x0} - a,$$

$$u_{yi} = u_{y0} + b.$$

a and b are the result of the rotation φ_{z0}. Due to the rotation, point i moves through an arc length $\varphi_{z0} r_i$ along the circle with radius r_i and centre O′. For a and b it applies that

$$a = r_i \left\{ \cos\alpha_i - \cos(\alpha_i + \varphi_{z0}) \right\},$$

$$b = r_i \left\{ \sin(\alpha_i + \varphi_{z0}) - \sin\alpha_i \right\}.$$

The expressions are much simplified when the rotation is small. If $\varphi_{z0} \ll 1$

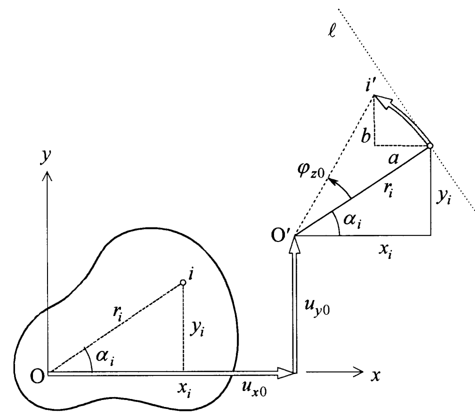


Figure 15.13 The displacement of point i due to a translation u_{x0}; u_{y0} and a *large* rotation φ_{z0}.

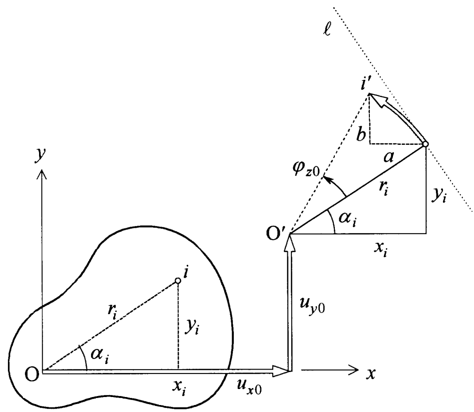


Figure 15.13 The displacement of point i due to a translation u_{x0}; u_{y0} and a *large* rotation φ_{z0}.

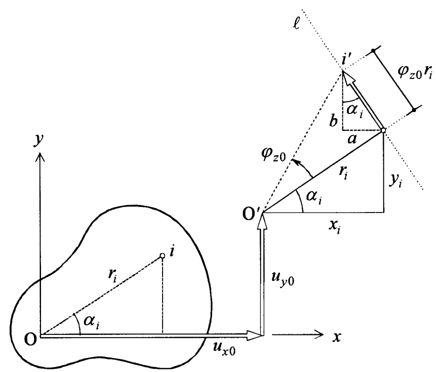


Figure 15.14 The displacement of point i due to a translation u_{x0}; u_{y0} and a *small* rotation φ_{z0}.

the circle can be replaced by its tangent ℓ (see Figure 15.14). The displacement of point i due to the rotation is then $\varphi_{z0}r_i$ with the following components:

$$a = \varphi_{z0}r_i \sin\alpha_i = \varphi_{z0}y_i,$$

$$b = \varphi_{z0}r_i \cos\alpha_i = \varphi_{z0}x_i.$$

For small rotations the following applies (ignoring the signs):

- *the horizontal displacement is equal to "rotation × vertical distance to the centre of rotation"*;
- *the vertical displacement is equal to "rotation × horizontal distance to the centre of rotation".*

Conclusion: *Due to a translation u_{x0}; u_{y0} and a small rotation φ_{z0} of the body the displacement of point i is*

$$u_{xi} = u_{x0} - a = u_{x0} - \varphi_{z0}y_i,$$

$$u_{yi} = u_{x0} + b = u_{y0} + \varphi_{z0}x_i.$$

Note that the geometric relationships between the various displacement quantities are linear for small rotations.

15.3.3 Virtual work equation

When a body is given a virtual displacement, the virtual work performed by all forces acting on it is

$$\delta A = \sum_i F_{xi}\delta u_{xi} + \sum_i F_{yi}\delta u_{yi}.$$

The virtual displacements δu_{xi}; δu_{yi} of point i can be expressed by three independent virtual displacements δu_{x0}; δu_{y0}; $\delta\varphi_{z0}$ of the body. The equation for δA assumes the form of expression G (see Section 15.3.1) only if the relationships between δu_{xi}; δu_{yi} and δu_{x0}; δu_{y0}; $\delta\varphi_{z0}$ are linear.

The geometric relationships appeared to be linear only for bodies subject to minor rotations. We can also say that the virtual displacements have to be very small.

In that case, the virtual work performed by the force at point i is

$$F_{xi}\delta u_{xi} + F_{yi}\delta u_{yi} = F_{xi}(\delta u_{x0} - y_i\delta\varphi_{z0}) + F_{yi}(\delta u_{y0} + x_i\delta\varphi_{z0}).$$

The total work performed by all the forces at the points $i, j, \ldots$ is

$$\delta A = \sum_i F_{xi}(\delta u_{x0} - y_i\delta\varphi_{z0}) + \sum_i F_{yi}(\delta u_{y0} + x_i\delta\varphi_{z0}).$$

This gives

$$\delta A = \delta u_{x0}\sum_i F_{xi} + \delta u_{y0}\sum_i F_{yi} + \delta\varphi_{z0}\sum_i (x_i F_{yi} - y_i F_{xi}).$$

The expression for virtual work δA is now in the same form as the expression for G; the quantities λ_1; λ_2; λ_3 have been replaced by the virtual displacements δu_{x0}; δu_{y0}; $\delta\varphi_{z0}$.

The demand

$$\delta A = \delta u_{x0}\sum_i F_{xi} + \delta u_{y0}\sum_i F_{yi} + \delta\varphi_{z0}\sum_i (x_i F_{yi} - y_i F_{xi}) = 0$$

for all arbitrary combinations of δu_{x0}; δu_{y0}; $\delta\varphi_{z0}$ (not all equal to zero) is a necessary and sufficient condition for equilibrium and is known as the *principle of virtual work*.

For simplicity, concentrated couples were not addressed. A concentrated couple can be replaced by a statically equivalent pair of forces. Assuming

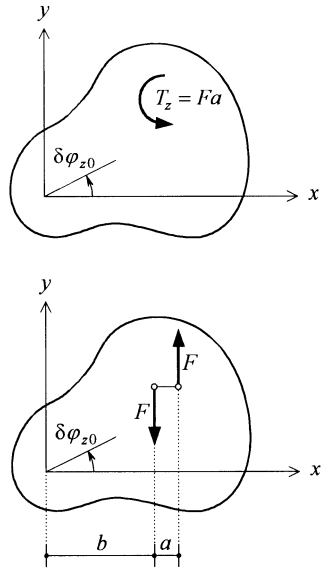


Figure 15.15 A concentrated couple can be replaced by a statically equivalent pair of forces. The virtual work performed by the couple is equal to the product of couple and rotation.

the pair of forces in Figure 15.15 we find

$$\delta A = -F \cdot b\delta\varphi_{z0} + F \cdot (a+b)\delta\varphi_{z0} = Fa \cdot \delta\varphi_{z0} = T_z \cdot \delta\varphi_{z0}.$$

The work performed by a couple is equal to the product of couple and rotation. The work is positive when the couple and rotation are in the same direction.

15.3.4 Virtual rotations in a mathematical sense

In deriving the virtual work equation, we found that the geometric relationships had to be linear in the virtual displacements. Mathematically, this means that the *first-order variation* has to be assumed for these virtual displacements.

We previously deduced (see Figure 15.13) that the following applies for the displacement of an arbitrary point i, due to a translation u_{x0}; u_{y0} and a rotation φ_{z0}:

$$u_{xi} = u_{x0} - r_i\{\cos\alpha_i - \cos(\alpha_i + \varphi_{z0})\},$$
$$u_{yi} = u_{y0} + r_i\{\sin(\alpha_i + \varphi_{z0}) - \sin\alpha_i\}.$$

The first-order variation of u_{xi} is defined as

$$\delta u_{xi} = \frac{\partial u_{xi}}{\partial u_{x0}}\delta u_{x0} + \frac{\partial u_{xi}}{\partial \varphi_{z0}}\delta\varphi_{z0}.$$

In the same way, the first-order variation of u_{yi} is defined as

$$\delta u_{yi} = \frac{\partial u_{yi}}{\partial u_{y0}}\delta u_{y0} + \frac{\partial u_{yi}}{\partial \varphi_{z0}}\delta\varphi_{z0}.$$

Elaborating these expressions (for $\varphi_{z0} = 0$) indeed leads to

$$\delta u_{xi} = \delta u_{x0} - r_i \sin\alpha_i \delta\varphi_{z0} = \delta u_{x0} - y_i \delta\varphi_{z0},$$

$$\delta u_{yi} = \delta u_{y0} + r_i \cos\alpha_i \delta\varphi_{z0} = \delta u_{y0} + x_i \delta\varphi_{z0}.$$

15.4 Virtual work equation for mechanisms

For mechanisms, the complete equilibrium equations can also be replaced by a virtual work equation. When the virtual displacement is chosen conform the freedom of movement at the joints (a *kinematically admissible virtual displacement*), the work performed by the interaction forces in the joints is zero and the virtual work equation includes only the work performed by the external forces.

When drawing the virtual displacements, one has to imagine that in the drawing the dimensions of the structure are considerably reduced and that the (very small) virtual displacements are considerably blown up. The magnitude of the angles (of rotation) is indicated by the so-called *orthogonal value*.

15.4.1 Virtual work equation

Mechanisms are systems of interconnected rigid bodies in which the joints are such that the bodies still have a certain degree of freedom with respect to one another.

Consider a system of two mutually hinged rigid bodies (1) and (2), loaded by a number of forces (see Figure 15.16a). There are acting interaction forces at the joint (joining forces). These occur always in pairs. In this case,

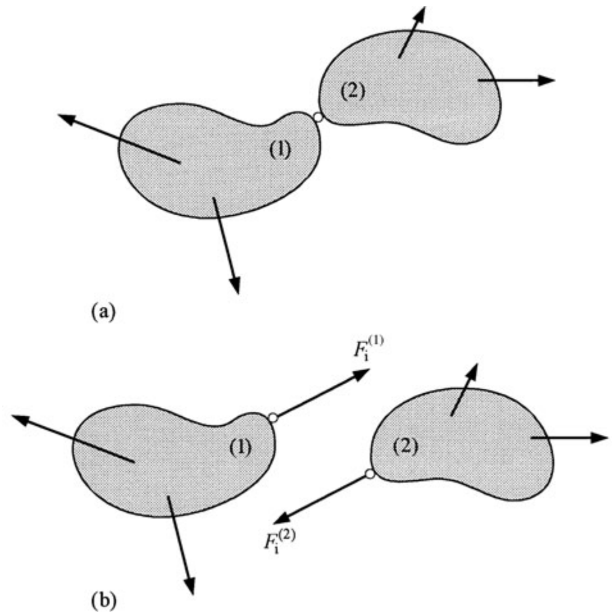


Figure 15.16 (a) Two bodies connected in a hinge. (b) The interaction forces at the hinged joint.

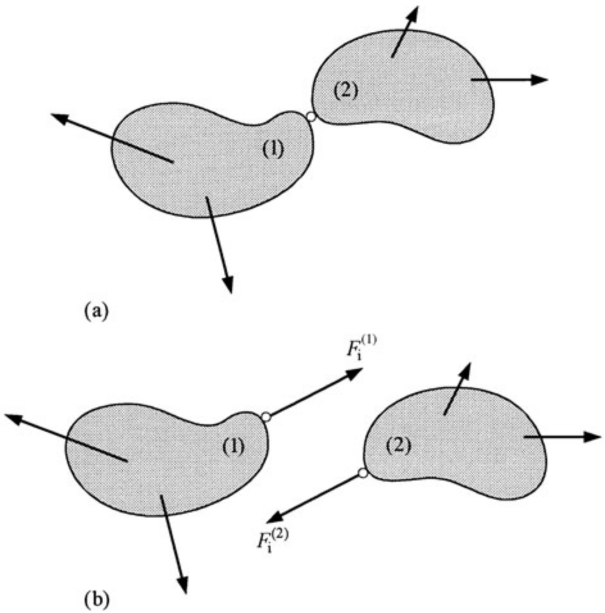


Figure 15.16 (a) Two bodies connected in a hinge. (b) The interaction forces at the hinged joint.

these forces are the two equal and opposite forces $F_i^{(1)}$ and $F_i^{(2)}$ (see Figure 15.16b).[1]

To investigate the equilibrium, the two bodies can be isolated from one another (see Figure 15.16b). The principle of virtual work can then be applied on each body. The virtual work $\delta A^{(1)}$ for a virtual displacement of body (1) is split into a part $\delta A_e^{(1)}$ due to the external forces[2] on body (1) and a part $\delta A_i^{(1)}$ due to the interaction force on body (1):

$$\delta A^{(1)} = \delta A_e^{(1)} + \delta A_i^{(1)} = 0.$$

The same applies for body (2):

$$\delta A^{(2)} = \delta A_e^{(2)} + \delta A_i^{(2)} = 0.$$

We can choose a virtual displacement for the connected bodies that is consistent with the freedom of movement in the joint. Such a displacement is referred to as a *kinematically admissible virtual displacement*. The virtual work equation is then

$$\delta A = \delta A^{(1)} + \delta A^{(2)} = \delta A_e^{(1)} + \delta A_e^{(2)} + \delta A_i^{(1)} + \delta A_i^{(2)} = 0.$$

The benefit of a kinematically admissible virtual displacement is that the interaction forces $F_i^{(1)}$ and $F_i^{(2)}$ together perform zero work. The points of application of these two equal and opposite forces always undergo the same

---

1 The upper index refers to the body, the lower index i refers to the interaction.

2 External in this sense does not refer to “the outside”, but rather to the cause of the force “from the outside”. The so-called external forces (also called loads), are independent forces in contrast to the interaction forces or joining forces (also referred to as internal forces), that are dependent forces.

displacement, so that

$$\delta A_{\mathrm{i}}^{(1)} + \delta A_{\mathrm{i}}^{(2)} = 0.$$

The virtual work equation now includes only the work performed by the external forces:

$$\delta A = \delta A_{\mathrm{e}}^{(1)} + \delta A_{\mathrm{e}}^{(2)} = 0.$$

Conclusion: *Due to a kinematically admissible virtual displacement of a mechanism in equilibrium, the virtual work performed by the (external) load equals zero. The fact that the work performed by the load is zero is a necessary and sufficient condition for the equilibrium of a mechanism.*

The approach can easily be expanded to include mechanisms of more than two bodies, as well as other than hinged joints.

15.4.2 The magnitude of the virtual displacements

Figure 15.17 shows a slide or a so-called *shear force hinge* with the interaction forces M; N, and a kinematically admissible virtual displacement δu; $\delta\varphi$.

We can see immediately that the pair of bending moments M does not perform any work. How different is it for the normal force pair N, that as a couple $N\delta u$ undergoes a (small) rotation $\delta\varphi$ and therefore, taking into account the directions shown in the figure, performs the work $-N\delta u\delta\varphi$. The geometric significance of $\delta u\delta\varphi$ can be seen on the figure.

When deriving the virtual work equation for a rigid body, the demand arose that the geometric relationships have to be linear in the virtual displacements. To achieve that, the virtual displacements have to be very small. Quadratic terms in the virtual displacements are therefore a degree smaller in the virtual work equation and can be discarded. Mathematically, this

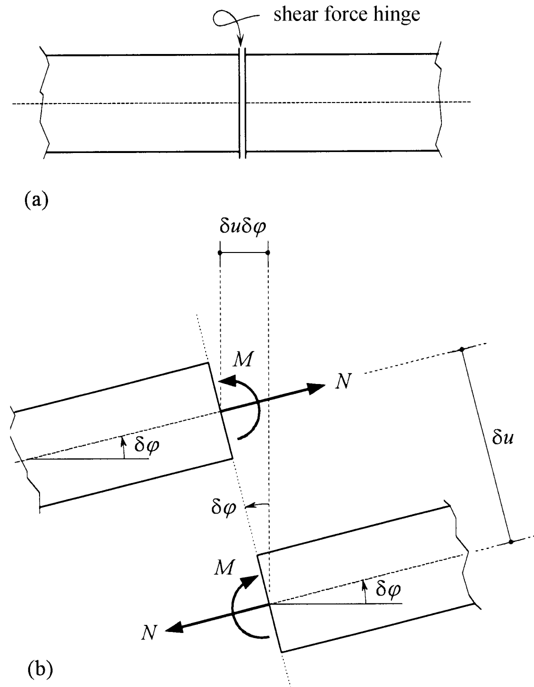


Figure 15.17 (a) A slide or a so-called shear force hinge with (b) the interaction forces after a kinematically admissible virtual displacement.

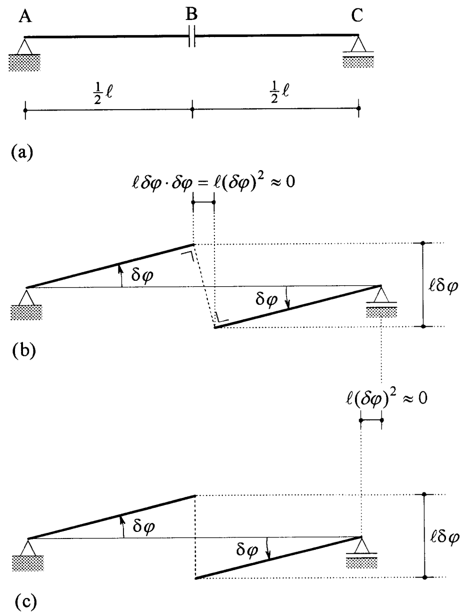


Figure 15.18 (a) The simply supported beam AC with a slide or shear force hinge at midspan B. (b) A kinematically admissible virtual displacement: the segments to the right and left of the slide remain parallel to one another. (c) The displacements in case the rotation $\delta\varphi$ is small and the quadratic terms in $\delta\varphi$ can be neglected.

demand is formulated by looking only at the first-order variation of the displacements in the virtual work equation (in other words, only the linear terms in the virtual displacements). This limitation with respect to the virtual displacements means that the $\delta u \delta\varphi$ is neglected, and the normal force pair N performs no virtual work.

In diagrams, the dimensions of the structure are greatly reduced and the displacements, even though they are infinitesimally small, are greatly enlarged. This can give rise to problems at first sight.

As an example, consider the simply supported beam ABC modelled as a line element in Figure 15.18a, with a *slide* or *shear force hinge* at midspan B. The mechanism has one degree of freedom. For a kinematically admissible virtual displacement, the displacement must be consistent with the freedom of movement at the supports and the slide. The latter means that beam segments AB and BC must remain parallel to one another on both sides of the shear force hinge. Figures 15.18b and 15.18c show the virtual displacement for the mechanism in two different ways. Figure 15.18b would seem to be the correct one, but this is not so.

One has to imagine that the virtual rotation $\delta\varphi$ is very small. In that case the horizontal displacement of the beam ends with respect to one another at B, $\ell(\delta\varphi)^2$, is a degree smaller than the vertical displacement $\ell\delta\varphi$. So the displaced beam ends at the shear force hinge B can be drawn directly above one another. Figure 15.18c gives therefore the correct representation of the virtual displacements in the mechanism.

Another example is beam ABC in Figure 15.19a, that has been changed into a mechanism by the introduction of the hinge at S. Figure 15.19b shows a virtual displacement: a bend $\delta\theta$ occurs at hinge S. The small angle (rotation) $\delta\theta$, that is drawn to a large scale, can be defined by the ratio $\delta a/\ell$. This value is known as the *orthogonal value*. The orthogonal value is not equal to the sine or tangent of the angle, nor to the value in radians.

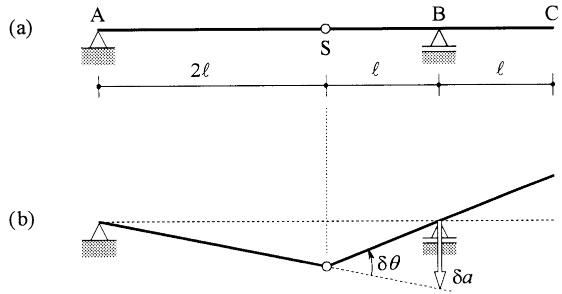


Figure 15.19 (a) Mechanism with (b) a kinematically admissible virtual displacement.

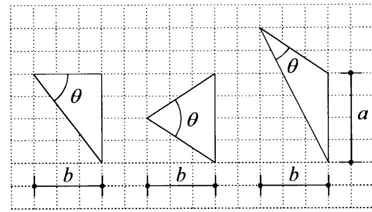


Figure 15.20 Three angles θ with the same orthogonal value a/b.

Figure 15.20 shows three angles θ that differ in magnitude from a trigonometric perspective, but for which the orthogonal value a/b is the same. Referring to Section 15.3.2 for the displacement due to a small rotation, it holds for all three cases that the vertical displacement a is equal to "*angle (of rotation)* $\theta \times$ *horizontal distance b to the centre of rotation*".

15.5 Calculating forces using virtual work

With statically determinate bar type structures, the support reactions and section forces follow directly from the *equilibrium*. When determining these forces, we can also use the *principle of virtual work* instead of the equilibrium equations. To do so, we have to change the structure into a mechanism in such a way that the force to be determined can perform work if the mechanism is subject to a virtual displacement. This is explained below by means of a number of examples.

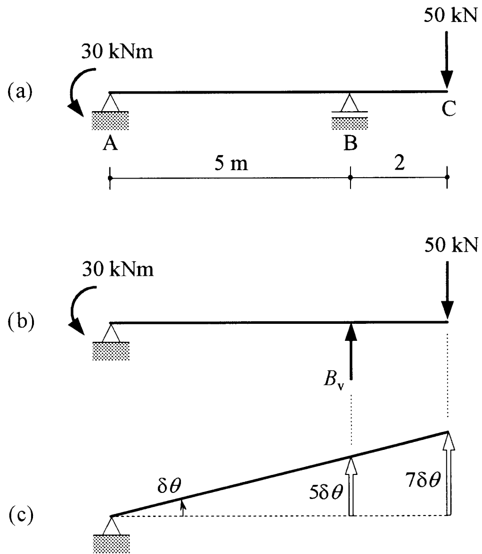


Figure 15.21 (a) Beam with (b) a mechanism for determining the vertical support reaction at B and (c) a kinematically admissible virtual displacement of the mechanism (displacements in m).

15.5.1 Support reactions

Example 1 – support reaction

We will now determine the vertical support reaction at B for the beam in Figure 15.21a using the principle of virtual work.

Solution:

Assume B_v, the vertical support reaction at B, has the direction as shown in Figure 15.21b. By removing the support at B we form a mechanism in which B_v can perform work. The mechanism has one degree of freedom.[1] Apply a virtual rotation $\delta\theta$ at A. Figure 15.21c shows the virtual displacement of the mechanism. The virtual displacements at B and C are equal to "*rotation* × *horizontal distance to centre of rotation A*" (see Section 15.3.2); they are respectively $5\delta\theta$ and $7\delta\theta$ m.

Note that the deformation of the beam is not taken into account.

Due to the virtual displacement, the couple and the forces perform virtual work. The total amount of work performed is

$$\delta A = +(30 \text{ kNm}) \times \delta\theta + B_v \times (5\delta\theta \text{ m}) - (50 \text{ kN}) \times (7\delta\theta \text{ m}).$$

The couple performs positive work, as does the support reaction B_v; the force at C in contrast performs negative work. For equilibrium the following applies

$$\delta A = 0$$

so that

[1] The position of the mechanism is fully defined by a single parameter, such as the rotation at A or the vertical displacement at B.

$$B_v = \frac{-(30 \text{ kNm}) \times \delta\theta + (50 \text{ kN}) \times (7\delta\theta \text{ m})}{(5\delta\theta \text{ m})} = +64 \text{ kN}.$$

As expected, the support reaction B_v turns out to be independent of the magnitude and direction of the virtual rotation $\delta\theta$.

Example 2 – Fixed-end moment
The hinged beam in Figure 15.22a carries a uniformly distributed load over its entire length. Below we will determine the fixed-end moment using the principle of virtual work.

Solution:
Assume the fixed-end moment A_m acts in the direction shown in Figure 15.22b. If we replace the fixed-end support by a hinged support, we create a mechanism in which the, as yet unknown, fixed-end moment A_m can perform work.

We select the vertical displacement of the hinge at S as a degree of freedom. Assume that the hinge is subject to a virtual displacement δu. Figure 15.22c shows the virtual displacement of the mechanism. Using "*vertical displacement = rotation × horizontal distance*" we can express the virtual rotation $\delta\theta$ at A in the virtual displacement δu:

$$\delta\theta = \frac{\delta u}{a}.$$

In equilibrium equations, a load on a rigid body can be replaced by its resultant. This also applies to the formulation of the virtual work equation. The distributed loads on AS and SB are replaced by their resultants:

$$R^{AS} = qa \text{ and } R^{SB} = 2qa.$$

The virtual displacements at the point of these resultants are easy to determine, and are both $\frac{1}{2}\delta u$.

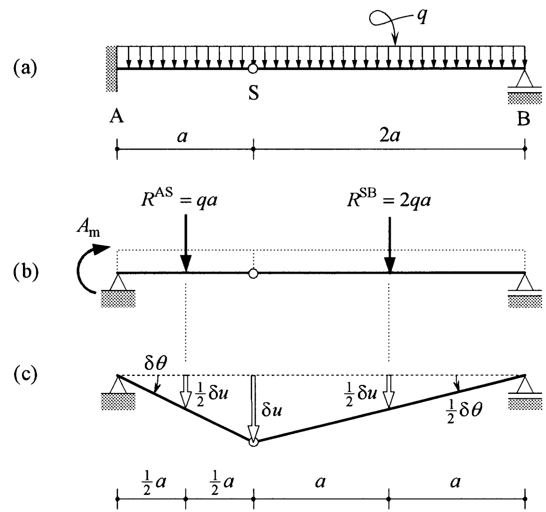


Figure 15.22 (a) Hinged beam with (b) a mechanism for determining the fixed-end moment at A and (c) a kinematically admissible virtual displacement of the mechanism.

For equilibrium the virtual work is zero:

$$\begin{aligned}\delta A &= +A_\mathrm{m} \cdot \delta\theta + R^\mathrm{AS} \cdot \tfrac{1}{2}\delta u + R^\mathrm{SB} \cdot \tfrac{1}{2}\delta u \\ &= +A_\mathrm{m} \cdot \frac{\delta u}{a} + qa \cdot \tfrac{1}{2}\delta u + 2qa \cdot \tfrac{1}{2}\delta u = 0\end{aligned}$$

so that

$$A_\mathrm{m} = -\tfrac{3}{2}qa^2.$$

In reality, A_m therefore acts in the direction opposite to the one assumed in Figure 15.22b.

Note: It is incorrect to use the resultant of the total distributed load over ASB. This gives a different (and incorrect) result. Check it!

15.5.2 Section forces

Example 1 – Bending moment
The simply supported beam in Figure 15.23a carries a uniformly distributed load over its entire length. Here we will calculate the bending moment M_C at cross-section C, at a third of the span.

Solution:
Change the structure into a mechanism (Figure 15.23b) by applying a hinge at C. Since a hinge cannot transfer bending moments, the bending moment M_C at C is applied to the mechanism as an external load. We have to take into account that the bending moment is an interaction force and therefore occurs as a pair of moments: one moment M_C acts on the left-hand part and another moment M_C, equal and opposite, acts on the right-hand part. The direction of M_C in Figure 15.23b is an assumption.

Assume C is subjected to a vertical virtual displacement δu. Figure 15.23c shows the virtual displacement of the mechanism. AC undergoes a rotation

$\delta\theta^{AC}$, and CB undergoes a rotation $\delta\theta^{CB}$. Both rotations can be expressed in terms of δu:

$$\delta\theta^{AC} = \frac{\delta u}{a} \text{ and } \delta\theta^{CB} = \frac{\delta u}{2a}.$$

Having replaced the distributed loads on AC and CB by their resultants, we find the virtual work equation:

$$\delta A = -M_C \cdot \delta\theta^{AC} - M_C \cdot \delta\theta^{CB} + qa \cdot \tfrac{1}{2}\delta u + 2qa \cdot \tfrac{1}{2}\delta u$$

$$= -M_C \cdot \frac{\delta u}{a} - M_C \cdot \frac{\delta u}{2a} + qa \cdot \tfrac{1}{2}\delta u + 2qa \cdot \tfrac{1}{2}\delta u = 0$$

so that

$$M_C = +qa^2.$$

The plus sign indicates that the bending moment acts in the direction assumed in Figure 15.23b.

Of course, the result is the same if we select a virtual displacement δu at C upwards instead of downwards. We have to realise that the virtual displacement of the mechanism has nothing to do with the actual deformation of the beam. In the virtual work equation, the actual deformation of the beam is neglected and all beam segments are considered entirely rigid.

In the deformed mechanism, beam segments AC and CB bend with respect to one another at the joint by $\delta\theta_C$. This is also referred to as a "*gap*".

From the geometry of the deformed mechanism in Figure 15.23c we find

$$\delta\theta_C = \delta\theta^{AC} + \delta\theta^{CB}.$$

Looking back to the virtual work equation, we see that the contribution by

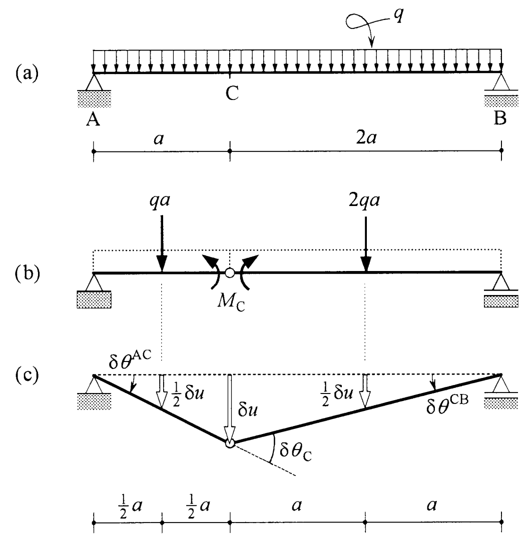


Figure 15.23 (a) Simply supported beam with (b) a mechanism for determining the bending moment at C, and (c) a kinematically admissible virtual displacement of the mechanism.

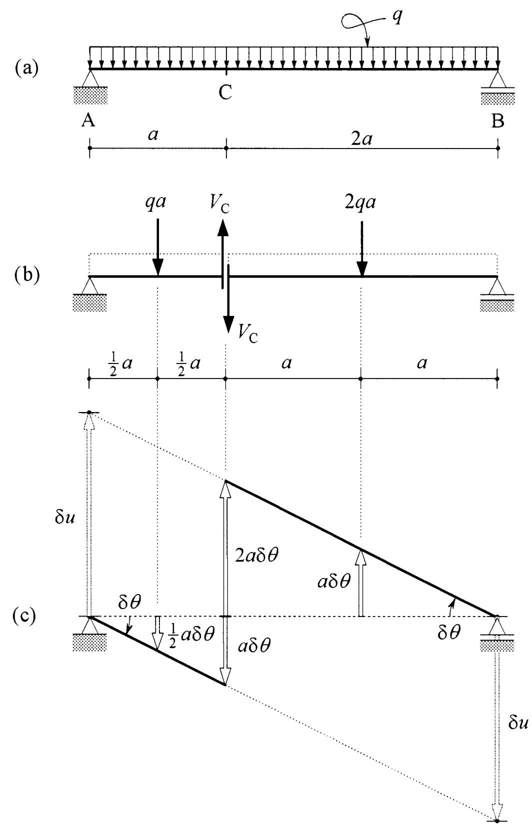


Figure 15.24 (a) Simply supported beam with (b) a mechanism for determining the shear force at C and (c) a kinematically admissible virtual displacement of the mechanism.

the pair of moments M_C, regardless of the sign, is equal to

$$\delta A(\text{due to } M_C) = \text{“bending moment} \times \text{gap”}$$
$$= M_C \cdot \delta\theta_C.$$

The sign is determined by the directions in which we choose the bending moment and the virtual displacement.

Example 2 – Shear force
We will now derive the shear force V_C at C for the same beam as in Example 1 (see Figure 15.24a).

Solution:
Change the structure into a mechanism (Figure 15.24b) by creating a *slide* or *shear force hinge* at C that cannot transfer shear forces.

The shear force is applied to the mechanism at C as an (external) load. Since the shear force is an interaction force it acts as a *pair of forces*: one shear force V_C acts on the left-hand segment and another equal and opposite shear force V_C acts on the right-hand segment. The direction of V_C in Figure 15.24b is an assumption.

Let the mechanism undergo a virtual displacement by displacing beam segments AC and CB at the shear force hinge over a distance δu with respect to one another. Both beam segments remain parallel to one another and are subject to the same rotation $\delta\theta$. We find the relationship between δu and $\delta\theta$ from the geometry of the deformed mechanism in Figure 15.24c:

$$\delta u = 3a\delta\theta.$$

After replacing the distributed loads on AC and CB by their resultants, we find the following for the virtual work equation:

$$\delta A = -V_C \cdot a\delta\theta - V_C \cdot 2a\delta\theta + qa \cdot \tfrac{1}{2}a\delta\theta - 2qa \cdot a\delta\theta = 0$$

so that

$$V_C = -\tfrac{1}{2}qa.$$

The minus sign indicates that the direction of V_C is opposite to the direction assumed in Figure 15.24b.

In the virtual work equation, the contribution of the *shear force* V_C, regardless of the sign, is equal to

$$\delta A(\text{due to } V_C) = \text{"shear force} \times \text{displacement in the shear force hinge"}$$
$$= V_C \cdot \delta u.$$

The sign is determined by the directions in which we assume the shear force and the virtual displacement.

Example 3 – Forces in a hinged beam
For the hinged beam in Figure 15.25a we will look for the mechanisms to determine the support moment at B, the shear force directly to the left of B and the shear force directly to the right of B.

Solution:
In order to find the support at B we convert the beam into a mechanism by introducing a hinge at B (Figure 15.25b). The bending moment is allowed to act on the mechanism as a load (as a *pair of moments*). The direction of M_B and the direction of the virtual displacement $\delta\theta$ can be chosen arbitrarily.

Figure 15.25c shows the mechanism for the shear force directly to the left of B. At this point, a slide or shear force hinge has been fitted into the beam. The shear force V_B has been applied on the mechanism as a load (a pair of

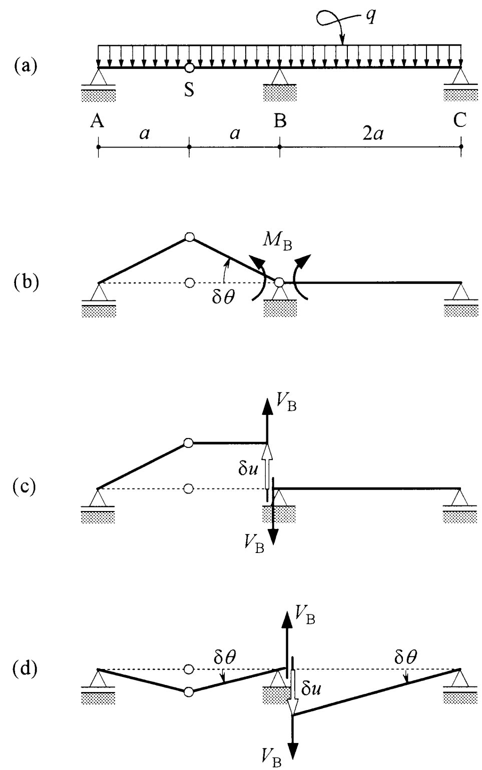


Figure 15.25 (a) Hinged beam with the mechanisms for determining (b) the support moment at B, (c) the shear force directly to the left of B and (d) the shear force directly to the right of B.

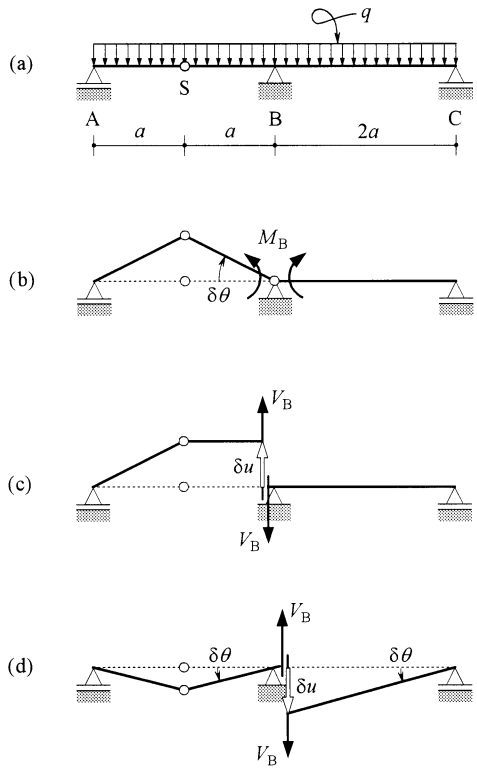


Figure 15.25 (a) Hinged beam with the mechanisms for determining (b) the support moment at B, (c) the shear force directly to the left of B and (d) the shear force directly to the right of B.

forces). At the shear force hinge, segments SB and BC can displace only with respect to one another, and cannot turn with respect to one another. Segments SB and BC therefore remain parallel. BC is fixed in a horizontal position due to the supports at B and C. With the vertical displacement δu, SB therefore remains horizontal.

Figure 15.25d gives the mechanism for the shear force directly to the right of B. Here too, SB and BC remain parallel. Due to the displacement δu at the shear force hinge, BC undergoes a rotation $\delta\theta = \delta u/2a$. SB undergoes the same rotation.

If the beam carries a uniformly distributed load q over its entire length (Figure 15.25a) the support moment at B is

$$M_B = -qa^2.$$

The shear force directly to the left of B is

$$V_B = +\tfrac{3}{2}qa.$$

The shear force directly to the right of B is

$$V_B = -\tfrac{3}{2}qa.$$

Check the answers using these mechanisms.

Example 4 – Normal force

Here we will derive the normal force in the truss in Figure 15.26a for the member DE using the principle of virtual work.

Solution:

Convert the truss into a mechanism by introducing a connection in member DE that cannot transfer normal forces. Such a telescopic connection is also referred to as a *normal force hinge*. At the normal force hinge, the normal force N is applied to the mechanism as a load (a pair of forces). In

Figure 15.26b it has been assumed that the normal force is a tensile force.

Figure 15.26c shows the virtual displacement for the mechanism. The mechanism consists of two self-contained bodies ACD and BCE that can respectively rotate about A and B and are hinged at C.

With the rules

"*horizontal displacement* $=$ *rotation* $\times$ *vertical distance*", and

"*vertical displacement* $=$ *rotation* $\times$ *horizontal distance*",

we can determine the rotation of the parts ACD and BCE and the displacements of the joints. Figure 15.26c all the relevant quantities are expressed in terms of the vertical displacement δw of joint C.

The member ends in the normal force hinge move with respect to one another over a distance δu that is equal to the distance that the joints D and E move towards one another:

$$\delta u = \tfrac{1}{4}\delta w + \tfrac{1}{2}\delta w = \tfrac{3}{4}\delta w.$$

We write down the virtual work equation:

$$F \cdot \tfrac{1}{2}\delta w + N \cdot \delta u = F \cdot \tfrac{1}{2}\delta w + N \cdot \tfrac{3}{4}\delta w = 0$$

so that

$$N = -\tfrac{2}{3}F.$$

The normal force in member DE is therefore a tensile force.

In the virtual work equation, the contribution by the *normal force* N, regardless of the sign, is equal to

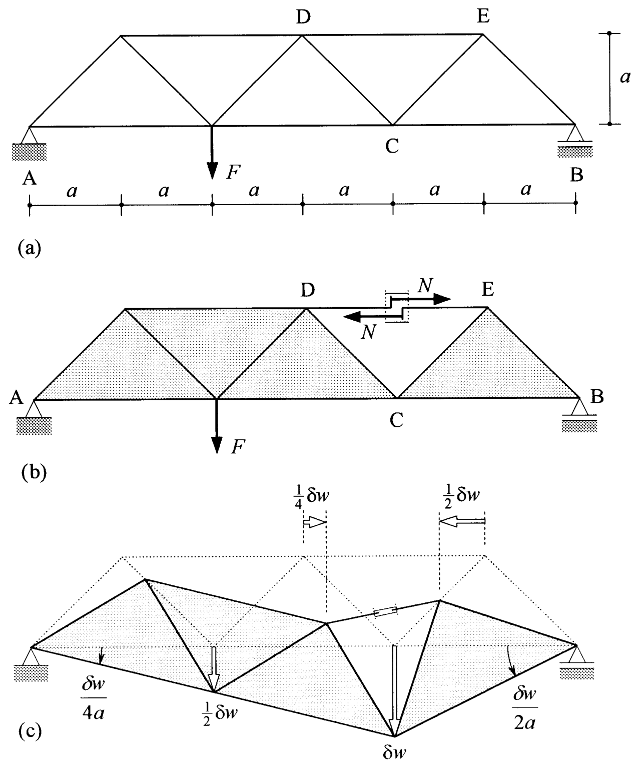


Figure 15.26 (a) Truss with (b) a mechanism for determining the normal force in DE and (c) a kinematically admissible virtual displacement of the mechanism.

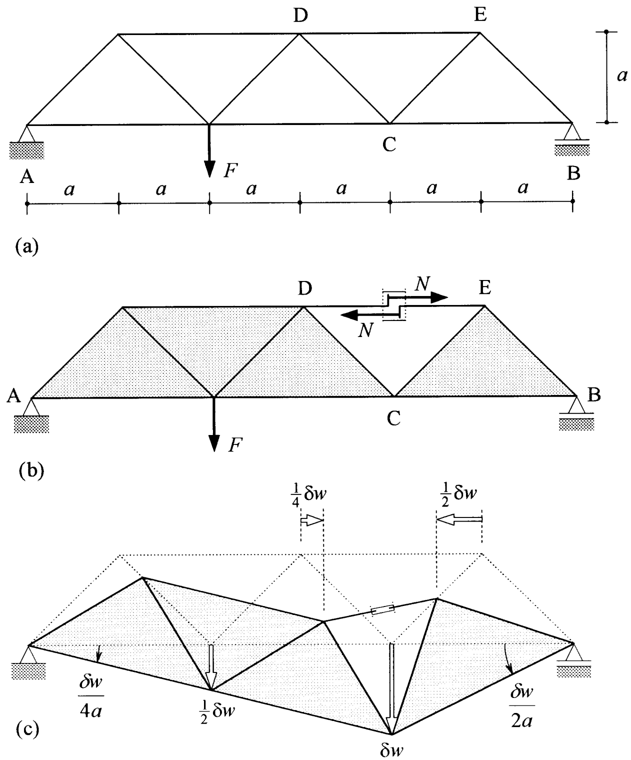


Figure 15.26 (a) Truss with (b) a mechanism for determining the normal force in DE and (c) a kinematically admissible virtual displacement of the mechanism.

$$\delta A(\text{due to } N) = \text{“normal force} \times \text{displacement in the normal force hinge”}$$
$$= N \cdot \delta u.$$

The sign is determined by the directions of the normal force and virtual displacement.

This example shows that the initial simplicity of the virtual work equation to replace the equilibrium equations is somewhat overshadowed by the more complicated geometry of the deformed mechanism.

15.6 Problems

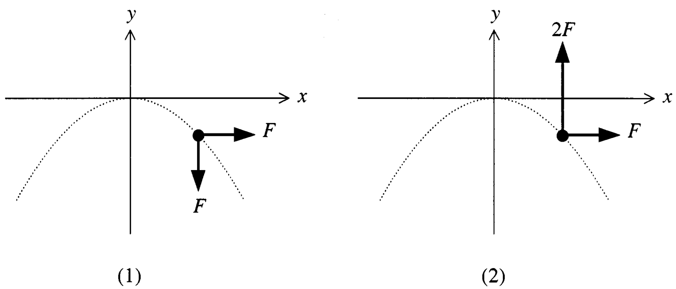
General comment: All calculations must be performed using virtual work.

Virtual work – mixed problems

15.1: 1–2 A particle is compelled to follow the following parabolic path in the xy plane:

$$y = -\frac{x^2}{2a}.$$

The particle is loaded by a horizontal and vertical force, as shown in the figure. There is no friction.



Questions:

a. At which point(s) is the particle in equilibrium? Give the coordinates for this/these point(s).
b. Can you (intuitively) say anything about the reliability (stability) of the equilibrium at this/these point(s)?

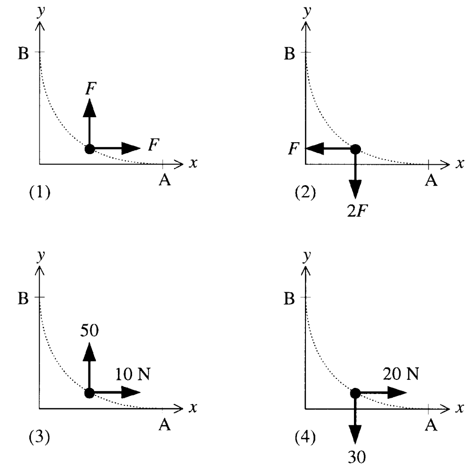
15.2: 1–4 A particle is compelled to follow a frictionless path in the xy plane between A and B with the following definition:

$$\sqrt{x} + \sqrt{y} = \sqrt{a}.$$

Here $a = 9$ m. The particle is loaded by the forces shown in the figure.

Questions:

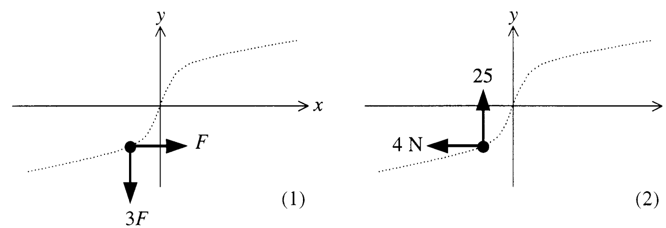
a. At which position(s) is the particle in equilibrium between A and B? Give the coordinates for this/these point(s).
b. Can you (intuitively) say anything about the reliability (stability) of the equilibrium at this/these position(s)?



15.3: 1–2 A particle is compelled to follow the frictionless path of a cubic parabola:

$$y = \sqrt[3]{x}.$$

The particle is loaded by the forces shown in the figure.



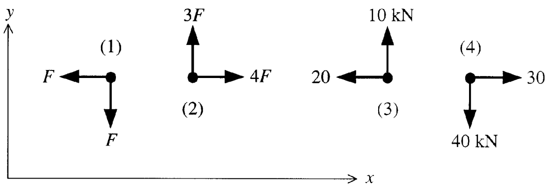
Questions:

a. At which position(s) is the particle in equilibrium? Give the coordinates for this/these point(s).
b. Can you (intuitively) say anything about the reliability (stability) of the equilibrium at this/these position(s)?

15.4: 1–4 A particle is compelled to follow an elliptical path:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

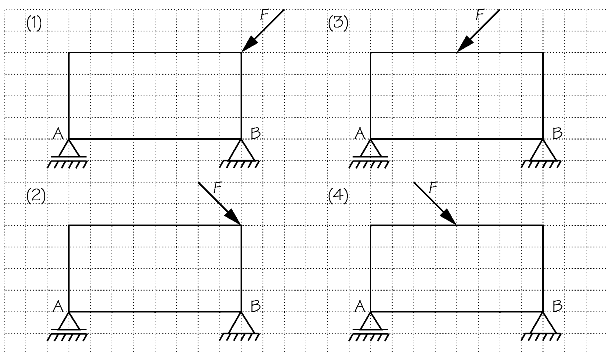
with $a = 2$ m and $b = 4$ m. The path is frictionless. The particle is loaded by the forces shown in the figure.



Questions:

a. Draw the path of the particle.
b. At which position(s) is the particle in equilibrium? Give the coordinates for this/these point(s).
c. Can you (intuitively) say anything about the reliability (stability) of the equilibrium in this/these position(s)?

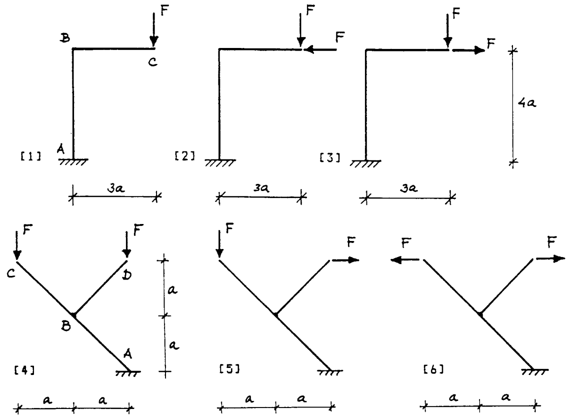
15.5: 1–4 A block is supported on a roller at A and a hinge at B, and is loaded by a force $F = 20\sqrt{2}$ kN in four different ways.



Questions:

a. Determine the support reaction at A.
b. Determine the vertical component of the support reaction at B.
c. Determine the horizontal component of the support reaction at B.

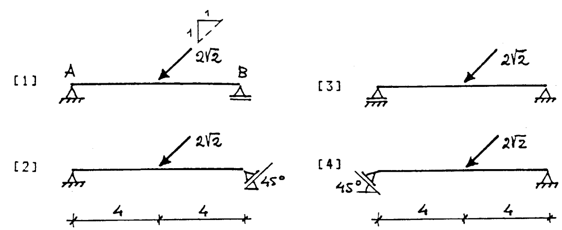
15.6: 1–6 You are given a number of structures fixed at A.



Questions:

a. Determine the horizontal support reaction at A.
b. Determine the vertical support reaction at A.
c. Determine the fixed-end moment at A.

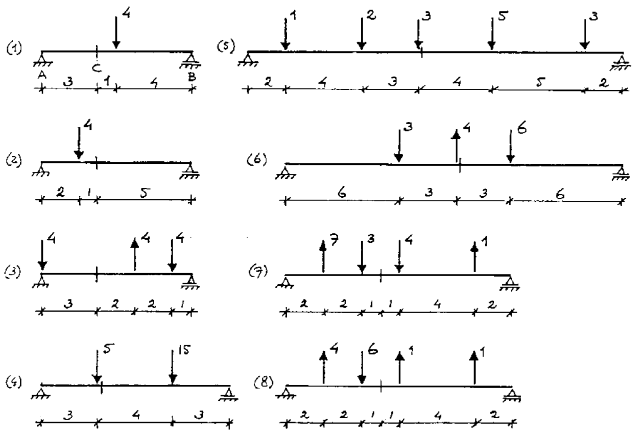
15.7: 1–4 A number of beams are supported on a hinge and a roller. The dimensions are in m, the forces are in kN.



Questions:

a. Determine (the components of) the support reaction at A.
b. Determine (the components of) the support reaction at B.

15.8: 1–8 A number of beams, simply supported at A and B, are composed of segments AC and BC that are rigidly joined at C. The location of joint C is shown in the figure by means of a vertical dash. The forces are given in kN, the lengths are in m.



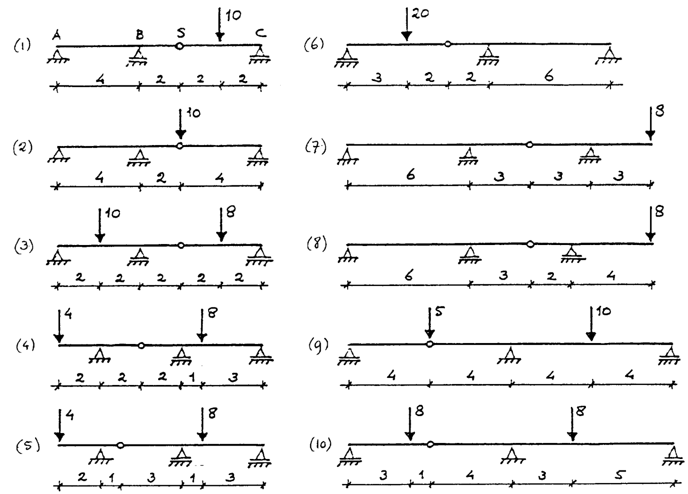
Questions:

a. Determine the support reaction at A.
b. Determine the support reaction at B
c. Determine the shear force at C.
d. Determine the bending moment at C.

15.9: 1–10 For hinged beam ABC you are given the lengths in metres and forces in kN.

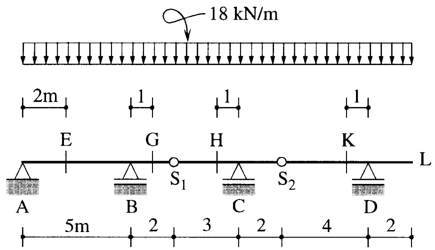
Questions:

a. Determine the support reaction at A.
b. Determine the support reaction at B.
c. Determine the support reaction at C.
d. Determine the support moment at B.
e. Determine the shear force directly to the left of B.
f. Determine the shear force directly to the right of B.



15.10: 1–10 As problem 15.9, but replace the concentrated loads by a uniformly distributed load of 10 kN/m over the entire length of the beam.

15.11 The hinged beam shown is subject to a uniformly distributed load of 18 kN/m.



Questions:

a. Determine the shear force at E.
b. Determine the bending moment at E.
c. Determine the shear force at G.
d. Determine the bending moment at G.
e. Determine the shear force at H.
f. Determine the bending moment at H.
g. Determine the shear force at K.
h. Determine the bending moment at K.