Cables, Lines of Force and Structural Shapes

14

A special type of tension-loaded line element is the entirely *flexible cable*. Cables have no “natural” shape, and adapt to the load.

In Section 14.1.1, we look at the behaviour of cables subject to a system of parallel forces.

In Section 14.1.2, we show that the shape of a cable with respect to its chord, due to a number of parallel forces, is similar to the bending moment diagram of a simply supported beam with the same span and the same load.

After deriving the *cable equation* from the equilibrium of a small cable element in Section 14.1.3, we apply this in Section 14.1.4 to a cable with a uniformly distributed full load (*force per horizontally measured length*). In this case, the cable is a *parabola*.

Next, the cable equation in Section 14.1.5 is applied to a cable loaded exclusively by its dead weight (*force per length measured along the cable*). The associated cable shape is a *catenary*.

In Section 14.2 we come back to the concept of *centre of force*, the point of application of the resultant of all normal stresses in the cross-section, or in other words, the point of application of the resultant of N and M (see Section 10.1.1). The centres of force in all consecutive cross-sections together form the *line of force*. If the line of force coincides with the member axis, the bending moments (and shear forces) are zero and the force flow occurs via normal forces.

In bending, the material in the cross-section is used less efficiently than in extension. To ensure maximum efficient use of material, the structural shape (member axis) should preferably be chosen in such a way that there is no bending and the force flow occurs via normal forces. In Section 14.3, on the basis of the cable shape and line of force, we look for structural shapes in which the force flow through bending remains limited.

14.1 Cables

Cables are line elements in which the resistance to bending is so small that it can be ignored. A fully flexible cable cannot transfer bending moments nor transverse forces. The force flow occurs entirely via normal forces, namely tensile forces.[1]

Cables are often used in structures with large spans such as suspension bridges and suspended roofs, but also in high-voltage cables, cableways, and the mooring of high structures such as radio and TV masts.

Cables do not have their own shape – they adapt to the load. Here, we assume that the axial stiffness of the cable is infinite. Therefore the cable has the same length before and after loading. The shape of the cable and the cable forces can then be deduced directly from the equilibrium equations.

In Section 14.1.1, we deduce the shape of the cable and cable forces directly from the equilibrium for a cable loaded by a number of parallel point loads.

---

[1] If there are compressive forces in the cable, the equilibrium is unstable (unreliable). In order to restore the equilibrium following a minor disruption in the cable shape under the given load, bending moments have to develop in the cable. Since this is not possible in an entirely flexible cable, the equilibrium is lost after a minor disruption in the cable shape.

In Section 14.1.2, we show that the *cable shape* is similar to the shape of the bending moment diagram of a simply supported beam with the same span and load.

We follow with a mathematical description of the relationship between cable force, cable shape and load in Section 14.1.3. We derive the so-called *cable equation* from the equilibrium equations for a small cable element.

Using the cable equation as basis, we calculate the cable shape in Section 14.1.4 due to a uniformly distributed load. The associated cable shape is a *parabola*.

At that point, the distributed load is a *force per horizontally measured length*. In Section 14.1.5, we calculate the cable shape due to its *dead weight*. The dead weight is a *force measured along the length of the cable*. The cable shape resulting from the dead weight is a *catenary*.

If the (vertical) sag of the cable with respect to its chord is small compared with the (horizontal) span, then the catenary can be approximated by the simpler parabola.

Finally, in Section 14.1.6, we present a number of examples.

14.1.1 Cables with point loads

Calculating the cable shape and cable forces from the equilibrium is illustrated using the cable in Figure 14.1a, supported at the fixed points A and B, and loaded by the vertical forces $F_C = 75$ kN and $F_D = 30$ kN. The cable has a (horizontal) span $\ell = 60$ m and a difference in elevation between supports A and B of $h = 9$ m. The distances between the supports and the lines of action of the forces are shown in the figure. The z coordinate of the cable of point E is also given: $z_E = 22$ m.

The dead weight of the cable is so small compared to the load that it can be ignored.

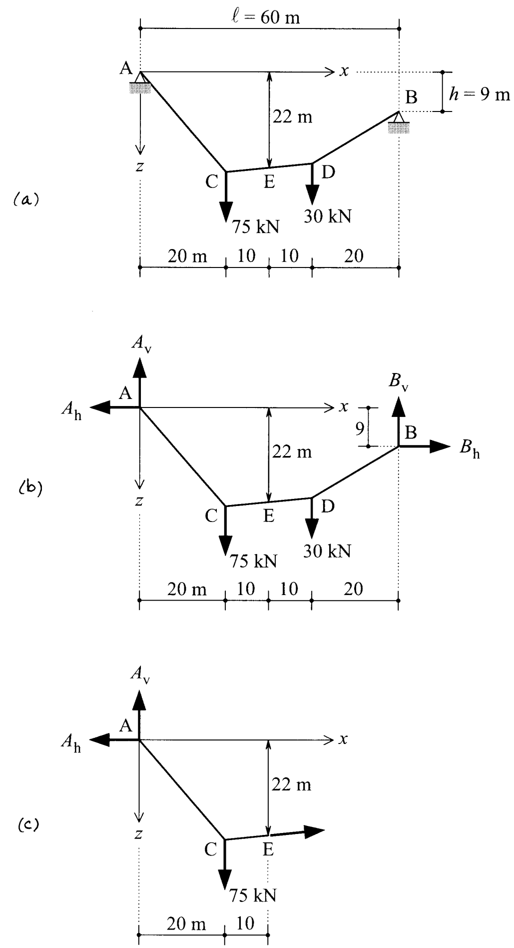


Figure 14.1 (a) Cable AB loaded by two vertical forces. (b) The isolated cable AB. (c) The isolated cable part AE.

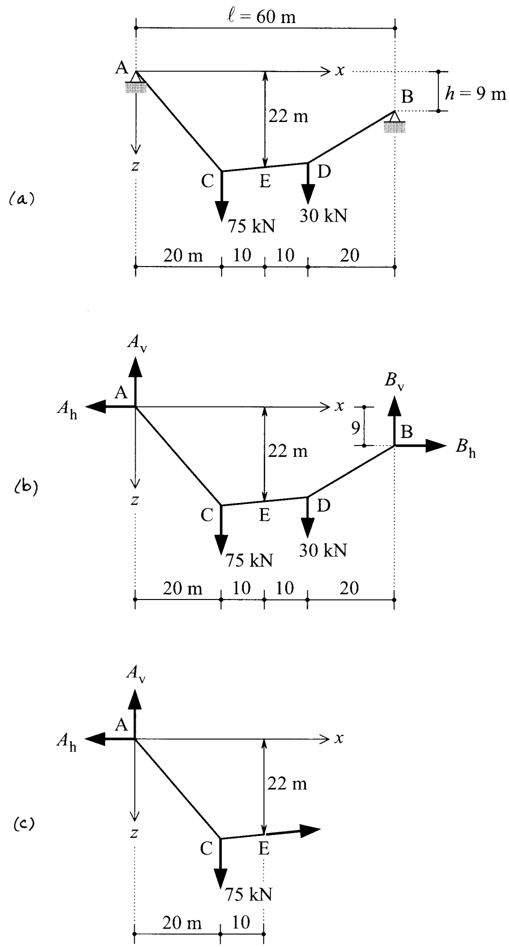


Figure 14.1 (a) Cable AB loaded by two vertical forces. (b) The isolated cable AB. (c) The isolated cable part AE.

Questions:

a. Determine the cable shape, or in other words, the z coordinates of C and D where kinks occur in the cable.

b. Determine the maximum and minimum cable force.

Solution:

a. With fully flexible cables, no bending moments can be transferred, and the cable remains straight between the places where forces are applied. Each straight part of the cable can be seen as a line element subject to a tensile force N, the cable force.

In Figure 14.1b the cable has been isolated. There are four unknown support reactions: A_h, A_v, B_h and B_v. There are three equilibrium equations:

$$\sum F_x = -A_h + B_h = 0, \tag{1}$$

$$\sum F_z = -A_v - B_v + (75 \text{ kN}) + (30 \text{ kN}) = 0, \tag{2}$$

$$\begin{aligned}\sum T_y|\mathrm{B} &= +A_h(9 \text{ m}) - A_v(60 \text{ m}) \\ &\quad + (75 \text{ kN})(40 \text{ m}) + (30 \text{ kN})(20 \text{ m}) = 0.\end{aligned} \tag{3}$$

For a unique solution to these three equations with four unknowns, we need a fourth equation. This is found from the moment equilibrium of the part of the cable to the right or left of E, the point where the z coordinate of the cable is given. Here, we select the part to the left of E, as this equilibrium equation contains only the unknowns A_h and A_v and in combination with Equation (3) leads to the quicker result (see Figure 14.1c):

$$\sum T_y|\mathrm{E} = +A_h(22 \text{ m}) - A_v(30 \text{ m}) + (75 \text{ kN})(10 \text{ m}) = 0. \tag{4}$$

From (3) and (4) we find

$$A_h = 60 \text{ kN},$$

$$A_v = 69 \text{ kN}.$$

From (1) and (2) we then find

$$B_h = 60 \text{ kN},$$

$$B_v = 36 \text{ kN}.$$

If the z coordinate of E (or of another point on the cable) is not given, the result remains undetermined, and the cable can assume various shapes, such as the two dotted shapes in Figure 14.2. The final shape of the cable is determined by the length of the cable. The approach via a given cable length is considerably more complicated than that with a locally given z coordinate of the cable, such as that at point E.

Hereafter we assume that the axial stiffness of the cable is infinite, so that the cable has the same length before and after loading. In that case, the cable shape and cable forces can be derived directly from the equilibrium. Since the cable cannot stretch, an increase in the load with a certain factor does not cause a change in the shape of the cable.

The z coordinates of respectively C and D are found from the moment equilibrium about C and D of the left-hand or right-hand part of the cable.

The following holds for the part to the left of C (see Figure 14.3a):

$$\sum T_y|C = +(60 \text{ kN}) \times z_C - (69 \text{ kN})(20 \text{ m}) = 0$$

so that

$$z_C = 23 \text{ m}.$$

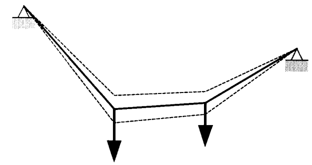


Figure 14.2 The equilibrium conditions are satisfied by an infinite number of cable shapes. The final shape is determined by the (developed) length of the cable.

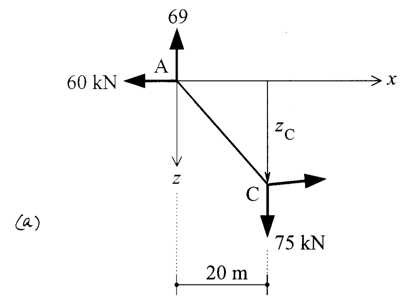


Figure 14.3 (a) The z coordinate follows from the moment equilibrium about C of AC.

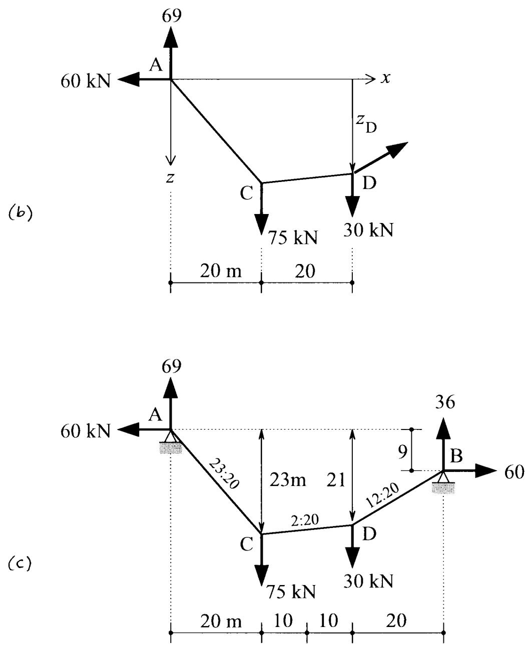


Figure 14.3 (b) The z coordinate follows from the moment equilibrium of ACD about D. (c) Support reactions and cable shape.

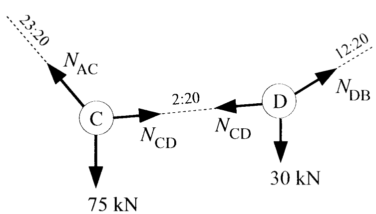


Figure 14.4 The cable forces can be determined using the force equilibrium for joints C and D.

For the part to the left of D applies (see Figure 14.3b)

$$\sum T_y|\text{D} = +(60\ \text{kN}) \times z_\text{D} - (69\ \text{kN})(40\ \text{m}) + (75\ \text{kN})(20\ \text{m}) = 0$$

so that

$$z_\text{D} = 21\ \text{m}.$$

The z coordinates of C and D fix the cable shape (see Figure 14.3c).

b. Cable forces N (in the straight parts) can now be calculated from the force equilibrium of joints C and D (see Figure 14.4).

Since the cable is loaded exclusively by vertical forces, it is easier to use the fact that the tensile force N in the cable has a horizontal component H that is constant over the entire length of the cable. This follows directly from the horizontal force equilibrium of an arbitrary part of the cable:

$$H = A_\text{h} = B_\text{h} = 60\ \text{kN}.$$

Assuming α is the angle that the cable makes with the horizontal, then (see Figure 14.5)

$$N = \frac{H}{\cos\alpha} = H\sqrt{1 + (\tan\alpha)^2}. \tag{5}$$

The maximum force in the cable occurs where $\tan\alpha$ is a maximum, that is where the slope of the cable is largest.

The geometry of the deformed cable gives

$$N^{\text{AC}} = H\sqrt{1 + (\tan\alpha^{\text{AC}})^2} = (60\ \text{kN})\sqrt{1 + (23/20)^2} = 91.44\ \text{kN},$$

$$N^{\text{CD}} = H\sqrt{1 + (\tan\alpha^{\text{CD}})^2} = (60\ kN)\sqrt{1 + (2/20)^2} = 60.30\ \text{kN},$$

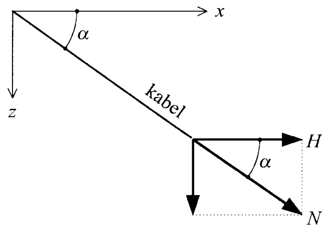


Figure 14.5 For cable force N it holds that $N = H/\cos\alpha = H\sqrt{1+(\tan\alpha)^2}$.

$$N^{\mathrm{DB}} = H\sqrt{1+(\tan\alpha^{\mathrm{DB}})^2} = (60\ \mathrm{kN})\sqrt{1+(12/20)^2} = 69.97\ \mathrm{kN}.$$

Check: The cable forces in AC and DB can also be found directly from the support reactions at A and B respectively:

$$N^{\mathrm{AC}} = \sqrt{A_{\mathrm{h}}^2 + A_{\mathrm{v}}^2} = \sqrt{(60\ \mathrm{kN})^2 + (69\ \mathrm{kN})^2} = 91.44\ \mathrm{kN},$$

$$N^{\mathrm{DB}} = \sqrt{B_{\mathrm{h}}^2 + B_{\mathrm{v}}^2} = \sqrt{(60\ \mathrm{kN})^2 + (36\ \mathrm{kN})^2} = 69.97\ \mathrm{kN}.$$

14.1.2 Relationship between cable shape and bending moment diagram

Figure 14.6a shows a simply supported cable with compression bar, loaded by n vertical point loads $F_1, F_2, \ldots, F_n$. The cable has a (horizontal) span ℓ with a difference h between the support elevations at A and B.

The place of the roller support B is fixed by the *compression bar* AB so that the cable shape can be determined as if A and B were immovable supports.

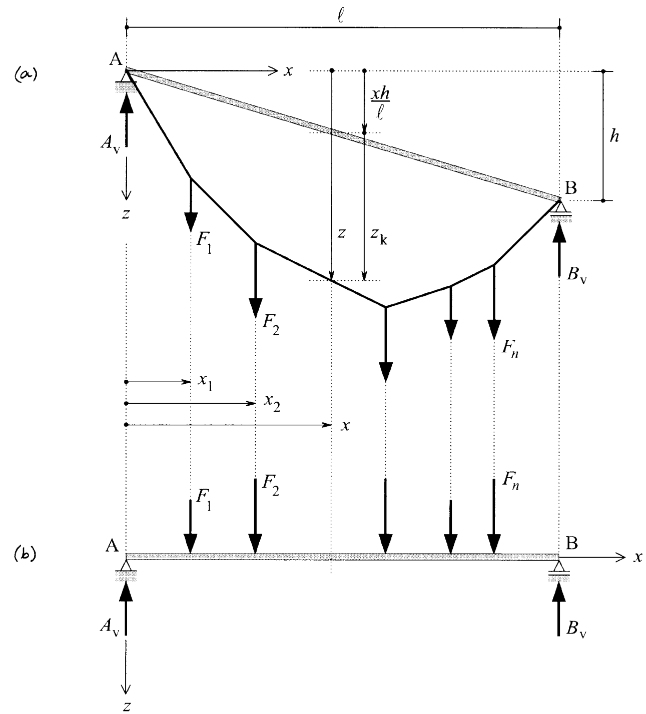


Figure 14.6 (a) A simply supported cable with compression bar loaded by a number of parallel forces. (b) A simply supported beam with the same span and load.

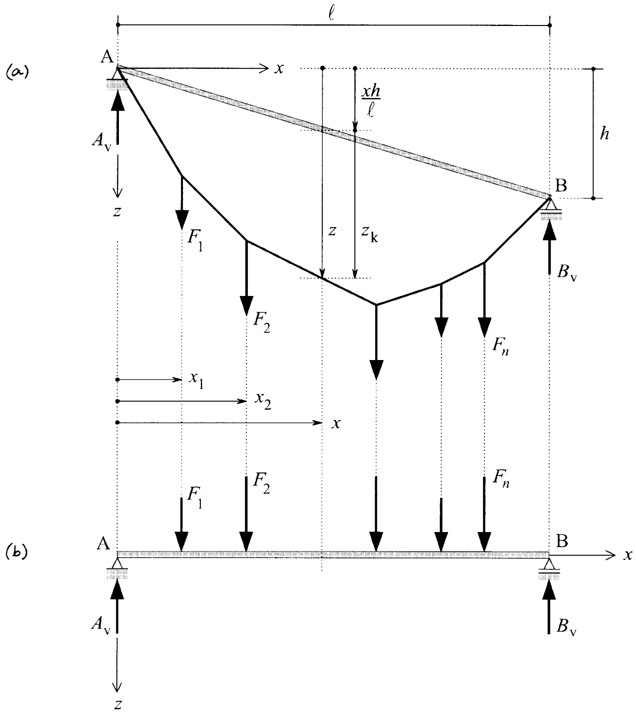


Figure 14.6 (a) A simply supported cable with compression bar loaded by a number of parallel forces. (b) A simply supported beam with the same span and load.

For the cable shape applies

$$z = z(x)$$

in which z is the distance from the x axis to the cable. The distance from the chord (compression bar) AB to the cable is hereafter indicated by

$$z_k = z_k(x).$$

From Figure 14.6a we can deduce that

$$z_k = z - \frac{x}{\ell}h.$$

Figure 14.6b shows a simply supported beam AB with the same span ℓ and the same load.

The cable with compression bar, and the simply supported beam have the same support reactions A_v and B_v at A and B respectively. There are no horizontal support reactions. That the support reactions are equal for cable and beam can easily be checked by calculation. In this way, the vertical support reaction A_v follows in both cases from the moment equilibrium about B of the structure as a whole:

$$\sum T_y|B = -A_v\ell + \sum_{i=1}^{n} F_i(\ell - x_i) = 0$$

so that

$$A_v = \frac{\sum_{i=1}^{n} F_i(\ell - x_i)}{\ell}.$$

In both cases, the vertical force equilibrium about B gives the following result for the vertical support reaction B_v:

$$B_\mathrm{v} = \frac{\sum_{i=1}^{n} F_i x_i}{\ell}.$$

In Figure 14.7, the part to the left of an arbitrary (vertical) section x has been isolated for both the cable with compression bar and the simply supported beam.

Since the cable is loaded exclusively by vertical forces, the tensile forces in the cable have a constant horizontal component H (see Section 14.1.1). From the horizontal force equilibrium of the isolated part in Figure 14.7a we find that the compressive force in bar AB has the same horizontal component H.

In addition to the horizontal forces H, there are also the vertical forces V and Hh/ℓ in the section, components of the tensile force in the cable and the compressive force in the bar (a two-force member) respectively. On the basis of the vertical equilibrium of the isolated section, it holds that

$$\sum F_z = -A_\mathrm{v} + \sum_{i=1}^{2} F_i + V - \frac{Hh}{\ell} = 0.$$

The vertical forces in the section are therefore

$$V - \frac{Hh}{\ell} = A_\mathrm{v} - \sum_{i=1}^{2} F_i. \tag{6}$$

In Figure 14.7b, there is a bending moment M and a shear force V^beam at the cross-section of the beam. The vertical equilibrium of the isolated part of the beam gives the shear force:

$$V^\mathrm{beam} = A_\mathrm{v} - \sum_{i=1}^{2} F_i. \tag{7}$$

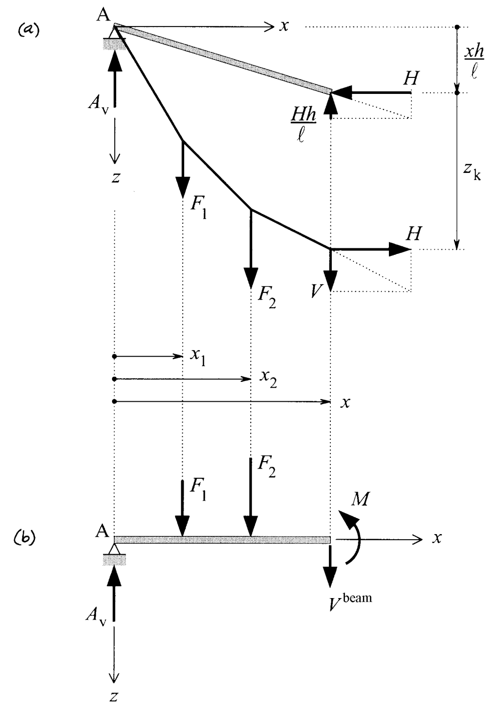


Figure 14.7 The isolated part to the left of section x of (a) the cable with a compression bar and (b) the beam.

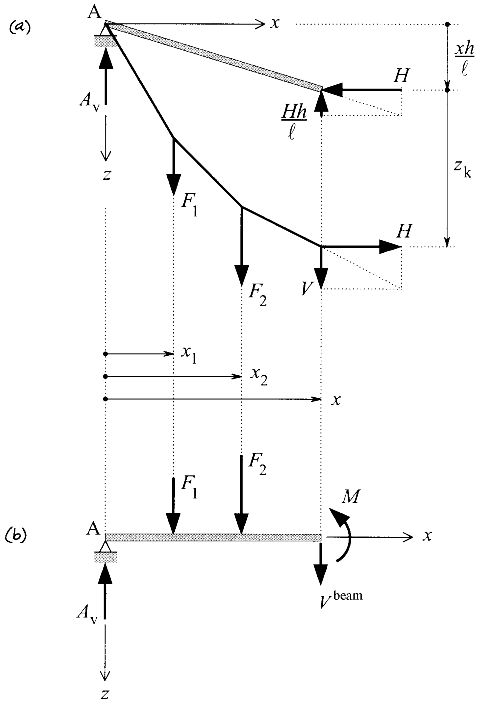


Figure 14.7 The isolated part to the left of section x of (a) the cable with a compression bar and (b) the beam.

From (6) and (7), we find the following relationship between the vertical component of the cable force and the shear force in the beam:

$$V - \frac{Hh}{\ell} = V^{\text{beam}}. \tag{8}$$

The term Hh/ℓ in (8) disappears when the supports of the cable are at equal elevations. The vertical component of the cable force is then equal to the shear force in the beam.

Conclusion: *The vertical component of the cable force is equal to the shear force in the beam (which can be read from the shear force diagram), only if the support reactions of the cable are at equal elevations.*

The moment equilibrium of the isolated part of the cable with compression bar about an arbitrary point in the section gives (see Figure 14.7a)

$$\sum T_y|\text{section} = -A_v x + \sum_{i=1}^{2} F_i(x - x_i) + Hz_k = 0$$

so that

$$Hz_k = A_v x - \sum_{i=1}^{2} F_i(x - x_i). \tag{9}$$

The moment equilibrium of the isolated part of the beam in Figure 14.7b gives

$$\sum T_y|\text{section} = -A_v x + \sum_{i=1}^{2} F_i(x - x_i) + M = 0$$

so that

$$M = A_v x - \sum_{i=1}^{2} F_i(x - x_i). \tag{10}$$

If we compare the equations (9) and (10) we find

$$H z_k = M. \tag{11}$$

Conclusion: *The product of the horizontal component H of the tensile force in the cable and the distance z_k from the chord (compression bar) AB to the cable is equal to the bending moment M in a simply supported beam with the same span and the same load.*

The horizontal component H of the tensile force in the cable is constant and therefore independent from x. In contrast, the cable shape $z_k = z_k(x)$, under the chord, and the bending moment $M = M(x)$ are functions of x. The equation

$$H z_k(x) = M(x)$$

shows that the cable shape under the chord (compression bar) AB has the same shape as the bending moment diagram. The force H can be seen as a scale factor.[1]

In the section in Figure 14.7, the left-hand part is subject to only two forces F_1 and F_2, and only these forces appear in the calculation. With an arbitrary alternative section, the number of forces on the left-hand side can be larger or smaller. The conclusions remain the same, however.

In Figure 14.8a, the compression bar AB has been isolated from the cable. In A and B, the compression bar is subject to horizontal forces H and vertical forces Hh/ℓ. In Figure 14.8b, equal but opposite forces act on the ends of the isolated cable, together with the forces A_v and B_v, which are equal to the support reactions of the beam AB in Figure 14.8c, with the same span and load.

[1] As a help for drawing bending moment diagrams, rule 16 in Section 12.1.6 already pointed out the relationship between cable shape and bending moment diagram.

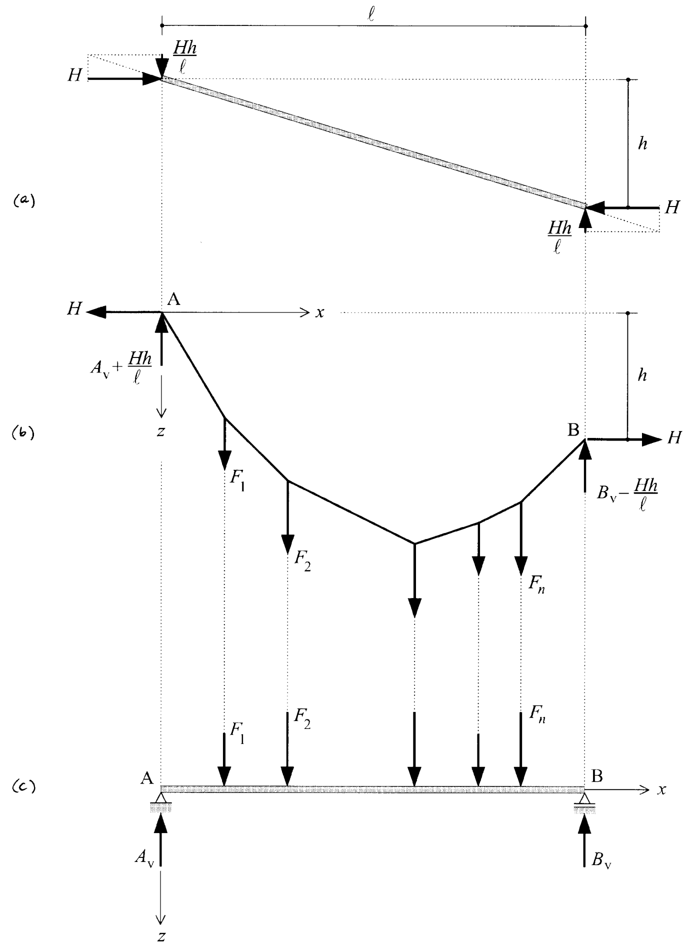


Figure 14.8 (a) The compression bar isolated from the cable. (b) The support reactions of the cable without compression bar. (c) The support reactions of a simply supported beam with the same span and load.

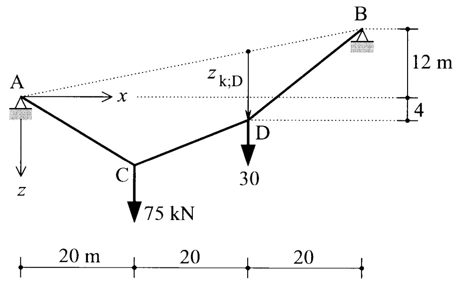


Figure 14.9 A cable loaded by two vertical forces.

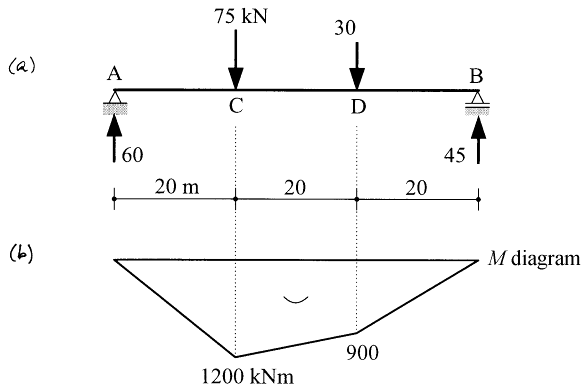


Figure 14.10 (a) A simply supported beam with the same span and load as the cable in Figure 14.9 with (b) the associated bending moment diagram.

The forces at A and B on the isolated cable in Figure 14.8b can be seen as the support reactions of a cable without compression member, supported at two fixed points.

The vertical support reactions A_v and B_v are equal to those of a beam with the same span and load only if the supports are at equal elevations.

With a difference h between both support elevations, the two horizontal support reactions H form a couple that leads to a change in the vertical support reactions of Hh/ℓ.

This is applied in the following two examples.

Example 1

The cable in Figure 14.9, supported at the fixed points A and B, is loaded in C and D by vertical forces of 75 and 30 kN respectively. Only the location of point D is given for the cable shape: it is 4 metres lower than support A.

Questions:

a. Determine the cable shape.
b. Determine the horizontal support reactions at A and B.
c. Determine the vertical support reactions at A and B.
d. Determine the maximum and minimum cable forces.

Solution:

a. Figure 14.10a shows a simply supported beam AB with the same (horizontal) span as the cable and the same vertical load. The support reactions are also shown. Figure 14.10b shows the associated M diagram. Under chord AB, the cable has the same shape as the M diagram; according to (11):

$$Hz_k = M.$$

The scale factor H is the horizontal component of the tensile force in the cable. At D, the distance from the chord AB to the cable is (see Figure 14.9)

$$z_{k;D} = \tfrac{2}{3}(12\ \text{m}) + (4\ \text{m}) = 12\ \text{m}.$$

This gives

$$H = \frac{M_D}{z_{k;D}} = \frac{900\ \text{kNm}}{12\ \text{m}} = 75\ \text{kN}.$$

At C, the distance between the chord and the cable is

$$z_{k;C} = \frac{M_C}{H} = \frac{1200\ \text{kNm}}{75\ \text{kN}} = 16\ \text{m}.$$

The cable shape is now determined (see Figure 14.11).

b. The horizontal support reactions at A and B are equal to the horizontal component H of the cable force determined above:

$$A_h = B_h = H = 75\ \text{kN}.$$

The horizontal support reactions are shown in Figure 14.11.

c. The vertical support reactions A_v and B_v in Figure 14.11 are equal to the vertical support reactions of the beam in Figure 14.10a, but since the horizontal support reactions act at different levels these have to be corrected by a force

$$\frac{Hh}{\ell} = \frac{(75\ \text{kN})(12\ \text{m})}{60\ \text{m}} = 15\ \text{kN}$$

so that

$$A_v = (60\ \text{kN}) - (15\ \text{kN}) = 45\ \text{kN},$$

$$B_v = (45\ \text{kN}) + (15\ \text{kN}) = 60\ \text{kN}.$$

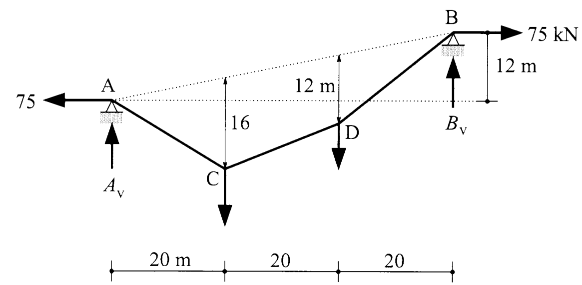


Figure 14.11 Under the chord, the cable has exactly the same shape as the bending moment diagram in Figure 14.10b. The scale factor is $H = 75$ kN.

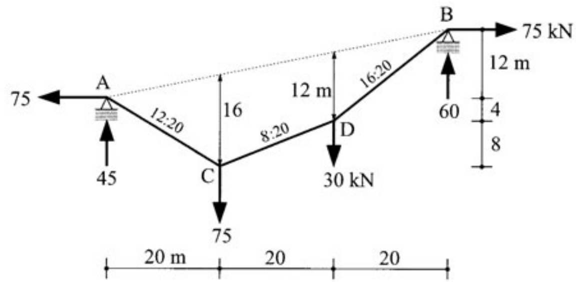


Figure 14.12 Support reactions and cable shape.

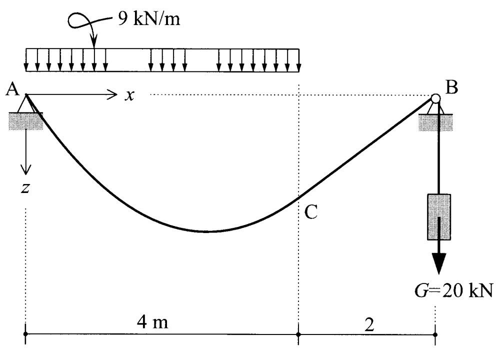


Figure 14.13 At B, cable AB passes over a pulley, and is kept under tension by a weight of 20 kN.

d. Figure 14.12 shows all the support reactions. This figure also includes the slopes of the straight cable parts. With (5)

$$N = H\sqrt{1 + (\tan\alpha)^2}$$

we find the cable force N in each of the straight cable parts:

$$N^{\mathrm{AC}} = (75\ \mathrm{kN})\sqrt{1 + (12/20)^2} = 87.5\ \mathrm{kN},$$

$$N^{\mathrm{CD}} = (75\ \mathrm{kN})\sqrt{1 + (8/20)^2} = 80.8\ \mathrm{kN},$$

$$N^{\mathrm{DB}} = (75\ \mathrm{kN})\sqrt{1 + (16/20)^2} = 96.0\ \mathrm{kN}.$$

The maximum cable force occurs in the steepest part DB:

$$N_{\mathrm{max}} = N^{\mathrm{DB}} = 96.0\ \mathrm{kN}.$$

The minimum cable force occurs in the shallowest part CD:

$$N_{\mathrm{min}} = N^{\mathrm{CD}} = 76.5\ \mathrm{kN}.$$

Check: N^{AC} and N^{DB} can also be found from the support reactions at A and B respectively:

$$N^{\mathrm{AC}} = \sqrt{A_{\mathrm{h}}^2 + A_{\mathrm{v}}^2} = \sqrt{(75\ \mathrm{kN})^2 + (45\ \mathrm{kN})^2} = 87.5\ \mathrm{kN},$$

$$N^{\mathrm{DB}} = \sqrt{B_{\mathrm{h}}^2 + B_{\mathrm{v}}^2} = \sqrt{(75\ \mathrm{kN})^2 + (60\ \mathrm{kN})^2} = 96.0\ \mathrm{kN},$$

Example 2

A uniformly distributed load of 9 kN/m acts over AC on the cable in Figure 14.13, of which the supports A and B are at equal elevations. At B, the cable runs over a frictionless pulley with negligible dimensions. A block of weight $G = 20$ kN is suspended at B.

Questions:

a. Determine the vertical component of the cable force in CB.
b. Determine the horizontal component of the cable force.
c. Determine the shape of the cable.
d. Determine the maximum and minimum forces in the cable and the places where these occur.
e. Determine the support reactions at A and B.

Solution:

a. Figure 14.14a shows a beam AB with the same span and load as the cable. Figures 14.14b and 14.14c also show the associated bending moment diagram and shear force diagram, with various details. The calculation is left to the reader.

$$V^{\text{CB}} = 12 \text{ kN}.$$

b. Since the pulley at B is frictionless, the following applies:

$$N^{\text{CB}} = \sqrt{H^2 + (V^{\text{CB}})^2} = G.$$

From this, we can find the horizontal component H of the cable force:

$$H = \sqrt{G^2 - (V^{\text{CB}})^2} = \sqrt{(20 \text{ kN})^2 - (12 \text{ kN})^2} = 16 \text{ kN}.$$

c. The cable has the same shape as the bending moment diagram in Figure 14.14b. According to (11):

$$H z_{\text{k}} = M$$

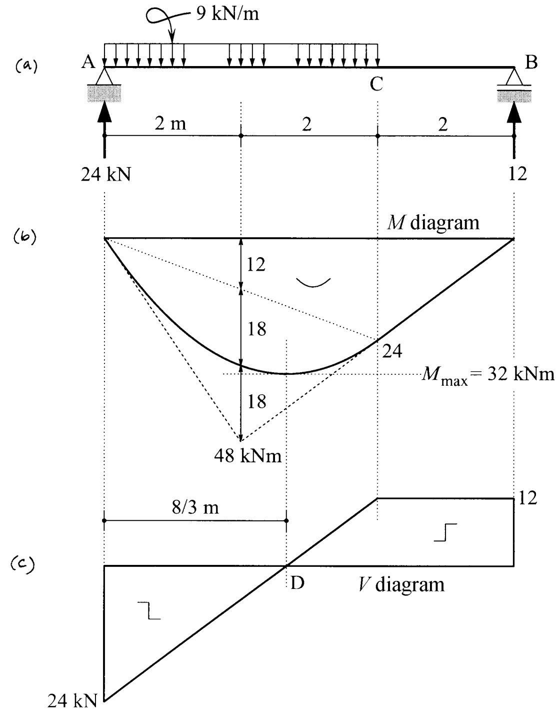


Figure 14.14 (a) A simply supported beam with the same span and load as the cable in Figure 14.13, with the associated (b) bending moment diagram and (c) shear force diagram.

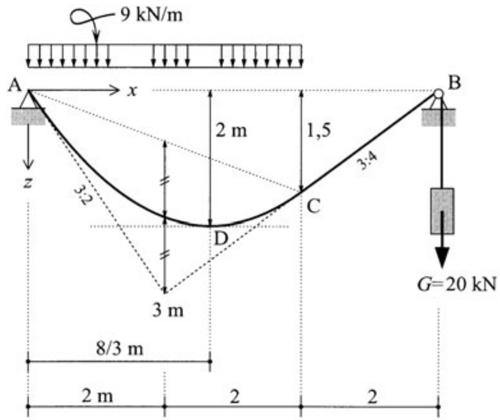


Figure 14.15 The cable is parabolic over AC. To sketch the shape of the cable, we can use the same auxiliary lines when drawing a parabolic bending moment diagram.

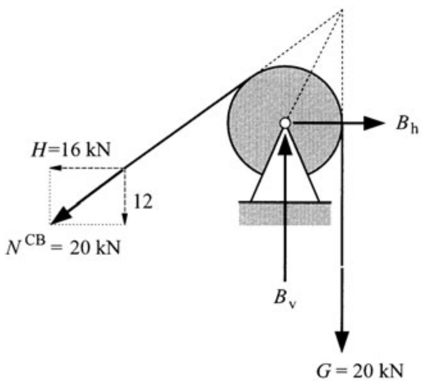


Figure 14.16 The forces acting in B on the trolley.

we find

$$z_{k;C} = z_C = \frac{M_C}{H} = \frac{24 \text{ kNm}}{16 \text{ kN}} = 1.5 \text{ m}.$$

In D, the bending moment is a maximum, and the cable sags most:

$$z_{k;D} = z_D = \frac{M_D}{H} = \frac{32 \text{ kNm}}{16 \text{ kN}} = 2 \text{ m}.$$

The cable shape over AC is parabolic. The auxiliary lines for drawing a parabolic M diagram (see Section 12.1.6) can also be used to draw the cable shape (see Figure 14.15).

d. The cable force is a maximum where the slope of the cable is a maximum. This is at A, as shown in Figure 14.15. With (5)

$$N = H\sqrt{1 + (\tan\alpha)^2}$$

we find

$$N_{max} = N_A = (16 \text{ kN})\sqrt{1 + (3/2)^2} = 28.84 \text{ kN}.$$

The cable force is a minimum at D, where the cable is horizontal and $V = 0$:

$$N_{min} = H = 16 \text{ kN}$$

e. The horizontal support reaction at A is equal to H:

$$A_h = H = 16 \text{ kN } (\leftarrow).$$

Since the cable supports are at equal elevations, the vertical support reaction at A is equal to the support reaction of the beam in Figure 14.14a:

$$A_v = 24 \text{ kN } (\uparrow).$$

Check: Figure 14.16 shows the forces acting on the pulley at B. The forces N^{CB} and G, both 20 kN, are known. The force equilibrium can be used to find the support reactions at B:

$$B_h = H = 16 \text{ kN } (\rightarrow),$$

$$B_v = (12 \text{ kN}) + (20 \text{ kN}) = 32 \text{ kN}) (\uparrow).$$

Check: The horizontal support reaction at B is equal to H. The vertical support reaction is equal to the support reaction at B of the beam in Figure 14.14a, increased with the vertical cable force G.

The support reactions at A and B are shown in Figure 14.17.

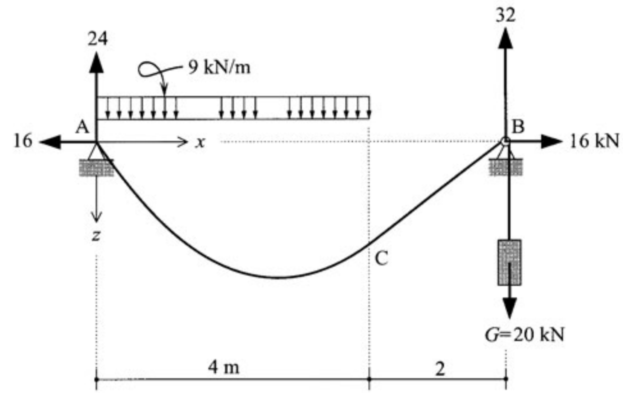


Figure 14.17 The support reactions in A and B.

14.1.3 Cable equation

Figure 14.18 shows a cable subject to a distributed load $q_z = q_z(x)$. The cable shape is $z = z(x)$.

In Figure 14.19, a small cable element of length Δx has been isolated from the deformed cable and blown up. As $\Delta x \to 0$ the distributed load q_z on the cable element can be considered to be uniformly distributed.

Assume the cable force at the left-hand section is a tensile force N, with a horizontal component H and a vertical component V.[1] The cable force at the right-hand section could have changed with respect to magnitude and

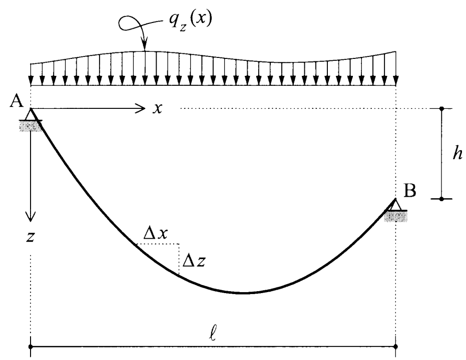


Figure 14.18 Cable with distributed load (force per horizontally measured length).

[1] V is not the transverse force in the cable, but the vertical component of the tensile force N. Instead of H and V we could also formally have written N_h and N_v.

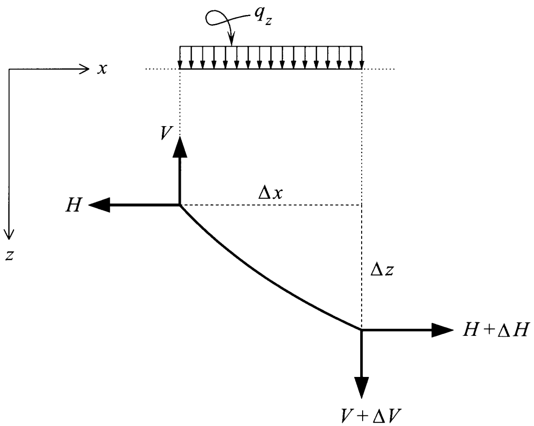


Figure 14.19 The enlarged element isolated from the cable.

direction. Assume for the right-hand part that the tensile force in the cable has the components $H + \Delta H$ and $V + \Delta V$.

There are three equilibrium equations for the cable element:

$$\sum F_x = -H + (H + \Delta H) = 0,$$

$$\sum F_z = -V + (V + \Delta V) + q_z \Delta x = 0,$$

$$\sum T_y|\text{right-hand section} = +H\Delta z - V\Delta x + q_z\Delta x\left(\tfrac{1}{2}\Delta x\right) = 0.$$

In the last equilibrium equation, the quadratic term in Δx is a degree smaller than the linear term in Δx and can be ignored as $\Delta x \to 0$. This leads to

$$\Delta H = 0,$$

$$\Delta V + q_z \Delta x = 0,$$

$$H\Delta z - V\Delta x = 0.$$

Divide each of these equations by Δx and proceed to the limit $\Delta x \to 0$; we generate three differential equations:

$$\frac{\mathrm{d}H}{\mathrm{d}x} = 0, \tag{12}$$

$$\frac{\mathrm{d}V}{\mathrm{d}x} = -q_z, \tag{13}$$

$$H\frac{\mathrm{d}z}{\mathrm{d}x} = V. \tag{14}$$

It follows from equation (12) that

$$H = \text{constant}.$$

The horizontal component H of cable force N is constant, or in other words, independent of x. This is in line with what we derived earlier in Section 14.1.1 for a cable subject to a system of vertical forces.

From equation (14) we find that the cable force N is directed along the tangent to the cable, as in Figure 14.20:

$$\tan\alpha = \frac{\mathrm{d}z}{\mathrm{d}x} = \frac{V}{H}.$$

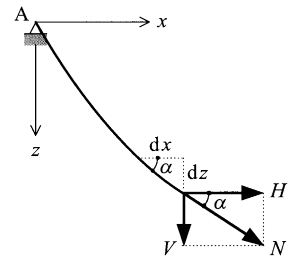


Figure 14.20 The cable force N is directed along the tangent to the cable: $\tan\alpha = \mathrm{d}z/\mathrm{d}x = V/H$.

The tensile force N in the cable is therefore

$$N = \sqrt{H^2 + V^2} = H\sqrt{1 + \left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)^2} = H\sqrt{1 + (\tan\alpha)^2}. \tag{15}$$

Tensile force N is largest where the slope $\mathrm{d}z/\mathrm{d}x$ of the cable is largest.

If we differentiate (14), in which H is constant, we find

$$H\frac{\mathrm{d}^2 z}{\mathrm{d}x^2} = \frac{\mathrm{d}V}{\mathrm{d}x}.$$

By substituting (13) in the equation above, we arrive at the so-called *cable equation*:

$$H\frac{\mathrm{d}^2 z}{\mathrm{d}x^2} = -q_z. \tag{16}$$

This differential equation, derived from the equilibrium of a cable element, forms a relationship between the horizontal component H of the cable force, the cable shape $z = z(x)$, and the distributed load $q_z = q_z(x)$.

The cable shape for a certain load $q_z = q_z(x)$ is the function $z = z(x)$ that satisfies the cable equation and the boundary conditions at the ends where the cable is suspended. In order to solve the cable equation, we have to know H. Sometimes H is not given, while the length of the cable is known. Finding the solution is far more complicated in that case.

Hereafter, we assume that the horizontal component H is known.

In Section 14.1.4, using the cable equation as basis, we determine the cable shape under a uniformly distributed load (*force per horizontally measured length*). The associated cable shape is a *parabola*.

In Section 14.1.5, we calculate the cable shape due to its dead weight (*force per length measured along the cable*). The shape of the cable under its dead weight is a *catenary*.

14.1.4 Cable with uniformly distributed load; parabola

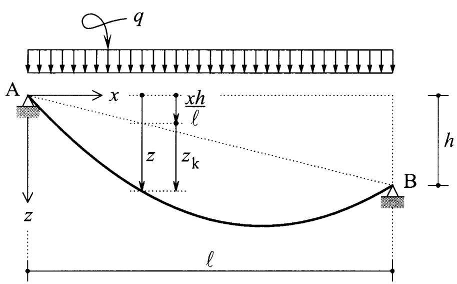


Figure 14.21 Cable with a uniformly distributed load q (force per horizontally measured length).

In Figure 14.21, cable AB, with span ℓ, carries a uniformly distributed load $q_z = q$. The difference in elevation of the supports A and B is h. From the cable equation we find

$$H\frac{\mathrm{d}^2 z}{\mathrm{d}x^2} = -q.$$

After integrating once, we find

$$H\frac{\mathrm{d}z}{\mathrm{d}x} = -qx + C_1,$$

while after integrating once more we find

$$Hz = -\tfrac{1}{2}qx^2 + C_1x + C_2.$$

The integration constants C_1 and C_2 follow from the boundary conditions at supports A and B:

$$x = 0; \ z = 0,$$

$$x = \ell; \ z = h.$$

Working out the boundary conditions gives

$$C_1 = H\frac{h}{\ell} + \tfrac{1}{2}q\ell,$$

$$C_2 = 0.$$

The cable shape is a parabola:

$$z = \frac{\tfrac{1}{2}qx(\ell - x)}{H} + \frac{h}{\ell}x.$$

This can be denoted as

$$z = z_{\text{k}} + \frac{h}{\ell}x$$

in which z_{k} is the distance from the chord to the cable (see Figure 14.21):

$$z_{\text{k}} = \frac{\tfrac{1}{2}qx(\ell - x)}{H} = \frac{M}{H}.$$

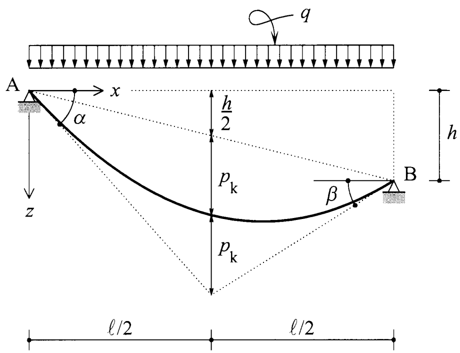


Figure 14.22 With uniformly distributed loads, the cable assumes the shape of a parabola. At A and B the tangents to the parabola are shown.

$M = \frac{1}{2}qx(\ell - x)$ is the bending moment in a simply supported beam with a uniformly distributed load (see Section 10.2.2, Example 1). The cable has the same parabolic shape under the chord as the M diagram; the *scale factor* is H.

Assume p_k is the sag of the parabola under the chord, that is the distance between the parabola and the chord at the middle of the span ℓ (see Figure 14.22):

$$p_k = z_k\left(x = \tfrac{1}{2}\ell\right) = \frac{\frac{1}{8}q\ell^2}{H}. \tag{17a}$$

If the sag p_k of the parabola under the chord is given, the horizontal component of the cable force follows from

$$H = \frac{\frac{1}{8}q\ell^2}{p_k}. \tag{17b}$$

The slope of the cable is then

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\frac{1}{2}q\ell}{H} - \frac{qx}{H} + \frac{h}{\ell}. \tag{18}$$

At the supports A ($x = 0$) and B ($x = \ell$) the slope is

$$\left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)_A = \frac{h}{\ell} + \frac{\frac{1}{2}q\ell}{H},$$

$$\left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)_B = \frac{h}{\ell} - \frac{\frac{1}{2}q\ell}{H}.$$

Check: These expressions can also be determined directly from Figure 14.22, where the tangents to the parabola at A and B are shown. Using (17a) we find

$$\tan\alpha = +\left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)_{\mathrm{A}} = \frac{2p_{\mathrm{k}} + \frac{1}{2}h}{\frac{1}{2}\ell} = \frac{\frac{1}{2}q\ell}{H} + \frac{h}{\ell}, \tag{19a}$$

$$\tan\beta = -\left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)_{\mathrm{B}} = \frac{2p_{\mathrm{k}} - \frac{1}{2}h}{\frac{1}{2}\ell} = \frac{\frac{1}{2}q\ell}{H} - \frac{h}{\ell}, \tag{19b}$$

The maxixum sag in the cable appears where $\mathrm{d}z/\mathrm{d}x = 0$. Here the parabola has its vertex. Equation (18) gives

$$x_{\mathrm{vertex}} = \tfrac{1}{2}\ell + \frac{Hh}{q\ell} \tag{20a}$$

or, using (17b)

$$x_{\mathrm{vertex}} = \tfrac{1}{2}\ell + \frac{\frac{1}{8}hl}{p_{\mathrm{k}}}. \tag{20b}$$

If we select the coordinate system at the vertex of the parabola, as in Figure 14.23, the formulas are far easier. With the boundary conditions ($x = 0$; $z = 0$) and ($x = 0$; $\mathrm{d}z/\mathrm{d}x = 0$) the cable shape is

$$z = -\frac{\frac{1}{2}qx^2}{H}. \tag{21}$$

The slope of the cable is then

$$\frac{\mathrm{d}z}{\mathrm{d}x} = -\frac{qx}{H}. \tag{22}$$

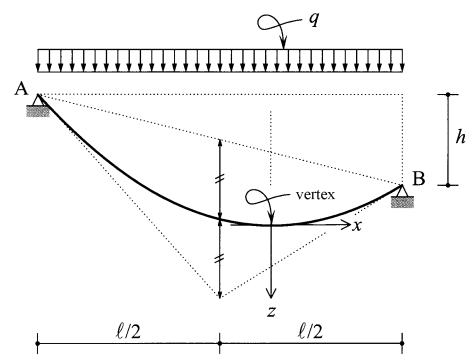


Figure 14.23 The origin of the xz coordinate system chosen at the vertex of the parabola.

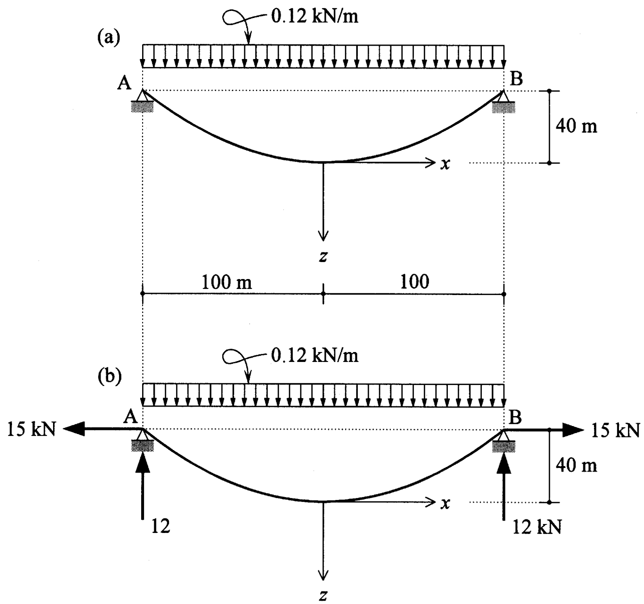


Figure 14.24 (a) Cable with the end supports at equal elevations, subject to a uniformly distributed load (force per horizontally measured length). (b) Support reactions.

The vertical component of the cable force is

$$V = H\frac{\mathrm{d}z}{\mathrm{d}x} = -qx. \tag{23}$$

For the tensile force in the cable we find

$$N = \sqrt{H^2 + V^2} = \sqrt{H^2 + (qx)^2}. \tag{24}$$

The use of these formulas is illustrated using an example.

Example

The cable in Figure 14.24a, with the end supports A and B at equal elevations, has a span of 200 metres and a sag of 40 metres at midspan. The cable carries a uniformly distributed load of 0.12 kN/m.

Questions:

a. Determine the horizontal component H of the cable force.
b. Determine the support reactions at A and B.
c. Determine the maximum cable force.

Solution:

a. The cable shape is a parabola of which the maximum sag, on the basis of symmetry, is at midspan. If we set the origin of the coordinate system here, then in accordance with (21)

$$H = -\frac{\frac{1}{2}qx^2}{z}.$$

Using the known coordinates of B ($x = 100$ m; $z = -40$ m) we find

$$H = -\frac{\frac{1}{2}(0.12\ \mathrm{kN/m})(100\ \mathrm{m})^2}{(-40\ \mathrm{m})} = 15\ \mathrm{kN}.$$

Of course we could also use the coordinates of A.

b. The horizontal support reactions at A and B are equal to H (see Figure 14.24b). The vertical support reactions A_v and B_v follow from the equilibrium of the cable as a whole. On the basis of symmetry, each support carries half of the total load:

$$A_v = B_v = \tfrac{1}{2}(200\text{ m})(0.12\text{ kN/m}) = 12\text{ kN }(\uparrow).$$

c. According to (24), the tensile force N in the cable is

$$N = \sqrt{H^2 + (qx)^2}.$$

The tensile force is a maximum at the supports A and B, with $x = \pm 100$ m

$$N_{\max} = \sqrt{(15\text{ kN})^2 + \{(0.12\text{ kN/m})(\pm 100\text{ m})\}^2} = 19.2\text{ kN}.$$

Check:

$$N_A = N_{\max}\sqrt{A_h^2 + A_v^2} = \sqrt{(15\text{ kN})^2 + (12\text{ kN})^2} = 19.2\text{ kN}.$$

Of course the same applies at B.

14.1.5 Cable subject to its dead weight; catenary

Figure 14.25 shows a cable under its uniformly distributed dead weight g. In the cable equation

$$H\frac{d^2z}{dx^2} = -q.$$

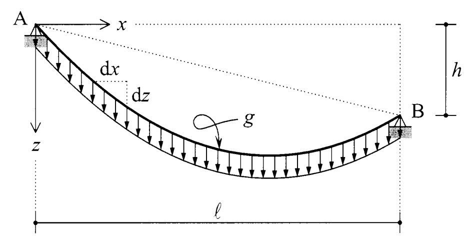


Figure 14.25 Cable loaded by its dead weight g (force per length measured along the cable).

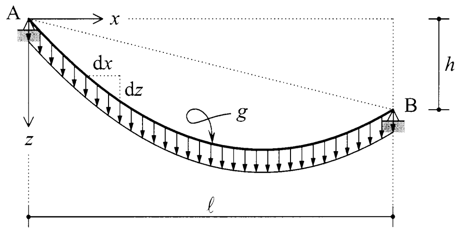


Figure 14.25 Cable loaded by its dead weight g (force per length measured along the cable).

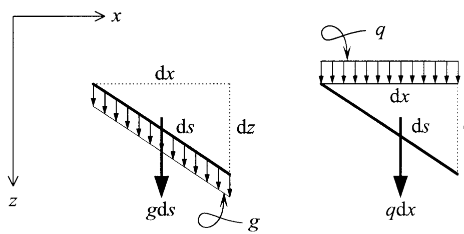


Figure 14.26 Replacing g (force per length measured along the cable) by q (force per horizontally measured length): $q = g\,\mathrm{d}s/\mathrm{d}x$.

q is a *vertical force per horizontally measured length*. The dead weight g of the cable is a *vertical force per length measured along the cable*.[1] In order to replace the dead weight g by the load q, Figure 14.26 shows an infinitesimally small cable element with length $\mathrm{d}s$. From the figure we find

$$g\,\mathrm{d}s = q\,\mathrm{d}x$$

and so

$$q = g\frac{\mathrm{d}s}{\mathrm{d}x} = g\frac{\sqrt{(\mathrm{d}x)^2 + (\mathrm{d}z)^2}}{\mathrm{d}x} = g\sqrt{1 + \left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)^2}.$$

The cable equation is now:

$$H\frac{\mathrm{d}^2 z}{\mathrm{d}x^2} = -g\sqrt{1 + \left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)^2}. \qquad (25)$$

To solve this second degree differential equation we assume:[2]

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \sinh u \qquad (26)$$

---

1 The symbol g is used for the dead weight of the cable, instead of the formal q_{dw}. By doing so, we avoid the recurring index "dw" and maintain the distinction with q (force per horizontally measured length). There should not be any confusion with the gravitational field strength g in this section.

2 The hyperbolic functions $\sinh(u)$ and $\cosh(u)$ are defined as follows:

$$\sinh(u) = \tfrac{1}{2}(\mathrm{e}^{+u} - \mathrm{e}^{-u}),$$

$$\cosh(u) = \tfrac{1}{2}(\mathrm{e}^{+u} + \mathrm{e}^{-u}).$$

in which u is a new variable. Substitute (26) in (25):

$$H\frac{\mathrm{d}}{\mathrm{d}x}(\sinh u) = -g\sqrt{1+\sinh^2 u}.$$

Calculating the terms on both sides of the equals sign gives

$$H\cosh u \cdot \frac{\mathrm{d}u}{\mathrm{d}x} = -g\cosh u$$

so that

$$H\frac{\mathrm{d}u}{\mathrm{d}x} = -g.$$

By integrating this first degree equation in u we find

$$u = -\frac{gx}{H} + C_1.$$

Substitution in (26) gives

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \sinh u = \sinh\left(-\frac{gx}{H} + C_1\right). \tag{27}$$

Integrating with respect to x gives

$$z = -\frac{H}{g}\cosh\left(-\frac{gx}{H} + C_1\right) + C_2. \tag{28}$$

The integration constants C_1 and C_2 follow from the boundary conditions.

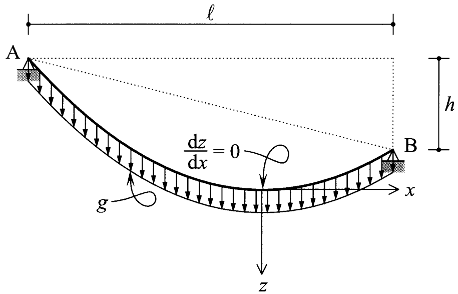


Figure 14.27 The origin of the xz coordinate system chosen at the point where $\mathrm{d}z/\mathrm{d}x = 0$.

If we choose the origin of the coordinate system at the point in the cable where $\mathrm{d}z/\mathrm{d}x = 0$, the boundary conditions are (see Figure 14.27)

$$x = 0; \quad z = 0,$$

$$x = 0; \quad \frac{\mathrm{d}z}{\mathrm{d}x} = 0.$$

With these boundary conditions, (27) and (28) give $C_1 = 0$ and $C_2 = H/g$ and the cable shape is[1]

$$z = -\frac{H}{g}\left(\cosh\frac{gx}{H} - 1\right). \tag{29}$$

The slope of the cable is

$$\frac{\mathrm{d}z}{\mathrm{d}x} = -\sinh\frac{gx}{H}. \tag{30}$$

The vertical component of the cable force is

$$V = H\frac{\mathrm{d}z}{\mathrm{d}x} = -H\sinh\frac{gx}{H}. \tag{31}$$

The tensile force in the cable is

$$N = \sqrt{H^2 + V^2} = H\sqrt{1 + \sinh^2\frac{gx}{H}} = H\cosh\frac{gx}{H}. \tag{32a}$$

---

1 Hereafter, we use the properties $\cosh(-u) = \cosh(+u)$ and $\sinh(-u) = -\sinh(+u)$.

According to (29)

$$H \cosh \frac{gx}{H} = H - gz$$

so that the tensile force can also be written as

$$N = H - gz. \tag{32b}$$

The use of the derived formulas is illustrated by an example.

Example
The cable in Figure 14.28a, of which the supports A and B are at equal elevations, has a span of 200 metres and a sag of 40 metres at the middle of the span. The cable is carrying only its dead weight of 0.12 kN/m.

Questions:
a. Determine the horizontal component H of the cable force.
b. Determine the maximum cable force.
c. Determine the support reactions at A and B.

Solution:
a. On the basis of symmetry, the cable is horizontal at midspan. If we assume here the origin of the coordinate system, the cable shape according to (29) would be

$$z = -\frac{H}{g}\left(\cosh \frac{gx}{H} - 1\right).$$

By substituting the known coordinates of A or B, we obtain an equation that allows us to calculate H. With the coordinates of B ($x = 100$ m; $z = -40$ m), for example, we find

$$(-40 \text{ m}) = -\frac{H}{(0.12 \text{ kN/m})}\left(\cosh \frac{(0.12 \text{ kN/m})(100 \text{ m})}{H} - 1\right).$$

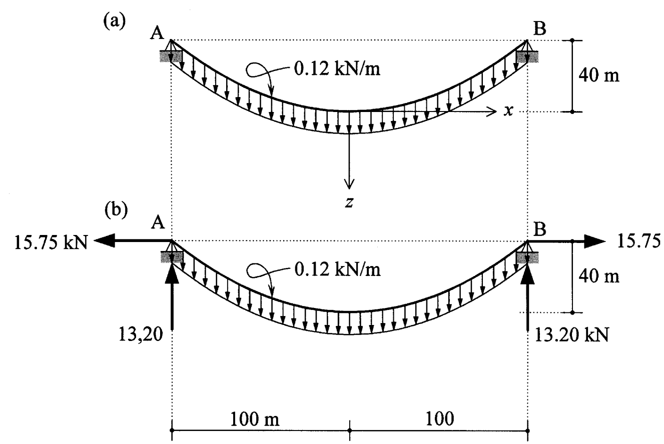


Figure 14.28 (a) Cable with the end supports at equal elevations, loaded by its dead weight (force per length measured along the cable). (b) Support reactions.

Table 14.1

H (kN)	f_1	f_2
15.00	1.320	1.337
15.25	1.315	1.326
15.50	1.310	1.315
15.75	1.305	1.305
16.00	1.300	1.295

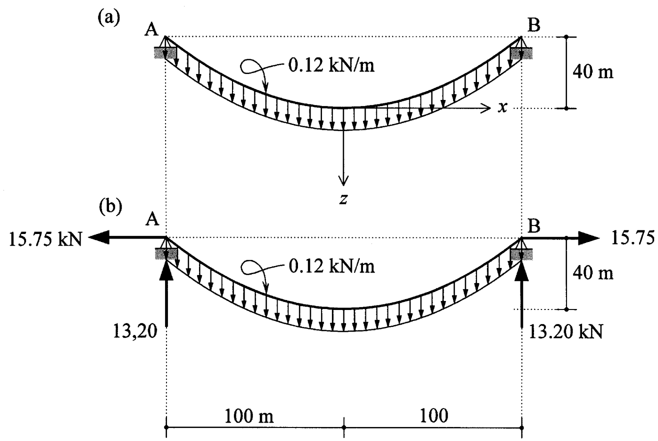


Figure 14.28 (a) Cable with the end supports at equal elevations, loaded by its dead weight (force per length measured along the cable). (b) Support reactions.

This can be converted into

$$1 + \frac{4.8\text{ kN}}{H} = \cosh\frac{12\text{ kN}}{H}.$$

To solve H we assume

$$f_1 = 1 + \frac{4.8\text{ kN}}{H}$$

and

$$f_2 = \cosh\frac{12\text{ kN}}{H}.$$

For various values of H we now calculate the function values f_1 and f_2. The calculation is performed in Table 14.1.

We are looking for the value of H for which $f_1 = f_2$. The table gives

$$H = 15.75\text{ kN}.$$

b. According to (32b), the tensile force in the cable is

$$N = H - gz.$$

The tensile force is a maximum at A and B, where $z = -40$ m:

$$N_{max} = (15.75\text{ kN}) - (0.12\text{ kN/m})(-40\text{ m}) = 20.55\text{ kN}.$$

c. The horizontal support reactions are equal to the horizontal component H of the tensile force in the cable (see Figure 14.28b):

$$A_h = B_h = H = 15.75\text{ kN}.$$

The vertical support reactions at A and B are equal to the vertical component V of the tensile force in the cable. According to (31)

$$V = -H \sinh \frac{gx}{H}.$$

At the supports, with $x = \pm 100$ m, we find (see Figure 14.28b)

$$A_\text{v} = B_\text{v} = |V_{x=\pm 100\text{ m}}|$$
$$= \left| -(15.75\text{ kN}) \sinh \frac{(0.12\text{ kN/m})(\pm 100\text{ m})}{15.75\text{ kN}} \right| = 13.20\text{ kN } (\uparrow).$$

Note: If we replace the uniformly distributed *load along the cable* by an equal uniformly distributed *load along the chord*, we obtain the situation of the example in Figure 14.24 (see Section 14.1.4). In that case the cable shape is a parabola. In Table 14.2, the results are compared for a parabola and a catenary, both with $\ell = 200$ m and $p_\text{k} = 40$ m.

The differences are relatively minor. In the example, the ratio between the sag and span is

$$\frac{p_\text{k}}{\ell} = \frac{40\text{ m}}{200\text{ m}} = 0.2.$$

The differences decrease sharply as the ratio p_k/ℓ decreases.

For *taut cables* ($p_\text{k}/\ell \ll 1$), the catenary can be approximated by a parabola, for which the distributed load *along the cable* is replaced by an equal distributed load *along the chord* (see Figure 14.29).

Table 14.2

forces in kN	parabola	catenary
H	15	15.75
V_{max}	12	13.20
N_{max}	19.2	20.55

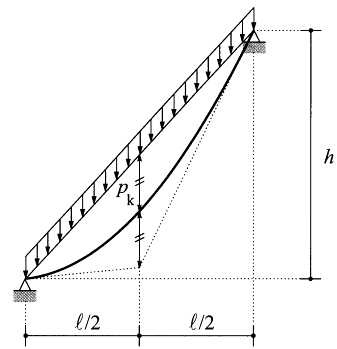


Figure 14.29 For taut cables ($p_\text{k}/\ell \ll 1$) the distributed load along the cable can be approximated by an equal distributed load along the chord.

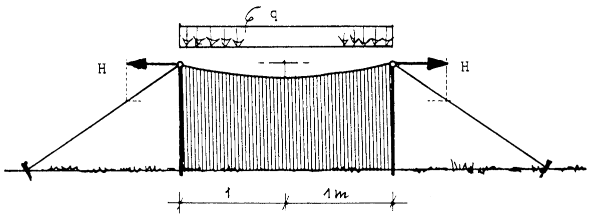


Figure 14.30 The load on the ridge of a shelter.

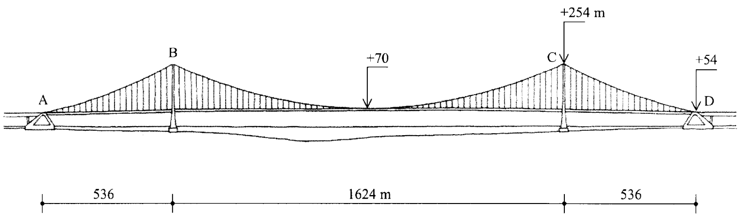


Figure 14.31 Suspension bridge over the Storebaelt (Large Belt) in Denmark.

14.1.6 Examples

Example 1

Figure 14.30 represents the longitudinal section of a shelter. The vertical load on the ridge is $q = 125$ N/m. The horizontal component of the force that the guys exert on the ridge is $H = 500$ N.

Question:
Determine the sag of the ridge in the middle of the shelter.

Solution:
The following applies for the sag:

$$p_{\mathrm{k}} = \frac{\frac{1}{8}q\ell^2}{H} = \frac{\frac{1}{8}(125\ \mathrm{N/m})(2\ \mathrm{m})^2}{500\ \mathrm{N}} = 125\ \mathrm{mm}.$$

Example 2

The dimensions for the suspension bridge in Figure 14.31 are derived from the bridge over the Storebaelt (Large Belt) in Denmark. The load on the cable, consisting of the dead weight of the cable, bridge deck and traffic load is set at 250 kN/m. The structure is designed in such a way that there is no bending in the towers.

Questions:

a. Determine the horizontal component H of the cable force in middle span BC.
b. Determine the maximum cable force in the middle span.
c. Determine the forces that cables BC and CD at C exert on the tower.
d. Determine the forces that cable CD at D exerts on the foundation block.
e. Determine the maximum cable force in end span CD.
f. Determine the ratio p_{k}/ℓ for the middle span and the end spans.

Solution:
Since the load is a force per horizontally measured length, the cable shapes in the middle span and the end spans are parabolas.

a. For middle span BC

$$p_\mathrm{k}^\mathrm{BC} = (254\ \mathrm{m}) - (70\ \mathrm{m}) = 184\ \mathrm{m}.$$

This gives

$$H^\mathrm{BC} = \frac{\frac{1}{8}q(\ell^\mathrm{BC})^2}{p_\mathrm{k}^\mathrm{BC}} = \frac{\frac{1}{8}(250\ \mathrm{kN/m})(1624\ \mathrm{m})^2}{184\ \mathrm{m}} = 448\ \mathrm{MN}.$$

b. The vertical component of the tensile force in cable BC is a maximum at B and C, as the cable is steepest here:

$$V_\mathrm{max}^\mathrm{BC} = \tfrac{1}{2}q\ell^\mathrm{BC} = \tfrac{1}{2}(250\ \mathrm{kN/m})(1624\ \mathrm{m}) = 203\ \mathrm{MN}.$$

The cable force is also a maximum here:

$$N_\mathrm{max}^\mathrm{BC} = \sqrt{(H^\mathrm{BC})^2 + (V_\mathrm{max}^\mathrm{BC})^2}$$
$$= \sqrt{(448\ \mathrm{MN})^2 + (203\ \mathrm{MN})^2} = 492\ \mathrm{MN}.$$

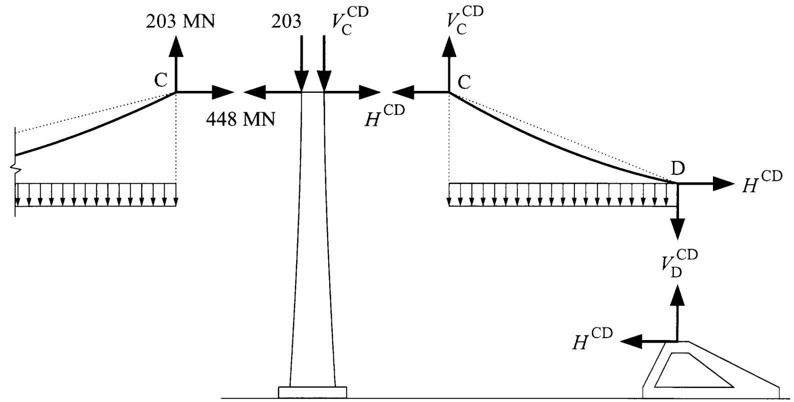


Figure 14.32 Cables BC and CD isolated from tower C and foundation block D.

c. In Figure 14.32, cables BC and CD have been isolated at C from the tower. If there is no bending in the tower, the resulting horizontal force on the tower must be zero. This means that the horizontal component of the cable force in an end span is equal to that in the middle span:

$$H^\mathrm{CD} = H^\mathrm{BC} = 448\ \mathrm{MN}.$$

In Figure 14.33, cable CD has been isolated. The resultant R of the uniformly distributed load is

$$R = (250\ \mathrm{kN/m})(536\ \mathrm{m}) = 134\ \mathrm{MN}.$$

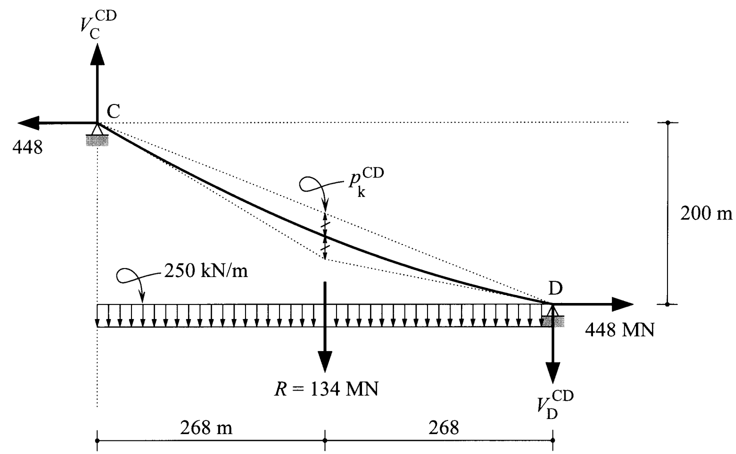


Figure 14.33 The isolated cable CD.

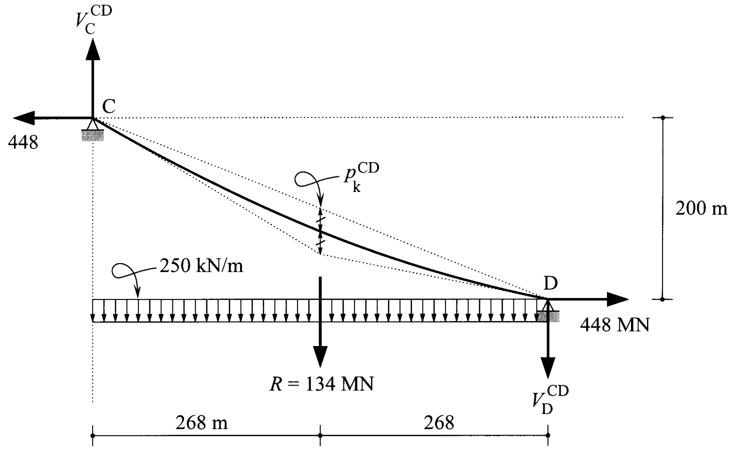


Figure 14.33 The isolated cable CD.

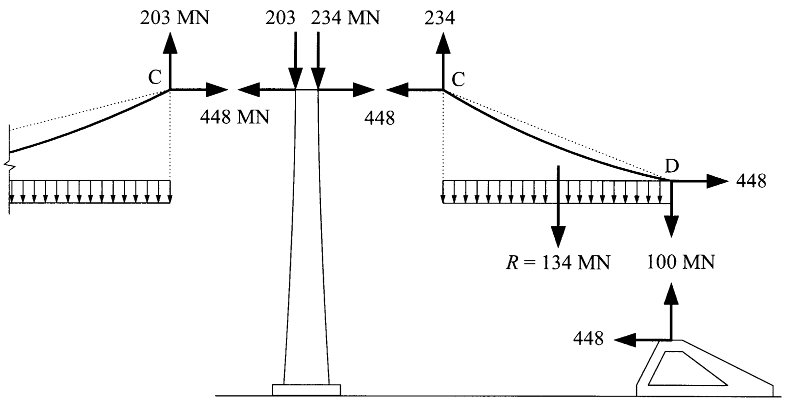


Figure 14.34 The forces on tower C and foundation block D.

The moment equilibrium of cable CD gives

$$\sum T|D \circlearrowright = (448\text{ MN})(200\text{ m}) - V_C^{CD}(536\text{ m}) + (134\text{ MN})(268\text{ m}) = 0$$

so that

$$V_C^{CD} = \frac{(448\text{ MN})(200\text{ m}) + (134\text{ MN})(268\text{ m})}{536\text{ m}} = 234\text{ MN}.$$

Next we find from the vertical force equilibrium

$$V_C^{CD} = V_C^{CD} - R = (234\text{ MN}) - (134\text{ MN}) = 100\text{ MN}.$$

In Figure 14.34, cables BC and CD have been isolated from the tower at C and the foundation block at D. The tower is loaded at C by a vertical compressive force:

$$V_C^{BC} + V_C^{CD} = (203\text{ MN}) + (234\text{ MN}) = 437\text{ MN}.$$

d. The foundation block in D is subject to a horizontal force of 448 MN and an upward vertical force of 100 MN (see Figure 14.34).

e. The maximum force in cable CD occurs at C, where the cable is steepest:

$$N_{\max}^{CD} = \sqrt{(H^{CD})^2 + (V_{\max}^{CD})^2} = \sqrt{(448\text{ MN})^2 + (234\text{ MN})^2} = 505\text{ MN}.$$

f. For middle span BC, it applies that

$$\frac{p_k^{BC}}{\ell^{BC}} = \frac{184 \text{ m}}{1624 \text{ m}} = 0.113.$$

The maximum (vertically measured) distance from cable CD to the chord is

$$p_k^{CD} = \frac{\frac{1}{8}q(\ell^{CD})^2}{H^{CD}} = \frac{\frac{1}{8}(250 \text{ kN/m})(536 \text{ m})^2}{448 \times 10^3 \text{ kN}} = 20 \text{ m}$$

so that for the end spans the following applies:

$$\frac{p_k^{CD}}{\ell^{CD}} = \frac{20 \text{ m}}{536 \text{ m}} = 0.037.$$

Example 3

Cable AB in Figure 14.35 has a span of 60 m and is carrying a number of pipelines with a total weight of 20 kN/m. The difference in elevation of the end supports at A and B is 12 m. C is the lowest point of the cable and is 4 m below B.

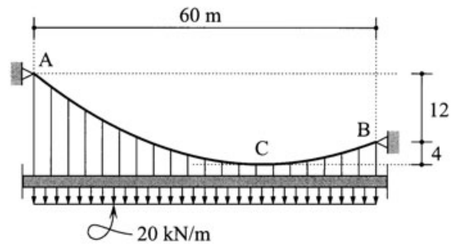


Figure 14.35 Cable AB is carrying a number of pipelines with a weight of 20 kN/m.

Questions:

a. Determine the horizontal component of the cable force.
b. Determine the support reactions at A and B.
c. Determine the maximum cable force.

Solution:

a. We can assume a coordinate system at A or C, and use the formulas derived above. Here we will use a different approach. In Figure 14.36, cable parts AC and BC have been isolated and all acting forces are shown. At C, the cable is horizontal and only force H acts. The lengths ℓ_A and ℓ_B are still unknown. The moment equilibrium of AC about A gives

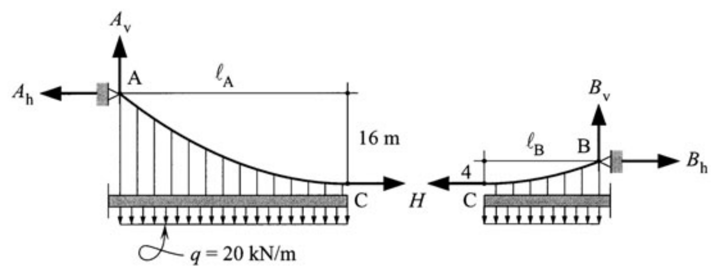


Figure 14.36 Cable parts AC and BC isolated at the lowest point C.

$$H \times (16 \text{ m}) = \tfrac{1}{2}q\ell_A^2. \qquad \text{(a)}$$

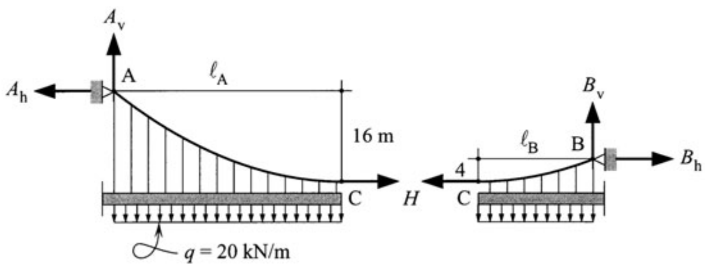


Figure 14.36 Cable parts AC and BC isolated at the lowest point C.

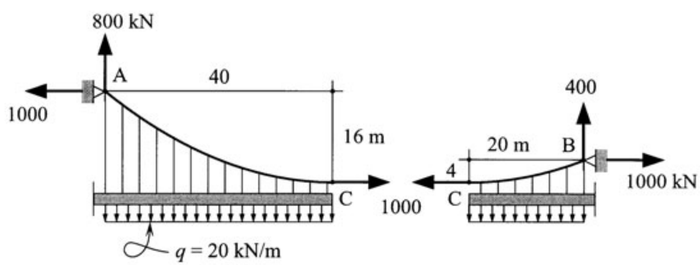


Figure 14.37 The isolated cable sections AC and BC with their dimensions and support reactions.

The moment equilibrium of BC about B gives

$$H \times (4 \text{ m}) = \tfrac{1}{2}q\ell_B^2. \qquad \text{(b)}$$

From (a) and (b) we can derive

$$\frac{\ell_A^2}{\ell_B^2} = 4 \Rightarrow \ell_A = 2\ell_B.$$

With $\ell_A + \ell_B = 60$ m we find

$$\ell_A = 40 \text{ m},$$

$$\ell_B = 20 \text{ m}.$$

Substituting $\ell_A = 40$ m in (a) gives, with $q = 20$ kN/m,

$$H = \frac{\frac{1}{2}q\ell_A^2}{16 \text{ m}} = \frac{\frac{1}{2}(20 \text{ kN/m})(40 \text{ m})^2}{16 \text{ m}} = 1000 \text{ kN}.$$

b. The horizontal support reactions follow from the horizontal force equilibrium of AC and CB (see Figure 14.37):

$$A_h = B_h = H = 1000 \text{ kN}.$$

The vertical support reactions follow from the vertical force equilibrium of AC and CB (see Figure 14.37):

$$A_v = q\ell_A = (20 \text{ kN/m})(40 \text{ m}) = 800 \text{ kN},$$

$$B_v = q\ell_B = (20 \text{ kN/m})(20 \text{ m}) = 400 \text{ kN}.$$

c. The maximum cable force occurs at A, where the slope of the cable is steepest:

$$N_{\max} = N_{\mathrm{A}} = \sqrt{A_{\mathrm{h}}^2 + A_{\mathrm{v}}^2} = \sqrt{(1000\ \mathrm{kN})^2 + (800\ \mathrm{kN})^2} = 1281\ \mathrm{kN}.$$

Example 4

A boat has cast its anchor in 30 m deep water (see Figure 14.38). The horizontal force that the boat exerts on the anchor chain is 3.5 kN. The anchor chain has a dead weight of 24 N/m. The upward water pressure on the chain is 3 N/m.

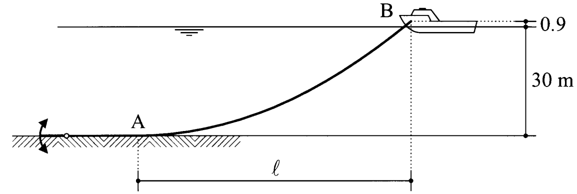


Figure 14.38 A boat has cast its anchor in 30 m deep water.

Question:
Determine the (horizontally measured) length ℓ for which the chain is free from the bottom, and the maximum force in the chain.

Solution:
The (uniformly) distributed load on the chain is equal to the dead weight minus the upward water pressure:

$$q = (24\ \mathrm{N/m}) - (3\ \mathrm{N/m}) = 21\ \mathrm{N/m}.$$

This load is a force per length measured along the chain. The anchor chain will therefore assume the shape of a catenary.[1] In Figure 14.39, the anchor chain has been isolated. At A, the cable is tangent to the bottom, and only a horizontal force H acts. The horizontal force equilibrium gives

$$H = 3.5\ \mathrm{kN}.$$

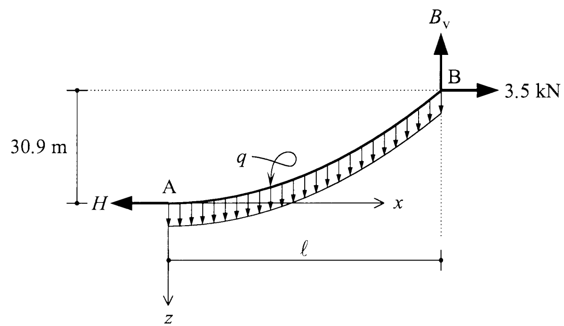


Figure 14.39 The isolated anchor chain with a uniformly distributed load along the chain. The anchor chain has the shape of a catenary.

For the further calculation, we use the formulas derived in Section 14.1.5. For the catenary in the coordinate system given in Figure 14.39, it applies

[1] We ignore the fact that the upward water pressure is missing over the small part that the chain is above the water.

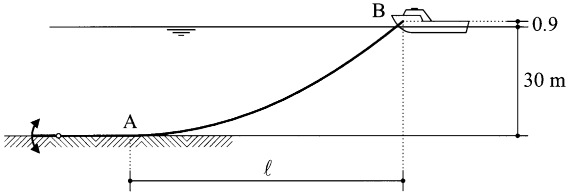


Figure 14.38 A boat has cast its anchor in 30 m deep water.

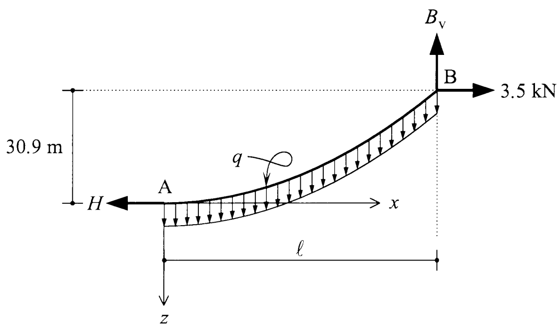


Figure 14.39 The isolated anchor chain with a uniformly distributed load along the chain. The anchor chain has the shape of a catenary.

that

$$z = -\frac{H}{q}\left(\cosh\frac{qx}{H} - 1\right)$$

or

$$\cosh\frac{qx}{H} = 1 - \frac{qz}{H}$$

so that

$$x = \frac{H}{q}\cosh^{-1}\left(1 - \frac{qz}{H}\right).$$

At B, $x = \ell$; $z = -30.9$ m applies, which gives the following (be aware of the units!):

$$\ell = \frac{3500\text{ N}}{21\text{ N/m}}\cosh^{-1}\left(1 - \frac{(21\text{ N/m})(-30.9\text{ m})}{3500\text{ N}}\right) = 100\text{ m}$$

For the vertical force at B

$$B_v = -V_{x=\ell} = H\sinh\frac{q\ell}{H}$$
$$= (3500\text{ N})\sinh\left(\frac{21\text{ N/m})(100\text{ m})}{3500\text{ N}}\right) = 2228\text{ N} \approx 2.23\text{ kN}.$$

The maximum force in the anchor chain is

$$N_{max} = N_B = \sqrt{H^2 + B_v^2} = \sqrt{(3.5\text{ kN})^2 + (2.23\text{ kN})^2} = 4.15\text{ kN}.$$

Alternative solution:
The load q is a *vertical force measured per length along the cable*. If the an-

chor chain is taut, this load can be approximated by an equal *vertical force per length measured along the chord* (see Figure 14.40). The associated shape of the anchor chain is then a parabola.

The resultant of the distributed load is

$$R = q\sqrt{\ell^2 + h^2}.$$

The moment equilibrium of the isolated chain gives

$$\sum T|B = Hh - R \cdot \tfrac{1}{2}\ell = Hh - \tfrac{1}{2}q\ell\sqrt{\ell^2 + h^2} = 0$$

so that

$$\frac{2Hh}{q} = \ell\sqrt{\ell^2 + h^2}.$$

After squaring

$$\left(\frac{2Hh}{q}\right)^2 = \ell^2(\ell^2 + h^2)$$

we find

$$\ell^4 + h^2\ell^2 - \left(\frac{2Hh}{q}\right)^2 = 0.$$

With $h = 30.9$ m, $H = 3500$ N and $q = 21$ N/m this leads to

$$\ell^4 + (30.9\text{ m})^2\ell^2 - \left(\frac{2 \times (3500\text{ N})(30.9\text{ m})}{21\text{ N/m}}\right)^2 = 0$$

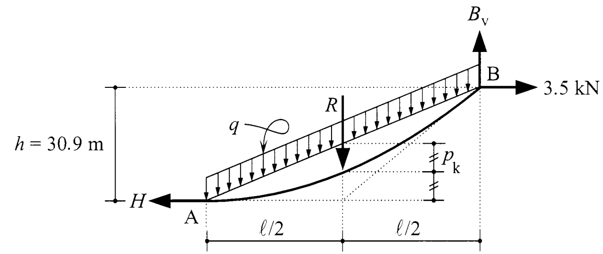


Figure 14.40 If the anchor chain is taut ($p_k/\ell \ll 1$), the distributed load along the chain can be replaced by an equal distributed load along the chord. The anchor chain now has the shape of a parabola.

so that

$$\ell^4 + (9.54.81\ \text{m}^2)\ell^2 - (106.09 \times 10^6\ \text{m}^4) = 0.$$

The solution is

$$\ell^2 = 9.834 \times 10^3\ \text{m}^2,$$
$$\ell = 99.17\ \text{m}.$$

We find for the vertical force at B:

$$B_\text{v} = R = q\sqrt{\ell^2 + h^2}$$
$$= (21\ \text{N/m})\sqrt{(99.17\ \text{m})^2 + (30.9\ \text{m})^2} = 2181\ \text{N} \approx 2.18\ \text{kN}.$$

This gives the maximum force in the chain:

$$N_\text{max} = N_\text{B} = \sqrt{H^2 + B_\text{v}^2} = \sqrt{(3.5\ \text{kN})^2 + (2.18\ \text{kN})^2} = 4.13\ \text{kN}.$$

The values found for ℓ and N_max deviate some 0.5% from those found using the exact calculation. The load *along the chord* (and a parabolic shape of the anchor chain) is therefore a good substitute for the load *along the anchor chain* (and a catenary). The ratio p_k/ℓ is

$$\frac{p_\text{k}}{\ell} = \frac{(30.9\ \text{m})/4}{99.17\ \text{m}} = 0.078 \ll 1.$$

Example 5
A simply supported concrete beam with length $\ell = 12$ m and a rectangular cross-section $A = bh = 300 \times 800\ \text{mm}^2$ carries a variable load $q_\text{q} = 16$ kN/m (see Figure 14.41). The dead weight of concrete is 25 kN/m^3.

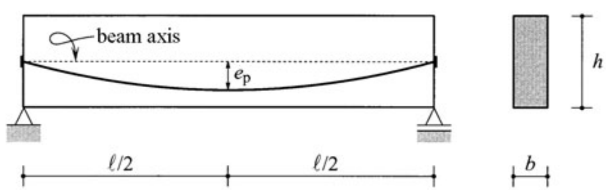


Figure 14.41 A concrete beam with a parabolic tendon.

Parabolic *post-tensioned cables* have been applied to the beam. After the concrete has been poured and hardened, the cables are tensioned by means of screw jacks, and anchored at the beam ends. The anchors are located at the beam axis. At midspan, the cables have an eccentricity of $e_p = 240$ mm with respect to the beam axis.

The (total) prestressing force in the cables is $F_p = 1050$ kN.[1]

Question:
Determine and draw the support reactions and the M diagram and V diagram the *prestressed beam* as a result of the following:

a. the dead weight only;
b. the dead weight and variable load.

Solution:
Before being tensioned, the cables are located in cylindrical canals. When the cables are tensioned, they will be pressed against the upper side of the canals (see Figure 14.42). The cables are in equilibrium because the beam exerts on the cables a uniformly distributed load q_p. In Figure 14.43 the two post-tensioned cables are replaced by a single tendon and all the forces acting on it are shown.

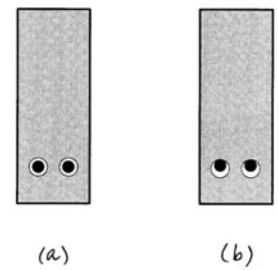


Figure 14.42 The location of the prestressing cables in their canals: (a) before and (b) after tensioning.

From the parabolic shape of the tendon we find

$$\tan\alpha = \frac{2e_p}{\frac{1}{2}\ell} = \frac{4e_p}{\ell} = \frac{4 \times (0.24 \text{ m})}{12 \text{ m}} = 0.08.$$

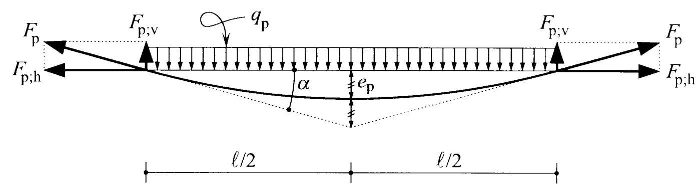


Figure 14.43 The isolated parabolic tendon with all the forces acting on it. The tangents to the tendon are shown at the ends.

The prestressing force F_p has components:

$$F_{p;h} = F_p \cos\alpha = 0.0968 \times F_p = -0.9968 \times (1050 \text{ kN}) = 1046.6 \text{ kN},$$

$$F_{p;v} = F_p \sin\alpha = 0.0797 \times F_p = -0.0797 \times (1050 \text{ kN}) = 83.7 \text{ kN}.$$

[1] The index p refers to prestressing.

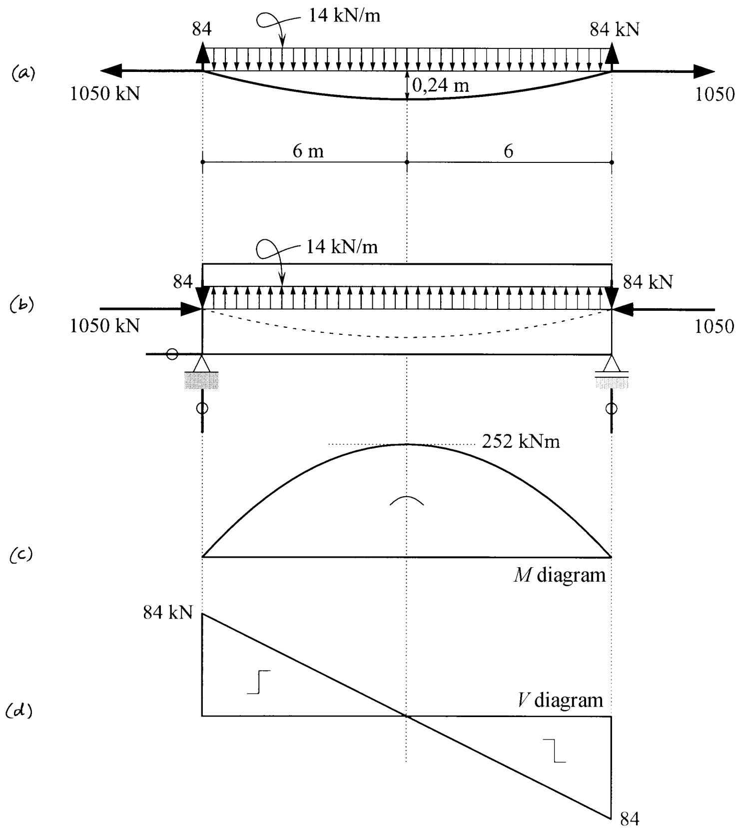


Figure 14.44 (a) The forces that the beam exerts on the tendon. (b) The forces that the tendon exerts on the beam. (c) The bending moment diagram and (d) shear force diagram of the beam due to the prestressing.

Since for prestressing cables in general $e_p/\ell \ll 1$, α is very small, so that we can make the following approximations (see Figure 14.43):

$$\cos\alpha \approx 1,$$
$$\sin\alpha \approx \tan\alpha = \frac{4e_p}{\ell} = 0.08.$$

In that case

$$F_{p;h} = F_p = 1050\ \text{kN},$$
$$F_{p;v} = \frac{4e_p}{\ell} F_p = 0.08 \times (1050\ \text{kN}) = 84\ \text{kN}.$$

We will use these values in further calculations.

In the tendon

$$F_{p;h}e_p = \tfrac{1}{8} q_p \ell^2$$

so that

$$q_p = \frac{8F_{p;h}e_p}{\ell^2} = \frac{8 \times (1050\ \text{kN})(0.24\ \text{m})}{12\ \text{m})^2} = 14\ \text{kN/m}.$$

In Figure 14.44a all the forces acting on the isolated tendon are shown again, this time with their values. All these forces are exerted by the concrete beam on the tendon: the concentrated forces via the anchors, the distributed forces directly via the beam. On the basis of the principle of action and reaction, the concrete beam is subject to equal and opposite forces (see Figure 14.44b). In Figures 14.44c and 14.44d, the associated M and V diagrams of the beam are shown. Since the forces form an equilibrium system (the vertical anchor forces are in equilibrium with the vertical distributed forces), the support reactions at A and B are zero, and not equal to the shear force here.

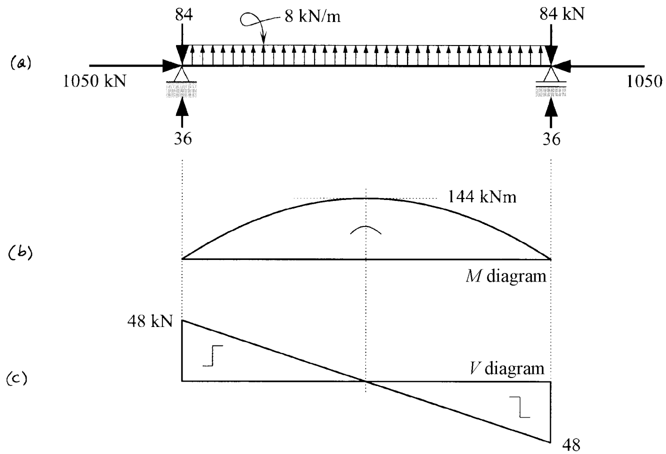


Figure 14.45 (a) All the forces acting on the beam due to the prestressing and dead weight. (b) The associated bending moment diagram and (c) the shear force diagram.

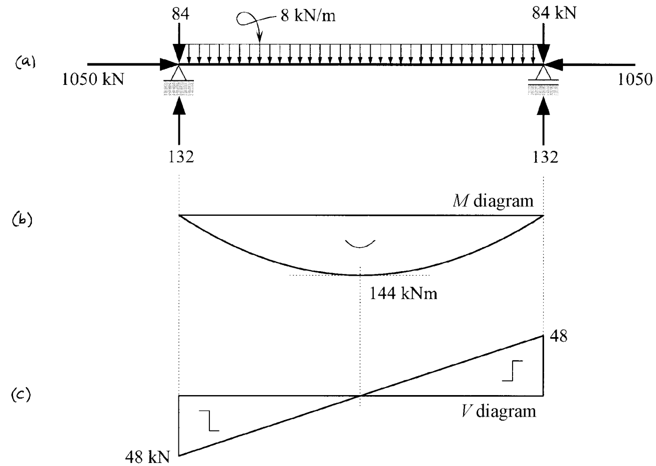


Figure 14.46 (a) All the forces acting on the beam due to the prestressing, dead weight and variable load. (b) The associated bending moment diagram and (c) the shear force diagram.

a. Figure 14.45a shows the forces due to the prestressing and dead weight for the beam modelled as a line element. For the dead weight, it applies that

$$q_{\text{dw}} = (0.3 \text{ m})(0.8 \text{ m})(25 \text{ kN/m}^3) = 6 \text{ kN/m}.$$

The resulting distributed load acts upwards:

$$q_{\text{p}}\ (\uparrow) + q_{\text{dw}}\ (\downarrow) = (14 - 6)(\text{kN/m})\ (\uparrow) = 8 \text{ kN/m}\ (\uparrow).$$

Figures 14.45b and 14.45c show the associated M and V diagrams.

b. Figure 14.46a shows the forces on the beam modelled as a line element due to the prestressing, dead weight and variable load. The resulting

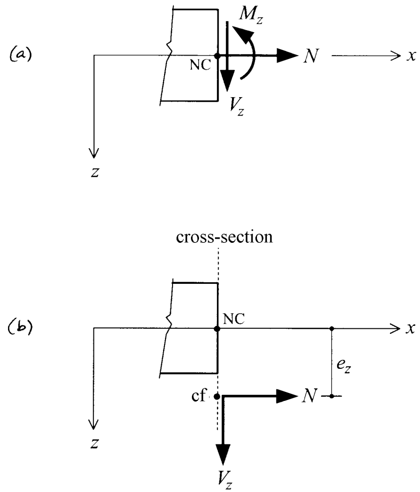


Figure 14.47 (a) The bending moment M_z and normal force N at the normal force centre NC are statically equivalent to (b) a single force N at the centre of force cf, with an eccentricity $e_z = M_z/N$ with respect to the member axis.

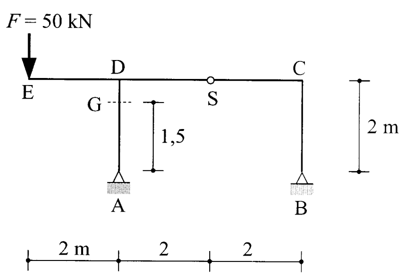


Figure 14.48 Three-hinged frame loaded by a vertical force at E.

distributed load is now acting downwards:

$$(q_{\mathrm{dw}} + q_{\mathrm{q}})\ (\downarrow) + q_{\mathrm{p}}\ (\uparrow) = (6 + 16 - 14)(\mathrm{kN/m})\ (\downarrow) = 8\ \mathrm{kN/m}\ (\downarrow).$$

Figures 14.46b and 14.46c show the associated M and V lines.

Note that in both Figure 14.45 and 14.46 the shear forces directly adjacent to the supports are not equal to the support reactions.

14.2 Centre of force and line of force

In a cross-section (cs) acts a bending moment M_z, normal force N and shear force V_z (see Figure 14.47a). The shear force is the resultant of all shear stresses in the cross-section; the bending moment and the normal forces are the resultants of the normal stresses (see also Section 10.1).

Assume that the resultant of all the normal stresses is a force N with eccentricity e_z with respect to the member axis (see Figure 14.47b). By shifting the normal force N normal to its line of action to the normal centre NC on the member axis, we create the bending moment M_z in Figure 14.47a:

$$M_z = N e_z.$$

The point in the cross-section where the resultant of all the normal stresses is transferred is known as the *centre of force* (cf). The centre of force can also be described as the intersection of the cross-section with the line of action of the resultant of all the forces that the cross-section has to transfer (see Section 10.1.1). This will be clarified in the two examples at the end of the section.

For the z coordinate of the centre of force, indicated by means of e_z, it holds that

$$e_z = \frac{M_z}{N}.$$

The centres of force at all consecutive cross-sections together form a line known as the *line of force*.

If the normal force is a tensile force ($N > 0$), we also refer to *centres of tension* and *lines of tension* instead of centres of force and lines of force. For a compressive force ($N < 0$), we refer to *centres of pressure* and *lines of pressure*.

The following can be said about lines of force:

- There is no line of force in areas where the normal force is zero. For $N \to 0$ it always holds that $e_z \to \infty$, and the centre of force is at infinity.
- If the bending moment is zero, $e_z = 0$ applies, and the centre of force is on the member axis. This means that in structures with hinges, the line of force always passes through the hinges, as the bending moment is zero there.
- Where the line of force intersects the member axis (or coincides with it), $e_z = 0$ and the bending moment is zero.

Example 1
Figure 14.48 shows a three-hinged portal frame, loaded by a vertical force $F = 50$ kN at E. Figure 14.49a shows the support reactions and Figures 14.49b and 14.49c shows the M and N diagrams. The calculation is left to the reader.

In Figure 14.49a, the lines of action of force F and the support reactions at A and B are shown. These lines of action intersect in one point. This offers a graphical check of the moment equilibrium of the three-hinged frame (see Section 5.3, Example 1).

Questions:

a. Determine the centre of force in cross-section G of post AD, 1.50 m above support A.
b. Determine the line of force for the parts AD, THE, DSC and BC.

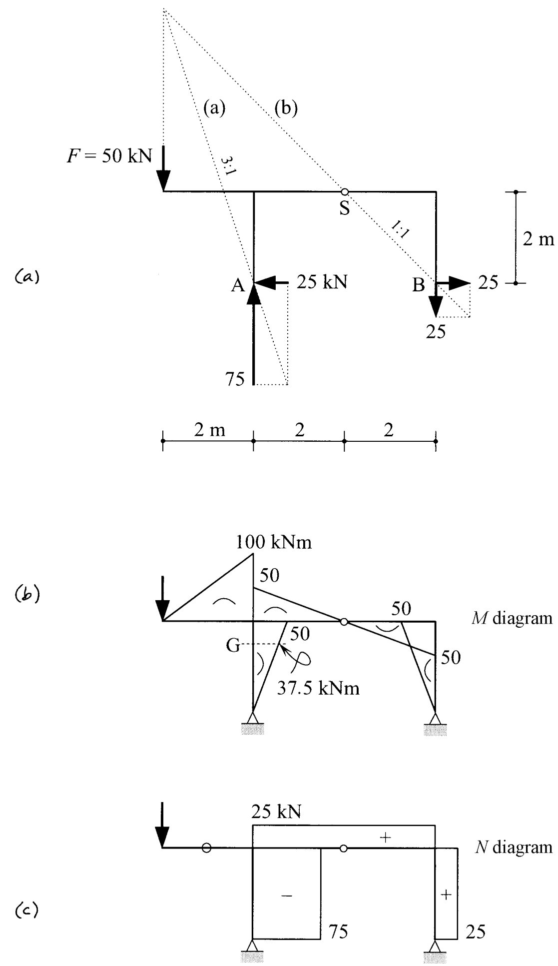


Figure 14.49 (a) Support reactions and line of action figure. (b) Bending moment diagram. (c) Normal force diagram.

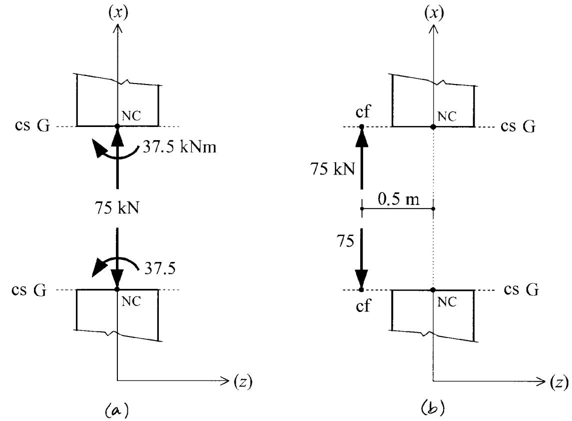


Figure 14.50 (a) Bending moment and normal force at cross-section G. They are statically equivalent to (b) the eccentric compressive forces to the left of the member axis.

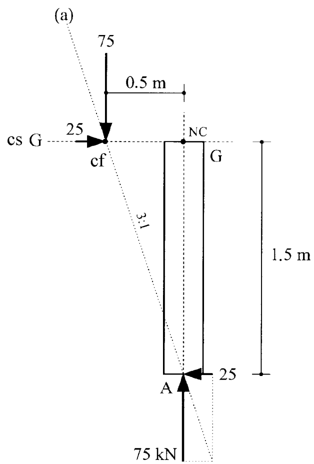


Figure 14.51 Isolated part AG. The centre of force cf in cross-section G is located on the line of action of the support reaction at A; that is, the force cross-section G has to transfer.

Solution:

a. Figure 14.50a shows the bending moment and the normal force at cross-section (cs) G. From the M diagram for post AD we can derive that the bending moment at the cross-section is $(1.5/2.0) \times (50\text{ kNm}) = 37.5\text{ kNm}$, with tension at the "inside" of the frame. From the N diagram it follows that there is a compressive force of 75 kN at the cross-section.

The section forces in Figure 14.50a are statically equivalent to the eccentric compressive forces in Figure 14.50b. The centre of force (cf) will be to the left of the member axis since a compressive force to the left of the member axis causes (with respect to the normal force centre NC) a moment that has the same direction as the bending moment in Figure 14.50a. The magnitude of the eccentricity e is:

$$e = \frac{|M|}{|N|} = \frac{37.5\text{ kNm}}{75\text{ kN}} = 0.5\text{ m}.$$

The location of the centre of force can also be calculated formally in a local coordinate system. In order to find the correct sign for e_z we do have to use the correct signs for N and M_z. For the xz coordinate system in Figure 14.50

$$M_z = +37.5\text{ kNm} \quad \text{and} \quad N = -75\text{ kN}$$

so that

$$e_z = \frac{M_z}{N} = \frac{+37.5\text{ kNm}}{-75\text{ kN}} = -0.5\text{ m}.$$

The centre of force is indeed to the left of the member axis.

b. In Figure 14.51, the part AG has been isolated. The support reactions act at A. If there is a equilibrium, a force has to act at cross-section G that has the same magnitude as the resulting force at A, and has the same line of action, but has to act in the opposite direction. In other words, the centre of force at cross-section G is located on the line of action (a) of the resulting force at A (see also Figure 14.49a). This leads to the following statement: the centre of force is the intersection of the cross-section with the line of action of the resultant of all forces that the cross-section has to transfer.

Since all the cross-sections in AD have to transfer the same resulting force at A, the centres of force in those cross-sections are on the same line of action (a). The line of force for AD therefore coincides with line of action (a) (see Figure 14.52a). Since the normal force in AC is a compressive force, the line of force is a line of pressure.

In the same way, the line of force of BC coincides with line of action (b) of the support reaction at B (see Figure 14.52a). Since the normal force is a tensile force, the line of force is here a line of tension.

If we look at a section in girder DSC, this, seen from the left, has to transfer the resultant of force F and the support reactions at A, and, seen from the right, the support reaction at B. Both have the same line of action (b), as shown by the line of action figure (see Figure 14.52b). The line of force for DSC coincides with line of action (b) and is a line of tension.

Since the normal force is zero, there exists no line of force for DE. All cross-sections between D and E have to transfer the same vertical force F. The line of action of F is parallel to the cross-sections, so that there are no intersections and therefore no centres of force.

Note: If the normal force in a beam is constant, the figure that is enclosed between the line of force and the member axis has the same shape as the M diagram. This is not surprising, for[1]

$$M = Ne.$$

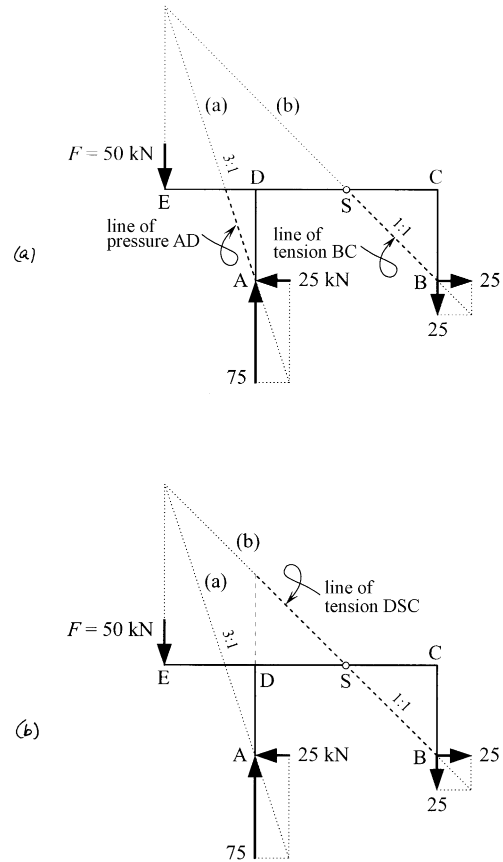


Figure 14.52 (a) The lines of force for AD and BC coincide with the lines of action of the support reactions at A and B respectively. (b) The line of force for DSC coincides with the line of action of the support reaction at B.

1 Without taking into account the coordinate system and signs.

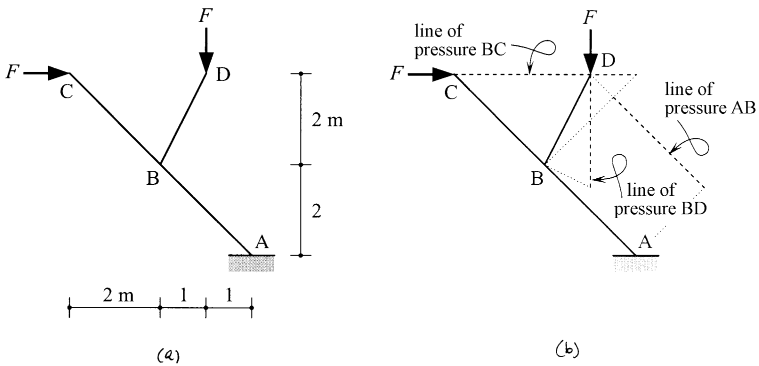


Figure 14.53 (a) Fixed bar type structure, loaded by two forces, with (b) the lines of force.

The scale factor is N. If N is a tensile force, the line of force is on the same side of the member axis as the M diagram. If N is a compressive force, the line of force and the M diagram are not on the same side. It is up to the reader to verify this, using the bending moment diagram in Figure 14.49b and the lines of force in Figure 14.52.

Example 2

The structure ABCD in Figure 14.53a is fixed at A, and loaded by a horizontal force F at C and a vertical force F at D.

Question:
Determine the lines of force for AB, BC and BD.

Solution:
The lines of force are shown in Figure 14.53b.

All cross-sections between C and B have to transfer the horizontal force F. The line of force for BC therefore coincides with the line of action of this horizontal force.

All cross-sections between D and B have to transfer the vertical force F. The line of force for BD coincides with the line of action of this vertical force.

All cross-sections between B and A have to transfer the resultant of the forces F at C and D. The line of force for AB coincides with the line of action of this resultant. The line of action figure shows that this line of action passes through D and is parallel to ABC.

Since the normal force is a compressive force everywhere, all the lines of force are lines of pressure.

14.3 Relationship between cable, line of force and structural shape

The bending moment and the normal force are the resultants of the normal stresses in a cross-section. Figure 14.54 shows the stress distribution due to a bending moment M and a normal force N.[1]

A characteristic of stress distribution in bending is that the outermost fibres of the cross-section are most heavily loaded, while the fibres in the environment of the member axis are virtually unloaded. In contrast, the stresses due to a normal force are constant over the cross-section. In extension, all fibres are therefore loaded equally.

If we compare the stress distributions due to bending and extension, the material in the cross-section is used far more efficiently in extension than in bending. With bending, the strength capacity of the fibres around the member axis is not used, and the small stresses only marginally contribute to the bending moment. For beams loaded by bending, one often sees an adaptation of the cross-section by omitting the less active material in the cross-section. In this way, a rectangular cross-section may become a *tubular section* or an I-section (see Figure 14.55).

In addition, when designing structures, designers look for shapes in which the bending moments remain as small as possible, and in which the force flow preferably occurs by extension. This is achieved by ensuring the member axis and line of force coincide as much as possible.

Since a cable cannot transfer bending moments, it assumes a shape in which the line of force coincides with its axis everywhere. Taking the cable shape and line of force as basis, in the following four examples, we look for

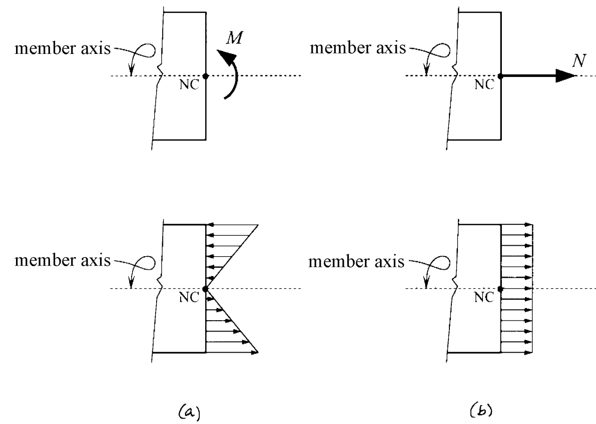


Figure 14.54 The distribution of normal stresses in a cross-section due to (a) a bending moment M and (b) a normal force N.

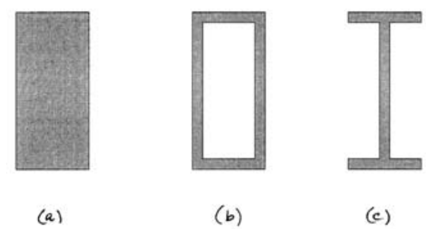


Figure 14.55 If a beam with (a) a rectangular cross-section is loaded by bending, the fibres around the member axis remain virtually unloaded. With (b) a tubular section or (c) an I-section the material is more effectively distributed across the cross-section.

[1] In Volume 2, *Stresses, Deformations, Displacements*, we take a closer look at the exact development of the normal stresses in a cross-section and at the conditions under which the stress distribution in Figure 14.54 applies.

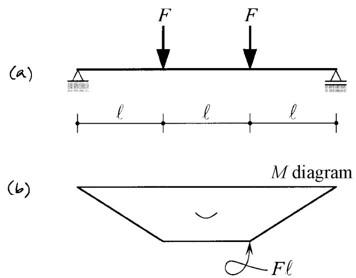


Figure 14.56 (a) A beam subject to bending by two forces with (b) the associated bending moment diagram.

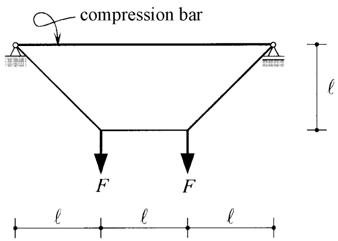


Figure 14.57 Cable with compression bar loaded by two forces. All the parts are subject to extension (tension and compression).

structural shapes in which the bending moments are as small as possible.

Example 1
The beam in Figure 14.56a is subject to bending by the two forces F. The M diagram is shown in Figure 14.56b.

In Figure 14.57, the same load is carried by a cable with compression bar. The cable and compression bar transfer normal forces only. The cable has the same shape as the M diagram in Figure 14.56b. With a cable sag the scale factor is

$$H = \frac{F\ell}{\ell} = F.$$

H is the (compressive) force in the bar that is equal to the horizontal component of the (tensile) force in the cable.

In Figure 14.58a the straight cable parts have been replaced by bars. Plus and minus signs indicate whether the bar forces are tensile or compressive. The structure can be considered a kind of *arch under tension* that is kept together by a *compression bar*. If the bar structure in Figure 14.58a is "turned over" with equal loads as shown in Figure 14.58b, all the signs in the bars change. The structure has now changed into an *arch under compression* with tension bar (*tie rod*).

In the position shown in Figure 14.58b, the bar structure is in equilibrium. However, the equilibrium is unstable (unreliable): a small change in position will cause the equilibrium to fail and the bar structure will collapse.[1] The bar structure is kinematically indeterminate. This collapse can be pre-

[1] To prove this we have to investigate the equilibrium of the structure in its deformed state. However, this topic is beyond the scope of this book. Here we assume that the reader is acquainted with this phenomenon of instbility on the basis of some practical experience.

vented by making the structure kinematically determinate, for example by introducing bracing members, and changing the bar structure into a truss (see Figures 14.59 and 14.60b). If we calculate the member forces for the given load, we find that all the interior members are zero members.

The cable in Figure 14.57 and the bar structure in Figure 14.58a are also kinematically indeterminate. The equilibrium is stable (reliable) in this case as the load makes the structure go back into the original equilibrium position after a disruption.

In Figure 14.60a, the bar structure in Figure 14.58a has been changed into a kinematically determinate truss. All the interior members are zero. As in contrast to the cable, the truss has the benefit that the shape does not change when the load changes.

In Figure 14.61, the forces on the truss are shifted to the horizontal plane through the supports. The verticals are no longer zero members. These

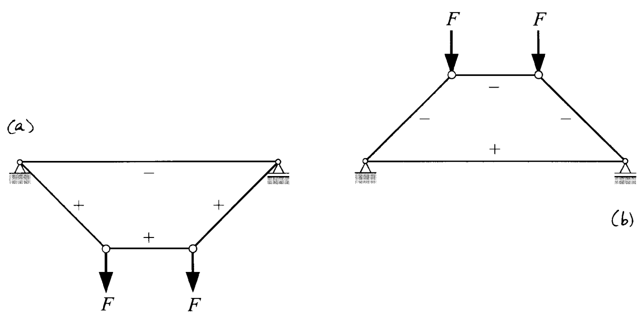


Figure 14.58 (a) The cable replaced by a bar structure. The bar structure is kinematically indeterminate, but the equilibrium is stable (reliable). (b) If the bar structure is folded over, the signs change in all the bars. The equilibrium is now unreliable (unstable).

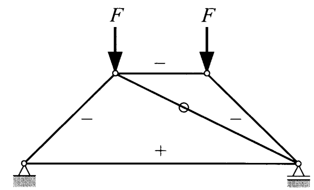


Figure 14.59 By applying an additional bar, the kinematically indeterminate bar structure changes into a kinematically and statically determinate truss.

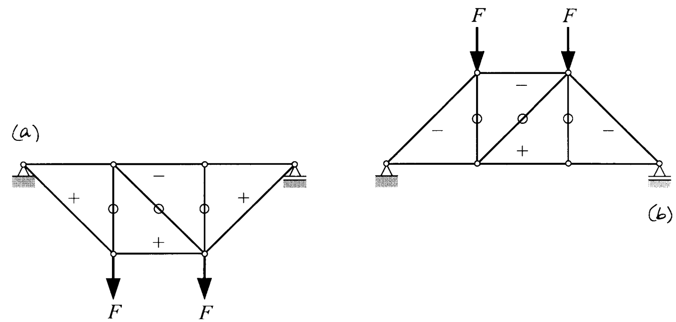


Figure 14.60 The bar structures in Figure 14.58 changed into trusses.

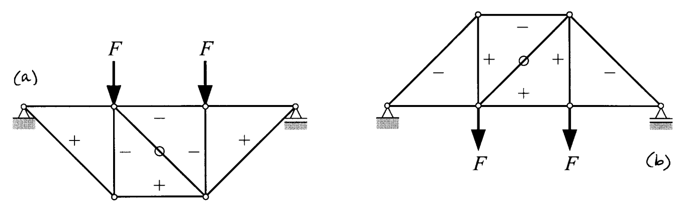


Figure 14.61 These trusses are an alternative for the beam subject to bending in Figure 14.56.

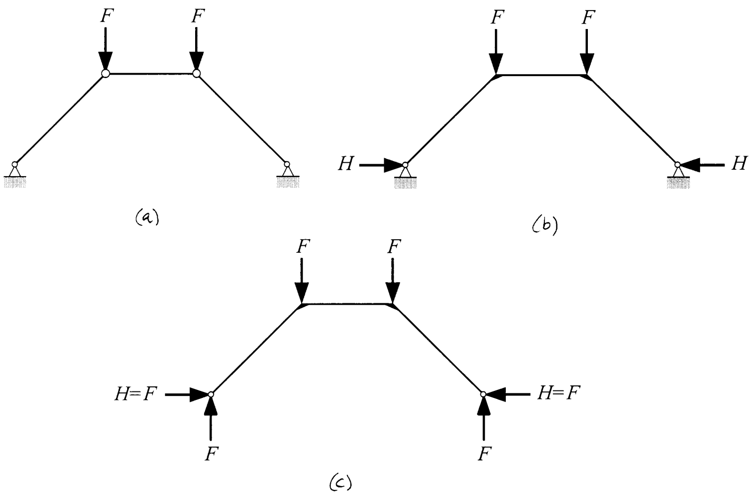


Figure 14.62 (a) Kinematically indeterminate bar structure, is (b) changed into a two-hinged frame. (c) The support reactions of the statically indeterminate two-hinged frame if axial deformation is ignored; the line of force coincides with the bent member axis: there is no bending.

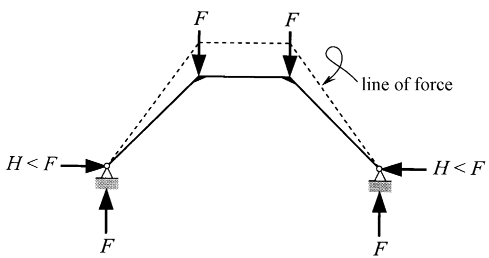


Figure 14.63 If we take into account the axial deformation, the horizontal support reactions are smaller and the line of force no longer coincides with the (bent) member axis: bending is generated.

trusses, in which all the members are subject to extension (or are zero members), can be an alternative for the beam subject to bending in Figure 14.56.

Figure 14.62a shows the bar structure from Figure 14.58b, but now without tension member. This kinematically indeterminate structure can be made kinematically determinate not only by changing it into a truss, but also by replacing the hinged joints between the bars by rigid joints. The structure then becomes a bent member, recognisable in Figure 14.62b as a two-hinged frame.

One problem is that the frame is statically indeterminate to the first degree. As such, it is not possible to determine the horizontal support reaction H directly from the equilibrium. The deformation of the frame also has to be taken into account. If the deformation by normal forces is ignored (as it was in the cable), it is possible to show that no bending occurs under the given load, and that the normal forces in the frame are equal to the forces in the two-force members in Figure 14.62a. In Figure 14.62c, the frame has been isolated and all the forces acting on it are shown. The line of force coincides everywhere with the bent member axis and there is no bending anywhere.

In reality, there is always some axial deformation due to normal forces. As such, the horizontal support reactions are somewhat smaller and, because the vertical support reactions remain equal, the line of force no longer coincides with the bent member axis (see Figure 14.63). Axial deformation therefore induces bending in the two-hinged frame.

Since statically indeterminate structures are more sensitive to settling and temperature, statically determinate structures are generally preferable, because the force distribution is more manageable. In Figure 14.64, the statically indeterminate two-hinged frame has been changed into a statically determined three-hinged frame. With the given load, the line of force coincides everywhere with the bent member axis and there is no bending anywhere.

Example 2

On girder CSD, the three-hinged portal frame in Figure 14.65a is carrying a uniformly distributed load. In Figures 14.65b and 14.65c, the support reactions and the bending moment diagram are shown. The calculation is left to the reader.

Question:

How can one reduce the bending moment in the frame, without changing the given load?

Solution:

Figure 14.65d shows the line of force for girder CSD. Cross-section C has to transfer the support reaction at A; the centre of force for cross-section C is therefore at A. In the same way, the centre of force for cross-section D is at B. The line of force passes through hinge S. The line of force for CSD has the same shape as a cable under a uniformly distributed load, which is a parabola. Since the normal force in girder CSD is a compressive force, "the parabolic cable is upside down" and the line of force is a line of pressure.

Check: Regard the line of force as an "upside-down cable" (see Figure 14.65d):

$$H = \frac{\frac{1}{8}q\ell^2}{p} = \frac{\frac{1}{8} \times (4\ \text{kN/m})(12\ \text{m})^2}{4\ \text{m}} = 18\ \text{kN}.$$

This force is indeed equal to the horizontal support reactions at A and B (see Figure 14.65b).

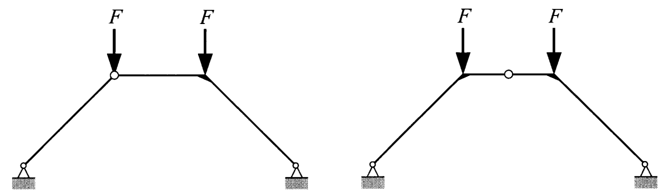


Figure 14.64 Three-hinged frames are statically determinate and therefore the force flow is less sensitive to axial deformations, settling and the influence of temperature.

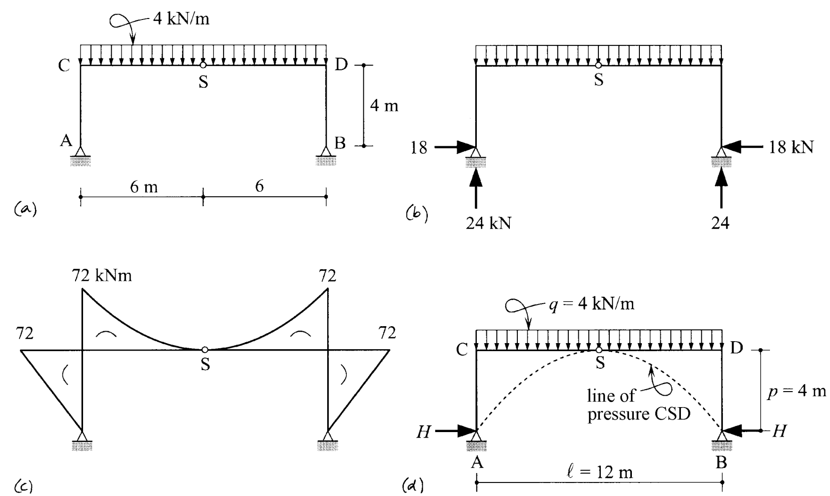


Figure 14.65 (a) Three-hinged frame with uniformly distributed load on girder CSD. (b) Support reactions. (c) Bending moment diagram. (d) The line of pressure for girder CSD is a parabola through A, S and B.

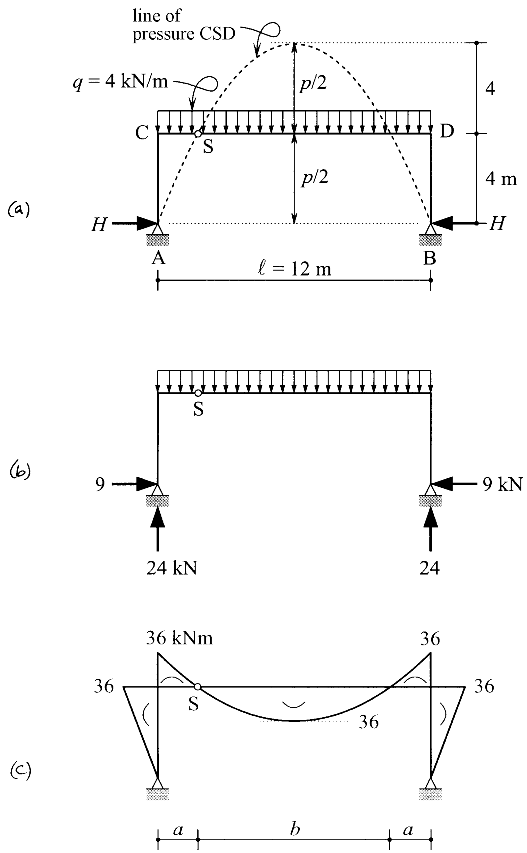


Figure 14.66 (a) The line of pressure can be changed by changing the location of hinge S. (b) The support reactions when the line of pressure is equally above and below the girder. (c) Bending moment diagram.

The bending moment in the girder can be influenced by the location of hinge S (see Figure 14.66a). The line of force maintains the shape of an "upside-down parabolic cable" through A, B and S.

Distance e from the line of pressure to the girder is a measure for the magnitude of the bending moment. It always holds that

$$|M| = |Ne| = |He|.$$

The moment distribution is most favourable when, in contrast to Figure 14.65d, the line of pressure is as much above the girder as below it. This situation is shown in Figure 14.66a. Here it holds that

$$H = \frac{\frac{1}{8}q\ell^2}{p} = \frac{\frac{1}{8} \times (4\ \text{kN/m})(12\ \text{m})^2}{8\ \text{m}} = 9\ \text{kN}.$$

Figures 14.66b and 14.66c show the support reactions and the bending moment diagram respectively. The distance b between the moment zeros follows from

$$\tfrac{1}{8}qb^2 = 36\ \text{kNm}$$

in which $q = 4$ kN/m. We find

$$b = \sqrt{\frac{8 \times (36\ \text{kNm})}{4\ \text{kN/m}}} = \sqrt{72\ \text{m}^2} \approx 8.5\ \text{m}.$$

The location of hinge S is then

$$a = \frac{\ell - b}{2} \approx 1.75\ \text{m}.$$

By moving hinge S we adapt the line of pressure to the shape of the frame. Alternatively, the shape of the frame can also be adapted to the line of pressure. This is shown in Figure 14.67a.[1]

For the horizontal support reaction H it holds that

$$H = \frac{\frac{1}{8}q\ell^2}{p} = \frac{\frac{1}{8} \times (4\ \text{kN/m})(12\ \text{m})^2}{4\ \text{m}} = 18\ \text{kN}.$$

In Figure 14.67b the support reactions are shown. They are of equal magnitude to the support reactions of the three-hinged frame in Figure 14.65. Figure 14.67c shows the bending moment diagram.

The moment distribution is determined below for girder SD. For this reason, in Figure 14.68 a part directly adjacent to hinge S has been isolated. Only the horizontal compressive force H acts at hinge S; there is no vertical force. In the other section, there is a bending moment M, shear force V and normal force N. From the moment equilibrium about this section we find

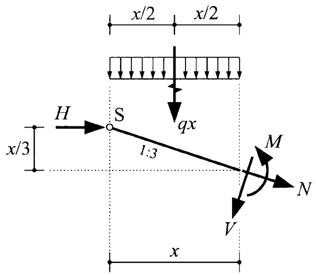


Figure 14.68 The forces on the isolated part of the frame to the right of S.

[1] Here we assume that the distributed load is still a force per horizontally measured length.

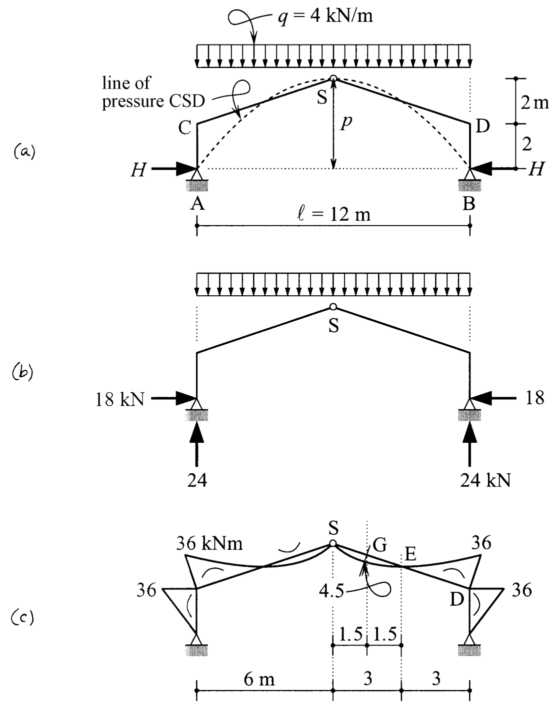


Figure 14.67 (a) A three-hinged frame whose shape has been somewhat adapted to the shape of the line of pressure. (b) Support reactions. (c) Bending moment diagram.

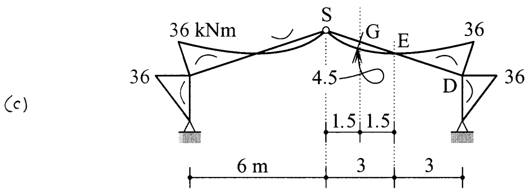


Figure 14.67 (c) Bending moment diagram.

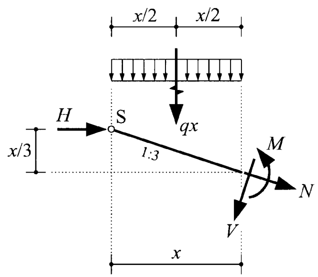


Figure 14.68 The forces on the isolated part of the frame to the right of S.

$$M = H \cdot \tfrac{1}{3}x - qx \cdot \tfrac{1}{2}x = \tfrac{1}{3}Hx - \tfrac{1}{2}qx^2 \qquad \text{(a)}$$

in which $H = 18$ kN and $q = 4$ kN/m.

Checking expression (a) for the bending moment at D, with $x = 6$ m

$$M = M_D = \tfrac{1}{3} \times (18 \text{ kN})(6 \text{ m}) - \tfrac{1}{2} \times (4 \text{ kN/m})(4 \text{ m})^2 = -36 \text{ kNm}.$$

This minus sign indicates that the bending moment at D acts opposite to the direction shown in Figure 14.68. This is in agreement with the M diagram in Figure 14.67c.

The bending moment is zero at E (see Figure 14.67c). With $M = M_E = 0$, it follows from (a) that:

$$x = x_E = \frac{2H}{3q} = \frac{2 \times (18 \text{ kN})}{3 \times (4 \text{ kN/m})} = 3 \text{ m}.$$

The field moment in SD is a maximum at G. Here $\mathrm{d}M/\mathrm{d}x = 0$. Differentiating expression (a) gives

$$\frac{\mathrm{d}M}{\mathrm{d}x} = \tfrac{1}{3}H - qx = 0$$

so that

$$x = x_G = \frac{\tfrac{1}{3}H}{q} = \frac{\tfrac{1}{3} \times (18 \text{ kN})}{4 \text{ kN/m}} = 1.5 \text{ m}.$$

From here, (a) gives

$$M = M_G = \tfrac{1}{3} \times (18 \text{ kN})(1.5 \text{ m}) - \tfrac{1}{2} \times (4 \text{ kN/m})(1.5 \text{ m})^2 = 4.5 \text{ kNm}.$$

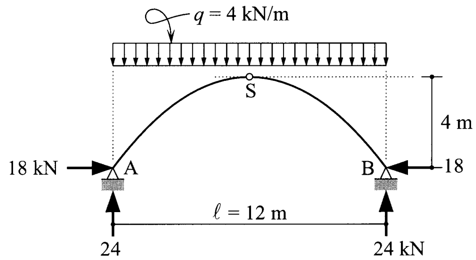


Figure 14.69 The adaptation to the line of force is optimal if the frame is in the shape of a parabola.

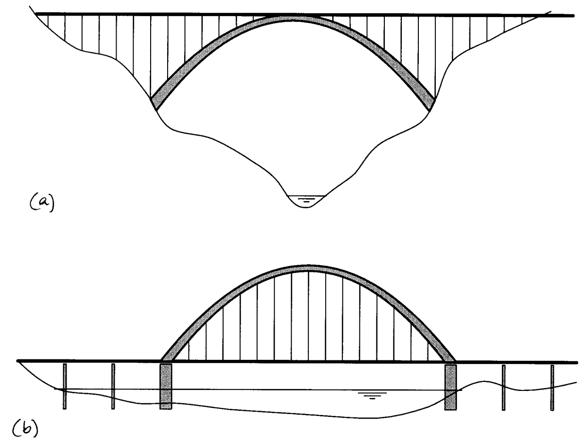


Figure 14.70 (a) Arch bridge with upper deck. The horizontal forces in the arch are directly transferred to the foundation. (b) Arch bridge with lower deck. The horizontal forces from the arch are transferred via the structure of the bridge deck, which acts as a tie rod.

The adaptation to the line of force is optimal if we give the frame the shape of a parabolic arch (see Figure 14.69). The line of force now coincides everywhere with the axis of the arch, and there is no bending anywhere. The support reactions are equal to those of the frames in Figures 14.65 and 14.67.

In Figure 14.70a, the arch has been used in a *bridge with upper deck*. This type of bridge is generally found in mountainous regions. The good foundation ground, generally rock, is capable of transferring the horizontal support reactions. In Figure 14.70b, the arch is used in a *bridge with lower deck*. By using the structure of the bridge deck as a *tie rod*, the piers are not subject to the horizontal forces from the arch.

Example 3

The third example demonstrating the concept of line of force concerns structures with *brickwork*. Brickwork effectively resists pressure, but is very poor at transferring tensile stresses. Since there is little tensile strength, this must not be relied on; the tensile strength has to be neglected in the calculation. Structures made of brickwork must therefore be designed so that no tension occurs.

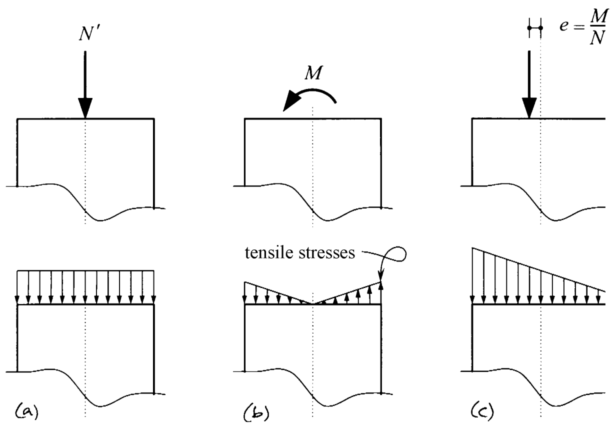


Figure 14.71 The distribution of the normal stresses in a rectangular cross-section due to (a) a centric compressive force, (b) a bending moment and (c) an eccentric compressive force.

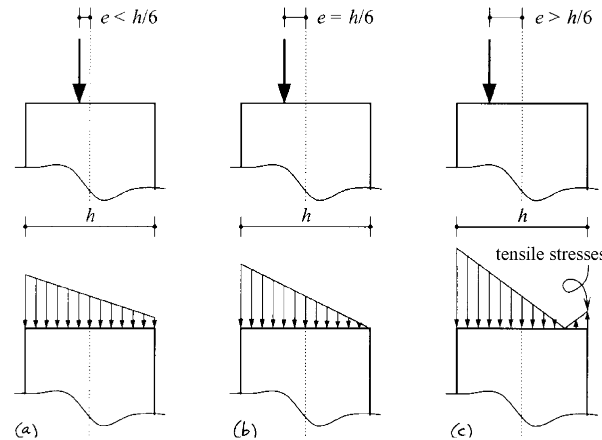


Figure 14.72 The normal stress distribution in a rectangular cross-section due to an eccentric compressive force. (a) With minor eccentricity, the entire cross-section is subject to compression. (b) If $e = h/6$, the normal stress is zero at the least compressed edge. (c) With major eccentricity, tensile stresses occur in the cross-section.

In Figures 14.71a and 14.71b the distribution of the normal stresses is shown for a rectangular cross-section due to a centric compressive force N' and a bending moment M. The centric compressive force and the bending moment are together statically equivalent to an eccentric compressive force N' (see Figure 14.71c). Due to the eccentricity of the compressive force, the compressive stresses increase on one side of the cross-section and decrease on the other.

When the eccentricity e of the compressive force is equal to one sixth of the depth h of the cross-section ($e = h/6$), the stress diagram is triangular (see Figure 14.72b). At one of the sides, the normal stress is zero. If the eccentricity is larger ($e > h/6$), then tensile stresses occur at that side (see

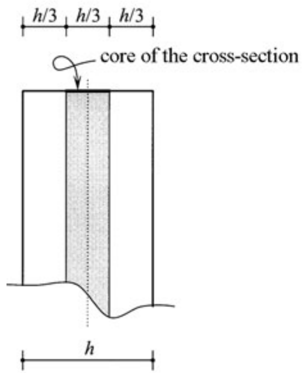


Figure 14.73 There are no tensile stresses if the centre of pressure is inside the core of the cross-section. With rectangular cross-sections this is the middle third.

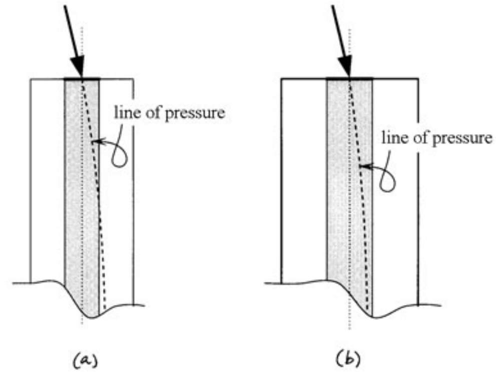


Figure 14.74 Due to the dead weight of a column, the line of pressure tends increasingly towards a line parallel to the member axis. (a) Tensile stresses occur where the line of pressure reaches outside the core (b) Increasing the column cross-section has two positive effects: the core increases in size while, thanks to the larger weight of the column, the eccentricity of the line of force decreases.

Figure 14.72c). In brickwork the joints cannot transfer tensile stresses and cracks will form. The cohesion of the cross-section is lost so that there is a danger of collapse.

To prevent tensile stresses in the cross-section, the centre of pressure (the point of application of the compressive force in the cross-section) for a rectangular cross-section must lie within the middle third of the cross-section. This area is known as the *core of the cross-section*[1] (see Figure 14.73).

Figure 14.74a shows a brickwork column, loaded at the top by an oblique force. At first, the line of force has the direction of the oblique force. Since the normal force increases downwards due to the weight of the column, the line of force is curved and increasingly tends towards a line parallel to the

1 In Volume 2, *Stresses, Deformations, Displacements*, we shall address this issue in further detail.

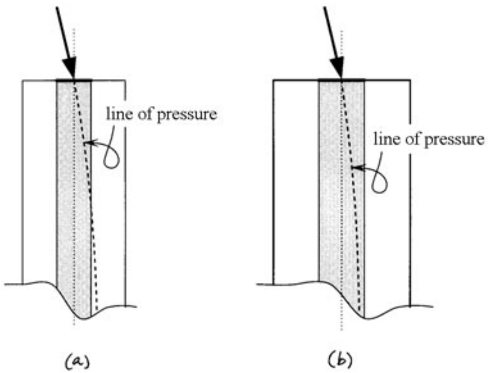


Figure 14.74 Due to the dead weight of a column, the line of pressure tends increasingly towards a line parallel to the member axis. (a) Tensile stresses occur where the line of pressure reaches outside the core (b) Increasing the column cross-section has two positive effects: the core increases in size while, thanks to the larger weight of the column, the eccentricity of the line of force decreases.

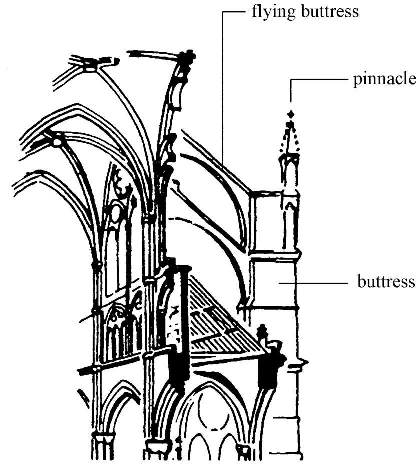


Figure 14.75 In gothic cathedrals, we can see how the tensile stresses in the buttresses are suppressed by using the weight of pinnacles.

column axis.

When there is a danger that the line of force will extend beyond the core of the cross-section, the column cross-section can be enlarged. First, due to the dead weight, the normal force will increase and the eccentricity of the centre of force will decrease: the line of pressure thereby moves towards the column axis. In addition, the core of the cross-section increases, and therefore the area in which the centre of pressure can lie also increases (see Figure 14.74b).

Increasing the compressive force N' in the cross-section, thereby reducing the eccentricity e of the line of force ($e = M/N'$), is also possible by increasing the height of the column to a greater height than is strictly necessary, or by introducing additional weight on the column by means of heavy statues or pinnacles. The latter is often used in gothic cathedrals (see Figure 14.75). Due to the introduction of aisles, flying buttresses were needed to support the main structure obliquely. The forces are transferred to buttresses, weighed down by pinnacles.

We have only looked at brickwork. There are many other materials that are unable to resist tension, such as *concrete*. Here, the issue of tensile stresses is approached differently.

In *reinforced concrete*, steel bars are placed in the area of tension. These reinforcement bars transfer the tensile stresses in the cross-section (see Figure 14.76).

In *prestressed concrete*, the tensile stresses are "suppressed" by introducing a prestress (see Figure 14.77). The prestress achieves the same effect as the weight of the pinnacles on the buttresses in Figure 14.75 (see also Section 13.1.5 and Section 14.1.6, Example 5).

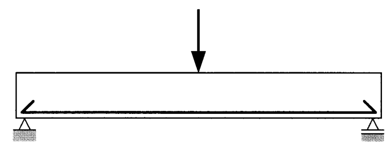


Figure 14.76 In reinforced concrete, the tensile stresses in the cross-section are transferred by reinforcement bars.

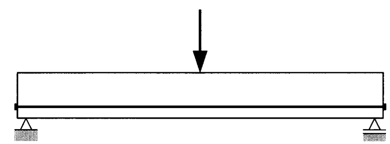


Figure 14.77 In prestressed concrete, the tensile stresses in the cross-section are "suppressed" by applying tendons

(a)

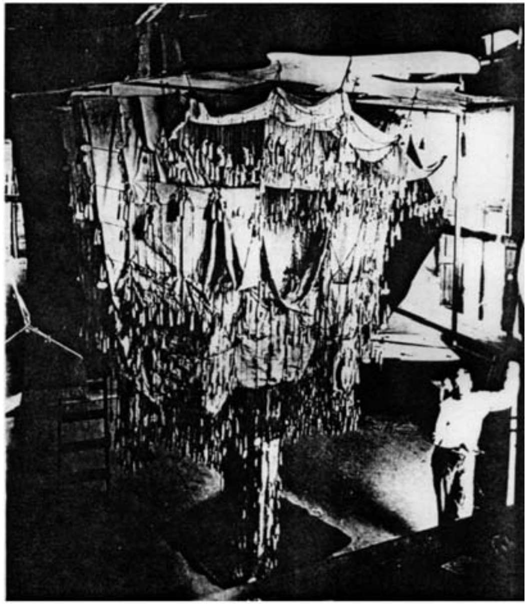
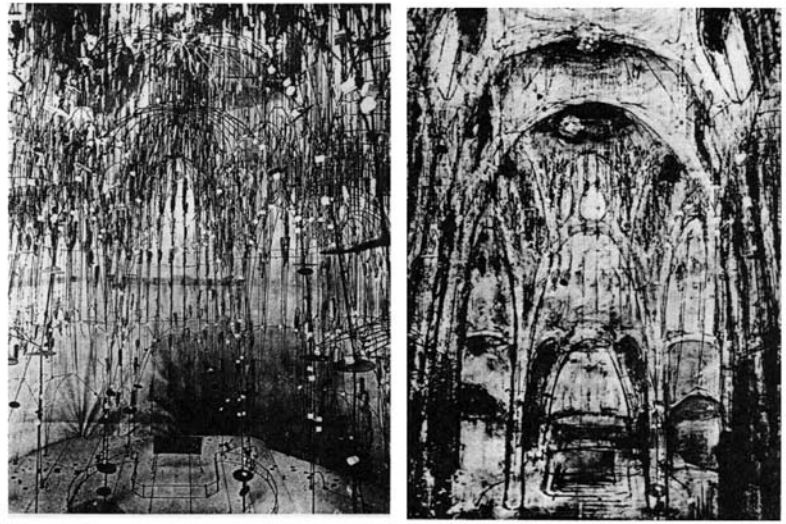


Figure 14.78 (a) Original photograph of Gaudi's suspension model for the Colonia Güell.

(b)



(c)

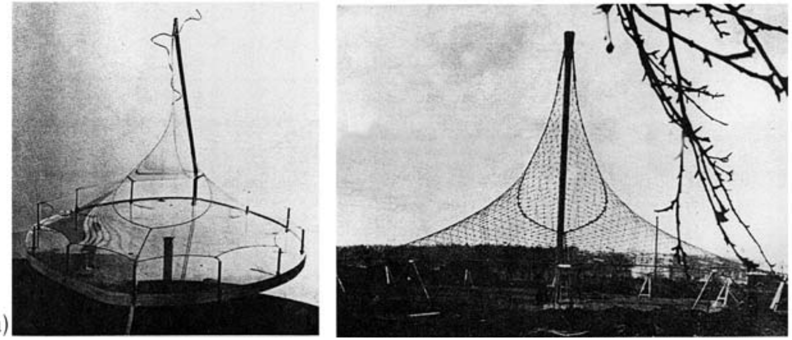
Figure 14.78 (b) Photograph of the inside of the model, rotated through 180°. The upside-down cables change into arches. (c) Interior sketch by Gaudi on the basis of the model.

Example 4

For the last example on lines of force, we refer back to the start: the cable shape.

The Spanish architect Gaudi[1] is known for his whimsical vaults in the Colonia Güell and the Sagrada Familia, two churches in Barcelona. He determined the shape of the vaults using a “suspension model”. Figure 14.78a shows an original photo of the model for the Colonia Güell, which was built between 1898 and 1908. Using bags of lead suspended from ropes, representing the dead weight, Gaudi determined the preferred shape of the arches. Figure 14.78b is a photo of the inside of the model, rotated through 180°. The upside-down cables are transformed into arches. Using this model, Gaudi drew the interior sketch in Figure 14.78c. Due to a lack of funds, the construction of the Colonia Güell had to be abandoned in 1914. Only the crypt was completed.

When designing cable structures subject to tension (cable networks) Frei Otto and his staff used the same experiment over 50 years later. In their method, soap membranes were the most important tool. To translate a *soap membrane model* in Figure 14.79a into an actual structure, it was meticulously photographed and measured. Figure 14.79b shows the actual structure being built. One of Frei Otto’s most famous buildings is the roof of the Olympic Stadium in Munich (1972) as a tent with saddle roof shapes (see Figure 14.80).



(a) (b)

Figure 14.79 (a) Frei Otto used the soap membrane model as a tool for designing cable networks. (b) The soap membrane model changed into an actual structure.

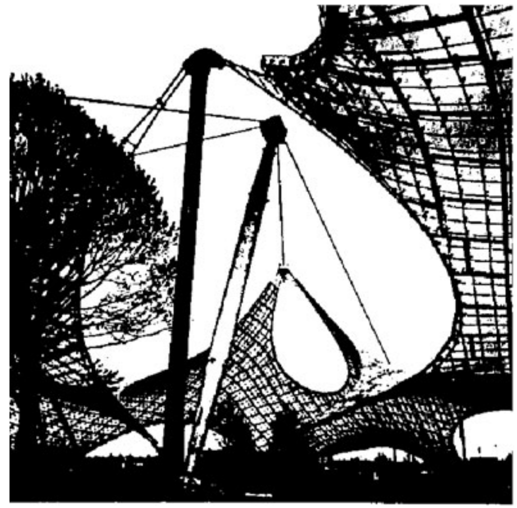


Figure 14.80 One of Frei Otto’s most famous buildings is the roof of the Olympic stadium in Munich (1972) in the shape of a tent with saddle roof shapes.

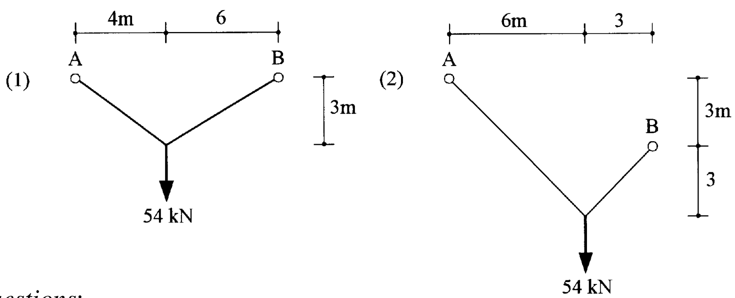
[1] Antoni Gaudi i Cornet (1852–1926), Spanish architect, studied and worked in Barcelona. He derived the shapes for his buildings from nature; this became known as the organic style. His most famous creation is the Sagrada Familia in Barcelona (construction started in 1883, not completed).

14.4 Problems

Unless indicated otherwise, the dead weight of the cable is ignored in the problems.

Cables, mixed problems (Section 14.1)

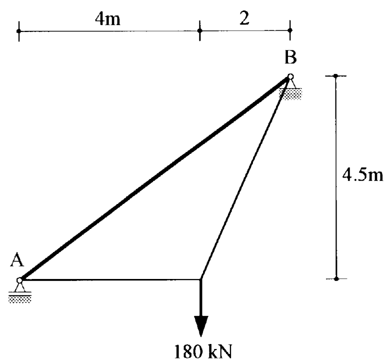
14.1: 1–2 Given two different cables, hung from fixed points A and B, are loaded by a single force.



Questions:

a. Determine the vertical support reaction at A.
b. Determine the horizontal support reaction at B.
c. Determine the maximum cable force.

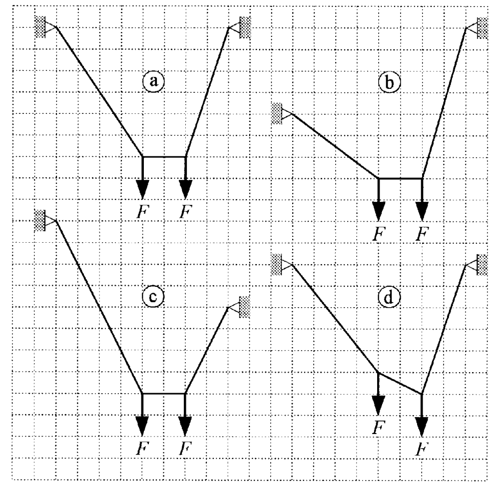
14.2 Given a cable with compression bar, loaded by a force of 180 kN.



Questions:

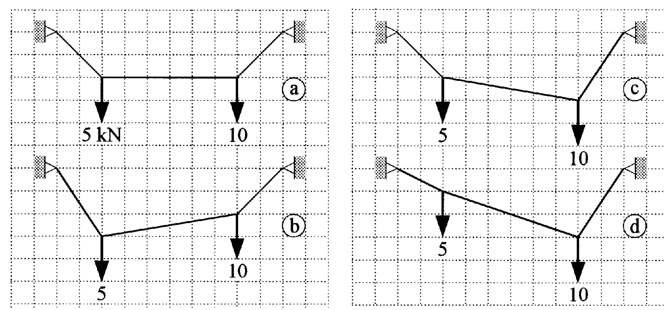
a. Determine the vertical component of the force in the compression bar.
b. Determine the horizontal component of the force in the compression bar.

14.3 A cable is loaded by two equal forces F.



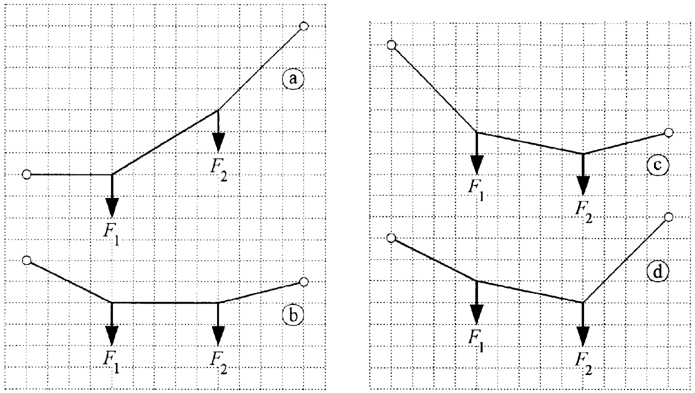
Question:
Which cable shape fits the load?

14.4 Two weights of respectively 5 and 10 kN are suspended from a cable.



Question:
Which cable shape fits the load?

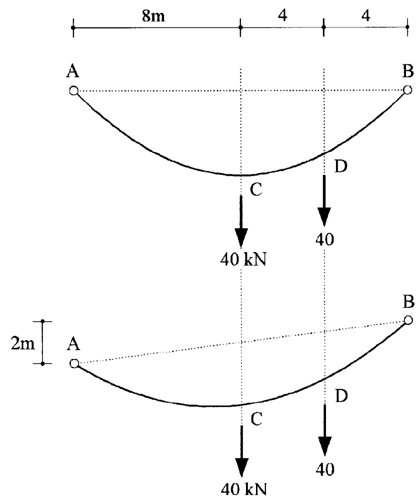
14.5 A cable, suspended between two fixed points, is subject to the forces F_1 and F_2, with $F_1 > F_2$.



Question:
Which cable shape does not fit this load?

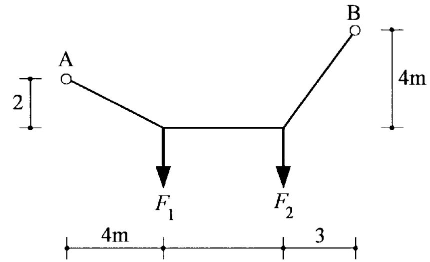
14.6: 1–2 A cable suspended between fixed points A and B is loaded at C and D by two forces of 40 kN. The horizontal measured lengths are those of the final position.

Question:
Find the ratio between the vertical distances from chord AB to the points C and D on the cable.



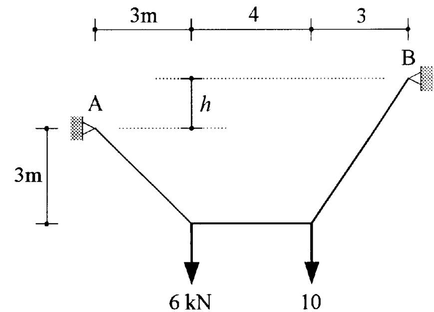
14.7 The cable shown is loaded by forces F_1 and F_2 in such a way that the middle part is horizontal.

Question:
Find the ratio F_1/F_2.



14.8 A cable is loaded by two forces of respectively 6 and 10 kN. The cable is horizontal between these forces.

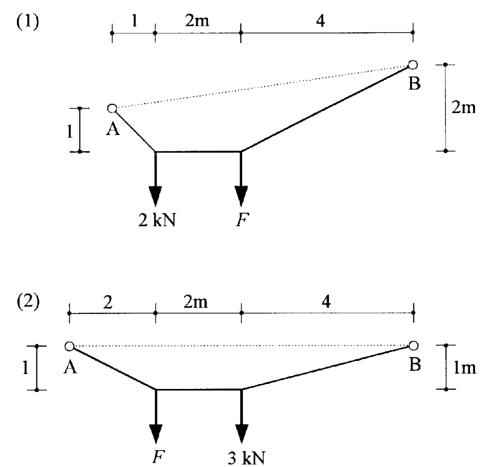
Question:
Determine the difference in height h between the suspension points A and B.



14.9: 1–2 Two cables, suspended at the fixed points A and B, are loaded by two forces. The magnitude of force F is unknown.

Questions:

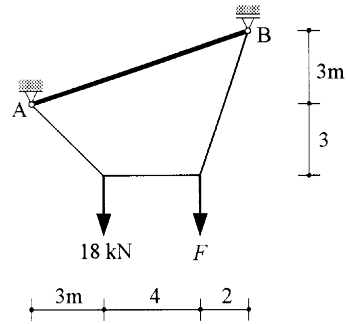
a. Determine the vertical support reaction at A.
b. Determine the horizontal component of the cable force.
c. Determine the magnitude of F.
d. Determine the maximum cable force.



14.10 Given a cable with compression bar. The cable is loaded by a force of 18 kN and an unknown force F, and assumes the shape shown.

Questions:

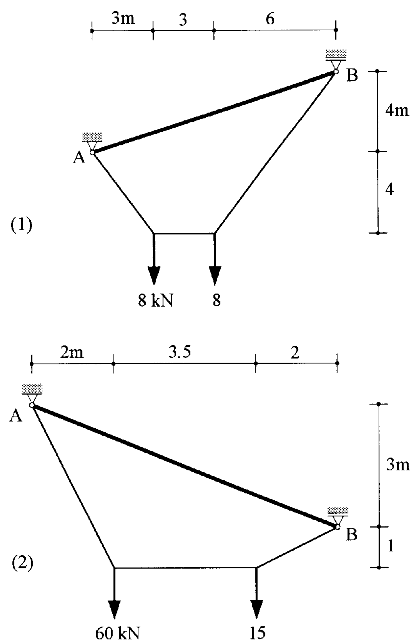
a. Determine the horizontal component of the cable force.
b. Determine the normal force in compression bar AB.
c. Determine the magnitude of F.



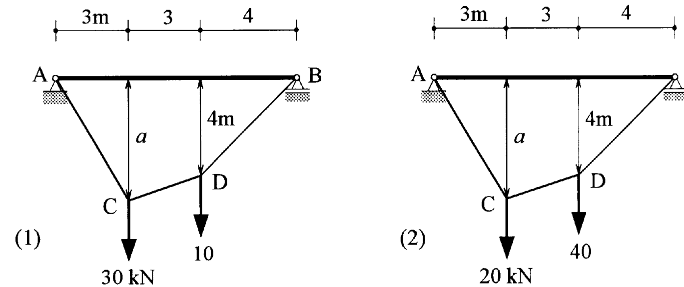
14.11: 1–2 Two cables with compression member are loaded in such a way that the middle part of the cable is horizontal.

Questions:

a. Determine the support reactions at A and B.
b. Determine the vertical component of the force in member AB.
c. Determine the maximum cable force.



14.12: 1–2 Two different cables with compression bar AB are loaded at C and D by forces.



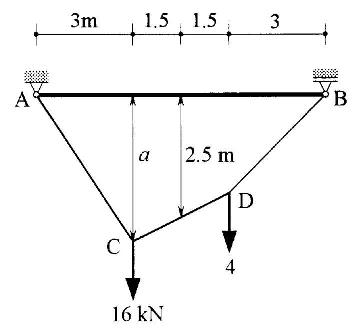
Questions:

a. Determine the force in compression bar AB.
b. Determine the distance a from point of application C to bar AB.
c. Determine the maximum cable force.

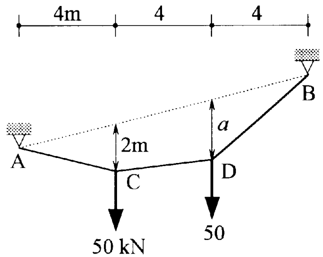
14.13 A cable with compression bar is loaded at C and D by two forces of 16 and 4 kN respectively. At midspan the distance between the cable and compression bar AB is 2.5 metres.

Questions:

a. Determine the force in the compression member.
b. Determine the distance a from point of application C to member AB
c. Determine the maximum cable force.



14.14 A cable suspended between A and B is loaded at C and D by two equal forces of 50 kN. The distance from point of application C to chord AB is 2 metres.



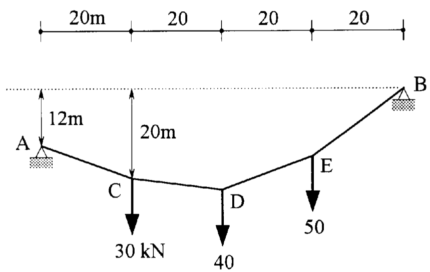
Questions:

a. Determine the horizontal component of the cable force.
b. Determine the distance *a* from point of application D to chord AB.

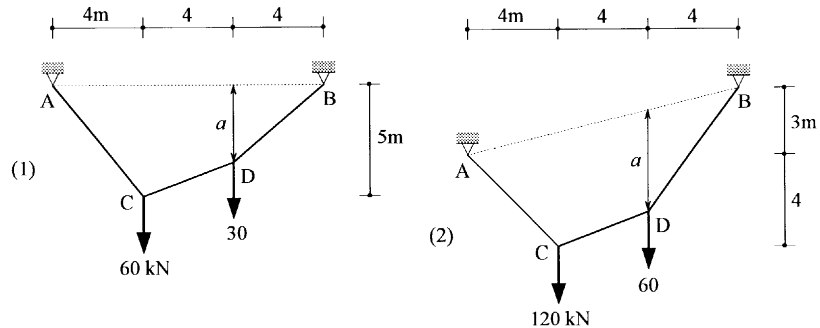
14.15 The cable shown is loaded at C, D and E by forces of 30, 40 and 50 kN respectively. The difference in elevation of the end supports at A and B is 12 metres. The distance from point of application C to the horizontal through B is 20 metres.

Questions:

a. Determine the distance from point of application D to the horizontal through B.
b. Determine the distance from point of application E to the horizontal through B.
c. Determine the support reactions at A.
d. Determine the support reactions at B.



14.16: 1–2 Two cables are suspended between points A and B, and are loaded in C and D by forces. The horizontal measured lengths are equal in both cases.



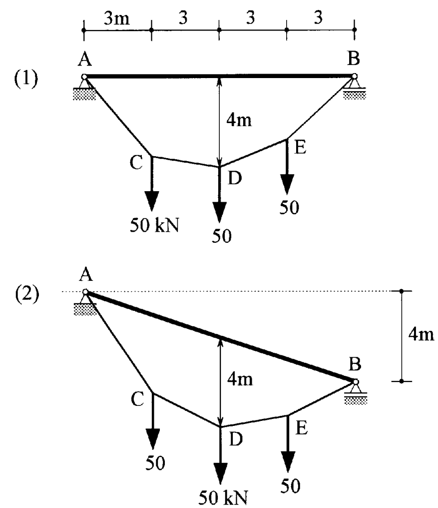
Questions:

a. Determine the horizontal component of the cable force.
b. Determine the distance *a* of point of application D to chord AB.
c. Determine the maximum cable force.

14.17: 1–2 Given two cables with compression bar, loaded by three forces of 50 kN.

Questions:

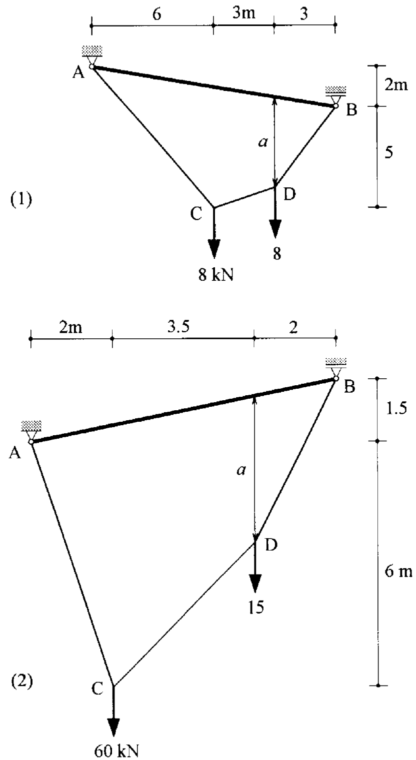
a. Determine the force in the compression bar.
b. Draw the cable shape to scale.
c. Determine the minimum cable force.
d. Determine the maximum cable force.



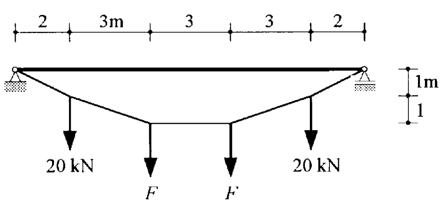
14.18: 1–2 Two cables with compression bar AB are loaded at C and D by forces.

Questions:

a. Determine the horizontal component of the cable force.
b. Determine the normal force in member AB.
c. Determine the distance a from point of application D to bar AB.
d. Determine the maximum cable force.



14.19 A cable with compression bar, loaded by two forces of 20 kN and two unknown forces F.

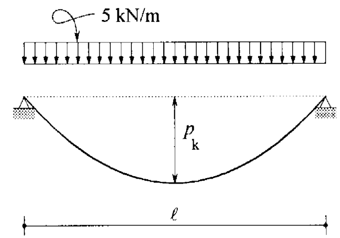


Questions:

a. Determine the force in the compression bar.
b. Determine the magnitude of forces F.

14.20: 1–3 A cable is carrying a uniformly distributed load of 5 kN/m and thereby assumes the shape shown.

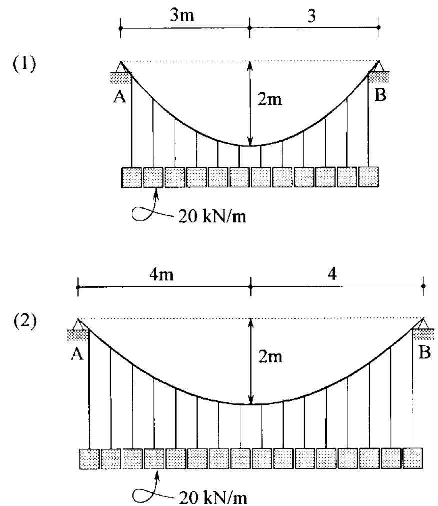
Question:
Determine the maximum cable force when:
(1) $\ell = 16$ m and $p_k = 3.20$ m.
(2) $\ell = 18$ m and $p_k = 4.50$ m
(3) $\ell = 20$ m and $p_k = 4$ m.



14.21: 1–2 Two cables with a uniformly distributed load of 20 kN/m.

Questions:

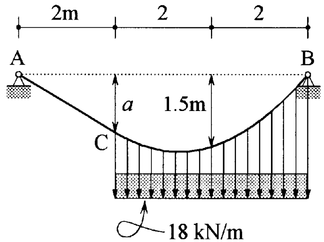
a. Determine the horizontal component of the cable force.
b. Determine the maximum cable force.
c. Determine the support reactions at A and B.



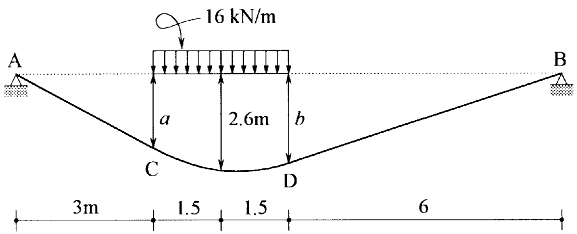
14.22 Due to the uniformly distributed load of 18 kN/m on CB cable AB assumes the shape shown.

Questions:

a. Determine the horizontal component of the cable force.
b. Determine the maximum cable force.
c. Determine the support reactions at A and B.
d. Determine the sag a at C.
e. Determine the maximum sag of the cable.



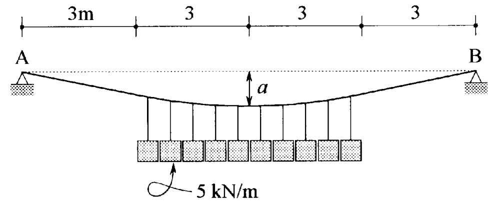
14.23 A uniformly distributed load of 16 kN/m is acting on cable AB between C and D. The cable sags 2.6 metres at the middle of the distributed load.



Questions:

a. Determine the horizontal component of the cable force.
b. Determine the sag a at C.
c. Determine the sag b at D.
d. Determine the maximum sag of the cable.

14.24 In the middle of cable AB a uniformly distributed load $q = 5$ kN/m is acting over a length of 6 metres. The horizontal component of the cable force is 75 kN.



Questions:

a. Determine the support reactions at A and B.
b. Determine the maximum cable force.
c. Determine the sag a in the middle of the cable.

14.25 See the figure for problem 14.24. In the middle part of cable AB a uniformly distributed load $q = 5$ kN/m acts over a length of 6 metres. The cable assumes the shape shown with $a = 1.25$ m.

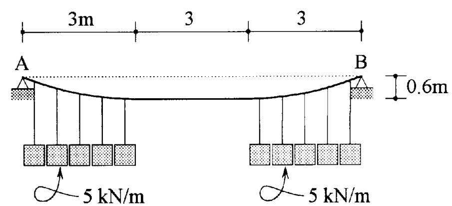
Questions:

a. Determine the horizontal component of the cable force.
b. Determine the support reactions at A and B.
c. Determine the maximum cable force.

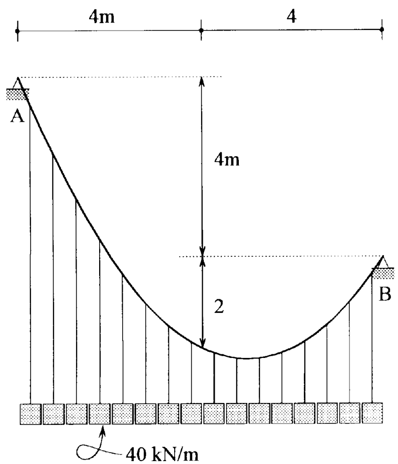
14.26 A uniformly distributed load $q = 5$ kN/m acts on cable AB at both ends over a length of 3 metres. The middle part is unloaded. The cable assumes the shape shown.

Questions:

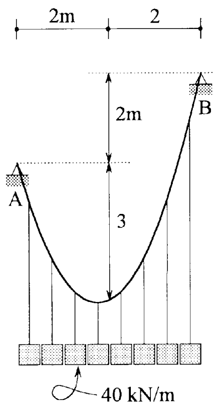
a. Determine the horizontal component of the cable force.
b. Determine the support reactions at A and B.
c. Determine the maximum cable force.



14.27: 1–2 Given two cables with a uniformly distributed load of 40 kN/m.



(1)

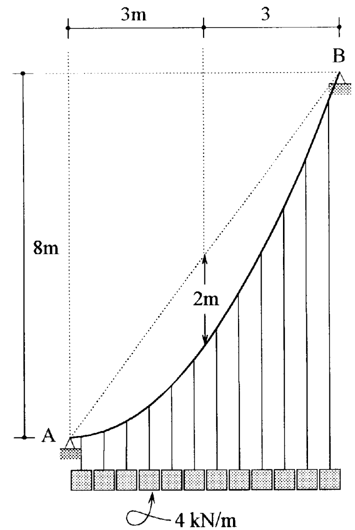


(2)

Questions:

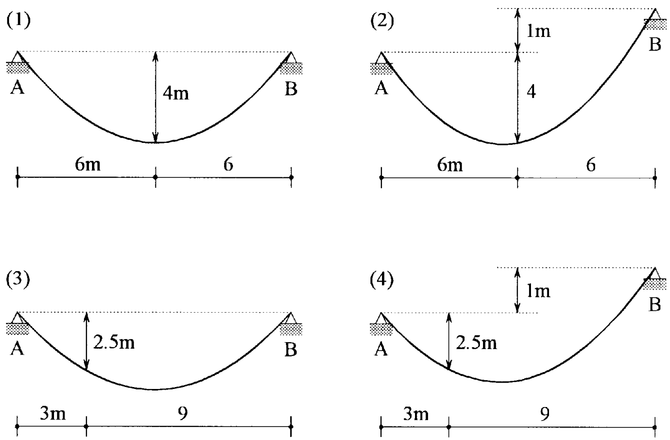
a. Determine the horizontal component of the cable force.
b. Determine the maximum cable force.
c. Determine the support reactions at A and B.
d. Where is the lowest point of the cable?
e. Determine the difference in height between support A and the lowest point of the cable.

14.28 Given a cable with a uniformly distributed load of 4 kN/m.



Question:
Determine the support reactions at A and B.

14.29: 1–4 Given four cables that under the influence of a uniformly distributed load $q = 36$ kN/m assume the shape shown.



Questions:
a. Determine the horizontal component of the force in the cable.
b. Determine the support reactions at A and B.
c. Determine the maximum cable force.

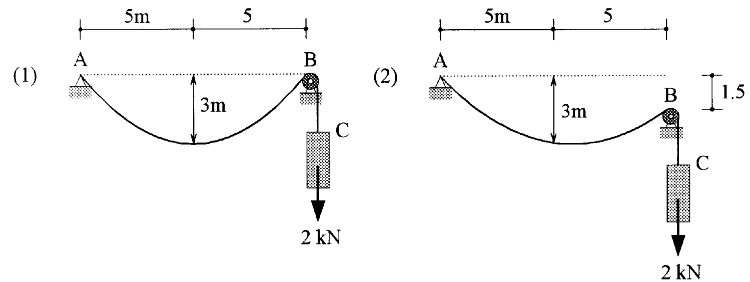
14.30: 1–4 As problem 14.29, but now with a uniformly distributed load q of 7.2 kN/m.

14.31: 1–4 Given four cables that under the influence of a uniformly distributed load q assume the shape in problem 14.29. The horizontal component of the cable force is 54 kN.

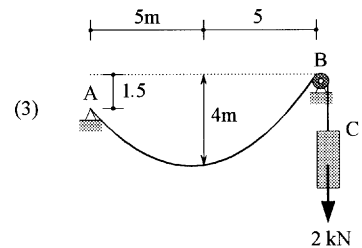
Questions:
a. Determine the magnitude of the distributed load q.
b. Determine the support reactions at A and B.
c. Determine the maximum cable force.

14.32: 1–4 As problem 14.31, but now the horizontal component of the cable force is 135 kN.

14.33: 1–3 At B, cable ABC passes over a pulley and is kept in equilibrium by a counterweight of 2 kN at C. The pulley is frictionless and has negligibly small dimensions. Assume that the cable has a parabolic shape due to of its dead weight. Ignore the dead weight of cable part BC.



Questions:
a. Determine the dead weight of the cable AB.
b. Determine the support reactions at A.
c. Determine the support reactions at B.



14.34 Given a cable.

Question:
Which load has to act on the cable to give it the following shapes:

a. Parabola.
b. Catenary.
c. Circle.

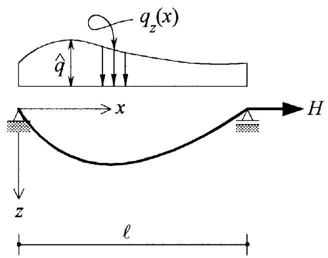
14.35: 1–4 A cable with span ℓ is loaded by four different distributed loads $q_z(x)$. Here $\hat{q}$ is the top value of the distributed load. The horizontal component of the cable force is H. In your calculation use $\ell = 15$ m, $\hat{q} = 8$ kN/m and $H = 75$ kN.

(1) $q_z = \hat{q}\dfrac{x}{\ell}$

(2) $q_z = 4\hat{q}\dfrac{x}{\ell}\left(1 - \dfrac{x}{\ell}\right)$

(3) $q_z = \hat{q}\sin\dfrac{\pi x}{\ell}$

(4) $q_z = \hat{q}\left(1 - \sin\dfrac{x}{\ell}\right)$



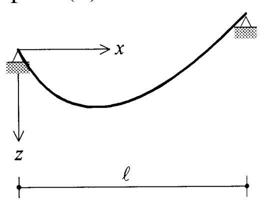
Questions:
a. Draw the load diagram.
b. Determine the cable shape as a function of x.
c. Determine the maximum sag in the cable.
d. Determine the maximum cable force.

14.36: 1–4 As problem 14.35, but now with values $\ell = 20$ m, $\hat{q} = 6$ kN/m and $H = 120$ kN.

14.37: 1–2 Under the influence of a distributed load $q_z(x)$ a cable with span ℓ assumes one of the following cable shapes $z(x)$:

(1) $z(x) = -(7.5\text{ m})\left(\dfrac{x}{\ell}\right)^2 + (5\text{ m})\left(\dfrac{x}{\ell}\right)$

(2) $z(x) = -(8\text{ m})\left(\dfrac{x}{\ell}\right)^3 + (6\text{ m})\left(\dfrac{x}{\ell}\right)$

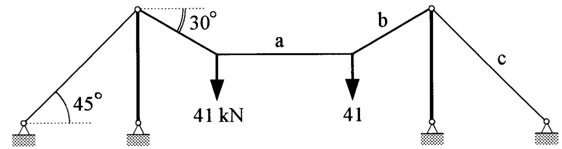


The horizontal component of the cable force is H. In your calculation use $\ell = 12$ m and $H = 60$ kN.

Questions:
a. Draw the cable shape.
b. Determine the maximum sag of the cable.
c. Determine the distributed load as a function of x.
d. Determine the maximum cable force.

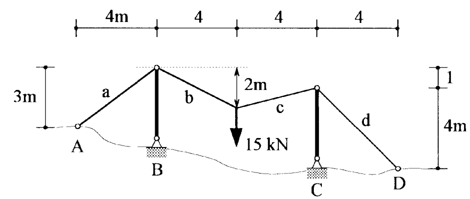
14.38: 1–2 As problem 14.37, but now with $\ell = 15$ m and $H = 90$ kN.

14.39 A symmetrical cable structure consists of two bar supports and a number of cables. The structure is loaded as shown by two forces of 41 kN.



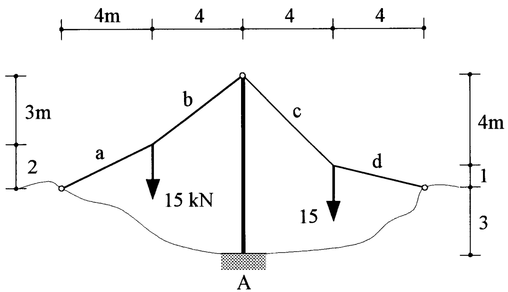
Questions:
a. Which of the cables a to c is most heavily loaded?
b. Determine the force in the most heavily loaded cable.
c. Determine the normal force in a bar support.

14.40 The cable structure shown consists of two bar supports and the four cables a, b, c and d. A force of 15 kN acts at midspan.



Questions:
a. Which cable is most heavily loaded?
b. Determine the force in the most heavily loaded cable.
c. Determine the support reactions at A to D and check the force and moment equilibrium of the structure as a whole.

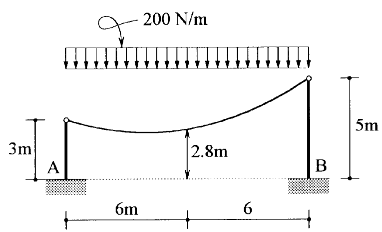
14.41 The cable structure shown consists of a column fixed at A and the four cables a, b, c and d. The structure is loaded as shown by two forces of 15 kN.



Questions:

a. Which cable is most heavily loaded?
b. Determine the support reactions at A, and draw them as they are acting in reality.

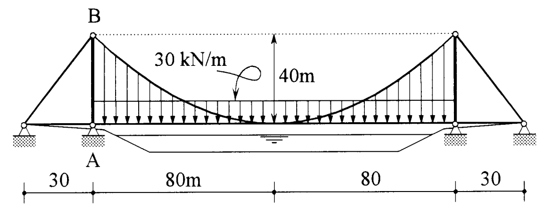
14.42 Party decorations made of coloured lamps and flags are suspended on a cable between two fixed columns of varying height. With a uniformly distributed load of 200 N/m the cable assumes the shape as shown.



Questions:

a. Determine the largest cable force.
b. Determine the support reactions at A. Draw them as they are acting in reality.
c. Determine the support reactions at B. Draw them as they are acting in reality.
d. Determine the free height under the lowest point of the cable.

14.43 Using the cable structure shown, a number of pipelines are led across a river. The load on the cable is uniformly distributed and is 30 kN/m.



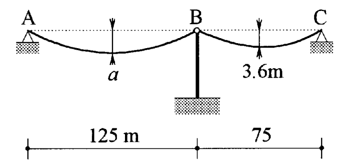
Question:

Determine the normal force in tower AB.

14.44 Two electricity cables, each with the same dead weight, (force per length), are attached to a mast at B. The maximum sag in field BC is 3.60 metres. We can assume that the cables have a parabolic shape due to their dead weight.

Questions:

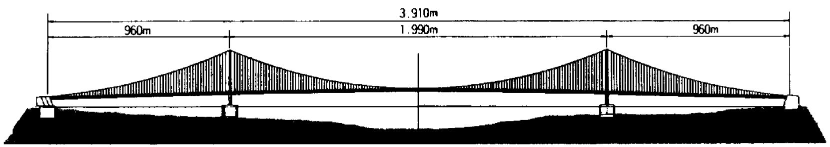
a. Determine the maximum sag a in field AB so that the total horizontal force that the cables at B exert on the mast is zero.
b. Determine the support reactions at A and C in case the dead weight of the cables is 12 N/m.



14.45 The Akashi Kaikyo Bridge in Japan was the largest suspension bridge in the world when it was opened in 1998/1999. The main span is 1990 m, and the side spans are 960 m. The following values apply for the position of the cable with respect to the average seawater level in Tokyo Bay:

Location	Level in metres
End anchors	+53
Towers	+297
Centre main span	+96

The load on the cable, consisting of the dead weight (of the cable, hangers and deck structure) and traffic loading is 450 kN/m.



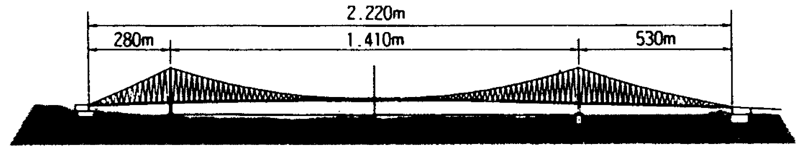
Questions:

a. Determine the forces that the cable in the middle span exerts on the towers.
b. Determine the forces that the cables in the end spans exert on the towers.
c. Determine the total load on a tower.
d. Determine the forces that the cables in the end spans exert on the anchor blocks.
e. Determine the maximum cable force in the middle span.
f. Determine the maximum cable force in the end spans.

14.46 On 17 July 1981, the suspension bridge over the Humber in Hull, England was opened, the longest suspension bridge in the world at the time. The bridge has a main span of 1410 m and end spans of respectively 290 and 530 m. A remarkable feature of this bridge is the major difference in length between both end spans. The following (estimated) values apply for the location of the cable:

Location	Level in metres
End anchors	+32.5
Towers	+162.5
Centre main span	+60.5

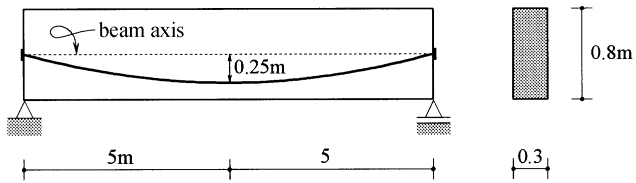
Assume that the load on the cable, consisting of the dead weight (of cable, hangers and deck structure) and the traffic loading, is 200 kN/m.



Questions:

a. Determine the forces that the cable in middle field exerts on the towers.
b. Determine the forces that the cable in the left-hand end field exerts on the tower.
c. Determine the total load on the left-hand tower.
d. Determine the forces that the cable in the left-hand end field exerts on the anchor block.
e. Determine the forces that the cable in the right-hand end field exerts on the tower.
f. Determine the total load on the right-hand tower.
g. Determine the forces that the cable in the right-hand end field exerts on the anchor block.
h. Determine the maximum cable force in the middle field.
i. Determine the maximum cable force in the left-hand end field.
j. Determine the maximum cable force in the right-hand end field.
k. The cable in the short end field was designed stronger than the cable in the other fields. Is this in line with your calculation?

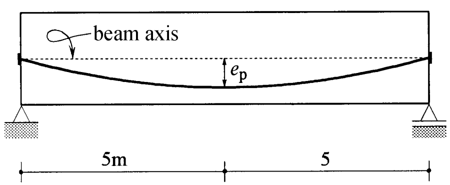
14.47 A simply supported prestressed beam has a length of 10 metres and a rectangular cross-section of $0.3 \times 0.8\ \text{m}^2$. The prestressing cable is parabolic with a maximum eccentricity of 0.25 m. The prestressing force is 1000 kN. The dead weight of the beam is $2500\ \text{kN/m}^3$. The variable load is 30 kN/m.



Questions:

a. Determine the forces that the prestressing cable exerts on the beam.
b. Determine the *M* and *V* diagrams for the beam resulting from only the prestressing.
c. Determine the *M* and *V* diagrams due to the prestressing and dead weight.
d. Determine the *M* and *V* diagrams due to the prestressing, dead weight and variable load.

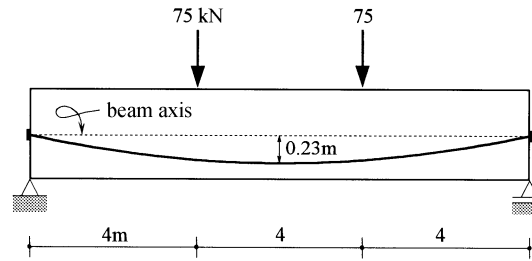
14.48 A simply supported prestressed beam with a length of 10 metres has a dead weight of 6 kN/m. The variable load is 24 kN/m. The parabolic tendon has a maximum eccentricity e_p. The prestressing force is F_p.



Questions:

a. How large must the product $F_p e_p$ be so that the maximum bending moment due to the prestressing and dead weight (in an absolute sense) is equal to the maximum bending moment due to the prestressing, dead weight and variable load?
b. How large must the product $F_p e_p$ be so that the maximum bending moment due to the prestressing and dead weight (in an absolute sense) is 5/7 the magnitude of the maximum bending moment due to the prestressing, dead weight and variable load?

14.49 A simply supported prestressed beam with a length of 12 metres has a dead weight of 7 kN/m. The variable load consists of two forces of 75 kN. The parabolic prestressing cable has an eccentricity of 0.23 m. The prestressing force is 1200 kN.

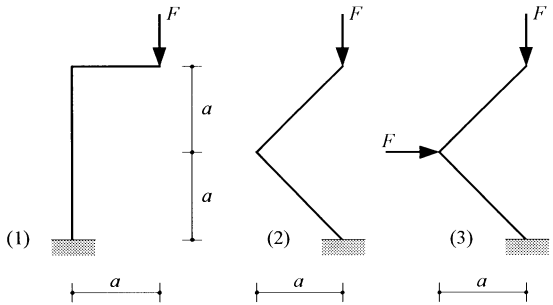


Questions:

a. Determine the forces that the prestressing cable exerts on the beam.
b. Determine the *M* and *V* diagrams for the beam due to only the prestressing.
c. Determine the *M* and *V* diagrams due to the prestressing and dead weight.
d. Determine the *M* and *V* diagrams due to the prestressing, dead weight and variable load.

Centre of point and line of force (Section 14.2)

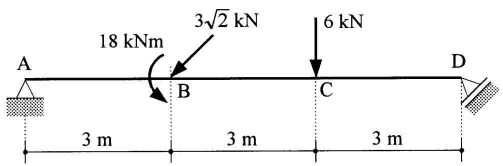
14.50: 1–3 Three fixed and bent beams are loaded by forces F.



Questions:

a. For each part of the structure draw the line of force and indicate whether it is a line of tension or a line of pressure.
b. From the line of force derive the variation of the normal force, shear force and bending moment. Draw the N, V and M diagrams for the entire structure.

14.51 Beam ABCD is supported at A on a hinge and at D on a roller; the roller track is inclined at an angle of 45°. A couple acts at B, as well as a 45° force. A vertical force acts at C. Dimensions and load can be found in the figure.



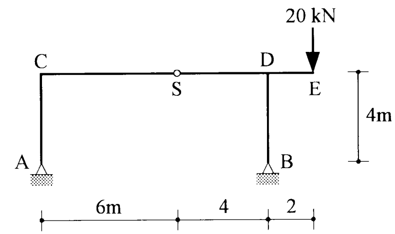
Questions:

a. Calculate and draw the support reactions at A and D.
b. Calculate and draw the M diagram, V diagram and N diagram, with the deformation symbols. Write down the relevant values.
c. For ABCD, draw the line(s) of force, and indicate whether they refer to tension or compression.
d. Where in cross-section C is the centre of force?

14.52 A three-hinged frame with overhang, is loaded by a force of 20 kN on the overhang.

Questions:

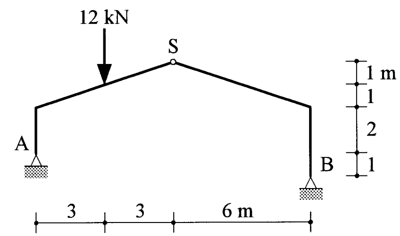
a. Determine the line of force for all parts of the frame and indicate whether they refer to tension or compression.
b. Determine the centre of force at cross-section C on column AC.
c. Determine the centre of force at cross-section C of girder CS.



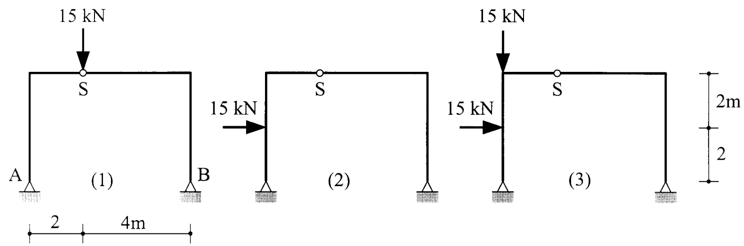
14.53 Pitched roof portal frame ASB is loaded by a vertical force of 12 kN.

Questions:

a. Determine the support reactions at A and B. Draw them as they act in reality and include relevant values.
b. Draw the M diagram, with the deformation symbols. Include relevant values.
c. Draw the lines of force for all parts of the frame. Indicate clearly whether they refer to tension or compression.



14.54: 1–3 The same three-hinged frame ASB is loaded in various ways by forces of 15 kN.



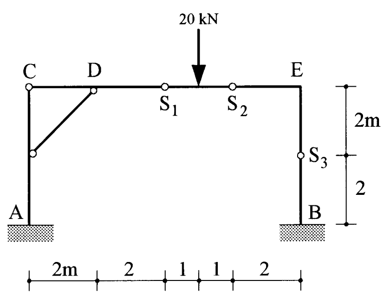
Questions:

a. Draw the lines of force for all parts of the frame. Indicate clearly whether they refer to tension or compression.
b. Draw the *M* and *N* diagrams for all parts of the frame.

14.55 Given a compound frame.

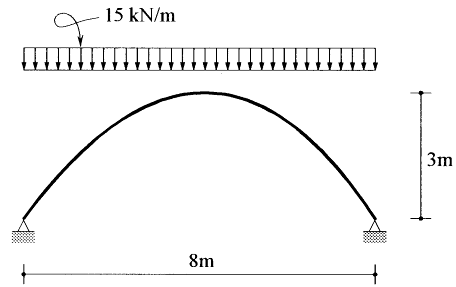
Questions:

a. Draw the *M* diagram for CDE.
b. Draw the line of force for DE.
c. How large is the normal force in DE?
d. Draw the line of force for CD.



Relationship between cable, line of force and structural shape (Section 14.3)

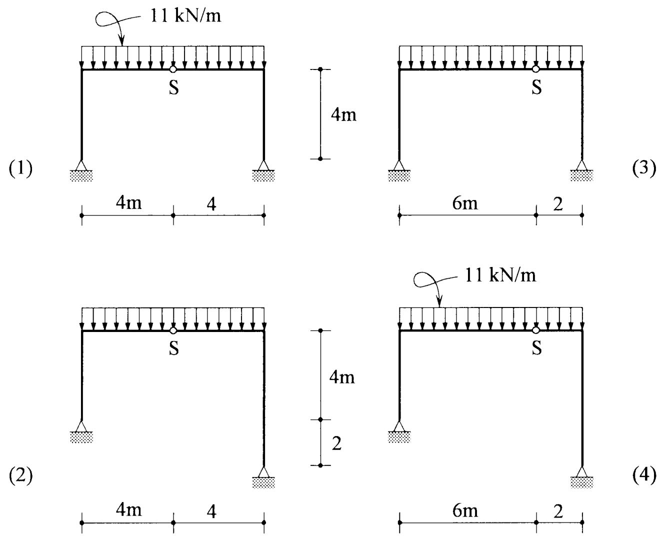
14.56 A parabolic arch has a uniformly distributed load. There is no axial deformation.



Questions:

a. Draw the line of force for the arch. Does it refer to tension or compression?
b. Determine and draw the horizontal support reactions.
c. Determine and draw the vertical support reactions.
d. Determine the maximum normal force in the arch.

14.57: 1–4 Four three-hinged frames have the same uniformly distributed load of 11 kN/m.



Questions:

a. Sketch the line of force for the girder.
b. Determine the normal force in the girder from this line of force.
c. Determine and draw the support reactions.
d. Draw the M, V and N diagrams for the entire structure.