11 Mathematical Description of the Relationship between Section Forces and Loading

In the previous chapter, a direct approach was used to determine the variation of section forces. Section forces were determined from the equilibrium of the isolated member part on the one or other side of the section. Usually the support reactions have to be determined first.

This chapter introduces a more *mathematical approach* based on the equilibrium of a small member segment with length Δx that approaches zero ($\Delta x \to 0$).

In Section 11.1, we derive the *differential equations for the equilibrium* of such an infinitesimal member segment.

Using examples, Sections 11.2 and 11.3 show how to determine the variation of the section forces. The examples in Section 11.2 relate to *extension* (relationship between N and q_x); those in Section 11.3 relate to *bending* (relationship between M_z, V_z and q_z).

Since no misunderstanding is possible, we will omit the index z in M_z and V_z to simplify the writing.

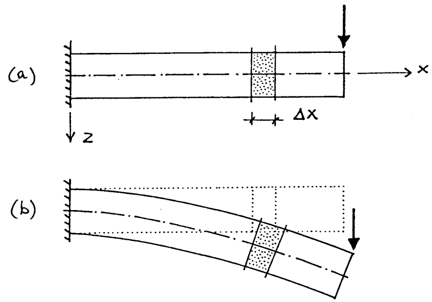


Figure 11.1 Member with (a) non-deformed geometry and (b) deformed geometry.

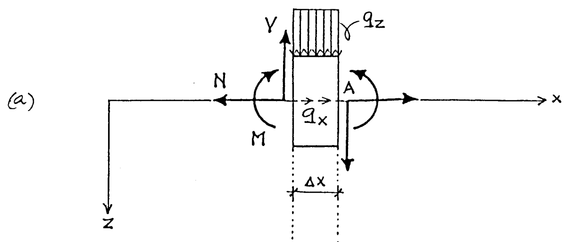


Figure 11.2 (a) Section forces N, V and M acting on the left-hand sectional plane of a small member segment with length Δx ($\Delta x \to 0$).

11.1 Differential equations for the equilibrium

The differential equations for equilibrium are derived from the equilibrium of a small member segment with a length Δx that approaches zero ($\Delta x \to 0$). We assume that the displacements due to the deformation of the member are negligible small. Therefore the equilibrium, including that of a small member element, can be related to the *non-deformed geometry* (see Figure 11.1).

In Figure 11.2a, a small segment with length Δx has been isolated from a member and greatly magnified. The member segment is subjected to q_x and q_z. The loads act on the member axis (this has not been drawn as such for q_z for the sake of clarity).

If length Δx of the member segment is sufficiently small ($\Delta x \to 0$), the distributed loads q_x and q_z can be considered *uniformly distributed*.

The (unknown) section forces on the left and right-hand sectional planes are shown in their positive direction. The section forces are a function of x, the location of the cross-section, and are generally different in both sectional planes.

In Figure 11.2a, it is assumed that the forces on the left-hand section are N, V and M. If the section forces increase over a distance Δx in the x direction by amounts ΔN, ΔV and ΔM, respectively (see Figures 11.2b to 11.2d), the forces on the right-hand sectional plane are then $N + \Delta N$, $V + \Delta V$ and $M + \Delta M$ (see Figure 11.2e).

From the *force equilibrium* of the small member segment it follows that

$$\sum F_x = -N + (N + \Delta N) + q_x \Delta x = 0, \qquad \text{(a)}$$

$$\sum F_z = -V + (V + \Delta V) + q_z \Delta x = 0. \qquad \text{(b)}$$

From the *moment equilibrium* it follows that (we have selected the moment sum about point A on the right-hand sectional plane)

$$\sum T_y|\mathrm{A} = -M - V\Delta x + (M + \Delta M) + q_z \Delta x \cdot \tfrac{1}{2}\Delta x = 0. \qquad \text{(c)}$$

With the three equilibrium conditions (a) to (c) this gives

$$\Delta N + q_x \Delta x = 0,$$

$$\Delta V + q_z \Delta x = 0,$$

$$\Delta M - V\Delta x + \tfrac{1}{2} q_z (\Delta x)^2 = 0.$$

After dividing by Δx we find

$$\frac{\Delta N}{\Delta x} + q_x = 0,$$

$$\frac{\Delta V}{\Delta x} + q_z = 0,$$

$$\frac{\Delta M}{\Delta x} - V = -\tfrac{1}{2} q_z \Delta x.$$

$\Delta N/\Delta x$ is the increase in the normal force per length in the x direction (see Figure 11.2b). In the limit $\Delta x \to 0$ this is known as the derivative from N with respect to x and is written $\mathrm{d}N/\mathrm{d}x$:

$$\lim_{\Delta x \to 0} \frac{\Delta N}{\Delta x} = \frac{\mathrm{d}N}{\mathrm{d}x}.$$

In the same way:

$$\lim_{\Delta x \to 0} \frac{\Delta V}{\Delta x} = \frac{\mathrm{d}V}{\mathrm{d}x},$$

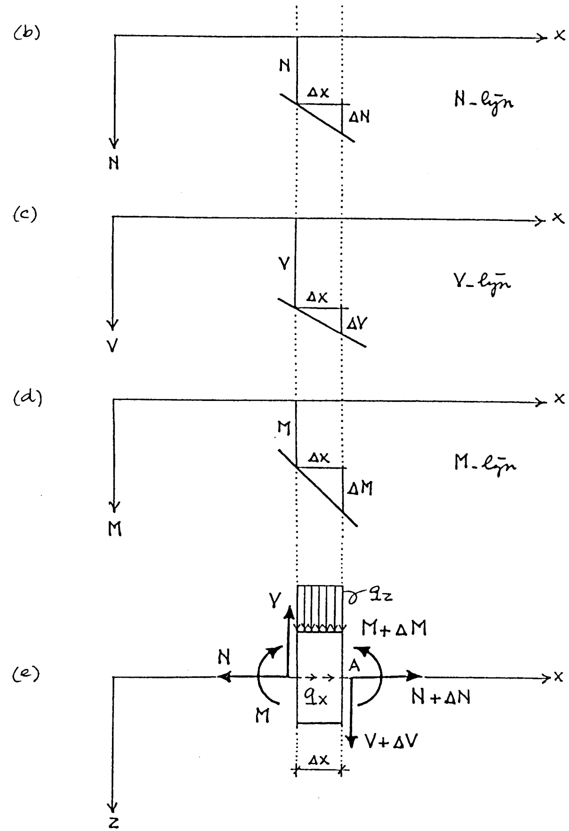


Figure 11.2 (b) to (d) Over a distance Δx in the x direction the section forces increase by amounts ΔN, ΔV and ΔM respectively. (e) The section forces on the right-hand sectional plane are then $N + \Delta N$, $V + \Delta V$ and $M + \Delta M$.

$$\lim_{\Delta x \to 0} \frac{\Delta M}{\Delta x} = \frac{\mathrm{d}M}{\mathrm{d}x}.$$

The three equations for the equilibrium of an elementary member segment with length Δx in the limit $\Delta x \to 0$ are

$$\frac{\mathrm{d}N}{\mathrm{d}x} + q_x = 0,$$
$$\frac{\mathrm{d}V}{\mathrm{d}x} + q_z = 0,$$
$$\frac{\mathrm{d}M}{\mathrm{d}x} - V = 0.$$

In the last equation, with $\Delta x \to 0$, the term $\frac{1}{2} q_z \Delta x$ has disappeared. This is justified since the contribution by q_z in the equation for the moment equilibrium is *one order smaller* than the contribution of the other terms.

The formulas derived give important *general information* about the variation of N, V and M in member segments (*fields*) where no concentrated forces and/or couples are acting. In Sections 11.2 and 11.3 this general information is translated into rules that allow us to easily draw N, V and M diagrams.

The first-order differential equation equation

$$\frac{\mathrm{d}N}{\mathrm{d}x} + q_x = 0 \quad (\textit{extension}) \tag{a}$$

provides a direct relationship between the (distributed) load q_x acting in the direction of the member axis, and the normal force N. This is known as the *equilibrium equation for extension*.

The equations

$$\frac{\mathrm{d}V}{\mathrm{d}x} + q_z = 0, \tag{b}$$

$$\frac{\mathrm{d}M}{\mathrm{d}x} - V = 0 \tag{c}$$

indicate the relationship between the (distributed) load q_z acting normal to the member axis, the shear force V and the bending moment M.

The shear force V can be eliminated by differentiating (c) to x and adding it to (b). This gives the second-order differential equation

$$\frac{\mathrm{d}^2 M}{\mathrm{d}x^2} + q_z = 0 \quad (\textit{bending}). \tag{d}$$

This equation provides the direct relationship between the (distributed) load q_z normal to the member axis and the bending moment M. This is known as the *equilibrium equation for bending*.

The variation of the normal force N depends only on the load in the direction of the member axis, q_x. The variation of the bending moment M and the shear force V depends only on the load q_z normal to the member axis. For a member, this means that the equilibrium equations for *extension* (only normal forces due to axial loads) and *bending* (only bending moments and shear forces due to loads normal to the member axis) can be treated separately.[1]

Comment 1: In Sections 10.2.1 to 10.2.3 we discussed the fact that axial loads give only normal forces (*extension*) and that loads normal to the mem-

[1] The loads have to be applied on the member axis.

ber axis (including couples) give only shear forces and bending moments (*bending*).

Comment 2: The derivation is not applicable if a concentrated force or a concentrated couple acts on the member segment. In that case, there is a "step change" or "abrupt change in slope" in the N, V and/or M diagram; see the examples in Section 10.2.1. N, V and/or M are, as functions of x, no longer continuous and/or continuously differentiable. In such a case, the member can be split into a number of *fields*, so that the differential equation is applicable for each individual field (see Section 11.2, Example 2 and Section 11.3, Example 4).

11.2 Mathematical elaboration of the relationship between N and q_x (extension)

We derived for the relationship between the normal force N and the distributed axial load q_x

$$\frac{\mathrm{d}N}{\mathrm{d}x} + q_x = 0$$

or in other words

$$\frac{\mathrm{d}N}{\mathrm{d}x} = -q_x.$$

By integrating once, we find the variation of the normal force N:

$$N = -\int q_x \,\mathrm{d}x.$$

With the exception of a constant, we have determined the *indefinite integral* (or *primitive function*) of q_x, and therefore the variation of the normal force N.

The unknown integration constant is found using a known (prescribed) value of N at one of the member *ends*. This is referred to as an *end condition*.

It is sometimes necessary to divide the member into a number of segments (*fields*), as in Figure 11.3. In that case, we also have to formulate conditions for the *joining* from one field to another. These conditions are referred to as *joining conditions*.

Purely mathematically both the end conditions and joining conditions can be regarded as boundary conditions for a specific field. They can be derived from the equilibrium of a small member segment with length Δx ($\Delta x \rightarrow 0$) at the boundaries of the field (an end and/or a joining).

For a statically determinate member, there are always sufficient boundary conditions to find the normal force variation *without previously determining the support reactions*.

We will illustrate this by means of two examples previously covered in Section 10.2.3:

- A column subject to its dead weight.
- A simply supported member that is loaded over two-thirds of its length by a uniformly distributed axial load along the member axis.

Example 1

Figure 11.4a gives the model of a column and its load.

Question:

Determine the variation of the normal force (the N diagram) from the differential equations for the equilibrium.

Solution:

The units used are m and kN; they are omitted hereafter from the calculation.

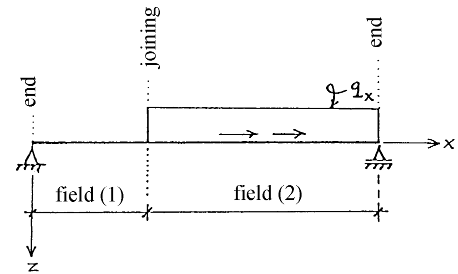


Figure 11.3 Since the distributed load is not acting along the entire length, the member has to be divided into two fields.

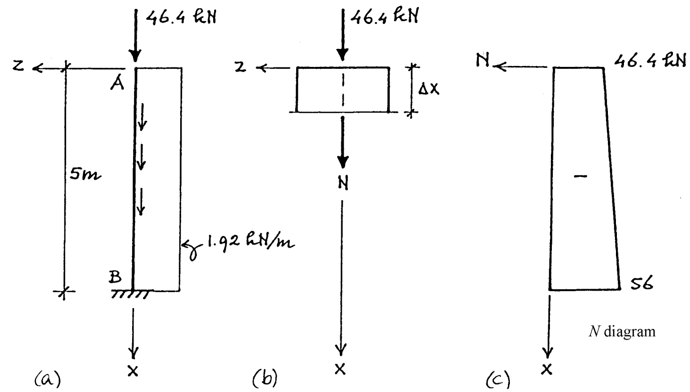


Figure 11.4 (a) Model of a column and its load. (b) The boundary condition (end condition) $N = -46.4$ kN follows from the force equilibrium in the x direction of the small end segment in A (with $\Delta x \rightarrow 0$). (c) N diagram.

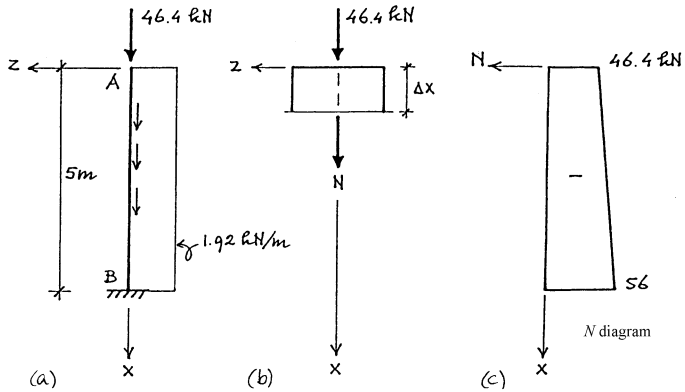


Figure 11.4 (a) Model of a column and its load. (b) The boundary condition (end condition) $N = -46.4$ kN follows from the force equilibrium in the x direction of the small end segment in A (with $\Delta x \rightarrow 0$). (c) N diagram.

In the given coordinate system:

$$q_x = 1.92 \;\; \text{kN/m}$$

so that

$$N = -\int q_x \, \text{d}x = -\int 1.92 \, \text{d}x = (-1.92x + C) \;\; \text{kN}. \qquad \text{(a)}$$

The *integration constant* C is found from the fact that there is a compressive force of 46.4 kN at the top of the column. The boundary condition (end condition) is therefore

$$x = 0 : N = -46.6 \text{ kN}. \qquad \text{(b)}$$

This boundary condition can also be derived more formally from the force equilibrium in the x direction of a small member segment with length Δx ($\Delta x \rightarrow 0$) at the top of the column (see Figure 11.4b):

$$\sum F_x = 46.4 + N = 0 \rightarrow N = -46.4 \text{ kN}.$$

With $\Delta x \rightarrow 0$ the contribution of $q_x \Delta x$, due to the distributed load, disappears.

Substitute the values of x and N from (b) in (a) and we find

$$C = -46.6 \text{ kN}.$$

This gives the variation of the normal force N:

$$N = (-1.92x - 46.4) \text{ kN}.$$

The normal force diagram is shown in Figure 11.4c. The results agree with what we found earlier in Section 10.2.3, Example 1.

Example 2
In Figure 11.5a, a uniformly distributed axial load q is acting on segment BC of member ABC which is simply supported at A and C.

Question:
Determine the variation of the normal force (the N diagram) from the differential equations for the equilibrium.

Solution:
Since the uniformly distributed load q acts only on part of the member, we have to distinguish between two segments:

- segment AB ($0 < x < a$), hereafter known as field (1).
- segment BC ($a < x < 3a$), hereafter known as field (2).

The normal force variation is determined per field. The field number is used as upper index for units that are field-dependent.

Field (1):

$$\frac{\mathrm{d}N^{(1)}}{\mathrm{d}x} = -q_x^{(1)} = 0 \rightarrow N^{(1)} = C^{(1)}.$$

In an unloaded field, the normal force is constant.

Field (2):

$$\frac{\mathrm{d}N^{(2)}}{\mathrm{d}x} = -q_x^{(2)} = -q \rightarrow N^{(2)} = -qx + C^{(2)}.$$

In a field with a uniformly distributed load, the normal force is linear.

Per field, there is one unknown integration constant; with two fields there is a total of two integration constants, $C^{(1)}$ and $C^{(2)}$. There are two boundary conditions available to solve these constants: a *joining condition* at B ($x = a$) and an *end condition* at C ($x = 3a$).

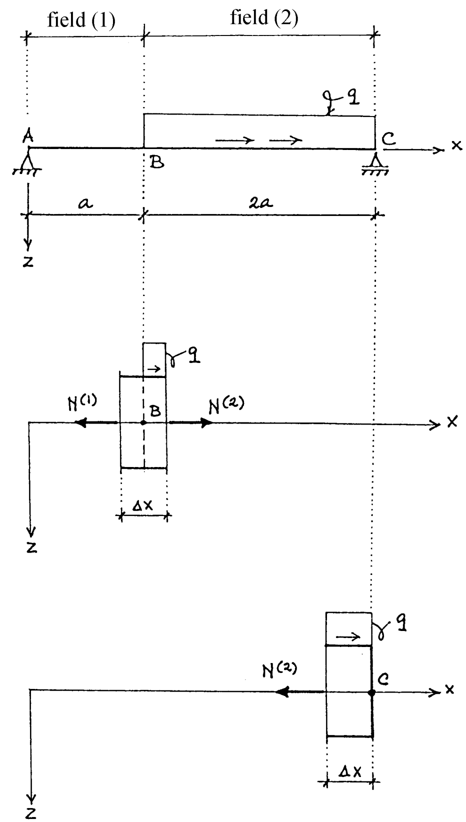


Figure 11.5 (a) A simply supported member with a uniformly distributed axial load on section BC. (b) The boundary condition (joining condition) at B, $N^{(1)} = N^{(2)}$, follows from the force equilibrium in the x direction of a small member segment at the joining at B (with $\Delta x \rightarrow 0$). (c) The boundary condition (end condition) at C, $N^{(2)} = 0$, follows from the force equilibrium in the x direction of a small end segment at C (with $\Delta x \rightarrow 0$).

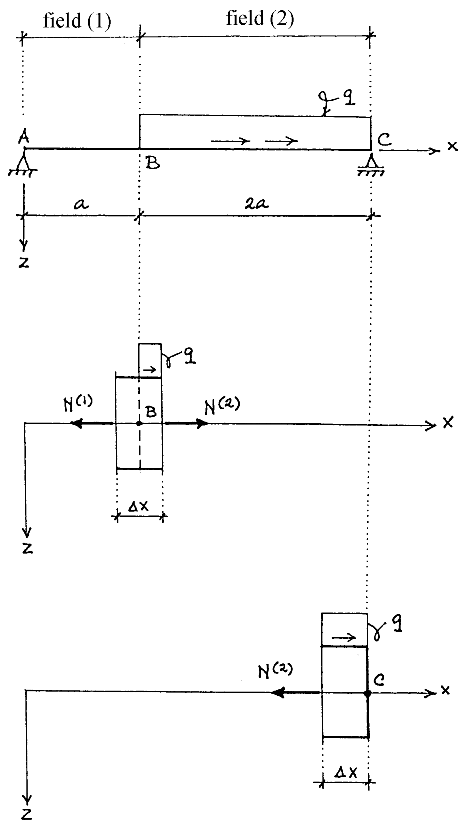


Figure 11.5 (a) A simply supported member with a uniformly distributed axial load on section BC. (b) The boundary condition (joining condition) at B, $N^{(1)} = N^{(2)}$, follows from the force equilibrium in the x direction of a small member segment at the joining at B (with $\Delta x \to 0$). (c) The boundary condition (end condition) at C, $N^{(2)} = 0$, follows from the force equilibrium in the x direction of a small end segment at C (with $\Delta x \to 0$).

- The *joining condition* at B ($x = a$)
  At B, the normal force in field (1) is equal to the normal force in field (2):

  $$x = a : N^{(1)} = N^{(2)}. \quad \text{(a)}$$

  This boundary condition can be derived directly from the force equilibrium in the x direction of the small member segment with length Δx ($\Delta x \to 0$) at the joining in B (see Figure 11.5b). The contribution of q disappears from the equilibrium equation as $\Delta x \to 0$.

- The *end condition* at C ($x = 3a$)
  At a roller support C the horizontal support reaction is zero, as is therefore the normal force:

  $$x = 3a : N^{(2)} = 0. \quad \text{(b)}$$

  This boundary condition can also be derived from the equilibrium of a small member segment at the end of the member (see Figure 11.5c). Here also, the contribution of q disappears in the equilibrium equation as $\Delta x \to 0$.

Elaboration of the conditions (a) and (b) leads to two equations with two unknowns:

$$C^{(1)} - C^{(2)} = -qa,$$

$$C^{(2)} = 3qa.$$

The solution is

$$C^{(1)} = 2qa,$$

$$C^{(2)} = 3qa.$$

This results in the normal force variation for both fields:

Field (1):

$$N^{(1)} = 2qa \quad (0 \leq x < a).$$

Field (2):

$$N^{(2)} = -qx + 3qa \quad (a < x \leq 3a).$$

Figure 11.6 shows the N diagram. The results agree with those found previously in Section 10.2.2, Example 2.

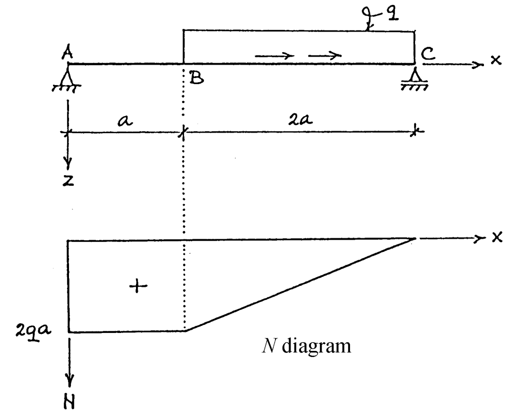


Figure 11.6 The loaded member with its N diagram.

11.3 Mathematical elaboration of the relationship between M, V and q_z (bending)

We derived the following for the relationship between M, V and a distributed load q_z normal to the member axis:

$$\frac{\mathrm{d}V}{\mathrm{d}x} + q_z = 0,$$

$$\frac{\mathrm{d}M}{\mathrm{d}x} - V = 0.$$

Eliminating the shear force leads to a direct relationship between the bending moment M and the distributed load q_z:

$$\frac{\mathrm{d}^2 M}{\mathrm{d}x^2} + q_z = 0$$

or in other words:

$$\frac{\mathrm{d}^2 M}{\mathrm{d}x^2} = -q_z.$$

On the basis of this latter equation, we find the shear force V after integrating once:

$$\frac{\mathrm{d}M}{\mathrm{d}x} = V = -\int q_z \,\mathrm{d}x$$

and after integrating again we find the variation of the bending moment M:

$$M = \int V \,\mathrm{d}x = -\int \left(\int q_z \,\mathrm{d}x \right) \mathrm{d}x.$$

With each integration, an integration constant appears. This means that the expression for the shear force V contains one unknown (C_1), and that for the bending moment M contains two (C_1 and C_1).

The two constants C_1 and C_2 follow from *end conditions and/or joining conditions* relating to V and M. They can be derived from the equilibrium of a small member segment with length Δx ($\Delta x \to 0$) on the boundaries (end or joining) of the field.

For statically-determinate members, there are always sufficient end and/or joining conditions to find the variation of the shear force and bending moment *without previously determining the support reactions*. This is illustrated using the following four examples:

1. A fixed beam, loaded at its free end by a concentrated load.
2. A simply supported beam and a beam fixed at one of its ends, both with a uniformly distributed load along its entire length.
3. A simply supported beam with a triangular load.
4. A simply supported beam with overhang (cantilever beam) and a uniformly distributed load along its entire length.

Example 1
Figure 11.7a shows a beam AB fixed at A and of length ℓ. At its free end B the beam is loaded normal to its axis by a force F.

Question:
Determine the V and M diagrams using the differential equations for equilibrium.

Solution:
For $0 < x < \ell$ it holds that

$$\frac{d^2M}{dx^2} = -q_z = 0.$$

Repeated integration gives

$$\frac{dM}{dx} = V = C_1, \qquad \text{(a)}$$

$$M = C_1x + C_2. \qquad \text{(b)}$$

In an unloaded field the shear force is constant and the bending moment is linear.

The integration constants C_1 and C_2 are found from the boundary conditions at the free end B. Here both V and M have a prescribed value: the shear force is equal to F (pay attention to the sign), and the bending moment is zero:

$$x = \ell : V = +F, \qquad \text{(c)}$$

$$x = \ell : M = 0. \qquad \text{(d)}$$

The boundary conditions can also be derived from the force and moment equilibrium of a small member segment with length Δx ($\Delta x \to 0$) at the

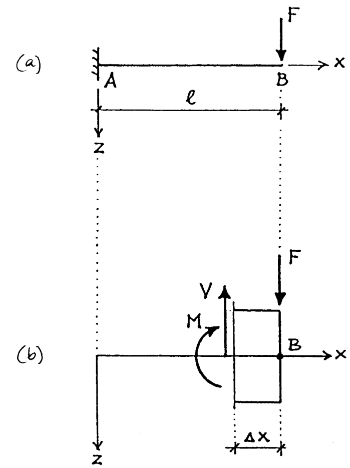


Figure 11.7 (a) A beam fixed at A and loaded at its free end B by a force F normal to the beam axis. (b) The boundary conditions $V = +F$ and $M = 0$ are found from the force and moment equilibrium of a small boundary segment at B (with $\Delta x \to 0$).

free end B (see Figure 11.7b). The section forces V and M, which here are acting on a negative section plane, have to be drawn in accordance with their positive directions. The equilibrium of the member segment gives

$$\sum F_z = -V + F = 0 \Rightarrow V = +F,$$

$$\sum T_z|\mathrm{B} = -M - V\Delta x = 0 \text{ (with } \Delta x \to 0) \;\Rightarrow M = 0.$$

Substitute (c) in (a) and (d) in (b); elaboration of the boundary conditions leads to

$$C_1 = +F,$$

$$C_2 = -F\ell.$$

This gives the variation of shear force V and bending moment M for beam AB:

$$V = F,$$

$$M = F_x - F\ell = -F(\ell - x).$$

The V and M diagrams are shown in Figure 11.8. The shear force V is constant. The bending moment M is negative everywhere and is linear. The bending moment (in the absolute sense) has its maximum at the fixed end:

$$|M|_{\max} = F\ell.$$

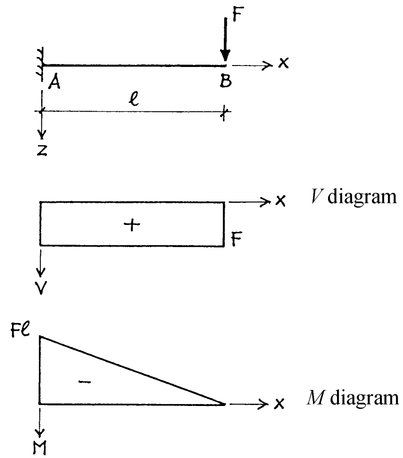


Figure 11.8 The loaded beam and its V and M diagrams.

Example 2
In Figure 11.9, a uniformly distributed load q is acting normal to the beam axis over the entire length ℓ of beam AB. The method of support in A and/or B is given below.

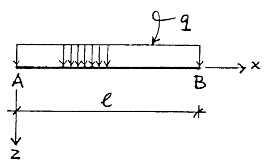


Figure 11.9 A beam for which the type of support at A and/or B is still unknown, with a uniformly distributed load normal to the member axis.

Question:
For the following three cases, determine the V and M diagrams using the differential equations for equilibrium (see Figure 11.10):

(a) The beam is simply supported at A and B;
(b) The beam has a fixed support at A and a roller support at B;
(c) The beam has a fixed support at B and a roller support at A.

Solution:
In all three cases, with $q_z = q$ the following applies:

$$\frac{d^2M}{dx^2} = -q,$$

$$\frac{dM}{dx} = V = -\int q\,dx = -qx + C_1,$$

$$M = \int V\,dx = \int (-qx + C_1)\,dx = -\tfrac{1}{2}qx^2 + C_1x + C_2.$$

Due to a uniformly distributed load, the shear force is linear, and the bending moment is quadratic (parabolic).

The constants C_1 and C_2 are determined by the boundary conditions (associated with the type of support) at A and/or B.

The boundary conditions are

case (a)	case (b)	case (c)
$x = 0 : M = 0$	$x = \ell : V = 0$	$x = 0 : V = 0$
$x = \ell : M = 0$	$x = \ell : M = 0$	$x = 0 : M = 0$

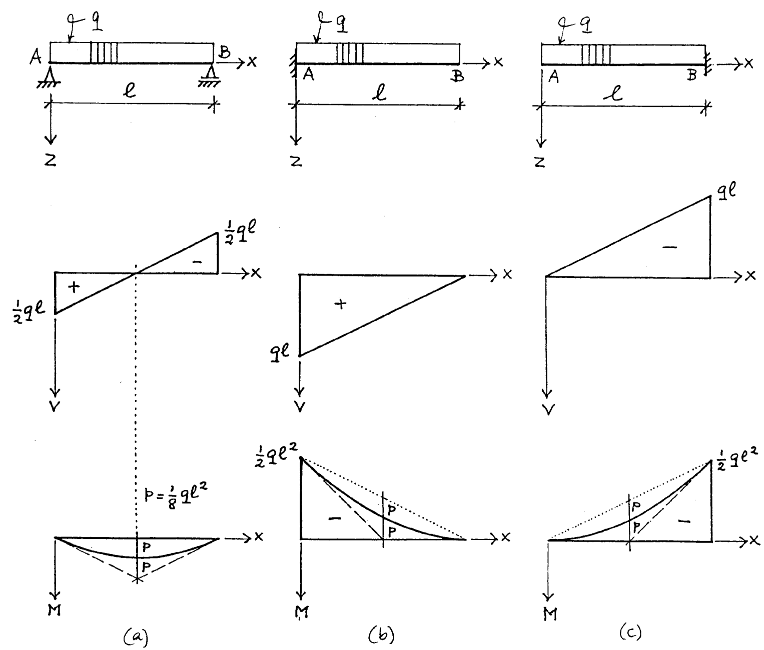


Figure 11.10 The same beam supported in three different ways, with the associated V and M diagrams: (a) simply supported at A and B, (b) fixed at A and (c) fixed at B.

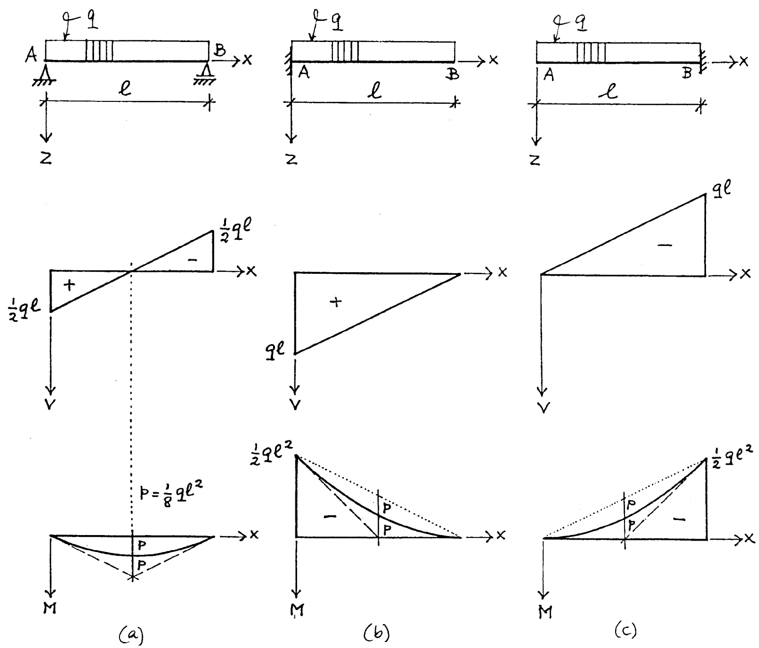


Figure 11.10 The same beam supported in three different ways, with the associated V and M diagrams: (a) simply supported at A and B, (b) fixed at A and (c) fixed at B.

- Elaboration of the boundary conditions in case (a):

$$x = 0 : M = C_2 = 0 \Rightarrow C_2 = 0,$$

$$x = \ell : M = -\tfrac{1}{2}q\ell^2 + C_1\ell + C_2 = 0 \Rightarrow C_1 = \tfrac{1}{2}q\ell$$

from which it follows that

$$V = -qx + \tfrac{1}{2}q\ell = \tfrac{1}{2}q(\ell - 2x),$$

$$M = -\tfrac{1}{2}qx^2 + \tfrac{1}{2}q\ell x = \tfrac{1}{2}qx(\ell - x).$$

- Elaboration of the boundary conditions in case (b):

$$x = \ell : V = -q\ell + C_1 = 0 \Rightarrow C_1 = q\ell,$$

$$x = \ell : M = -\tfrac{1}{2}q\ell^2 + C_1\ell + C_2 = 0 \Rightarrow C_2 = -\tfrac{1}{2}q\ell^2$$

from which it follows that

$$V = -qx + q\ell = q(\ell - x),$$

$$M = -\tfrac{1}{2}qx^2 + q\ell x - \tfrac{1}{2}q\ell^2 = -\tfrac{1}{2}q(\ell - x)^2.$$

- Elaboration of the boundary conditions in case (c):

$$x = 0 : V = C_1 = 0 \Rightarrow C_1 = 0,$$

$$x = 0 : M = C_2 = 0 \Rightarrow C_2 = 0$$

from which it follows that

$$V = -qx,$$

$$M = -\tfrac{1}{2}qx^2.$$

Figure 11.10 shows the V and M diagrams for all three cases. The tangents to the M diagram are also shown at A and B. These intersect in $x = \frac{1}{2}\ell$, at mid-span. In the figure, an important variable p is shown: $p = \frac{1}{8}q\ell^2$. We will make use of p in Chapter 12.

Below we again show how, for two cases, the boundary conditions (end conditions) can be derived from the equilibrium of a small element with length Δx ($\Delta x \to 0$) at the beam ends.

Boundary condition at the hinged support A in case (a)

In Figure 11.11a, a member segment with length Δx ($\Delta x \to 0$) has been isolated at the hinged support at A. The figure shows all the forces acting on it, including the unknown vertical support reaction at A. Moment equilibrium about A requires

$$\sum T_y|\mathrm{A} = M - V\Delta x - q\Delta x \cdot \tfrac{1}{2}\Delta x = 0.$$

For $\Delta x \to 0$ the terms with Δx disappear and we find the boundary condition at A:

$$M = 0.$$

Boundary conditions at the free member end B in case (b)

In Figure 11.11b, the small "last" member segment at the free end B is shown, with all the forces acting on it. The element has a length Δx.

The equations for the equilibrium are

$$\sum F_z = -V + q\Delta x = 0,$$

$$\sum T_y|\mathrm{B} = -M - V\Delta x + q\Delta x \cdot \tfrac{1}{2}\Delta x = 0.$$

For $\Delta x \to 0$ the terms with Δx disappear in both equations and we find the boundary conditions at the free member end B:

$$V = 0,$$

$$M = 0.$$

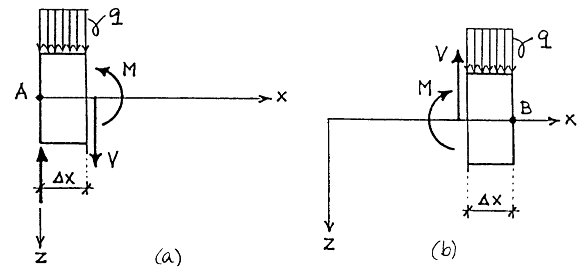


Figure 11.11 (a) Small end segment at hinged support A (see Figure 11.10a). (b) Small end segment at free end B (see Figure 11.10b).

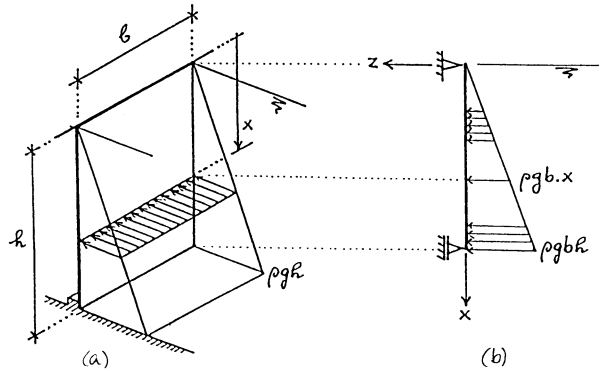


Figure 11.12 (a) The water pressure on a water-retaining slide (b) modelled as a line load on a line element.

Example 3

Figure 11.12a shows a water-retaining slide of width b and height h, which is supported by a hinge at the top and supported against a sill below. The mass density of water is ρ.

Question:
Model the slide as a line element with a line load, and use the differential equations for equilibrium to find the variation of the shear force and the bending moment.

Solution:
The water pressure on the slide at a depth x is

$$p = \rho g x,$$

in which g is the gravitational field strength. The water pressure increases linearly with the depth.

In Figure 11.12b the slide with width b is modelled as a line element (beam). The support at the base is considered a roller support. At a depth x the load on the slide is

$$q_z = pb = \rho g b x.$$

It holds

$$\frac{d^2 M}{dx^2} = -q_z = -\rho g b x,$$

$$V = \frac{dM}{dx} = -\int q_z \, dx = -\int \rho g b x \, dx = -\tfrac{1}{2}\rho g b x^2 + C_1,$$

$$M = \int V \, dx = \int \left(-\tfrac{1}{2}\rho g b x^2 + C_1\right) dx = -\tfrac{1}{6}\rho g b x^3 + C_1 x + C_2.$$

Due to a linear distributed load, the shear force is a quadratic (parabolic) function in x, and the bending moment is a third degree (cubic) function in x.

The constants C_1 and C_2 follow from the boundary conditions that the bending moment at both the top and the base is zero:

$$x = 0 : M = 0,$$

$$x = h : M = 0.$$

Elaboration of the boundary conditions leads to

$$C_1 = \tfrac{1}{6}\rho g b h^2,$$

$$C_2 = 0.$$

The expressions for the shear force and the bending moment are therefore

$$V = \tfrac{1}{6}\rho g b(h^2 - 3x^2), \qquad \text{(a)}$$

$$M = \tfrac{1}{6}\rho g b(h^2 x - x^3). \qquad \text{(b)}$$

The V diagram is a second degree curve (parabola); the M diagram is a third degree curve (cubic).

Both diagrams are shown in Figure 11.13. At A and B tangents to the V and M diagrams are also shown. Note that the tangents to the M diagram intersect at $x = \frac{2}{3}h$, the location where the resultant R of the *triangular load* acts. We will make use of this in the next chapter.

The bending moment is extreme when

$$\frac{\mathrm{d}M}{\mathrm{d}x} = V = 0.$$

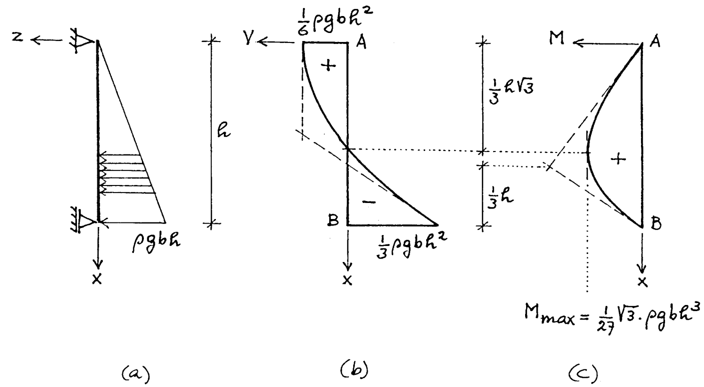


Figure 11.13 (a) The water-retaining slide modelled as a beam with its (b) shear force diagram and (c) bending moment diagram.

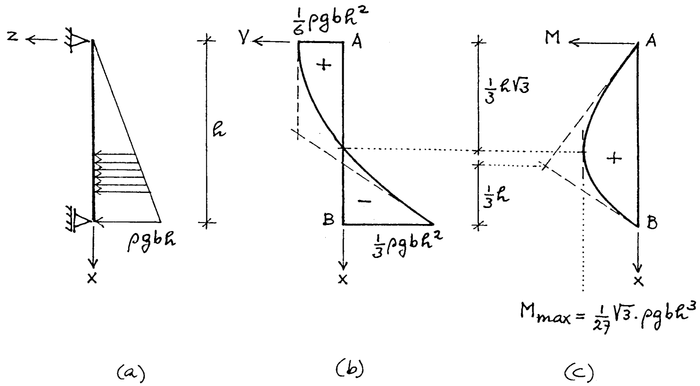


Figure 11.13 (a) The water-retaining slide modelled as a beam with its (b) shear force diagram and (c) bending moment diagram.

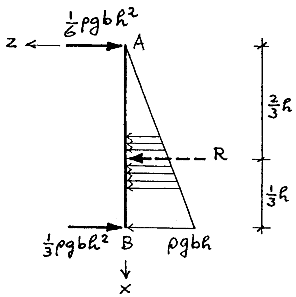


Figure 11.14 The support reactions are found from the V diagram as the shear forces at the member ends.

In other words: the bending moment is extreme where the shear force is zero. This location can be found from (a):

$$V = \tfrac{1}{6}\rho g b(h^2 - 3x^2) = 0 \Rightarrow x = \tfrac{1}{3}h\sqrt{3}.$$

Substituting this value of x in the expression for M gives the maximum bending moment:

$$M_{\max} = M_{(x=\frac{1}{3}h\sqrt{3})} = \tfrac{1}{6}\rho g b\left\{h^2\left(\tfrac{1}{3}h\sqrt{3}\right) - \left(\tfrac{1}{3}h\sqrt{3}\right)^3\right\}$$

$$= \frac{\sqrt{3}}{27}\rho g b h^3 = 0.064\rho g b h^3.$$

The support reactions can be found from the V diagram as the shear forces on the beam ends:

$$x = 0 : V = +\tfrac{1}{6}\rho g b h^2,$$

$$x = h : V = -\tfrac{1}{3}\rho g b h^2.$$

These shear forces on the boundaries of the beam are shown in Figure 11.14. As a check, one can be examine whether the beam as a whole is in equilibrium. The resultant $R = \frac{1}{2}\rho g b h^2$ of the distributed load acts at $x = \frac{2}{3}h$. This gives

$$\sum F_z = R - \tfrac{1}{6}\rho g b h^2 - \tfrac{1}{3}\rho g b h^2 = 0,$$

$$\sum T_y|\mathrm{B} = R \cdot \tfrac{1}{3}h - \tfrac{1}{6}\rho g b h^2 \cdot h = 0.$$

Force and moment equilibrium therefore are satisfied.

Example 4

Cantilever beam ABC in Figure 11.15a is simply supported at A and B, and has an overhang BC at B. The beam is carrying a uniformly distributed load of 40 kN/m over its entire length. The dimensions are given in the figure.

Question:

Using the differential equations for equilibrium, determine the variation of the shear force and the bending moment.

Solution:

The as yet unknown support reaction at B gives a discontinuity in the distributed load on the isolated member. At this point, the differential equations for the equilibrium are not valid (see Section 11.1). The beam therefore has to be split into two parts or *fields*:

- part AB with (0 m) $< x <$ (5 m), hereafter known as field (1).
- part BC with (5 m) $< x <$ (7 m), hereafter known as field (2).

The differential equations for the equilibrium are elaborated per field. For the units that are field-dependent, the field number is used as upper index.

All units are expressed in m and kN. The units are hereafter omitted from the calculation.

For field (1) with (0 m) $< x <$ (5 m):

$$q_z = 40\text{ kN/m},$$

$$\frac{\mathrm{d}^2 M^{(1)}}{\mathrm{d}x^2} = -q_z = -40\text{ kN/m},$$

$$V^{(1)} = \frac{\mathrm{d}M^{(1)}}{\mathrm{d}x} = -\int 40\,\mathrm{d}x = (-40x + C_1^{(1)})\text{ kN},$$

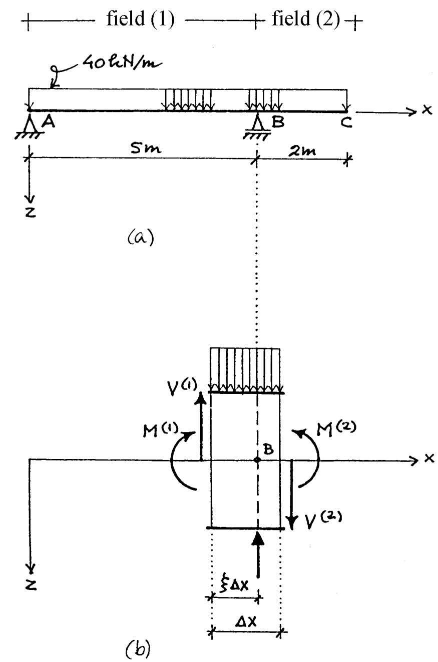


Figure 11.15 (a) For the cantilever beam with a uniformly distributed load along the entire length we have to distinguish two fields. (b) The joining condition, $M^{(1)} = M^{(2)}$, at support B is found from the moment equilibrium of a small beam segment with length $\Delta x \to 0$.

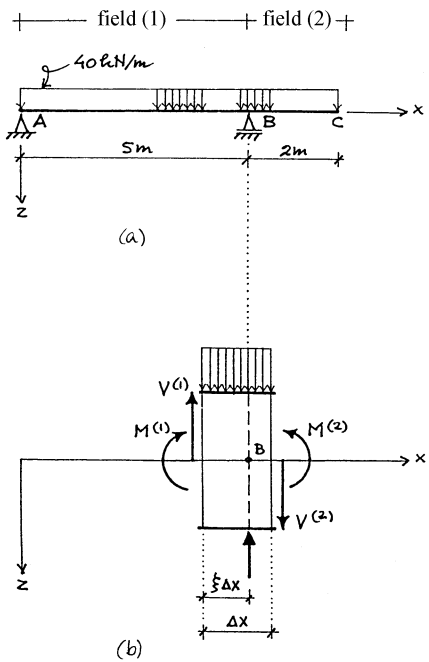


Figure 11.15 (a) For the cantilever beam with a uniformly distributed load along the entire length we have to distinguish two fields. (b) The joining condition, $M^{(1)} = M^{(2)}$, at support B is found from the moment equilibrium of a small beam segment with length $\Delta x \to 0$.

$$M^{(1)} = \int V^{(1)}\,\mathrm{d}x = \int (-40x + C_1^{(1)})\,\mathrm{d}x$$
$$= (-20x^2 + C_1^{(1)}x + C_2^{(1)})\text{ kNm}.$$

For field (2) with (5 m) < x < (7 m):

$$q_z = 40\text{ kN/m},$$
$$\frac{\mathrm{d}^2 M^{(2)}}{\mathrm{d}x^2} = -q_z = -40\text{ kN/m},$$
$$V^{(2)} = \frac{\mathrm{d}M^{(2)}}{\mathrm{d}x} = -\int 40\,\mathrm{d}x = (-40x + C_1^{(2)})\text{ kN},$$
$$M^{(2)} = \int V^{(2)}\,\mathrm{d}x = \int (-40x + C_1^{(2)})\,\mathrm{d}x$$
$$= (-20x^2 + C_1^{(2)}x + C_2^{(2)})\text{ kNm}.$$

There are four boundary conditions available for solving the total of four unknown integration constants $C_1^{(1)}$, $C_2^{(1)}$, $C_1^{(2)}$ and $C_2^{(2)}$:

1. end condition at A: $x = 0;\ M^{(1)} = 0.$
2. joining condition at B: $x = 5;\ M^{(1)} = M^{(2)}.$
3. end condition at C: $x = 7;\ V^{(2)} = 0.$
4. end condition at C: $x = 7;\ M^{(2)} = 0.$

For the joining condition at B, we will show below how this can be derived from the equilibrium of a member segment with length Δx ($\Delta x \to 0$) at the joining of the two fields.

Figure 11.15b shows the small member segment with the four section forces acting on it and the unknown support reaction at B. If B is not located in the middle of the element, but at a distance $\xi \Delta x$ from the left-hand section plane, respectively $(1-\xi)\Delta x$ from the right-hand section plane $(0 < \xi < 1)$, then the equation for the moment equilibrium about B is

$$\sum T_y|\mathrm{B} =$$
$$-M^{(1)} + M^{(2)} - V^{(1)}\xi\Delta x - V^{(2)}(1-\xi)\Delta x + q_z\Delta x\left(\xi - \tfrac{1}{2}\right)\Delta x = 0.$$

As $\Delta x \to 0$ all terms with Δx disappear and we are left with

$$M^{(1)} = M^{(2)}.$$

This is the joining condition we are looking for.

The derivation is considerably simpler if, as is standard, the beam element is chosen such that B is in the middle. In that case $\xi = \frac{1}{2}$, and

$$\sum T_y|\mathrm{B} = -M^{(1)} + M^{(2)} - V^{(1)}\tfrac{1}{2}\Delta x - V^{(2)}\tfrac{1}{2}\Delta x = 0.$$

As $\Delta x \to 0$ this again gives the joining condition we are looking for.

Elaboration of the end conditions and the joining condition leads to a set of four equations and four unknowns:

1. end condition at A:
$$C_2^{(1)} = 0.$$
2. joining condition at B:
$$-20 \times 5^2 + C_1^{(1)} \times 5 + C_2^{(1)} = -20 \times 5^2 + C_1^{(2)} \times 5 + C_2^{(2)}.$$
3. end condition at C:
$$-40 \times 7 + C_1^{(2)} = 0.$$
4. end condition at C:
$$-20 \times 7^2 + C_1^{(2)} \times 7 + C_2^{(2)} = 0.$$

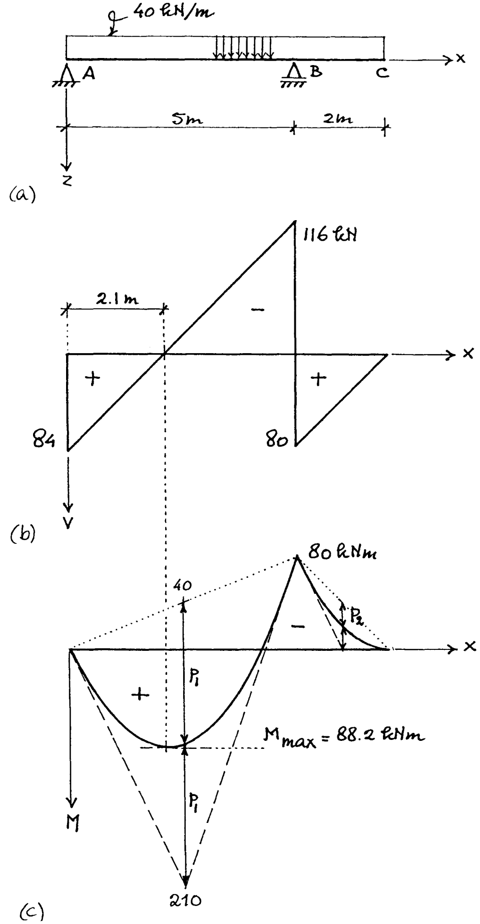


Figure 11.16 (a) A cantilever beam with a uniformly distributed load, and the associated (b) V diagram and (c) M diagram. The bending moment M is extreme where the shear force V is zero or changes sign.

Or more neatly put

$$\begin{aligned} C_2^{(1)} &= 0,\\ 5C_1^{(1)} + C_2^{(1)} - 5C_1^{(2)} - C_2^{(2)} &= 0,\\ C_1^{(2)} &= 280,\\ 7C_1^{(2)} + C_2^{(2)} &= 980. \end{aligned}$$

The solution to the set is

$$C_1^{(1)} = 84 \text{ kN},$$

$$C_2^{(1)} = 0,$$

$$C_1^{(2)} = 280 \text{ kN},$$

$$C_2^{(2)} = -980 \text{ kNm}.$$

From this it follows for field (1) with $(0 \text{ m}) < x < (5 \text{ m})$ that

$$V^{(1)} = (-40x + 84) \text{ kN},$$

$$M^{(1)} = (-20x^2 + 84x) \text{ kNm},$$

and for field (2) with $(5 \text{ m}) < x < (7 \text{ m})$ that

$$V^{(2)} = (-40x + 280) \text{ kN},$$

$$M^{(2)} = (-20x^2 + 280x - 980) \text{ kNm},$$

Figure 11.16 shows the V and M diagrams. At A, B and C, the tangents to the M diagram are also shown. These intersect at the middle of each field.

Note that $p_1 = \frac{1}{8} \times 40 \times 5^2 = 125$ kNm and $p_2 = \frac{1}{8} \times 40 \times 2^2 = 20$ kNm, or in other words, for each field: "$p = \frac{1}{8} q\ell^2$".

The bending moment in field (1) is a maximum where the tangent to the M diagram is horizontal, or where

$$\frac{dM}{dx} = V = -40x + 84 = 0 \Rightarrow x = 2.1 \text{ m}.$$

The maximum bending moment therefore occurs to the left of the middle of AB. Substituting $x = 2.1$ in the expression for $M^{(1)}$ gives the value of the maximum bending moment:

$$M_{\max} = -20 \times 2.1^2 + 84 \times 2.1 = 88.2 \text{ kNm}.$$

The support reactions at A and B are shown in Figure 11.17a. Their magnitude and direction can be found directly from the shear force diagram (see Figure 11.17b). This is shown below for the support reaction at B.

Figure 11.17c shows (only) the shear forces directly to the left and right of joint B. The vertical force equilibrium of joint B gives

$$B_v = 116 + 80 = 196 \text{ kN}.$$

The support at B is carrying 116 kN from the left-hand field and 80 kN from the right-hand field. *The support reaction at B is exactly the same magnitude as the "step change" in the shear force diagram.*

The support reactions derived from the shear force diagram can be checked using the equilibrium of the beam as a whole.

With $R = 7 \times 40 = 280$ kN (see Figure 11.18)

$$\sum F_z = 280 - 84 - 196 = 0,$$

$$\sum T_y|A = -280 \times 3.5 + 196 \times 5 = 0.$$

The beam as a whole therefore satisfies force and moment equilibrium.

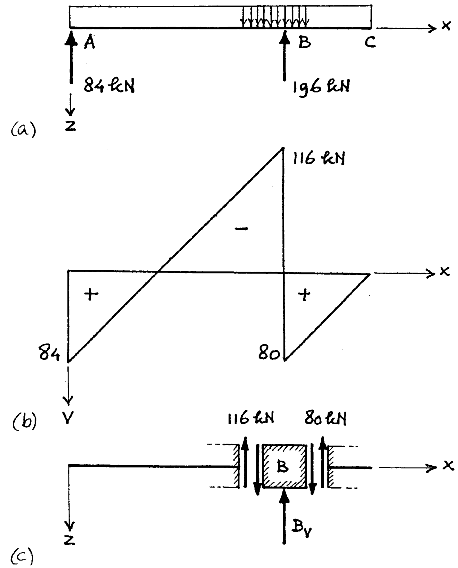


Figure 11.17 (a) The magnitude and direction of the support reactions at A and B follow from (b) the shear force diagram. (c) The shear forces directly to the left and right of joint B. The support reaction at B is the same magnitude as the step change in the shear force diagram.

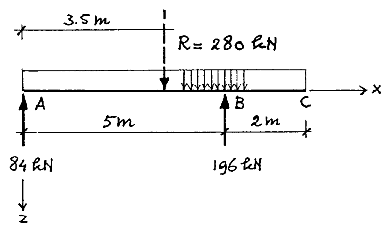


Figure 11.18 The isolated beam with all the forces acting on it.

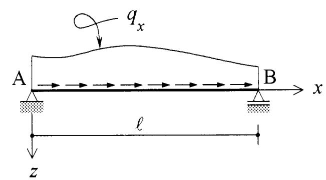
11.4 Problems

Differential equations for the equilibrium (Section 11.1)

11.1 Member AB with length ℓ is subjected to extension by a distributed axial load $q_x = q_x(x)$.

Questions:

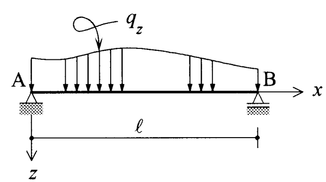
a. Isolate a small segment of length Δx ($\Delta x \to 0$) from the member and draw all the forces acting on it.
b. From the equilibrium of the member segment, derive the relationship between the normal force in the member and the distributed load.



11.2 Beam AB with length ℓ is subjected to bending by a distributed load $q_z = q_z(x)$, normal to the beam axis.

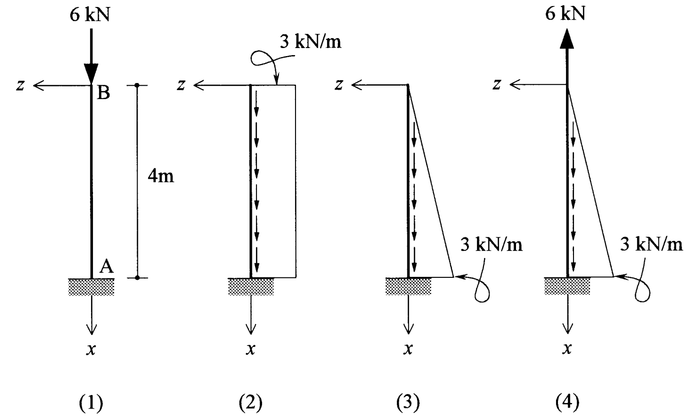
Questions:

a. Isolate a small segment of length Δx ($\Delta x \to 0$) from the member and draw all the forces acting on it.
b. From the equilibrium of the member segment, derive the relationship between the bending moment and the shear force.
c. From the equilibrium of the member segment, derive the relationship between the shear force and the distributed load.
d. From the equilibrium of the member segment, derive the relationship between the bending moment and the distributed load.



Mathematical elaboration of the relationship between N and q_x (extension) (Section 11.2)

11.3: 1–4 A four-metre high column AB is subjected to extension by four different axial loads.



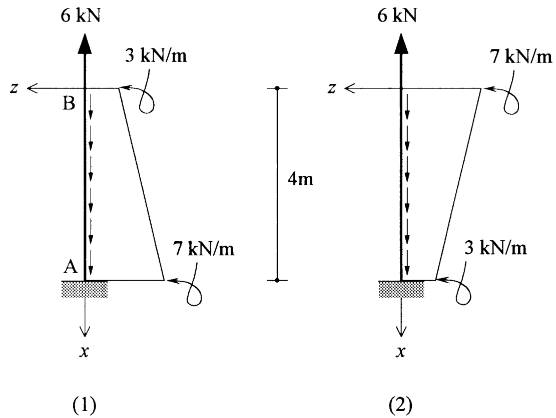
Questions:

a. By integrating the differential equations for the equilibrium, determine the normal force as a function of x, without previously calculating the vertical support reaction at A.
b. Draw the normal force diagram.
c. Calculate the vertical support reaction at A from the equilibrium of the column as a whole and check whether this agrees with the normal force diagram found.

11.4: 1–2 Column AB, 4 m high, is subjected to extension by two different axial loads.

Questions:

a. Write down the distributed load as a function of x.



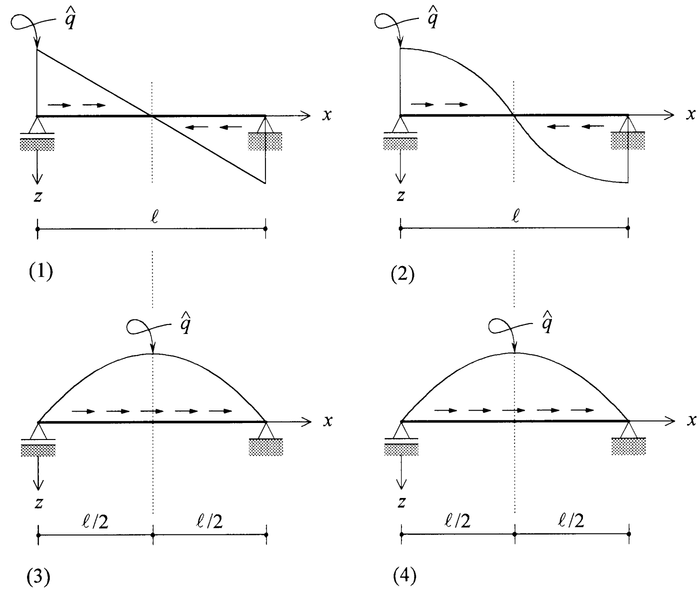
b. Determine the variation of N as a function of x by integration of the differential equation for the equilibrium (without previously calculating the vertical support reaction at A).
c. Draw the N diagram.
d. At which height is the normal force in the column zero?
e. Calculate the vertical support reaction at A from the equilibrium of the column as a whole and check whether this is in agreement with the N diagram found.

11.5: 1–4 A simply supported member with length ℓ is subjected to extension by four different distributed loads $q(x)$ with top value $\hat{q}$:

(1) $q(x) = \hat{q} \cdot \left(1 - 2\dfrac{x}{\ell}\right)$, (2) $q(x) = \hat{q} \cos \dfrac{\pi x}{\ell}$,

(3) $q(x) = 4\hat{q} \cdot \left(\dfrac{x}{\ell} - \dfrac{x^2}{\ell^2}\right)$, (4) $q(x) = \hat{q} \sin \dfrac{\pi x}{\ell}$.

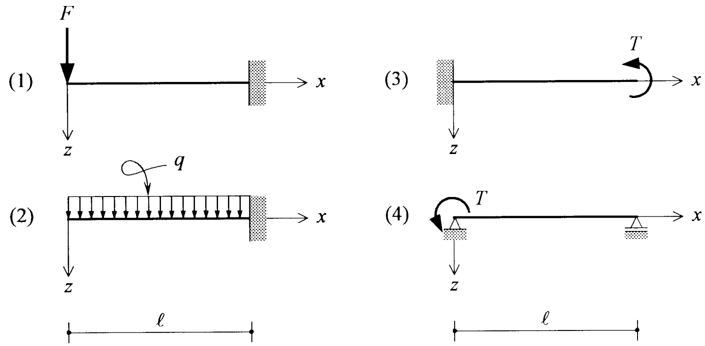
In the calculation use $\ell = 5$ m and $\hat{q} = 2.4$ kN/m.



Questions:
a. Using the differential equation for the equilibrium, determine the variation of N as a function of x.
b. Draw the N diagram. Include the numerical values.
c. Where is N extreme, and what is this extreme value?
d. Determine the support reactions, and draw them as they are really acting on the member.

Mathematical elaboration of the relationship between M, V and q_z (bending) (Section 11.3)

11.6: 1–4 Four beams subjected to bending.



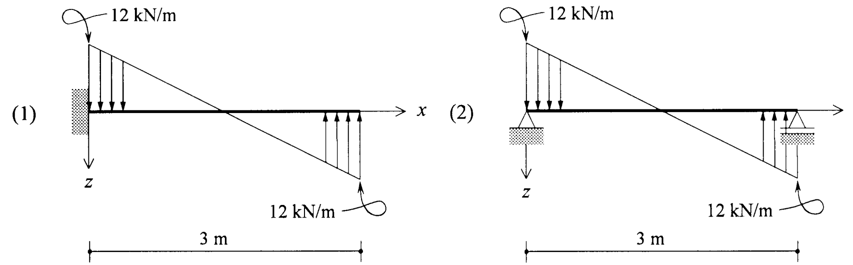
Questions:

a. By integrating the differential equations for the equilibrium, determine the variation of the shear force V and the bending moment M as a function of x, without previously determining the support reactions.
b. Draw the V and M diagrams.
c. Use the V and M diagrams to determine the magnitude and direction of the support reactions. Draw them as they act on the beam and check their values on the basis of the equilibrium of the beam as a whole.

11.7: 1–2 A beam with a linearly distributed load is supported in two different ways.

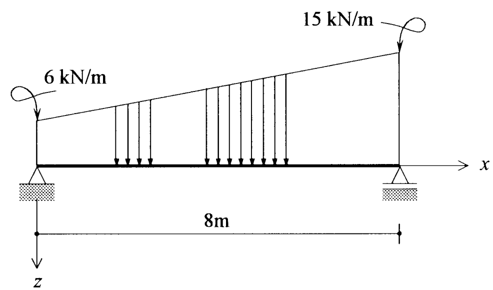
Questions:

a. Write down the distributed load as a function of x.
b. Without previously calculating the support reactions, use the differential equations for the equilibrium to determine the variation of V and M as a function of x.



c. Draw the V and M diagram and include their values and signs.
d. In which cross-sections are V and M extreme, and what are their extreme values?
e. Using the V and M diagrams, determine the magnitude and direction of the support reactions. Draw them as they act on the beam, and check their values on the basis of the equilibrium of the beam as a whole.

11.8 A beam subjected to bending by a trapezoidal load.



Questions:

a. Write down the distributed load as a function of x.
b. Without previously calculating the support reactions, use the differential equations for the equilibrium to determine V and M as a function of x.
c. Draw the V and M diagrams and include their values and signs.

d. In which cross-section is M extreme, and what is its value?
e. Using the V diagram determine the magnitude and direction of the support reactions. Draw them as they act on the beam and check their values on the basis of the equilibrium of the beam as a whole.

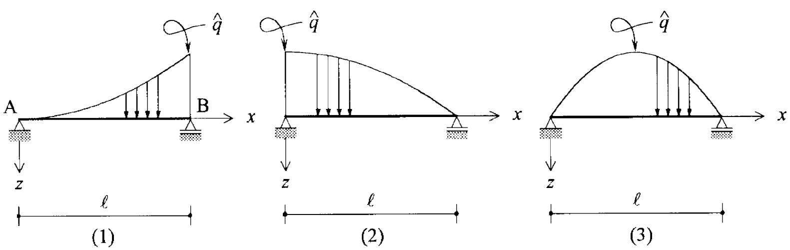
11.9: 1–3 A simply supported beam AB with length ℓ is subjected to bending by three different parabolic distributed loads with the same top value $\hat{q}$:

$$(1)\quad q(x) = \hat{q}\frac{x^2}{\ell^2},$$

$$(2)\quad q(x) = \hat{q}\cdot\left(1 - \frac{x^2}{\ell^2}\right),$$

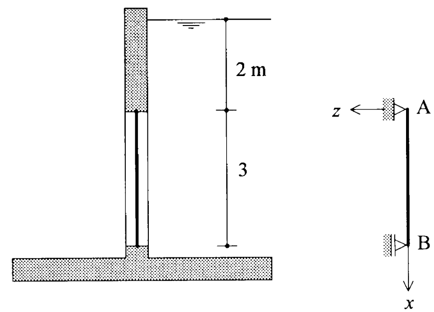
$$(3)\quad q(x) = 4\hat{q}\cdot\left(\frac{x}{\ell} - \frac{x^2}{\ell^2}\right).$$

In the calculation use $\ell = 4$ m and $\hat{q} = 30$ kN/m.



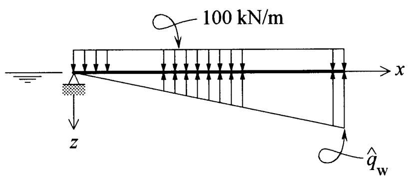
Questions:
a. Determine M and V as a function of x.
b. Draw the M and V diagrams. Include the values and signs.
c. Determine the location and magnitude of the maximum bending moment.
d. Using the V diagram, determine the support reactions at A and B, and draw them as they actually act on the beam.

11.10 An opening in a dam is closed by means of a 3 metre high slide. The top of the slide is two metres below water level. A one metre wide strip from the slide is modelled as the simply supported beam AB. The specific weight of water is 10 kN/m^3.



Questions:
a. Write down the distributed load on AB due to the water pressure as a function of x.
b. Without previously calculating the support reactions, use the differential equations for the equilibrium to determine V and M as a function of x.
c. Draw the V and M diagram and include their values and signs.
d. In which cross-section is M extreme, and what is its value?
e. Using the V diagram, determine the magnitude and direction of the support reactions. Draw them as they act on the beam and check their values on the basis of the equilibrium of beam AB as a whole.

11.11 A 30-m long ship is stranded on rock just below the water level. The figure shows a rough model of the situation. The ship is modelled as a line element with a weight of 100 kN/m. The rock is acting as a hinged support. The upward water pressure on the ship is modelled as line load and varies linearly from zero at the rock to the top value $\hat{q}_w$ at the free-floating end, where the ship is deepest.



Questions:

a. Using the equilibrium of the ship as a whole, determine the value of $\hat{q}_w$.
b. Write down the total distributed load on the ship as a function of x.
c. Use the differential equations for the equilibrium to determine V and M as a function of x.
d. Draw the V and M diagrams, and include the values and signs.
e. In which cross-sections are V and M extreme, and what are their values?
f. Give an assessment of the reality of this model.