Probability and Observation Theory : Old Exams

	23-06-2005	31-08-2005	08-07-2006	30-08-2006	02-07-2007
1	✓		✓		
2	✓	✓	✓	✓	✓
3	✓	✓	✓	✓	✓
4	✓	✓	✓	✓	✓
5	✓	✓	✓	✓	✓
6	✓	✓	✓	✓	✓
7	✓	½	✓	✓	✓
8	✓	✓	✓	✓	✓
9	✓	✓		✓	✓
10	✓	✓	✓	✓	
11					✓
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24					

Exam 23-06-2005

1) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

$P(A) = 5/6$

$P(B) = 3/36$

$P(A \cap B) = 2/36$

$\hookrightarrow P(A \cup B) = \frac{30+3-2}{36} = \frac{31}{36} \rightarrow$ (B)

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2) $P(H|A) = \frac{P(A|H) \cdot P(H)}{P(A)}$ $\rightsquigarrow$ A = warning light on, H = oil pressure too low

$\hookrightarrow P(A|H) = 0{,}99$

$P(H) = 0{,}1$

$P(A) = 0{,}1 \cdot 0{,}99 + 0{,}9 \cdot 0{,}02$

$P(H|A) = \frac{0{,}099}{0{,}117} = 0{,}8462$

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3) $p(\text{not detected within 3 cycles}) < 0{,}001 = p_c^3$

$p_c = \sqrt[3]{0{,}001} = 0{,}1 \rightarrow p = 1 - 0{,}1 = 0{,}9 \rightarrow r = 5$ km

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4) $h_1 \sim N(7000, 150^2)$

$h_2 \sim N(8000, 150^2)$

$dh = (h_2 - h_1) \sim N(1000, \sqrt{(150^2 + 150^2)})$

$\hookrightarrow$ st. dev $= 212{,}13$

$P(dh < 500) = \Phi\left(\frac{500-1000}{212{,}13}\right) = \Phi(-2{,}36) = 0{,}0091$

5) $f_\omega(\omega) = \frac{1}{b} \cdot \exp\left(\frac{\omega_0 - \omega}{b}\right)$ for $\omega \geq \omega_0$

$$\bar{\omega} = \int_{\omega_0}^{\infty} \frac{\omega}{b} \cdot \exp\left(\frac{\omega_0 - \omega}{b}\right) d\omega$$

$\hookrightarrow u = \frac{\omega}{b}$ $\quad v = -b \cdot \exp\left(\frac{\omega_0 - \omega}{b}\right)$

$du = \frac{1}{b}$ $\quad dv = \exp\left(\frac{\omega_0 - \omega}{b}\right)$

$$\bar{\omega} = \left[-\frac{\omega}{b} \cdot b \cdot \exp\left(\frac{\omega_0 - \omega}{b}\right)\right]_{\omega_0}^{\infty} + \int_{\omega_0}^{\infty} \frac{1}{b} \cdot b \cdot \exp\left(\frac{\omega_0 - \omega}{b}\right) d\omega$$

$$= (0 - -\omega_0) + \left[-b \cdot \exp\left(\frac{\omega_0 - \omega}{b}\right)\right]_{\omega_0}^{\infty}$$

$$= \omega_0 + (0 - -b) = \omega_0 + b$$

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6) $\underline{z} \sim N\left(10 \cdot 10, \frac{0{,}50^2}{12} \cdot 10\right)$

$\hookrightarrow$ st. dev $= \sqrt{0{,}20833} = 0{,}456$

$$P(\underline{z} < 99) = \Phi\left(\frac{99 - 100}{0{,}456}\right) = \Phi(-2{,}19) = 0{,}0143$$

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7) positively correllated ; when x_1 is big, the change that $(x_1 + x_2)$ is big has increased

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8) $f_{\underline{x}}(x) = 0{,}2 \cdot \exp(-0{,}2 \cdot x)$

$$\bar{x}_{(x>2)} = \int_2^{\infty} x \cdot \exp(-0{,}2 \cdot x)\, dx$$

$\rightarrow u = x$ $\quad v = -\exp(-0{,}2 \cdot x)$

$du = 1$ $\quad dv = 0{,}2 \cdot \exp(-0{,}2 \cdot x)$

$$\bar{x}_{(x>2)} = \left[-x \cdot \exp(-0{,}2 \cdot x)\right]_2^{\infty} + \int_2^{\infty} \exp(-0{,}2 \cdot x)\, dx$$

$$= (0 - -2 \cdot \exp(-0{,}4)) + \left[-5 \cdot \exp(-0{,}2 \cdot x)\right]_2^{\infty}$$

$$= 2 \cdot \exp(-0{,}4) + 5 \cdot \exp(-0{,}4) = 4{,}69 \text{ months.}$$

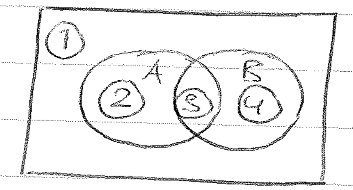
9) a Cauchy distribution

Exam 31-08-2005

1) $P(A) = 1/3$

$P(B) = 1/2$

$P(A \cup B) = 3/4$



$\hookrightarrow$ ② + ③ = 1/3 = 4/12

③ + ④ = 1/2 = 6/12

② + ④ = 3/4 = 9/12

$P(A^c \cup B^c) = P(A^c) + P(B^c) - P(A^c \cap B^c)$

$= (1 - ① - ④) + (1 - ① - ②) - ①$

$P(A \cap B) = P(A) + P(B) - P(A \cup B)$

$= 4/12 + 6/12 - 9/12 = 1/12$

$\hookrightarrow$ ③ = 1/12

② = 3/12

④ = 6/12

① = $1 - \frac{10}{12} = \frac{2}{12}$???

740 - 175 = 565

2825

282.5 + 175 = 457,5

2) $P(H|A_c) = \frac{P(A_c|H) \cdot P(H)}{P(A_c)}$ ⤳ A = light is on, H = oil pressure too low

$\hookrightarrow P(A_c|H) = 0{,}01$
$\hookrightarrow P(H) = 0{,}1$
$\hookrightarrow P(A_c) = 0{,}98 \cdot 0{,}9 + 0{,}1 \cdot 0{,}01$

$\Big\} \; P(H|A_c) = \frac{0{,}001}{0{,}883} = 0{,}00113$

3) Equation 2.31 ; $\bar{y} = g(\bar{x}) + \frac{1}{2} \cdot g''(x) \cdot \sigma_x^2$

$\hookrightarrow E(z) = 3 \cdot (3)^2 - 2 \cdot 5 + \frac{1}{2} \cdot 6 \cdot 2 = 23$

4) $\underline{X}_1 \sim N(2,5)$
$\underline{X}_2 \sim N(5,9)$ $\Big\} \; \underline{Z} = 3 \cdot \underline{X}_1 - 2 \cdot \underline{X}_2 + 1$

$\hookrightarrow z \sim N\left((3 \cdot 2 - 2 \cdot 5 + 1), (3^2 \cdot 5 + (-2)^2 \cdot 9)\right)$
$z \sim N(-3, 81)$

$P(-12 \leq \underline{z} \leq 6) = \Phi\left(\frac{6 - -3}{9}\right) - \Phi\left(\frac{-12 - -3}{9}\right) = \Phi(1) - \Phi(-1)$
$= 0{,}683$

5) $f_{\underline{X}_1}(x_1) = \lambda \cdot \exp(-\lambda x_1)$
$f_{\underline{X}_2}(x_2) = \lambda \cdot \exp(-\lambda x_2)$ $\Big\} \; \underline{Y} = \frac{\underline{X}_1}{\underline{X}_2}$

eq. 2.101 ; $f_{\underline{Y}}(y) = \int_{-\infty}^{\infty} |x_2| \cdot f_{\underline{X}_1 \underline{X}_2}(y \cdot x_2, x_2)\, dx_2$

$\hookrightarrow = \lambda^2 \cdot \exp(-\lambda \cdot y \cdot x_2 - \lambda \cdot x_2)$

$f_{\underline{Y}}(y) = \int_0^{\infty} x_2 \cdot \lambda^2 \cdot \exp(-\lambda y x_2 - \lambda x_2)\, dx_2$

$\hookrightarrow u = x_2 \qquad v = \frac{\lambda^2 \cdot \exp(-\lambda y x_2 - \lambda x_2)}{-\lambda y - \lambda}$

$du = 1 \qquad dv = \lambda^2 \exp(-\lambda y x_2 - \lambda x_2)$

$$f_y(y) = \left[\frac{x_2 \cdot \lambda^2 \cdot \exp(-\lambda y x_2 - \lambda x_2)}{-\lambda y - \lambda} \right]_0^\infty - \int_0^\infty \left(\frac{\lambda^2 \exp(-\lambda y x_2 - \lambda x_2)}{-\lambda y - \lambda} \right) dx_2$$

$$= (0-0) - \left[\frac{\lambda^2 \exp(-\lambda y x_2 - \lambda x_2)}{(-\lambda y - \lambda)^2} \right]_0^\infty$$

$$= -\left[\frac{\lambda^2 \exp(-\lambda y x_2 - \lambda x_2)}{(-\lambda)^2 (y+1)^2} \right]_0^\infty = -\left(0 - \frac{1}{(y+1)^2} \right) = \frac{1}{(y+1)^2}$$

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6) $f_{X_1}(x_1) = 1 \cdot \exp(-x_1)$

$f_{X_2}(x_2) = 3 \cdot \exp(-3 \cdot x_2)$

$f_{X_1,X_2} = 3 \cdot \exp(-x_1 - 3x_2)$

$$F_{X_1,X_2}(x_1, x_2) = \int_0^{x_2}\int_0^{x_1} 3 \cdot \exp(-x_1 - 3x_2)\, dx_1\, dx_2$$

$$= 1 + \exp(-x_1 - 3x_2) - \exp(-3x_2) - \exp(-x_1)$$

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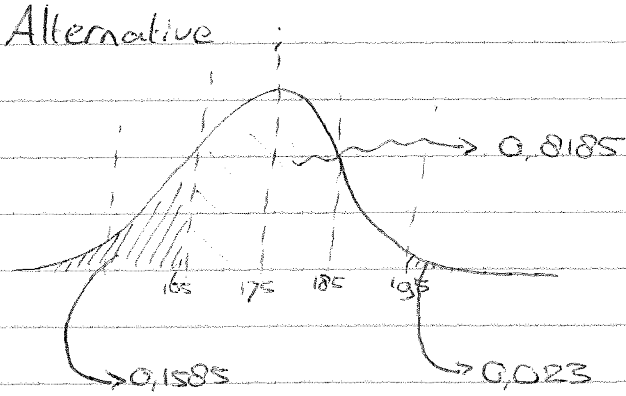
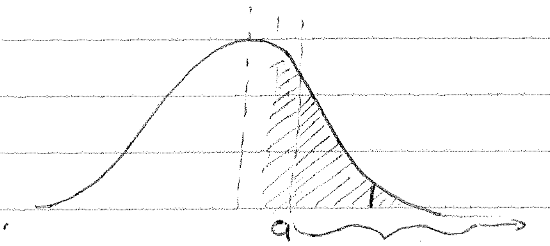
7) Alternative





$\hookrightarrow \frac{0,8185}{2} + 0,023 = 0,43225$

$P(X > a) = 0,43225$

$\hookrightarrow a = 0,17 \longrightarrow \bar{x}_{new} = 175 + 0,17 \cdot 10 = 176,7$

8) eq. 2.91 ; $f_{\underline{y}}(y) = f_{\underline{x}}(g^{-1}(y)) \cdot \left| \frac{dg^{-1}(y)}{dy} \right|$

$f_{\underline{x}} = \frac{1}{2}$, $y = g(x) = 3x^2 + 1$

$g^{-1}(y) = x = \sqrt{\frac{y-1}{3}}$

$$\frac{dg^{-1}(y)}{dy} = \left(\left(\frac{y-1}{3}\right)^{1/2}\right)' = 1/2 \cdot \left(\frac{y-1}{3}\right)^{-1/2} \cdot 1/3$$

$$= 1/6 \cdot \frac{1}{\sqrt{\frac{y-1}{3}}} = \frac{\sqrt{9}}{6} \cdot \frac{1}{\sqrt{9} \cdot \sqrt{\frac{y-1}{3}}}$$

$$= \frac{1}{2 \cdot \sqrt{3(y-1)}}$$

$$f_x(g^{-1}(y)) = 1/2$$

$$\hookrightarrow f_{\underline{y}}(y) = \frac{1}{2} \cdot \frac{1}{2 \cdot \sqrt{3(y-1)}} = \frac{1}{4\sqrt{3(y-1)}}$$

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9) y_2 and y_3

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10) $f_{\underline{x}}(x) = 0{,}4 \cdot \exp(-0{,}4 \cdot x)$

$$\bar{x} = \int_0^\infty x \cdot 0{,}4 \cdot \exp(-0{,}4 \cdot x)\, dx \quad \rightarrow u = x \quad v = -\exp(-0{,}4 x)$$

$$du = 1 \quad dv = 0{,}4 \cdot \exp(-0{,}4 x)$$

$$\bar{x} = \left[-x \cdot \exp(-0{,}4x) \right]_0^\infty + \int_0^\infty \exp(-0{,}4x)\, dx$$

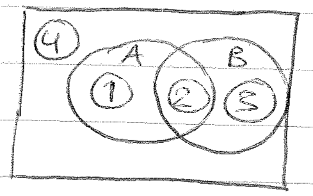
$$= (0 - -0) + \left[-\frac{\exp(-0{,}4x)}{0{,}4} \right]_0^\infty = (0 - - 1/0{,}4) = 2{,}5$$

5 * 2,5 = 12,5 months

[illegible]

[illegible] ;

Exam 02-07-2007



1) $P(A) = 1/3$

$P(B|A^c) = 1/4$

$P(A \cup B) = ?$

~~$P(A|B) = \frac{P(A \cap B)}{P(B)}$~~

$\hookrightarrow P(B|A^c) = \frac{P(B \cap A^c)}{P(A^c)}$

$①+② = 1/3$

$\frac{③}{1-①+②} = 1/4 \longrightarrow ③ = 1/4 \cdot (1-①+②)$

$③ \quad 1/4 \; 2/3 = 2/12 = 1/6$

$①+② = 1/3$

$P(A \cup B) = ①+③$

$③ = 1/6$ $\quad P(A \cup B) = ①+② + ②+③ - ②$

$①+② = 1/3$

$④ = 1/2$

2) 1/4

3) A and B are independent

" " not disjoint

4)

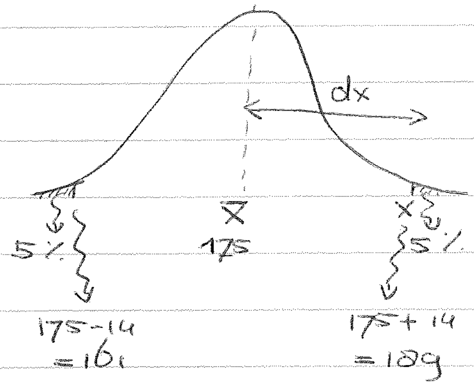
$$P = \frac{\exp(5{,}085 - 0{,}1156 \cdot 31)}{1 + \exp(5{,}085 - 0{,}1156 \cdot 31)} = P_{failure} = 0{,}8178$$

$$P_{no\ failure} = 1 - 0{,}8178 = 0{,}1822$$

$$p(k \geq 1) = 1 - p(k < 1) = 1 - p(0) = 1 - (0{,}1822)^6$$

$$= 0{,}99996$$

5)



$\rightarrow r_\alpha = 1{,}645$

$\hookrightarrow dx = 1{,}645 \cdot \sigma_x = 14$ cm

6) $T \sim N(0{,}79,\ 0{,}1^2)$

$S \sim N(1{,}13,\ 0{,}14^2)$

$$\hookrightarrow (S-T) \sim N(0{,}34,\ 0{,}1^2 + 0{,}14^2)$$

st. dev = 0,172

$$P((S-T) < 0) = \Phi\left(\frac{0 - 0{,}34}{0{,}172}\right) = \Phi(1{,}976) = 0{,}0239$$

7) $y = \begin{bmatrix} 0{,}2 & 0{,}8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$,

↳ A

$$Q_{xx} = \begin{bmatrix} 5{,}0^2 & -0{,}11 \cdot 5{,}0 \cdot 2{,}9 \\ -0{,}11 \cdot 5{,}0 \cdot 2{,}9 & 17{,}6^2 \end{bmatrix} = \begin{bmatrix} 25 & -1{,}595 \\ -1{,}595 & 309{,}76 \end{bmatrix}$$

$$Q_{yy} = \begin{bmatrix} 0{,}2 & 0{,}8 \end{bmatrix} \begin{bmatrix} 25 & -1{,}595 \\ -1{,}595 & 309{,}76 \end{bmatrix} \begin{bmatrix} 0{,}2 \\ 0{,}8 \end{bmatrix}$$

$$= \begin{bmatrix} 3{,}724 & 247{,}489 \end{bmatrix} \begin{bmatrix} 0{,}2 \\ 0{,}8 \end{bmatrix} = 198{,}74$$

↳ st.dev = 14,10

$$\sigma_z^2 = \sigma_{x_1}^2 \cdot \left(\frac{\partial g(\bar{x})}{\partial x_1}\right)^2 + \sigma_{x_2}^2 \cdot \left(\frac{\partial g(\bar{x})}{\partial x_2}\right)^2 + 2 \cdot \rho \cdot \sigma_{x_1} \sigma_{x_2} \cdot \frac{\partial g(\bar{x})}{\partial x_1} \frac{\partial g(\bar{x})}{\partial x_2}$$

$$g(x_1, x_2) = \underline{y} = 0{,}2 \cdot \underline{X}_1 + 0{,}8 \cdot \underline{X}_2$$

$$\frac{\partial y}{\partial x_1} = 0{,}2$$

$$\frac{\partial y}{\partial x_2} = 0{,}8$$

$$\sigma_z^2 = 2{,}9^2 \cdot 0{,}2^2 + 17{,}6^2 \cdot 0{,}8^2 + 2 \cdot (-0{,}11) \cdot 2{,}9 \cdot 17{,}6 \cdot 0{,}2 \cdot 0{,}8$$

$$= 196{,}78$$

$$\hookrightarrow \sigma_z = 14{,}03$$

8) ~~$a_1^2 x_2^2 + \sigma_2^2 x_1^2 + a_1^2 \sigma_2^2 = 0{,}4(0{,}1 \cdot 10)^2 + (0{,}1 \cdot 2)^2 + (0{,}1 \cdot 0{,}1)^2 = 1 + 0{,}04 + 1{,}0 \cdot 10^{-4}$~~

$$z = \pi r^2 \cdot h$$

$$\frac{\partial z}{\partial r} = 2\pi r h \rightarrow \frac{\partial z(\bar{x})}{\partial r} = 2 \cdot \pi \cdot 2 \cdot 10 = 40\pi$$

$$\frac{\partial z}{\partial h} = \pi \cdot r^2 \rightarrow \frac{\partial z(\bar{x})}{\partial h} = \pi \cdot 2^2 = 4\pi$$

$$\sigma_z^2 = \sigma_r^2 \cdot (40\pi)^2 + \sigma_h^2 (4\pi)^2 = 0{,}1^2 (40^2\pi^2 + 4^2\pi^2)$$

$\sigma_z^2 = 16{,}16 \cdot \pi^2 \rightarrow \sigma_z = \sqrt{16{,}16} \cdot \pi = 12{,}63 \text{ cm}^2$

9) ~~[illegible]~~ A = ~~[illegible]~~ diagnosis is positive
B = person suffers from decease

$P(B) = $ ~~0,1~~ $0{,}005$
$P(A^c|B^c) = 0{,}8$
$P(A|B) = 0{,}98$

$\hookrightarrow P(B^c|A) = \dfrac{P(A|B^c) \cdot P(B^c)}{P(A)}$

$P(A|B^c) = 0{,}2$
$P(B^c) = $ ~~0,5~~ $0{,}995$
$P(A) = $ ~~0,2·0,1 + 0,5·0,98~~
~~0,2·0,5 + 0,5·0,98~~
$0{,}2 \cdot 0{,}995 + 0{,}005 \cdot 0{,}98$

$\} \; P(B^c|A) = $ ~~0,695~~ $0{,}976$

10) X

11) $f_X(x) = 0{,}2 \cdot \exp(-0{,}2 \cdot x) \rightarrow P(A|B) = \dfrac{P(A \cap B)}{P(B)}$

$P(A \cap B) = \int_3^5 0{,}2 \cdot \exp(-0{,}2 \cdot x)\,dx$

$= \left[-\exp(-0{,}2\,x)\right]_3^5 = 0{,}1809$

$P(B) = \left[-\exp(-0{,}2\,x)\right]_3^6 = 0{,}2476$

$\} \; \dfrac{0{,}1809}{0{,}2476} = 0{,}731$

12) $f_{\underline{x}}(x) = 1 \quad , \underline{y} = -\ln \underline{x}$

$$g^{-1}(y) = \exp(-y) = \frac{1}{\exp(y)} = e^{-y}$$

$$\frac{\partial g^{-1}}{\partial y} = -e^{-y}$$

$\hookrightarrow f_{\underline{y}}(y) = 1 \cdot e^{-y}$

13)

$E(\underline{\omega}) = (3/2)\pi \quad \rightarrow \sin(E(\underline{\omega})) = -1$

$E(\sin(\underline{\omega}))$

$f_{\underline{\omega}}(\underline{\omega}) = \frac{1}{\pi} \quad , \underline{y} = \sin(\underline{\omega})$

$$\underline{\omega} = \arcsin(\underline{y}) = g^{-1}(y)$$

$$\frac{\partial \omega}{\partial y} = \frac{1}{\sqrt{1-y^2}}$$

$$f_{\underline{y}}(y) = \frac{1}{\pi} \cdot \left\{ \frac{1}{\sqrt{1-y^2}} \right\} = \frac{1}{\pi \sqrt{1-y^2}}$$

$$E(f_y(y)) = \frac{1}{\pi} \int_{-\infty}^{\infty} y \cdot (1-y^2)^{-1/2} \, dy = \frac{1}{\pi} \left[-\tfrac{1}{2} \cdot (1-y^2)^{1/2} \right]_{-1}^{1}$$

$=$

14) $\sigma_{\hat{x}}^2 = \dfrac{\sigma_x^2}{n}$

16)

$$Q_{\hat{x}\hat{x}} = (A^T Q_{yy}^{-1} A)^{-1}$$

↳ BLUE : $= \sigma^2 \cdot I_{33}$

$$Q_{yy}^{-1} = \begin{bmatrix} 1/\sigma^2 & 0 & 0 \\ 0 & 1/\sigma^2 & 0 \\ 0 & 0 & 1/\sigma^2 \end{bmatrix}$$

$$A^T \cdot Q^{-1} = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1/\sigma^2 & 0 & 0 \\ 0 & 1/\sigma^2 & 0 \\ 0 & 0 & 1/\sigma^2 \end{bmatrix} = \begin{bmatrix} 1/\sigma^2 & -1/\sigma^2 & 2/\sigma^2 \\ 0 & 2/\sigma^2 & 1/\sigma^2 \end{bmatrix}$$

$$A^T Q^{-1} A = \begin{bmatrix} 1/\sigma^2 & -1/\sigma^2 & 2/\sigma^2 \\ 0 & 2/\sigma^2 & 1/\sigma^2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 6/\sigma^2 & 0 \\ 0 & 5/\sigma^2 \end{bmatrix}$$

$$[A^T Q^{-1} A]^{-1} = \frac{\sigma^4}{30} \begin{bmatrix} 5/\sigma^2 & 0 \\ 0 & 6/\sigma^2 \end{bmatrix} = \begin{bmatrix} 1/6 \cdot \sigma^2 & 0 \\ 0 & 1/5 \cdot \sigma^2 \end{bmatrix}$$

$$\sigma_{\hat{x}_1}^2 = 1/6 \cdot \sigma^2 = 1 \rightarrow \sigma = \sqrt{6}$$

18) $y = A + Bt + Cz \rightarrow$

$$3 = 1 \cdot A + 1 \cdot B + 1 \cdot C$$
$$6 = 1 \cdot A + 0 \cdot B + 3 \cdot C$$
$$5 = 1 \cdot A + 2 \cdot B + 1 \cdot C$$
$$0 = 1 \cdot A + 0 \cdot B + 0 C$$

$$A^T \cdot A \hat{x} = A^T y$$

$$A^T A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 2 & 0 \\ 1 & 3 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 3 \\ 1 & 2 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 3 & 5 \\ 3 & 5 & 3 \\ 5 & 3 & 11 \end{bmatrix}$$

$$A^T y = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 2 & 0 \\ 1 & 3 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \\ 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 14 \\ 13 \\ 26 \end{bmatrix}$$

19) $x = x_0 + v_0 \cdot t$

$$\begin{bmatrix} 12,6 \\ 9,1 \\ 8,3 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_0 \\ v_0 \end{bmatrix}$$

$\hat{x} = (A^T \cdot A)^{-1} \cdot A^T y$ (eq. 3.12)

$$\hookrightarrow A^T \cdot A = \begin{bmatrix} 1 & 1 & 1 \\ -3 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 14 \end{bmatrix}$$

$$(A^T A)^{-1} = \begin{bmatrix} 14/42 & 0 \\ 0 & 3/42 \end{bmatrix} = \begin{bmatrix} 1/3 & 0 \\ 0 & 1/14 \end{bmatrix}$$

$$A^T y = \begin{bmatrix} 1 & 1 & 1 \\ -3 & 1 & 2 \end{bmatrix} \begin{bmatrix} 12,6 \\ 9,1 \\ 8,3 \end{bmatrix} = \begin{bmatrix} 30 \\ -12,1 \end{bmatrix}$$

$$\begin{bmatrix} x_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} 1/3 & 0 \\ 0 & 1/14 \end{bmatrix} \begin{bmatrix} 30 \\ -12,1 \end{bmatrix} = \begin{bmatrix} 10 \\ -12,1/14 \end{bmatrix}$$

$x = 10 - (12,1/14) \cdot t = 0$

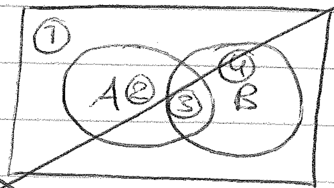
$\hookrightarrow t = \left(\frac{10 \cdot 14}{12,1}\right) = 11,57$ days

Exam 30-08-06

$P(A) = 0,6$

$P(A \cup B) = 0,8$

$P(A|B) = 0,5$



② ③ ② + ③ = 0,6

② + ④ = 0,8

$\frac{③}{③ + ④} = 0,5 \rightarrow ③ = 0,5(③ + ④)$

~~(b) = 0,5 (3) + 0,5 (4) → (3) = 1/3 (4)~~

1) $P(A) = 0{,}6$
$P(A \cup B) = 0{,}8$
$P(A|B) = 0{,}5$
} $P(B)$?

$$P(A|B) = \frac{P(A\cap B)}{P(B)} \rightarrow P(B) = \frac{P(A\cap B)}{P(A|B)}$$

$$P(A\cap B) = P(A) + P(B) - P(A\cup B)$$

$$P(B) = \frac{P(A) - P(A\cup B)}{P(A|B)} + \frac{P(B)}{P(A|B)}$$

$$P(B)\left(1 - \frac{1}{P(A|B)}\right) = \frac{P(A) - P(A\cup B)}{P(A|B)}$$

$$P(B) = \frac{\frac{P(A) - P(A\cup B)}{P(A|B)}}{1 - \frac{1}{P(A|B)}} = \frac{\frac{0{,}6 - 0{,}8}{0{,}5}}{1 - \frac{1}{0{,}8}} = \frac{-0{,}4}{-0{,}25} = \times$$

---

2) A = type A
B = type B
H = lasts > 3 months
} $P(A) = 1/3$
$P(B) = 2/3$
$P(H|A) = 3/5$
$P(H|B) = 4/5$

$$P(A|H) = \frac{P(H|A)\cdot P(A)}{P(H)} = \frac{3/5 \cdot 1/3}{3/5 \cdot 1/3 + 4/5 \cdot 2/3} = \frac{3/15}{11/15} = 3/11$$

---

3) $\exp(-\lambda)\cdot\frac{\lambda^k}{k!}$ $\quad P(k > 4) = 1 - P(k \leq 4) = 0{,}1847$

$$P(k\leq 4) = \exp(-3)\cdot\left(\frac{3^0}{1} + \frac{3^1}{1} + \frac{3^2}{2} + \frac{3^3}{6} + \frac{3^4}{24}\right) = 0{,}8153$$

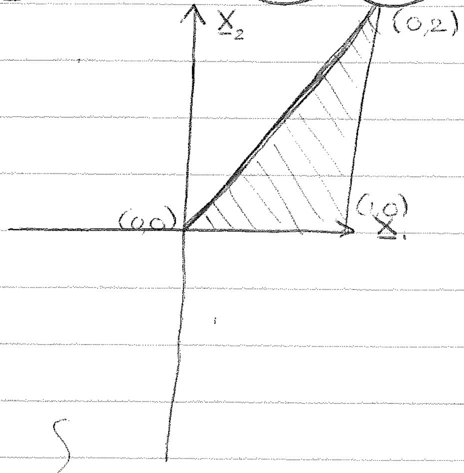
↳ 16,375

4) mean $= (9/5) \cdot 10 + 32 = 50$

variance $= (9/5)^2 \cdot 4 = 12{,}96 \rightarrow$ st.dev $= 3{,}6$

5) $F_x = \int_{-\infty}^{x} \left(-3/4 \cdot x^2 + 3/2 \cdot x\right) dx = \left[-1/4 \cdot x^3 + 3/4\, x^2 \right]_0^2$

6) $\underline{X}_1 < 1/2 \cdot \underline{X}_2$



$\hookrightarrow 0 \leq x_1 \leq 1/2 \cdot x_2$

$0 \leq x_2 \leq 2$

$\hookrightarrow \int_0^2 \int_0^{1/2 \cdot x_2} 1/10 \left(3 \cdot x_1^2 + 8 \cdot x_1 \cdot x_2\right) dx_1\, dx_2$

$\hookrightarrow \left[1/10 \left(x_1^3 + 4 \cdot x_1^2 \cdot x_2 \right) \right]_0^{1/2 \cdot x_2}$

$= 1/10 \cdot \left(1/8 \cdot x_2^3 + 4 \cdot 1/4 \cdot x_2^3 \right) = (9/80)\, x_2^3$

$\int_0^2 (9/80)\, x_2^3\, dx_2 = \left[(9/320) \cdot x_2^4 \right]_0^2 = \frac{144}{320} = 9/20$

7) $N \sim (20{,}0,\ 0{,}01^2)$

$\rightsquigarrow \Phi(1) - \Phi(-2) = 0{,}8185 \rightarrow 1 - 0{,}8185 = 0{,}1815$

8) $P(A|B) = \frac{P(A \cap B)}{P(B)}$

$f_{\underline{x}}(x) = 0{,}2 \cdot \exp(-0{,}2 \cdot x)$

$P(5 \leq x \leq 6) = \int_5^6 0{,}2 \cdot \exp(-0{,}2x)\,dx = [-\exp(-0{,}2x)]_5^6$

$= 0{,}06669$

$P(3 < x < 6) = [-\exp(-0{,}2 \cdot x)]_3^6 = 0{,}2476$

$\hookrightarrow \frac{0{,}06669}{0{,}2476} = 0{,}269$

---

9) X

---

10) $f_x = 0{,}5 \cdot \exp(-0{,}5 \cdot x)$

$P(x > 15) = \int_{15}^{\infty} 0{,}5 \cdot \exp(-0{,}5 \cdot x)\,dx = [-\exp(-0{,}5\,x)]_{15}^{\infty}$

$= 0 + 5{,}53 \cdot 10^{-4}$

$P(x < 15) = 1 - 5{,}53 \cdot 10^{-4}$

$(1 - 5{,}53 \cdot 10^{-4})^{365} = 0{,}8172 \rightarrow P(\text{flood}) = 1 - 0{,}81724$

$= 0{,}183$

---

12) $Q_{\hat{x}\hat{x}} = [A^T \cdot Q_{yy}^{-1} \cdot A]^{-1}$

$\hookrightarrow$ solving gives ; $\sigma_{\hat{x}_2}^2 = 1 = \left(\frac{3}{36}\right)\sigma^2 \rightarrow \sigma = \sqrt{12} = 3{,}46$

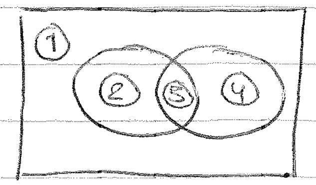
---

16 $A = a \cdot b \rightarrow$ eq. 2.67 ;

$\sigma_z^2 = \sigma_a^2 \cdot b^2 + \sigma_b^2 \cdot a^2 = 1^2 \cdot 500^2 + 1^2 \cdot 1000^2 = 1118\ cm^2$

Exam 03-07-2006

1)



$(2)+(3) = P(A) = 0{,}6$
$(3)+(4) = P(B) = 0{,}3$
$(3) = P(A \cap B) = 0{,}2$

$(4) = 0{,}3 - (3) = 0{,}1$
$(2) = 0{,}6 - (3) = 0{,}4$
$(3) = 0{,}2$

$\Big\}\ (1) = 1 - 0{,}2 - 0{,}4 - 0{,}1 = 0{,}3$

$P(A^c \cap B^c) = 1 - (P(A \cup B) + P(A \cap B))$

$\hookrightarrow = 1 - (P(A) + P(B) - 2P(A \cap B) + P(A \cap B))$
$= 1 - (0{,}6 + 0{,}3 - 2 \cdot 0{,}2 + 0{,}2)$
$= 1 - 0{,}7 = 0{,}3$

---

2) ~~P(A)=~~ A = type A $\quad P(A) = 1/3$
B = type B $\quad P(B) = 2/3$
H = lasts > 3 months $\quad P(H|A) = 3/5$
$P(H|B) = 4/5$

$P(H) = 1/3 \cdot 3/5 + 2/3 \cdot 4/5 = (3+8)/15 = 11/15$

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3) $P(\underline{X} = 3) = (0{,}95)^2 \cdot 0{,}05 = 0{,}045$

---

4) $f_{\underline{Y}}(y) = f_{\underline{X}}(g^{-1}(y)) \cdot \left| \dfrac{dg^{-1}(y)}{dy} \right|$

$\hookrightarrow g^{-1}(y) = \ln(y) (= x)$

$\dfrac{dg^{-1}(y)}{dy} = \dfrac{1}{y}$

$$f_{\underline{y}}(y) = \frac{1}{\sqrt{2\pi}\cdot\sigma\cdot y}\cdot \exp\left(-\frac{1}{2}\left(\frac{\ln(y)-\bar{x}}{\sigma}\right)^2\right)$$

5) $f_{\underline{x}}(x) = (3/2)x - (3/4)x^2$

$\hookrightarrow g(x) = y = \sqrt{x} \rightarrow g^{-1}(y) = y^2 = x$

$f_{\underline{y}}$ $\qquad \frac{dg^{-1}}{dy} = 2y$

$\hookrightarrow f_{\underline{y}}(y) = \left((3/2)y^2 - (3/4)y^4\right)\cdot 2y$

$= 3y^3 - (3/2)y^5 \rightarrow 0 \leq y \leq \sqrt{2}$

6) $f_{\underline{x}_1,\underline{x}_2}(x_1,x_2) = 1/10\cdot(3x_1^2 + 8\cdot x_1\cdot x_2)$

$\hookrightarrow$ page 37 ; $f_{\underline{x}_1}(x_1) = \int_0^2 \left(1/10\,(3\cdot x_1^2 + 8\cdot x_1 x_2)\right) dx_2$

$= \left[1/10\,(3x_1^2\cdot x_2 + 4\cdot x_1\cdot x_2^2)\right] = \left[(3/10)\cdot x_1^2\cdot x_2 + (2/5)\cdot x_1\cdot x_2^2\right]_0^2$

$= (3/5)\cdot x_1^2 + (8/5)\cdot x_1 \rightarrow 0 \leq x_1 \leq 1$

7) $\underline{x} \sim N(10, 0{,}01^2)$

$1 - 0{,}997 = 0{,}003 = 0{,}3\,\%$

8) $P(|\underline{x} - \bar{x}| \geq \epsilon) \leq \frac{\sigma_x^2}{\epsilon^2} = 0{,}1$

$\sigma_x^2 = \frac{\sigma_n^2}{n}$

$\frac{3/n}{0{,}5^2} = 0{,}1 \qquad \frac{3}{n} = 0{,}025$

$\hookrightarrow n = \frac{3}{0{,}025} = 120$

9) x

10)

17) 0,5

18) $P_A \cdot Q_{yy} \cdot P_A^T$

$\hookrightarrow P_A = A(A^T W A)^{-1} \cdot A^T \cdot W$

$\hookrightarrow W = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$\hookrightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 0 \end{bmatrix}$

$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 6 & 14 \end{bmatrix}$

$\frac{1}{20} \hookrightarrow \begin{bmatrix} 7/10 & -3/10 \\ -3/10 & 1/5 \end{bmatrix}$

4) $P(A) = 1/3 \rightarrow P(A^c) = 2/3$

$P(B|A^c) = 1/4$ ~~$\rightarrow P(A^c \cap B)$~~

$P(A \cup B) = ?$

$P(B \cap A^c) = P(B|A^c) \cdot P(A^c) = 1/4 \cdot 2/3 = 1/6$

$P(A \cup B) = P(B|A^c) + P(A|B^c)$

PDF Transformation Rule :

~~$f_y(y) = f_x(g^{-1}(y)) \cdot \left| \frac{dg^{-1}(y)}{dy} \right|$~~

2-21 $Z_1 = 2 \cdot X_1 + X_2$ $\rightarrow \det(\partial_{x^T} G(x)) = \det \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} = -2 - 1$

$Z_2 = X_1 - X_2$

$= -3$

$f_y(y) = f_x(x) \cdot |\det[\partial_{x^T} \cdot G(x)]|^{-1}$

~~$f_y(x) = f_{x_1, x_2}$~~

$f_{Z_1 Z_2}(z_1, z_2) = (1/3) \cdot f_{X_1 X_2}($

$Z_1 = 2 \cdot X_1 + X_2 \rightarrow Z_1 = 2 \cdot X_1 + X_1 - Z_2 \rightarrow \boxed{X_1 = \frac{Z_1 + Z_2}{3}}$

$Z_2 = X_1 - X_2 \rightarrow X_2 = X_1 - Z_2$

$\hookrightarrow X_1 = Z_2 + X_2 \rightarrow Z_1 = 2 \cdot Z_2 + 3 \cdot X_2$

$X_2 = \boxed{\frac{Z_1 - 2 \cdot Z_2}{3}}$

$f_{Z_1 Z_2} = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{4}} \cdot \frac{1}{\sqrt{4}} \cdot \frac{1}{3} \cdot \exp\left(-\frac{(Z_1 + Z_2)^2}{72} - \frac{(Z_1 - 2Z_2)^2}{72} \right)$

$\frac{-(Z_1^2 + 2 Z_1 Z_2 + Z_2^2) - (Z_1^2 - 4 \cdot Z_1 Z_2 + 4 \cdot Z_2^2)}{}$

$\frac{-2 \cdot Z_1^2 + 2 \cdot Z_1 \cdot Z_2 - 5 Z_2^2}{72}$

10) $Z_1 = X_1 + X_2 \rightarrow X_1 = Z_1 - X_2 \quad , \quad X_2 = Z_1 - X_1$

(23-06-05) $Z_2 = X_1 - 2\cdot X_2 \rightarrow Z_2 = Z_1 - 3X_2$

$$X_2 = \frac{Z_1 - Z_2}{3}$$

$$Z_2 = 3X_1 - 2\cdot Z_1$$

$$X_1 = \frac{2\cdot Z_1 + Z_2}{3}$$

$\hookrightarrow \det \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} = -3$

$\hookrightarrow f_{Z_1 Z_2} = \left(\frac{1}{3}\right)\cdot f_{X_1 X_2}\left(\frac{2\cdot Z_1 + Z_2}{3}, \frac{Z_1 - Z_2}{3}\right)$

$$= \frac{1}{3}\cdot\frac{1}{2\pi}\cdot\frac{1}{2}\cdot\frac{1}{\sqrt{1}}\cdot \exp\left(-\frac{1}{2}\left(\frac{\frac{2\cdot Z_1 + Z_2}{3}}{2}\right)^2 - \frac{1}{2}\left(\frac{Z_1 - Z_2}{3}\right)^2\right)$$

$\hookrightarrow (2\cdot Z_1 + Z_2)^2 = (4\cdot Z_1^2 + 4\cdot Z_1\cdot Z_2 + Z_2^2) * \left(-\frac{1}{72}\right)$

$\hookrightarrow (Z_1 - Z_2)^2 = (Z_1^2 - 2\cdot Z_1\cdot Z_2 + Z_2^2) * \left(-\frac{1}{18}\right)$

~~$\hookrightarrow \exp\left(-\frac{4}{36} Z_1^2 - \frac{4}{36} Z_1 Z_2 - \frac{1}{36} Z_2^2 - \frac{1}{18} Z_1^2 + \frac{1}{18} Z_1 Z_2 - \frac{1}{18} Z_2^2\right)$~~

~~$\exp\left(-\frac{3}{18} Z_1^2 - \frac{3}{36} Z_2^2\right)$~~

$\frac{1}{12\pi}\cdot \exp\left(\left(-\frac{4}{72} - \frac{1}{18}\right) Z_1^2 + \left(-\frac{4}{72} + \frac{2}{18}\right) Z_1 Z_2 + \left(-\frac{1}{72} - \frac{1}{18}\right) Z_2^2\right)$

$\hookrightarrow \left(\left(-\frac{1}{9}\right) Z_1^2 + \left(\frac{1}{18}\right)\cdot Z_1 Z_2 + \left(-\frac{5}{72}\right)\cdot Z_2^2\right.$

B

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 3 \\ 2 & 1 \end{bmatrix}$$

$$Q_{\hat{y}\hat{y}} = A \cdot \left(A^T Q_{yy}^{-1} \cdot A \right)^{-1} \cdot A^T$$

$$\begin{bmatrix} 1/\sigma^2 & 1/\sigma^2 & 2/\sigma^2 \\ 1/\sigma^2 & 3/\sigma^2 & 1/\sigma^2 \end{bmatrix} \begin{bmatrix} 1/\sigma^2 & 0 & 0 \\ 0 & 1/\sigma^2 & 0 \\ 0 & 0 & 1/\sigma^2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 6/\sigma^2 & 6/\sigma^2 \\ 6/\sigma^2 & 11/\sigma^2 \end{bmatrix}$$

$$\hookrightarrow [\;\;]^{-1} = \begin{bmatrix} 11/\sigma^2 & -6/\sigma^2 \\ -6/\sigma^2 & 6/\sigma^2 \end{bmatrix} * \frac{\sigma^4}{66 - 36} = \begin{bmatrix} \sigma^2 (11/30) & -\sigma^2 (1/5) \\ -\sigma^2 (1/5) & \sigma^2 (1/5) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 11/30\,\sigma^2 & -6/30\,\sigma^2 \\ -6/30\,\sigma^2 & 6/30\,\sigma^2 \end{bmatrix} = \begin{bmatrix} 1/6\,\sigma^2 & 0 \\ -7/30\,\sigma^2 & 12/30\,\sigma^2 \\ 12/30\,\sigma^2 & -6/30\,\sigma^2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} (1/6\,\sigma^2) & \cdots & \cdots \\ \cdots & \ddots & \cdots \\ \cdots & \cdots & (18/30)\sigma^2 \end{bmatrix}$$

$(1/6\,\sigma^2) \to \sigma_{\hat{y}_1}$; $(18/30)\sigma^2 \to \sigma_{\hat{y}_3}$

14) $\sigma_{\bar{x}}^2 = \frac{\sigma_x^2}{n} \leadsto 0{,}1 = \frac{2}{n} \to n = 20$

$$\begin{bmatrix} h_{12} \\ h_{23} \\ h_{34} \\ h_{41} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} h_2 \\ h_3 \\ h_4 \end{bmatrix}$$

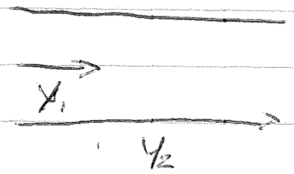
$h_{12} = h_2$

$h_{23} = h_3 - h_2$

$h_{34} = h_4 - h_3$

$h_{41} = -h_4$

$\underline{y} = A\underline{x}$



$$\hookrightarrow \begin{bmatrix} AB \\ BC \\ AC \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$Q_{\hat{y}\hat{y}} = (A^T Q_{yy}^{-1} A)^{-1}$$

$$= \begin{bmatrix} 1/3 & -1/3 & 0 \\ 0 & 1/3 & 1/3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{bmatrix}^{-1}$$

$$[\ ]^{-1} = \frac{1}{4/9 - 1/9} \quad \frac{9}{3} = 3 \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & & \\ & 2 & \\ & & 2 \end{bmatrix}$$

$$\hookrightarrow \sigma_{\hat{y}_3\, AC = y_3} = 2$$

$$\begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \\ h_5 \\ h_6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ & & & & \\ & & & & \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \\ h_5 \end{bmatrix}$$

1(b)

$$\begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} h_{12} \\ h_{23} \end{bmatrix}$$

$$\begin{bmatrix} 1/2 & 0 & -1/2 \\ 0 & 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1/2 \\ -1/2 & 1 \end{bmatrix}$$

~~$h_{12} = h_{12}$~~

~~$h_{23} = h_{23}$~~

$h_{31} =$ ~~$h_{12} - h_{23}$~~

$$\hookrightarrow \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix} \left(* \frac{1}{1 - 1/4} = * \left(\frac{4}{3}\right) \right)$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 4/3 & 2/3 \\ 2/3 & 2/3 \end{bmatrix} = \begin{bmatrix} 4/3 & 2/3 \\ 2/3 & 2/3 \\ -2/3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 4/3 & 2/3 \\ & \end{bmatrix}$$

$$\sigma^2_{\hat{h}_1} = 4/3$$

$h_2 + h_4 = h_6$
$h_4 = h_6 - h_2$

1(a)

$$\begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \\ h_5 \\ h_6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} h_{51} \\ h_{52} \\ h_{53} \\ h_{24} \\ h_{34} \end{bmatrix}$$

$$\begin{bmatrix} 1/2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & -1/2 & 0 & 1/2 \\ 0 & 0 & 1/2 & 0 & -1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1/2 & 1/2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 & 0 & 0 & 0 & 0 \\ 0 & 3/2 & 0 & 1/2 & -1/2 \\ 0 & 0 & 1 & 0 & -1/2 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & -1/2 & -1/2 & 0 & 1 \end{bmatrix}$$

9) $\underline{v} = \sqrt{v_x^2 + v_y^2 + v_z^2}$, $\sigma^2 = \frac{kT}{m}$

2.25 ; $\underline{u}_i = \frac{x_i}{\sigma}$ with $u_i \sim N(0,1)$

$$\underline{z} = \underline{x}_1^2 + \underline{x}_2^2 + \underline{x}_3^2 = \sigma^2(\underline{u}_1^2 + \underline{u}_2^2 + \underline{u}_3^2) = \sigma^2 \cdot \underline{y}$$

$$\underline{y} \sim \chi^2(3,0) \quad , \quad f_{\underline{y}}(y) = \frac{1}{\sqrt{2\pi}} \cdot \sqrt{y} \cdot \exp\left(-\frac{y}{2}\right)$$

$$\underline{z} = \sigma^2 \cdot y \longrightarrow y = \frac{1}{\sigma^2} \cdot z$$

$$\left|\frac{dy}{dz}\right| = \left|\frac{1}{\sigma^2}\right|$$

$$\hookrightarrow f_{\underline{z}}(z) = \frac{1}{\sigma^2} \cdot f_{\underline{y}}\left(\frac{z}{\sigma^2}\right)$$

$$\underline{u}_i = \frac{x_i}{\sigma}$$

$$z = \sqrt{x_1^2 + x_2^2 + x_3^2} = \sqrt{\sigma^2(u_1^2 + u_2^2 + u_3^2)} = \sigma \cdot \sqrt{y}$$

$$\hookrightarrow y = \chi^2(3,0) \quad , \quad f_{\underline{y}}(y) = \frac{1}{\sqrt{2\pi}} \cdot \sqrt{y} \cdot \exp\left(-\frac{y}{2}\right)$$

$$z = \sigma\, y^{1/2} \rightarrow (z)^{-1} = y = \left(\frac{z^2}{\sigma^2}\right) \longrightarrow \frac{d(z)^{-1}}{dz} = \frac{2 \cdot z}{\sigma^2}$$

$$f_{\underline{v}} = \frac{2 \cdot z}{\sigma^2} \cdot \frac{1}{\sqrt{2\pi}} \sqrt{\frac{z^2}{\sigma^2}} \cdot \exp\left(-\frac{z^2/\sigma^2}{2}\right)$$

$$= \frac{2 \cdot v \cdot m}{kt} \cdot \frac{1}{\sqrt{2\pi}} \cdot \frac{v\sqrt{m}}{\sqrt{kt}} \cdot \exp\left(-\frac{v^2 \cdot m}{kt}\right)$$

$$= \left(\frac{m}{2\pi kt}\right)^{3/2} \cdot 4\pi v^2 \cdot \exp\left(-\frac{mv^2}{2kT}\right)$$